INVERSE MAPPING THEOREM AND LOCAL FORMS OF CONTINUOUS MAPPINGS

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Abstract: We present a homological version of the Inverse Mapping Theorem for open and discrete continuous maps between oriented topological manifolds, with assumptions on the degree of the maps, but without any assumption on differentiability. We prove that this theorem is equivalent to the known homological version of the Implicit Mapping Theorem. Additionally, we study conditions for a map between oriented topological manifolds to be locally like an injection or a projection, via alternative notions of topological immersions and submersions.

Key words: Local degree, inverse mapping theorem, implicit mapping theorem, local immersion theorem, local submersion theorem.

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1 Introduction

In analysis, one of the major results is the Inverse Mapping Theorem, which gives conditions under which a differentiable map, between two differential manifolds of the same dimension, has a local differentiable inverse. Such Theorem is equivalent to the Implicit Mapping Theorem, which provides the theoretical underpinning for the technique of implicit differentiation. Generalizations of these theorems, in special of the second one, have been proposed in several approaches.

In [2], C. Biasi and E. L. dos Santos present a homological version of the Implicit Mapping Theorem for (not necessarily differentiable) continuous maps between well-behaved topological spaces. In such a context, the authors prove that, under certain assumption, such a map defines an implicit map on a neighborhood of a given point and that this map is continuous at the point, but they do not prove the local continuity of the implicit map. Skirting this trouble, in [1], the same authors, in collaboration with C. Gutierrez, preset a more complete version of the Implicit Mapping Theorem, for continuous maps, whose proof is grounded on the articles [4] of P. T. Church and [10] of J. Väisälä. Next we state that theorem using the terminology and the notations which are explained in Section 2.

Theorem 1.1. (Implicit Mapping Theorem – [1, Theorem 4.1]) *Let X be a locally path connected Hausdorff space (in particular a topological manifold) and let Y and Z be oriented connected topological n-manifolds. Let $f : X\times Y \to Z$ be a continuous map such that, for all $x \in X$, the map $f_x : Y \to Z$ given by $f_x(y) = f(x, y)$ is open and discrete. Suppose $|\deg(f_{x_0}; y_0)| = 1$ for a point $(x_0, y_0) \in X \times Y$ and* *put* $w_0 = f(x_0, y_0)$. *Then there exist an open neighborhood* V *of* x_0 *in* X *and a continuous map* $g : V \to Y$ *such that* $f(x, g(x)) = w_0$ *for all* $x \in V$.

In this article, specifically in Section 3, we prove the following version of the Inverse Mapping Theorem for continuous maps between topological manifolds, without any assumption on differentiability.

Theorem 1.2 (Inverse Mapping Theorem)**.** *Let* $f : X \to Y$ *be an open and discrete continuous map between oriented topological* n*-manifolds and let* $x_0 \in X$ *be a point. If* $|\deg(f; x_0)| = 1$, *then* f *is a local homeomorphism at* x_0.

Also in Section 3, after prove Theorem 1.2, we present a version of the Inverse Mapping Theorem (Theorem 3.1) for maps between topological manifolds which are topological groups. In such theorem we use the notion of degree at an image point.

Inspired by the equivalences between the Inverse Mapping Theorem and the Implicit Mapping Theorem in the differential approach, we prove that also the homological versions of these theorems are equivalent, for open and discrete continuous maps between oriented topological manifolds. Specifically, we prove:

Theorem 1.3. *Theorem 1.1 and Theorem 1.2 are equivalent providing, in Theorem 1.1, we ask* X *to be an orientable topological manifold and* f *to be open.*

Theorem 1.3 is proved in the end of Section 5 as a consequence of a characterization of the maps between topological manifolds which behaves locally like a projection. Such characterization is presented in Theorem 5.2 as an extension, for continuous maps, of the classical Local Submersion Theorem [7, p. 20]. Essentially, the assumption on differentiability is replaced by assumptions on the local degree of the maps. In this context, we present, in Section 4, a key result which we weave now: given topological n-manifolds X and Y and a continuous map $f : X \times Y \to Y$, we define a pairing map $F : X \times Y \to X \times Y$ by setting $F(x, y) = (x, f(x, y))$. Additionally, given a point $(x_0, y_0) \in X \times Y$, we consider the map $f_{x_0} : Y \to Y$ defined by $f_{x_0}(y) = f(x_0; y)$. Then we prove that $\deg(F; (x_0, y_0)) = \deg(f_{x_0}; y_0)$.

In Section 6, in order to complete the analogy with the differential approach, we characterize the maps between topological manifolds which behave locally like an injection, presenting in Theorem 6.3 an extension, for continuous maps, of the classic Local Immersion Theorem [7, p. 15].

2 Preliminary and auxiliary concepts

As we have said in the introduction, the proof of Theorem 1.1 proposed in [1] is grounded on [4] and [10]. We explain: Given a discrete continuous map $f : X \to Y$ between topological manifolds of the same dimension n, we define B_f to be the set of the points of X in which f fail to be a local homeomorphism. By [4] and [10], the map f is open if and only if $\dim B_f = \dim f(B_f) \leq n-2$. Thus, if f is open, then B_f does not disconnect X and the degree of f in each point of X is nonzero, by [1, Lemma 3.1]. That is the key point for the proof of Theorem 1.1. Therefore, we see that the lack of differentiability is essentially offset by assumptions on the degree of the maps, and for this, we need ask the maps to be discrete: A map $f : X \to Y$ is said to be *discrete at a point* $x_0 \in X$ if x_0 is an isolated point in $f^{-1}(f(x_0))$. If f is discrete at all points of X, we say that f is *discrete*. To avoid confusion on alternative definitions, we consider the following concepts of degree:

Definition 2.1. Let $f : X \to Y$ be a continuous map between oriented and connected topological manifolds of the same dimension $n \geq 1$. If f is discrete at $x_0 \in X$, put $y_0 = f(x_0)$ and consider an open neighborhood V of x_0 in X such that $V \cap f^{-1}(y_0) = \{x_0\}$. Additionally, consider $\alpha_{x_0} \in H_n(V, V - x_0)$ and $\beta_{y_0} \in H_n(Y, Y - y_0)$ to be the local orientation classes of X and Y at x_0 and y_0, respectively. We define:

(1) The *degree of f at x_0* to be the integer $\deg(f; x_0)$ satisfying the identity $f_*(\alpha_{x_0}) = \deg(f; x_0)\beta_{y_0}$, where $f_* : H_n(V, V - x_0) \to H_n(Y, Y - y_0)$ is the homomorphism induced by f.

