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An estimative for the number of limit cycles in a Liénard-like perturbation of a quadratic non-linear center

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Abstract The number of limit cycles which bifurcates from periodic orbits of a differential system with a center has been extensively studied recently using many distinct tools. This problem was proposed by Hilbert in 1900 and it is a difficult problem so only particular families of such systems were considered. In this paper we study the maximum number of limit cycles that can bifurcate from an integrable non-linear quadratic isochronous center, when perturbed inside a class of Liénard-like polynomial differential systems of arbitrary degree n. We apply the averaging theory of first order to this class of Liénard-like polynomial differential systems and we estimate that the number of limit cycles is $2[(n-2)/2]$, where [.] denotes the integer part function.

Mathematics Subject Classification (2000)
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1 Introduction

Finding an upper bound to the maximum number of limit cycles that the class of all polynomial vector fields with a fixed degree can have is posed by the second part of the 16th Hilbert‘s problem. As the 16th Hilbert problem turned out a strongly difficult one Smale [28] particularized it to Liénard polynomial differential equations in his list of problems for the present century. The classical Liénard systems is given by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where f is an analytical function and $g(x) = x$.

Many results on the number of limit cycles has been obtained for the generalized differential equation (1) being $f(x)$ and $g(x)$ polynomials in the variable x of degrees $n-1$ and m respectively.

System (1) were studied in 1977 by Lins, de Melo and Pugh [16] who stated the following conjecture: *if $f(x)$ has degree $n-1>0$ and $g(x)=x$, then* (1) *has at most* $[(n-1)/2]$ *limit cycles.* Here $[z]$ denotes the integer part function of $z \in \mathbb{R}$. They also proved the conjecture for $n = 2, 3$. For $n = 4$ this conjecture has been proved in 2012 (see [14]).

For $n \geq 7$ Dumortier, Panazzolo and Roussarie in [10] proved that this conjecture is not true, they show that these differential equations can have $[(n-1)/2]+1$ limit cycles. In 2011 De Maesschalck and Dumortier proved in [25] that the classical Liénard equation of degree $n \geq 6$ can have $[(n-1)/2]+2$ limit cycles. The conjecture for $n = 5$ is still open.

Results on the number of limit cycles for generalized Liénard polynomial differential equations can be found in [3,4,9,12,21,22,23,24,31].

In [18] the authors considered the Liénard system which is a perturbation of a linear center

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + \varepsilon(y f_n(x) + g_m(x)), \end{cases}$$

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where f_n and g_m are polynomials of degree n and m respectively. Applying the averaging theory of first order they showed that there are differential equations (1) having at least $\left[\frac{n}{2}\right]$ limit cycles.

The number of limit cycles which bifurcates from periodic orbits of a differential system with a center (linear or not) has been extensively studied recently using many distinct tools, the inverse integral factor method (see [13]), the methods of Hopf and homoclinic bifurcation theory (see [32]), Lyapunov constants (see [11]), Melnikov function (see [33]), averaging theory (see [5,19,17]).

Our aim is to study the maximum number of limit cycles that can bifurcates from periodic orbits of quadratic polynomial system having an isochronous center.

The classification of the quadratic polynomial differential systems having an isochronous center is due to Loud [20]. Loud proved that, except to an affine change of variables and a rescaling of the independent variable, any quadratic differential system having an isochronous center can be written in one of the following forms where H_j is its corresponding first integral:

Table 1 Quadratic isochronous centers and its corresponding first integrals

Isochronous centers	First integrals
$\dot{x} = -y + x^2 - y^2$, $\dot{y} = x(1+2y)$,	$H_1(x,y) = \frac{x^2+y^2}{1+2y}$
$\dot{x} = -y + x^2$, $\dot{y} = x + xy$,	$H_2(x,y) = \frac{x^2+y^2}{(1+y)^2}$
$\dot{x} = -y - \frac{4}{3}x^2$, $\dot{y} = x(1 - \frac{16}{3}y)$,	$H_3(x,y) = \frac{9(x^2+y^2)-24x^2y+16x^4}{-3+16y}$
$\dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2$ $\dot{y} = x(1+\frac{8}{3}y)$	$H_4(x,y) = \frac{9(x^2+y^2)+24y^3+16y^4}{(3+8y)^4}$

The next tables shows some level curves of the first integrals H_j defined in Table 1. Note that all the level curves in a neighborhood of the origin are closed.