(2) Provided $f^{-1}(y_0)$ is finite, the *degree of f at y_0* to be the integer

$$\deg(f; y_0) = \sum_{x \in f^{-1}(y_0)} \deg(f; x).$$

See [6] for a directly definition of $\deg(f; y_0)$ independently of $\deg(f; x)$. We choose this equivalent definition, according to [6, Proposition 4.7 on p. 269], to provide a more direct adaptation to our context.

In what follows, m and n are nonnegative integers and $B^k_r(c)$ denotes the open ball of radius $r > 0$ and center $c \in \mathbb{R}^k$. Based in the definition of *n-slice* proposed in [8, p. 101] and in the so common flatness property, we define:

Definition 2.2. A connected subset $S \subset \mathbb{R}^{m+n}$ (or $\mathbb{R}^n \times \mathbb{R}^m$) is called a *skew n-slice* if it satisfies the following locally flatness property: for each point $z \in S$ there exist an open neighborhood V of z in $\mathbb{R}^{m+n}$ (or $\mathbb{R}^n \times \mathbb{R}^m$) and a homeomorphism $h : B^n_1(0) \times B^m_1(0) \to V$ such that $h(0, 0) = z$ and $h(B^n_1(0) \times 0) = V \cap S$.

Obviously, Definition 2.2 remains the same if we replace $B^n_1(0) \times B^m_1(0)$ by $B^n_{r_1}(c_1) \times B^m_{r_2}(c_2)$ for any positive real r_1 and r_2 and points $c_1 \in \mathbb{R}^n$ and $c_2 \in \mathbb{R}^m$.

We remark that a skew n-slice $S \subset \mathbb{R}^{n+m}$ (or $\mathbb{R}^m \times \mathbb{R}^n$) is a topological n-submanifold of $\mathbb{R}^{n+m}$ (or $\mathbb{R}^m \times \mathbb{R}^n$). On the other hand, a topological n-manifold embedded into $\mathbb{R}^{n+m}$ is not necessarily a skew n-slice. For instance, a wild knot in $\mathbb{R}^3$ is not a skew 1-slice (but a tame knot is). See [5] for details in knot theory.

Of course, the graph of a continuous map $f : U \to \mathbb{R}^m$, with $U \subset \mathbb{R}^n$ open and connected, is a skew n-slice in $\mathbb{R}^n \times \mathbb{R}^m$.

We suggest compare the *skew n-slice* definition with the *n-slice* definition presented in [8, p. 101], to conclude that our definition is less restrictive.

3 Proof of the Inverse Mapping Theorem

In this section, we first prove Theorem 1.2, the Inverse Mapping Theorem for open and discrete continuous maps between oriented topological manifolds, and next we present an alternative version for that theorem for maps between topological manifolds which are, additionally, topological groups.

Proof of the Inverse Mapping Theorem. Let consider parametrizations

$$\varphi : A_\varphi \to U' \subset X \qquad \text{and} \qquad \psi : A_\psi \to V' \subset Y,$$

with A_φ and A_ψ open subsets of $\mathbb{R}^n$ containing the origin $0 \in \mathbb{R}^n$, such that $\varphi(0) = x_0 \in U'$ and $\psi(0) = y_0 = f(x_0) \in V'$. Since f is continuous and discrete, we may assume that $U' \cap f^{-1}(y_0) = \{x_0\}$ and $f(U') \subset V'$. Consider the composed map $\psi^{-1} \circ f \circ \varphi : A_\varphi \to A_\psi$ and define the map $F : A_\psi \times A_\varphi \to \mathbb{R}^n$ by setting $F(\hat{x}, \hat{y}) = (\psi^{-1} \circ f \circ \varphi)(\hat{y}) - \hat{x}$. We have:

(*i*) $F(0,0) = \psi^{-1} \circ f \circ \varphi(0) = \psi^{-1} \circ f(x_0) = \psi^{-1}(y_0) = 0$.

(*ii*) Since φ and ψ are homeomorphisms, it follows that the composition $\psi^{-1} \circ f \circ \varphi$ is an open and discrete continuous map. Then, for all $\hat{x} \in A_\varphi$, the (continuous) map $F_{\hat{x}} : A_\varphi \to \mathbb{R}^n$ defined by $F_{\hat{x}}(\hat{y}) = F(\hat{x}, \hat{y})$ is open and discrete.

(*iii*) Particularly, $F_0 : A_\varphi \to \mathbb{R}^n$ is given by $F_0(\hat{y}) = \psi^{-1} \circ f \circ \varphi(\hat{y})$ and we have $F_0^{-1}(0) = \{0\}$, so that $\deg(F_0; F_0(0)) = \deg(F_0; 0)$, where the left side refers to the degree defined in item (2) of Definition 2.1 and the right side refers to the degree defined in item (1) of Definition 2.1.