Fig. 1 Some level curves of H_1 and H_2.

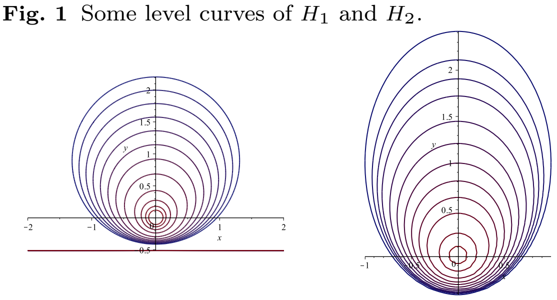
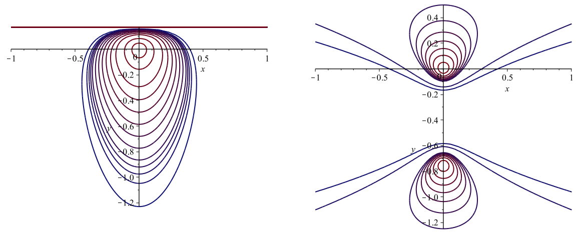


Fig. 2 Some level curves of H_3 and H_4.



In this paper we investigate the systems

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{cases} \tag{2}$$

where f_n, g_n are polynomials of degree n. Note that when $\epsilon = 0$ this system has a isochronous center as in the Table 1. The main tool employed in this investigation was the averaging method described in [5]

Chicone-Jacob [8] and Buica-Llibre [5] proved that any 2-degree polynomial perturbation of system

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy, \end{cases} \tag{3}$$

has at most 2 limit cycles which bifurcate from this center using different methods.

The main result we prove in this paper is the following.

Theorem 1 *Consider system*

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{cases} \tag{4}$$

where f_n, g_n are polynomials of degree n. Then the maximum number of limit cycles that can be bifurcates from the center ($\varepsilon = 0$ in (4)) using the averaging theory of first order is:
i) 1, *if* $n = 2, 3$,
ii) $2\left[\dfrac{n-2}{2}\right]$, $n \geq 4$, *where* $[.]$ *denotes the integer part function.*

Taking in account the mentioned result from [18] and Theorem 1 we can say that the number of limit cycles in a Liénard-like perturbation of a quadratic nonlinear center is always greater or equal than a Liénard-like perturbation of a linear center. The table 1 illustrates the number of limit cycles for both cases.

This paper is organized as follows. In Section 2 we describe the results of the averaging theory that we shall need to our purposes, and briefly compare the applications of the averaging theorems for linear and no linear

Table 2 Comparing the number of limit cycles in a Liénard-like perturbation of a linear and quadratic non-linear centers according to the degree n of the perturbation.

Center type	# cycles by n-degree perturbation							
	1	2	3	4	5	6	...	m
$\dot{x} = -y$ $\dot{y} = x$	0	1	1	2	2	3	...	$\left[\frac{m}{2}\right]$
$\dot{x} = -y + x^2$ $\dot{y} = x + xy$	1	1	1	2	2	4	...	$2\left[\frac{m-2}{2}\right]$

systems. Moreover, we define the special family of perturbation we shall consider. The proof of Theorem 1 is contained in Section 3. Section 4 is reserved to Conclusions.

2 Averaging theory

In this section we briefly describe some results on periodic averaging of first order. This is the simplest form of averaging, and is concerned with approximating solutions of a non-autonomous differential equation by solutions of an autonomous one. In particular, the first order averaging method is equivalent to the study of the first order Melnikov function (both are equivalent to the study of the displacement function). For more references on this, see [26].