(*iv*) By the well known properties of the degree we have

$$\begin{aligned}\deg(F_0; F_0(0)) &= \deg(\psi^{-1} \circ f \circ \varphi; 0) \\ &= \deg(\psi^{-1}; 0) \deg(f; x_0) \deg(\varphi; 0) = \deg(f; x_0),\end{aligned}$$

so that $|\deg(F_0; F_0(0))| = |\deg(f; x_0)| = 1$.

By Theorem 1.1, there exist an open neighborhood $V'' \subset A_\psi$ of 0 and a continuous map $\zeta : V'' \to A_\varphi$ such that $\psi^{-1} \circ f \circ \varphi \circ \zeta(\hat{x}) - \hat{x} = F(\hat{x}, \zeta(\hat{x})) = 0$ for all $\hat{x} \in V''$, which implies that

$$f \circ \varphi \circ \zeta(\hat{x}) = \psi(\hat{x}) \quad \text{for all} \quad \hat{x} \in V''. \tag{1}$$

Since ψ is a homeomorphism, it follows that ζ is injective. By the Domain Invariance Theorem (see [3]), ζ maps V'' homeomorphically onto its image $\zeta(V'') \subset A_\varphi$, which is open in $\mathbb{R}^n$. Put $V = \psi(V'') \subset V'$ and $U = \varphi(\zeta(V'')) \subset U'$. Note that U is an open subset of X and V is an open subset of Y. Consider the composed homeomorphism $g = \varphi \circ \zeta \circ \psi^{-1} : V \to U$. For each $y \in V$, there exists a unique $\hat{x} \in V''$ such that $y = \psi(\hat{x})$, and we have, by identity (1) above,

$$f \circ g(y) = f \circ \varphi \circ \zeta \circ \psi^{-1}(y) = f \circ \varphi \circ \zeta(\hat{x}) = \psi(\hat{x}) = y,$$

which proves that $f \circ g(y) = y$ for all $y \in V$. Since g is a homeomorphism from V into U, the restrict map $f|_U : U \to V$ is also a homeomorphism; indeed $f|_U$ is the inverse homeomorphism for g. □

Our second Inverse Mapping Theorem is for open and discrete continuous maps between topological n-manifolds which are, additionally, topological groups. Here we use the degree of a map defined in item (2) of Definition 2.1.

Theorem 3.1 (Inverse Mapping Theorem for topological groups)**.** *Let $f : X \to Y$ be an open and discrete continuous map between topological n-manifolds. Suppose X and Y are topological groups, let $x_0 \in X$ be a point and $y_0 = f(x_0)$. If $f^{-1}(y_0)$ is finite and $|\deg(f; y_0)| = 1$, then there exist an open neighborhood V of y_0 in Y and a continuous map $g : V \to X$, with $g(y_0) = x_0$, such that $f \circ g(y) = y$ for all $y \in V$.*

Proof. Let consider the (continuous) map $F : X \times Y \to Y$ defined by $F(x, y) = f(x) * y^{-1}$, where $*$ is the operation on the topological group Y and y^{-1} is the symmetric of y. Clearly $F(x_0, y_0) = e$, the identity of Y. For each given point $y \in Y$, we define a map $F_y : X \to Y$ by setting $F_y(x) = F(x, y)$. Since f is open and discrete, also F_y is open and discrete for all $y \in Y$. Moreover, $F_{y_0}^{-1}(e) = f^{-1}(y_0)$ is finite and we have

$$\left|\deg(F_{y_0}; e)\right| = \Big|\sum_{x \in F_{y_0}^{-1}(e)} \deg(F_{y_0}; x)\Big| = \Big|\sum_{x \in f^{-1}(y_0)} \deg(f; x)\Big| = 1.$$

By Theorem 1.1, there exist an open neighborhood V of y_0 in Y and a continuous map $g : V \to Y$ such that $F(g(y), y) = e$ for all $y \in V$, that is, $f(g(y)) * y^{-1} = e$ for all $y \in V$, which implies that $f \circ g(y) = y$ for all $y \in V$. □

4 On the degree of a pairing map

It is a known consequence of the Künneth Formula that, for maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between topological manifolds of the same dimension, we have

$$\deg(f_1 \times g_1; (x_1, x_2)) = \deg(f_1; x_1)\ \deg(f_2; x_2).$$

However, if we consider a map $f : X \times Y \to Y$ and we define $F : X \times Y \to X \times Y$ by setting $F(x, y) = (x, f(x, y))$, then F is not a product map and the Künneth Formula does not apply to express the degree of F. If we consider X and Y as $\mathcal{C}^1$ manifolds and ask f to be a $\mathcal{C}^1$ map, then it is well known that if the second partial derivative $\partial_2 f$ is an isomorphism at a point $z_0 = (x_0, y_0)$, then the Jacobian matrix $JF(z_0)$ is an isomorphism and F is a local diffeomorphism at z_0. This fact leads us to think that if $\deg(f(x_0, \cdot); y_0) = \pm 1$, then $\deg(F; z_0) = \pm 1$. In fact, this is a particular case of the main result of this section, which we state now:

Proposition 4.1. *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and let $z_0 = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ be a point. Consider the (continuous) maps $f_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$, defined by $f_{x_0}(y) = f(x_0, y)$, and $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$, defined by $F(x, y) = (x, f(x, y))$. If f_{x_0} is discrete at y_0, then F is discrete at z_0 and $\deg(F; z_0) = \deg(f_{x_0}; y_0)$.*

Since we consider just local concepts, as the local degree, the result of Proposition 4.1 remains true if we replace the euclidian spaces by topological manifolds.