The averaging theory was formalized by 1930, but some naive results were conjectured even in the 18th century. For a historical description we suggest [27].

The next theorem is the classical averaging theorem for periodic differential system.

Theorem 2 *Consider the following differential system*

$$x' = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\epsilon) \tag{5}$$

where $x \in D$ (D is an open subset of $\mathbb{R}$), $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f,g are T-periodic in the variable t. Suppose that f and g are maps of class C^2. Consider the average function of $f(t,x)$ with respect to t

$$f^0(y) = \int_0^T f(t,y)\,dt. \tag{6}$$

If $p \in D$ is such that $f^0(p) = 0$ and $Df^0(p) \neq 0$, then for every $|\varepsilon| > 0$ small, there exists a T-periodic solution $\varphi_\varepsilon(t)$ of system (5) such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.

Systems in the form of (5) are said to be in the standard form. We remark that the regularity condition that f,g are of class C^2 is not really necessary, but simplify the statement. For a proof and more comments, see chapter 6 of [27].

Theorem 2 is about the existence of periodic solutions for non-autonomous systems. The following construction allows us to use this theorem for proving the existence of limit cycles.

Consider the planar system

$$\begin{cases} \dot{x} = p(x,y) + \varepsilon P(x,y) \\ \dot{y} = q(x,y) + \varepsilon Q(x,y) \end{cases} \tag{7}$$

where $p,q,P,Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In the case $p(x,y) = -y$ and $q(x,y) = x$, for $\varepsilon = 0$, system (7) is the linear center. Using a polar change of coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, we obtain

$$\begin{cases} \dot{r} = \varepsilon\left[\cos(\theta)P(v) + \sin(\theta)Q(v)\right] \\ \dot{\theta} = 1 + (\varepsilon/r)\left[\cos(\theta)Q(v) - \sin(\theta)P(v)\right], \end{cases} \tag{8}$$

where $v = (r\cos(\theta), r\sin(\theta))$.

For small $\varepsilon > 0$, the second equation of (8) is not zero. Then we can reparametrize (8) to obtain

$$\begin{cases} \dot{r} = \varepsilon\left[\cos(\theta)P(v) + \sin(\theta)Q(v)\right] + o(\varepsilon^2) \\ \dot{\theta} = 1 \end{cases} \tag{9}$$

Taking θ as the new time, (9) turns into a non-autonomous differential equation

$$r' = \varepsilon\left[\cos(\theta)P(v) + \sin(\theta)Q(v)\right] + o(\varepsilon^2) \tag{10}$$

where the prime is derivative with respect to θ.

Note that we can apply Theorem 2 to system (10). Let

$$G(r) = \int_0^{2\pi} \left[\cos(\theta)P(v) + \sin(\theta)Q(v)\right]\,d\theta.$$

By Theorem 2, each simple zero of G is associated with a periodic solution of (10) and then with a limit cycle of (7).

If P,Q are polynomials with no constant terms, P with degree M and Q with degree N, for instance,

$$\begin{aligned} P(x,y) &= \sum_{k=1}^{M} \sum_{i+j=k} a_{i,j} x^i y^j, \\ Q(x,y) &= \sum_{k=1}^{N} \sum_{i+j=k} b_{i,j} x^i y^j, \end{aligned} \tag{11}$$

where $a_{i,j}$ and $b_{i,j}$ are real constants then $G(r)$ writes as

$$G(r) = \sum_{k=1}^{M} r^k \sum_{i+j=k} a_{i,j} \int_0^{2\pi} \cos^{i+1}(\theta)\sin^j(\theta)\,d\theta + \\ + \sum_{k=1}^{N} r^k \sum_{i+j=k} b_{i,j} \int_0^{2\pi} \cos^i(\theta)\sin^{j+1}(\theta)\,d\theta.$$

Note that $G(r)$ is a polynomial with degree max $\{M,N\}$, that is, the maximum number of limit cycles of system (7) with $p(x,y) = -y$, $q(x,y) = x$ and P, Q polynomials given by (11) is $\max\{M, N\}$. In particular, the number of limit cycles depends directly on the coefficients of P, Q: the coefficient of r^k depends just on $a_{i,j}, b_{i,j}$ with $i + j = k$.