In order to prove Proposition 4.1, we consider the following lemma:

Lemma 4.2. *Let $f : S^n \to S^n$ be a continuous self-map of the n-sphere, let $S^{n-1} \subset S^n$ be an equator and let E^n_+ and E^n_- be the corresponding closed hemispheres. If E^n_+ and E^n_- are invariant by f, then f restricts to a self-map $f|_{S^{n-1}}$ of the $(n-1)$-sphere S^{n-1} and we have $\deg(f) = \deg(f|_{S^{n-1}})$.*

Proof. This result is in [9, Exercise 3 on p. 207]. The proof follows by a straightforward application of Mayer-Vietoris sequence. □

In the proof of Proposition 4.1, B^k_r and D^k_r mean the open ball and the closed disc, respectively, of radius r and center at $0 \in \mathbb{R}^k$. Additionally, $S^{k-1}_r = \partial D^k_r$.

Proof of Proposition 4.1. Without loss of generality, we consider $z_0 = (0, 0)$ and $f(z_0) = 0$. We first prove the result for $m = 1$. Next, we prove the general case by induction on m.

Since f is continuous and f_0 is discrete at 0, there exists $0 < \varepsilon < 1$ such that

$$f((-2\varepsilon, 2\varepsilon) \times B^n_{2\varepsilon}) \subset D^n_1 \quad \text{and} \quad f_0^{-1}(0) \cap ((-2\varepsilon, 2\varepsilon) \times B^n_{2\varepsilon}) = \{0\}.$$

Since $F(x, y) = (0, 0)$ forces $x = 0$, we have $F^{-1}(0, 0) \cap ((-2\varepsilon, 2\varepsilon) \times B^n_{2\varepsilon}) = \{(0, 0)\}$, which implies that F is discrete at $(0, 0)$.

It follows that f_0 induces a map from the pair $(D^n_\varepsilon, D^n_\varepsilon - 0)$ into the pair $(D^n_1, D^n_1 - 0)$, whose induced homomorphism $f_{0*} : H_n(D^n_\varepsilon, D^n_\varepsilon - 0) \to H_n(D^n_1, D^n_1 - 0)$ determines the degree $\deg(f_0; 0)$. Consider the inclusion $l : S^{n-1}_\varepsilon \hookrightarrow D^n_\varepsilon$ and the radial retraction $r : D^n_1 \twoheadrightarrow S^{n-1}_1$. We have the following commutative diagram:

$$
\begin{array}{ccc}
H_n(D_\varepsilon^n, D_\varepsilon^n - 0) & \xrightarrow{\ f_{0*}\ } & H_n(D_1^n, D_1^n - 0) \\
\partial \big\downarrow \approx & & \approx \big\downarrow \partial \\
\widetilde{H}_{n-1}(D_\varepsilon^n - 0) & \xrightarrow{\ f_{0*}\ } & \widetilde{H}_{n-1}(D_1^n - 0) \\
\approx \big\uparrow l_* & & \approx \big\downarrow r_* \\
\widetilde{H}_{n-1}(S_\varepsilon^{n-1}) & \xrightarrow{\ \phi = r_* f_{0*} l_*\ } & \widetilde{H}_{n-1}(S_1^{n-1})
\end{array}
$$

It follows that $\deg(f_0; 0) = \deg(\phi)$.

Consider the suspension SS_ε^{n-1} consisting of all segments connecting the points of the sphere $0 \times S_\varepsilon^{n-1} \subset 0 \times \mathbb{R}^n$ to the points $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ in $\mathbb{R} \times \mathbb{R}^n$. Additionally, consider the analogous suspension SD_ε^n, so that $SS_\varepsilon^{n-1} = \partial SD_\varepsilon^n$.

Obviously, SS_ε^{n-1} is a homeomorphic n-sphere and SD_ε^n is a homeomorphic closed $(n+1)$-disc, both contained in the homeomorphic closed $(n+1)$-disc $K_\varepsilon^{n+1} = [\varepsilon, \varepsilon] \times D_\varepsilon^n$ in $\mathbb{R} \times \mathbb{R}^n$. Also $K_1^{n+1} = [-\varepsilon, \varepsilon] \times D_1^n$ is a homeomorphic closed $(n+1)$-disc in $\mathbb{R} \times \mathbb{R}^n$ whose homeomorphic n-sphere corresponding to its boundary we denote by Σ_1^n. Clearly, we have a (radial) retraction $r : K_1^{n+1} - (0,0) \twoheadrightarrow \Sigma_1^n$. By construction

$$
F(SD_\varepsilon^n) \subset K_\varepsilon^{n+1} \quad \text{and} \quad F(SS_\varepsilon^{n-1}) \subset F(SD_\varepsilon^n - (0,0)) \subset K_1^{n+1} - (0,0).
$$

As above, we have the following commutative diagram:

$$
\begin{array}{ccc}
H_{n+1}(SD_\varepsilon^n, SD_\varepsilon^n - (0,0)) & \xrightarrow{\ F_*\ } & H_{n+1}(K_1^{n+1}, K_1^{n+1} - (0,0)) \\
\partial \big\downarrow \approx & & \approx \big\downarrow \partial \\
\widetilde{H}_n(SD_\varepsilon^n - (0,0)) & \xrightarrow{\ F_*\ } & \widetilde{H}_n(K_1^{n+1} - (0,0)) \\
\approx \big\uparrow l_* & & \approx \big\downarrow r_* \\
\widetilde{H}_n(SS_\varepsilon^{n-1}) & \xrightarrow{\ \Phi = r_* F_* l_*\ } & \widetilde{H}_n(\Sigma_1^n)
\end{array}
$$

It follows that $\deg(F; (0,0)) = \deg(\Phi)$.

Now, it is easy to see that S_ε^{n-1} corresponds to the equator $0 \times S_\varepsilon^{n-1} \subset SS_\varepsilon^{n-1}$ and S_1^{n-1} corresponds to the equator $0 \times S_1^{n_1} \subset \Sigma_1^n$. Furthermore, since F keeps fixed the first coordinate, the composition

$$
SS_\varepsilon^{n-1} \hookrightarrow (SD_\varepsilon^n - (0,0)) \xrightarrow{F} K_1^{n+1} - (0,0) \twoheadrightarrow \Sigma_1^n
$$

maps the positive (respectively the negative) closed hemisphere of SS_ε^{n-1} into the positive (respectively the negative) closed hemisphere of Σ_1^n. Moreover, since $F(0,y) = f_0(y)$, the restriction of F on the equator $0 \times S_\varepsilon^{n-1}$ corresponds to the composition

$$
S_\varepsilon^{n-1} \hookrightarrow D_\varepsilon^n - 0 \xrightarrow{f_0} D_1^n - 0 \twoheadrightarrow S_1^{n-1}.
$$

Hence, Lemma 4.2 implies that $\deg(F; (0,0)) = \deg(f_0; 0)$. Therefore, we have proved the result for $m = 1$.

Now we prove the general case by induction on m. Suppose that the result holds true for a given $m \geq 1$ and let $f : \mathbb{R}^{m+1} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Define $F : \mathbb{R}^{m+1} \times \mathbb{R}^n \to \mathbb{R}^{m+1} \times \mathbb{R}^n$ by setting $F(x, y) = (x, f(x, y))$.

In what follows, we consider the natural identification $\mathbb{R}^{m+1} \equiv \mathbb{R} \times \mathbb{R}^m$ given by

$$(x_0, x_1, \ldots, x_m) \leftrightarrow (x_0, (x_1, \ldots, x_m)).$$

To simplify, for a given point $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$, we write $x^0 = (x_1, \ldots, x_m)$, so that the previous identification gives $\mathbb{R}^{m+1} \ni x \equiv (x_0, x^0) \in \mathbb{R} \times \mathbb{R}^m$.

Using this identification, we consider the origin in $\mathbb{R}^{m+1}$ as $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$. Thus, we have the map $f_{(0,0)} : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_{(0,0)}(y) = f(0, 0, y)$.

We should prove that $\deg(F; (0, 0, 0)) = \deg(f_{(0,0)}; 0)$.

Define $h : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by setting $h(x_0, x^0, y) = (x^0, f(x_0, x^0, y))$ and, corresponding to the origin $0 \in \mathbb{R}$, consider the map $h_0 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ given by $h_0(x^0, y) = h(0, x^0, y)$. Thus, $h_0(x^0, y) = (x^0, f(0, x^0, y))$ and the induction hypothesis implies that $\deg(h_0; (0, 0)) = \deg(f_{(0,0)}; 0)$.

On the other hand, using the identification $\mathbb{R}^{m+1} \times \mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ in the domain and in the range of the map F, we have

$$F(x_0, x^0, y) = (x_0, x^0, f(x_0, x^0, y)) = (x_0, h(x_0, x^0, y)).$$

Then, the first part of the proof applies for the map h and gives the identity $\deg(F; (0, 0, 0)) = \deg(h_0; (0, 0))$.

Therefore, $\deg(F; (0, 0, 0)) = \deg(f_{(0,0)}; 0)$, as we wanted to prove. □

5 Maps locally like a projection

In this section, we study continuous map, between euclidian spaces, which behave locally like a projection. Since the concepts involved here are all of local character, the results may be generalized for maps between oriented topological manifolds.

In what follows, given a map $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and a point $(x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$, we consider the map $f_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f_{x_0}(y) = f(x_0, y)$.

Definition 5.1. An open continuous map $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is called:

(*i*) a *straight topological submersion* at a given point $z_0 = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ if the map f_{x_0} is discrete and $|\deg(f_{x_0}; y_0)| = 1$.

(*ii*) a *skew topological submersion* at a given point $z_0 = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ if there exists a skew n-slice $S \subset \mathbb{R}^m \times \mathbb{R}^n$, containing the point z_0, such that $f|_S : S \to \mathbb{R}^n$ is discrete and $|\deg(f|_S; z_0)| = 1$.

Of course, this definition works also for a map $f : A \times B \to \mathbb{R}^n$ defined in a basic open subset $A \times B$ of $\mathbb{R}^m \times \mathbb{R}^n$.

After Theorem 5.2 we clarify that each straight or skew topological submersion is actually a *topological submersion* in the sense of [8, p. 89].

We remark that a straight topological submersion is a skew topological submersion (take $S = x_0 \times \mathbb{R}^n$), but the converse is not true; for instance, consider the open continuous map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = x$. Since f_0 is the zero constant map $y \mapsto 0$, the map f is not a straight topological submersion at the origin $(0, 0) \in \mathbb{R} \times \mathbb{R}$. However, if we take S to be the diagonal in $\mathbb{R} \times \mathbb{R}$, then S is a skew 1-slice and the restricted map $f|_S : S \to \mathbb{R}$ is the homeomorphism $(x, x) \mapsto x$, which shows that f is a skew topological submersion at $(0, 0)$. We note that f is also a submersion in the differential sense. For a more interesting example, consider the map $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by $f((x, y), z) = x^3(y^4 - z^2 + 1)$. This map is neither a straight topological submersion at the origin (since $f_{(0,0)}$ is the zero constant map) nor a submersion in the differential sense at the origin (since the gradient vector $\nabla f((0, 0), 0)$ is null). However, f is a skew topological submersion at the origin. In fact, for the skew 1-slice $S = \{((x, x), x^2) : x \in \mathbb{R}\} \subset \mathbb{R}^2 \times \mathbb{R}$, containing the origin, the restricted map $f|_S : S \to \mathbb{R}$ is the homeomorphism $((x, x), x^2) \mapsto x^3$.