Now we turn to the general case. Our aim is to put (7) in the standard form to apply the Averaging Theorem 2.

The polar coordinates do not help anymore - in fact, they just work when the system is a linear center for $\varepsilon = 0$.

From now, we assume that (7), for $\varepsilon = 0$, has a first integral H, and a continuous family of closed orbits between the level curves $H(x,y) = h$ with $h \in (h_1, h_2)$, i.e.

$$\{\Gamma_h\} \subset \{(x,y) : H(x,y) = h, h_1 < h < h_2\}.$$

This allow us to find a change of coordinates that put (7) in the standard form.

Assume that $xp(x,y) - yp(x,y) \neq 0$ for all (x,y) in $\bigcup \Gamma_h$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ be a continuous function such that

$$H(\rho(R,\phi)\cos(\phi), \rho(R,\phi)\sin(\phi)) = R^2 \tag{12}$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\phi \in [0, 2\pi)$.

The change of coordinates $x = \rho(R,\phi)\cos(\phi)$, $y = \rho(R,\phi)\sin(\phi)$ applied to system (7) give us

$$\begin{cases} \dot{R} = \varepsilon L(R,\phi), \\ \dot{\phi} = 1 + \varepsilon S(R,\phi), \end{cases}$$

for some smooth functions L, S. Now, using the same argument as in the linear center, we can obtain a non-autonomous system in the form

$$R' = \varepsilon L(R,\phi) + o(\varepsilon^2), \tag{13}$$

where the prime is the derivative with respect to ϕ. Note that (13) is in the standard form.

Applying the Averaging Theorem 2 to (13) and writing the expression of L, we obtain the following result:

Theorem 3 ([5], Theorem 5.2) *Consider that system (7) has a first integral H for $\varepsilon = 0$, and a continuous family of closed orbits*

$$\{\Gamma_h\} \subset \{(x,y) : H(x,y) = h, h_1 < h < h_2\}. \tag{14}$$

Let $\mu(x,y)$ be an integrating factor for system (7). If

$$F(R) = \int_0^{2\pi} \frac{\mu \cdot (x^2 + y^2) \cdot (Pq - Qp)}{2R \cdot (qx - py)}\, d\phi, \tag{15}$$

where μ, P, p, Q, q depends on $x = \rho(R,\phi)\cos(\phi)$ and $y = \rho(R,\phi)\sin(\phi)$, then each simple zero of $F(R)$ give us a limit cycle of (7).

Remark 1 The integrand of $F(R)$ is exactly the function $L(R,\phi)$ we defined above.

Now we workout an example that shows that the dependence of F on P, Q is not so directly as in the linear case.

Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + x^2 + \varepsilon c x y^2, \\ \dot{y} = x + xy + \varepsilon(a x^2 + b y^3), \end{cases} \tag{16}$$

where $\varepsilon > 0$ is a small parameter and $a, b, c \in \mathbb{R}$. For $\varepsilon = 0$, this system has a center at the origin.

The function F given by (15) is

$$F(Z) = \rho(Z)(2aZ^2 + (a + c - 3b)Z - 2c),$$

where $R = \sqrt{1 - Z^2}$ and ρ is a C^1 function without zeroes for $Z \in (0,1)$.

Note that the leader coefficient in the polynomial $d(Z) = \dfrac{F(Z)}{\rho(Z)}$ is $2a$, but a is not associated with the higher degree terms in (16); furthermore, the coefficient associated to the third order term y^3 in the second line of (16) appears just on the coefficient of the linear term of d.

This indirect dependence of F on the coefficients of the perturbation make difficult to consider general perturbations of non-linear centers.