Using the classical Local Submersion Theorem in [7, p. 20], it is easy to see that all differential submersion $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is actually a skew topological submersion, since such a map is locally like a projection. Indeed, the following theorem means that each straight or skew topological submersion is locally like a projection; therefore, this theorem is a version of the Local Submersion Theorem for (not necessarily differentiable) straight/skew topological submersions.

Theorem 5.2 (Local Topological Submersion Theorem)**.** *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a straight or skew topological submersion at a given point $z_0 = (x_0, y_0)$ and put $w_0 = f(z_0)$. Then there exist a (basic) open neighborhood $V \times W$ of (x_0, w_0) in $\mathbb{R}^m \times \mathbb{R}^n$ and a homeomorphism $\varphi : V \times W \to Z$ onto an open neighborhood Z of z_0 in $\mathbb{R}^m \times \mathbb{R}^n$, such that $f \circ \varphi(x, w) = w$ for all $(x, w) \in V \times W$.*

Proof. At first, we suppose that f is a straight topological submersion at the point z_0. Define $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by setting $F(x, y) = (x, f(x, y))$. Then F is continuous, open and discrete and, by Proposition 4.1, $|\deg(F; z_0)| = 1$. By Theorem 1.2, F is a local homeomorphism at z_0. Hence, there exist an open neighborhood Z of z_0 in $\mathbb{R}^m \times \mathbb{R}^n$ which is mapped homeomorphically by F onto a (basic) open neighborhood $V \times W$ of (x_0, w_0) in $\mathbb{R}^m \times \mathbb{R}^n$. Let $\varphi : V \times W \to Z$ be the inverse homeomorphism for $F|_Z : Z \to V \times W$. Since F keeps fixed the first coordinate, the homeomorphism φ has the same property, so that $\varphi(x, w) = (x, \varphi_2(x, w))$ for all $(x, w) \in V \times W$. Thus, for all $(x, w) \in V \times W$, we have

$$(x, w) = F \circ \varphi(x, w) = F(x, \varphi_2(x, w)) = (x, f(x, \varphi_2(x, w))) = (x, f \circ \varphi(x, w)).$$

Therefore, $f \circ \varphi(x, w) = w$ for all $(x, w) \in V \times W$.

Now, we suppose that f is a skew topological submersion at z_0. We take a skew n-slice S, as in Definition 5.1, and a homeomorphism $h : B_1^m(x_0) \times B_1^n(y_0) \to U$, onto an open neighborhood U of z_0 in $\mathbb{R}^m \times \mathbb{R}^n$, such that $h(x_0, y_0) = z_0$ and $h(x_0 \times B_1^n(y_0)) = U \cap S$. Then the composed map $f' = f \circ h : B_1^m(x_0) \times B_1^n(y_0) \to \mathbb{R}^n$ is a straight topological submersion at (x_0, y_0) with $f'(x_0, y_0) = w_0$. By the first part of the proof, there exist a (basic) open neighborhood $V \times W$ of (x_0, w_0) in $\mathbb{R}^m \times \mathbb{R}^n$ and a homeomorphism $\varphi' : V \times W \to Z'$ onto an open neighborhood $Z' \subset B_1^m(x_0) \times B_1^n(y_0)$ of z_0, such that $f' \circ \varphi'(x, w) = w$ for all $(x, w) \in V \times W$. Take $Z = h(Z')$. Then $\varphi = h \circ \varphi' : V \times W \to Z$ is a homeomorphism onto an open neighborhood of z_0 in $\mathbb{R}^m \times \mathbb{R}^n$ and we have $f \circ \varphi(x, w) = w$ for all $(x, w) \in V \times W$. □

Theorem 5.2 means that each straight or skew topological submersion $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a *topological submersion* as defined in [8, p. 89]. In fact, given such a straight or skew topological submersion at a point (x_0, y_0), with $f(x_0, y_0) = w_0$, take a homeomorphism $\varphi : V \times W \to Z$ as in Theorem 5.2. Then the map $\sigma : V \to Z$ defined by $\sigma(x) = \varphi(x, w_0)$ is a local section for f whose image contains (x_0, y_0).

By an example we show that Theorem 5.2 works for some differential maps which are not a submersion in the classical sense. Consider the open differential map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = y^3$. Then f is not a submersion at the origin, in the differential sense, and so the classical Local Submersion Theorem does not apply for f at $(0, 0)$. However, the restricted map $f_0 : \mathbb{R} \to \mathbb{R}$ is given by $y \mapsto y^3$, and so f_0 is a homeomorphism. Therefore, f is a straight topological submersion at $(0, 0)$ and Theorem 5.2 works.

The next theorem is a version of Theorem 1.1, the Implicit Mapping Theorem, for open maps between euclidian spaces. Due its local character, it may be extended for open maps between oriented topological manifolds. We remark that Theorem 1.1 does not refer to the uniqueness of the implicit map, but Theorem 5.3 does. Therefore, we require a little more, but we also found some more.

Theorem 5.3 (Implicit Mapping Theorem for straight topological submersions)**.** *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a straight topological submersion at a given point $z_0 = (x_0, y_0)$ and put $w_0 = f(z_0)$. Then there exist open neighborhoods V of x_0 in $\mathbb{R}^m$ and Z of z_0 in $\mathbb{R}^m \times \mathbb{R}^n$, for which there exists a unique continuous map $\xi : V \to \mathbb{R}^n$ such that, for each $x \in V$, one has $(x, \xi(x)) \in Z$ and $f(x, \xi(x)) = w_0$.*

Proof. Define the map $\xi : V \to \mathbb{R}^n$ by setting $\xi(x) = \varphi_2(x, w_0)$, where φ_2 is as in the first part of the proof of Theorem 5.2. Then ξ is continuous and we have $(x, \xi(x)) = \varphi(x, w_0) \in Z$ for all $x \in V$. Moreover, $f(x, \xi(x)) = f \circ \varphi(x, w_0) = w_0$ for all $x \in V$. Conversely, if $(x, y) \in Z$ and $f(x, y) = w_0$, then

$$(x, y) = \varphi \circ F(x, y) = \varphi(x, f(x, y)) = \varphi(x, w_0) = (x, \varphi_2(x, w_0)) = (x, \xi(x)),$$

so that $y = \xi(x)$, which proves the uniqueness of the map ξ. □

Theorem 5.3 does not hold true for skew topological submersion. In fact, consider the aforementioned map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by $f(x, y) = x$, which is a skew topological submersion at the origin $(0, 0) \in \mathbb{R} \times \mathbb{R}$. For each continuous map $\xi : (-\delta, \delta) \to \mathbb{R}$, we have $f(x, \xi(x)) = x$, so that, there is not such a map ξ satisfying $f(x, \xi(x)) = 0$ for all $x \in (-\delta, \delta)$. Therefore, Theorem 5.3 does not work for f.

We remark that Theorem 5.3, extended for maps between oriented topological manifolds (which is actually possible) is the version of Theorem 1.1 for open maps between oriented topological manifolds. Therefore, we have:

Proof of Theorem 1.3. We use directly Theorem 1.1 to prove Theorem 1.2. On the other hand, as a directly implication of Theorem 1.2 we prove Theorem 5.3, which is (after the aforementioned extension) exactly Theorem 1.1 for open maps between topological manifolds. □

To finalize this section, we use the proofs of Theorems 5.2 and 5.3 to highlight an important technical difference between the two concepts of submersion introduced in Definition 5.1. To be specific, we note that the essential facts used in the first part of the proof of Theorems 5.2, and so in the proof of Theorem 5.3, is that the pairing map F is continuous, open and discrete and $|\deg(F; z_0)| = 1$. The next proposition relates this facts with the two concepts of topological submersions.

Proposition 5.4. *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be an open continuous map and consider $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ defined by $F(x, y) = (x, f(x, y))$. Let $z_0 = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ be a point. If F is discrete and $|\deg(F; z_0)| = 1$, then f is a skew topological submersion. The converse is true if f is a straight topological submersion.*

Proof. In order to prove the first part of the proposition, we note that the proofs of Theorems 5.2 and 5.3 work with the assumption "F is open and discrete and $|\deg(F; z_0)| = 1$" instead the assumption "f is a straight topological submersion at the point z_0". Therefore, we may assume the results of Theorems 5.2 and 5.3. Consider positive real numbers ε and δ such that $B^m_\varepsilon(x_0) \subset V$ and $B^n_\delta(w_0) \subset W$. Define $h : B^m_\varepsilon(x_0) \times B^n_\delta(w_0) \to \mathbb{R}^m \times \mathbb{R}^n$ by setting $h(x, w) = \varphi(x, w)$. Then h restricts to a homeomorphism $h : B^m_\varepsilon(x_0) \times B^n_\delta(w_0) \to Z'$ onto an open neighborhood $Z' \subset Z$ of $z_0 = h(x_0, w_0)$. Hence $S = h(x_0 \times B^n_\delta(w_0))$ is a skew n-slice such that $z_0 \in S \subset U$. Moreover, each point in S is of the form $h(x_0, w) = (x_0, \varphi_2(x_0, w))$ with $w \in B^n_\delta(w_0)$. Let $h_1 : x_0 \times B^n_\delta(w_0) \to S$ be the homeomorphism obtained from h by the obvious restrictions of its domain and range. Then $f|_S \circ h_1(x_0, w) = w$ for all $w \in B^n_\delta(w_0)$. It follows that $f|_S$ is injective. We clarify that $f|_S$ is open: given an open subset $A \subset S$, we have $f|_S(A) = (f|_S \circ h_1)(h_1^{-1}(A)) = \pi_2(h_1^{-1}(A))$, where $\pi_2 : B^m_\varepsilon(x_0) \times B^n_\delta(w_0) \to B^n_\delta(w_0)$ is the projection onto the second coordinate; it follows that $f|_S(A)$ is open in $\mathbb{R}^n$. Therefore, we have proved that $f|_S : S \to \mathbb{R}^n$

is an open and injective continuous map, which implies that $f|_S$, and so f, maps S homeomorphically onto its image $f|_S(S) = f(S)$ in $\mathbb{R}^n$.

The converse follows from Proposition 4.1. □

The converse in Proposition 5.4 is not true, in general, if f is a skew topological submersion but is not a straight topological submersion. In fact: we take again $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x, y) = x$, which is a skew topological submersion at $(0, 0)$, respect to the skew 1-slice $S = \{(x, x) : x \in \mathbb{R}\}$. We have showed that f is not a straight topological submersion. Now, the corresponding map $F : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$ is given by $F(x, y) = (x, f(x, y)) = (x, x)$ and, therefore, F is not even discrete.

6 Maps locally like an injection

In this section, we study continuous maps between euclidian spaces, which behave locally like an injection. Since the concepts involved here are all of local character, the results may be generalized for maps between oriented topological manifolds.

In what follows, $\pi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the first projection. Given a map $f : X \to Y$ and subsets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$, we define $f|_A^B : A \to B$ to be the map obtained from f by restriction of its domain and range.

Definition 6.1. A discrete continuous map $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ is called:

(*i*) a *straight topological immersion* at a given point $x_0 \in \mathbb{R}^n$ if the composed map $\pi_1 \circ f$ is open and discrete and $|\deg(\pi_1 \circ f; x_0)| = 1$.

(*ii*) a *skew topological immersion* at a given point $x_0 \in \mathbb{R}^m \times \mathbb{R}^n$ if there exist an open neighborhood U of x_0 in $\mathbb{R}^n$ and a skew n-slice $S \subset \mathbb{R}^n \times \mathbb{R}^m$, with $f(U) \subset S$, such that the map $f|_U^S$ is open and discrete and $|\deg(f|_U^S; x_0)| = 1$.

It follows by Theorem 1.2 that a skew topological immersion, at a given point x_0, restricts to an embedding of an open neighborhood of x_0. Thus, each skew topological immersion, and so each straight topological immersion (see Proposition 6.2 below), is actually a *topological immersion* is the sense of [8, p. 88].

We remark that f and $\pi_1 \circ f$ are both discrete if and only if the pre-image by f of each affine plane $x \times \mathbb{R}^m \subset \mathbb{R}^n \times \mathbb{R}^m$ is discrete.

A skew topological submersion is not necessarily a straight topological immersion. In fact: the map $f : \mathbb{R} \to \mathbb{R}\times\mathbb{R}$ given by $f(x) = (0, x)$ is a skew topological immersion at the origin $0 \in \mathbb{R}$, respect to the skew 1-slice $S = \{(0, y) : y \in \mathbb{R}\} \subset \mathbb{R}\times\mathbb{R}$, but f is not a straight topological immersion, since $\pi_1 \circ f$ is the zero constant map. On the other hand, a straight topological immersion is a skew topological immersion, as we show in the next proposition.

Proposition 6.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ be a discrete continuous map. If f is a straight topological immersion at a given point $x_0 \in \mathbb{R}^n$, then f is a skew topological immersion at x_0.*

Proof. From the assumption, the composed map $\pi_1 \circ f$ is continuous, open and discrete and $|\deg(\pi_1 \circ f; x_0)| = 1$. By Theorem 1.2, $\pi_1 \circ f$ maps an open neighborhood U of x_0 in $\mathbb{R}^n$ homeomorphically onto an open neighborhood V of $\pi_1 \circ f(x_0)$ in $\mathbb{R}^n$. Let $\psi : V \to U$ be the corresponding inverse homeomorphism. Take S to be the component of $\pi_1^{-1}(V) \cap Im(f)$ containing $f(x_0)$. Then, we have $S = f(U)$. We claim that the restricted map $f|_U^S : U \to S$ is a homeomorphism and so $|\deg(f_U^S; x_0)| = 1$. In fact: at first, note that $f|_U^S$ is clearly continuous; second, note that $f|_U^S$ is open, since for an open set $A \subset U$, we have $f|_U^S(A) = S \cap \pi_1^{-1}(\pi_1 \circ f(A))$; and finally, note that $f|_U^S$ is a bijection with corresponding inverse $g : S \to U$ given by $g(z) = \psi \circ \pi_1(z)$. It is proved that f is a skew topological immersion at x_0. □

Using the classical Local Immersion Theorem in [7, p. 15], it is easy to prove that each differential immersion $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ is actually a skew topological immersion, since such a map is locally like an injection. Indeed, the following theorem means that each straight or skew topological immersion is locally like an injection; therefore, this theorem is a version of the Local Immersion Theorem for (not necessarily differentiable) straight/skew topological immersions.

Theorem 6.3 (Local Topological Immersion Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ be a straight or skew topological immersion at a point $x_0 \in \mathbb{R}^n$. Then there exist open neighborhoods W of 0 in $\mathbb{R}^n$, U' of x_0 in $\mathbb{R}^n$ and V' of $f(x_0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, and homeomorphisms $\psi : W \to U'$ and $\varphi : W \times B_1^m(0) \to V' \cap S$, such that $\varphi^{-1} \circ f \circ \psi(x) = (x, 0)$ for all $x \in W$.*

Proof. Let U be an open neighborhood of x_0 in $\mathbb{R}^n$ and let $S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a skew n-slice, with $f(U) \subset S$, such that the restricted map $f|_U^S : U \to S$ is open and discrete and $|\deg(f|_U^S; x_0)| = 1$. Consider a homeomorphism $h : B_1^n(0) \times B_1^m(0) \to V$ onto an open neighborhood V of $f(x_0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ such that $h(B_1^n(0) \times 0) = V \cap S$ and $h(0, 0) = f(x_0)$. For our purposes, we may suppose $f(U) \subset V$ so that $f(U) \subset V \cap S$. The (well defined) composed map $f' : U \to B_1^n(0)$ given by $f'(x) = \pi_1 \circ h^{-1} \circ f(x)$ is continuous, open and discrete, maps x_0 to 0, and $|\deg(f'; x_0)| = 1$. By Theorem 1.2, f' maps an open neighborhood $U' \subset U$ of x_0 homeomorphically onto an open neighborhood $W \subset B_1^n(0)$ of $0 \in \mathbb{R}^n$. Let $\psi : W \to U'$ be the corresponding inverse homeomorphism. Consider the open set $V' = h(W \times B_1^m(0)) \subset V$ and take the homeomorphism $\varphi : W \times B_1^m(0) \to V'$ to be h itself with the obvious restriction on its domain and range. We have $f(U') = \varphi(W \times 0) = V' \cap S$ and $\varphi^{-1}(f(U')) = W \times 0$. Moreover $\pi_1 \circ \varphi^{-1} \circ f \circ \psi(x) = f' \circ \psi(x) = x$ for all $x \in W$. Therefore $\varphi^{-1} \circ f \circ \psi(x) = (x, 0)$ for all $x \in W$. □

We observe that Theorem 6.3 works for some differential maps which are not a immersion in the classical sense. Consider the discrete differential map $f:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ given by $f(x)=(x^3,x^3)$. Then f is not an immersion at the origin, in the differential sense, and so the classical Local Immersion Theorem does not apply for f at 0. However, $\pi_1\circ f:\mathbb{R}\to\mathbb{R}$ is the homeomorphism $x\mapsto x^3$. Therefore, f is a straight topological immersion at 0 and Theorem 6.3 works.

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