As we want to study systems with perturbations of arbitrary degree, we fix a system, given by

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{cases}$$

where f_n, g_n are polynomials of degree n.

We call this perturbation Liénard like due to its similarity with the classical Liénard system.

3 Proof of Theorem 1

Recall system (4)

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{cases}$$

with f_n, g_n polynomials of degree n.

A first integral H and an integrating factor μ of the quadratic center (3) have the expression $H(x,y) = \dfrac{x^2 + y^2}{(1+y)^2}$ and $\mu = \dfrac{2}{(1+y)^3}$, respectively, for more details see [7]. For this system we note that $h_1 = 0$ and $h_2 = 1$, $\rho = \dfrac{R}{1 - R\sin(\psi)}$ is solution of $H(\rho\cos(\psi), \rho\sin(\psi)) = R^2$.

The integrand in expression (15), for this case, is given by

$$L(R,\psi) = \zeta(R,\psi) f_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right) + \eta(R,\psi) g_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right), \tag{17}$$

where

$$\zeta(R,\psi) = -R^2\sin(\psi) - R\cos^2(\psi) + R,$$

$$\eta(R,\psi) = R^2\sin(\psi) - 2R + R\cos^2(\psi) + \sin(\psi).$$

Finally, the first Melnikov function (given by (15)) is

$$F(R) = \int_0^{2\pi}\left(\zeta(R,\psi) f_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right) + \eta(R,\psi) g_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)\right) d\psi. \tag{18}$$

Note that we are interested in the isolated zeros of $F(R)$ with $R \in (0,1)$ (as the center is contained in $H^{-1}((0,1))$.

Put $f_n(x) = \sum_{j=1}^n a_j x^j$ and $g_n(x) = \sum_{j=1}^n b_j x^j$. Then $F(R)$ is given by

$$\begin{aligned} F(R) &= \int_0^{2\pi}\left(\zeta(R,\psi)\sum_{j=1}^n a_j\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)^j \right.\\ &\quad \left. + \eta(R,\psi)\sum_{j=1}^n b_j\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)^j\right) d\psi \\ &= \int_0^{2\pi}\left(\zeta(R,\psi)\sum_{j=1}^n a_j\frac{R^j\cos^j(\psi)}{(1-R\sin(\psi))^j}\right.\\ &\quad \left. + \eta(R,\psi)\sum_{j=1}^n b_j\frac{R^j\cos^j(\psi)}{(1-R\sin(\psi))^j}\right) d\psi \\ &= -\sum_{j=1}^n a_j R^{j+2}\int_0^{2\pi}\frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad -\sum_{j=1}^n a_j R^{j+1}\int_0^{2\pi}\frac{\cos^{j+2}(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad +\sum_{j=1}^n a_j R^{j+1}\int_0^{2\pi}\frac{\cos^{j}(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad +\sum_{j=1}^n b_j R^{j+2}\int_0^{2\pi}\frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad +\sum_{j=1}^n b_j R^{j+1}\int_0^{2\pi}\frac{\cos^{j+2}(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad -2\sum_{j=1}^n b_j R^{j+1}\int_0^{2\pi}\frac{\cos^{j}(\psi)}{(1-R\sin(\psi))^j}\,d\psi \\ &\quad +\sum_{j=1}^n b_j R^{j}\int_0^{2\pi}\frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\,d\psi. \end{aligned} \tag{19}$$

Denote

$$J_{\alpha,\beta,\gamma}(R) = \int_0^{2\pi}\frac{\sin^\alpha(\psi)\cos^\beta(\psi)}{(1-R\sin(\psi))^\gamma}\,d\psi. \tag{20}$$

The next lemmas make explicit expressions for $J_{1,j,j}(R)$, $J_{0,j+2,j}(R)$ and $J_{0,j,j}(R)$. We just present the proofs for two of them, the others are similar.

Lemma 1 *$J_{0,2l+1,2l+1}(R) \equiv 0$, for all l.*

Proof. Note that

$$\begin{aligned} J_{0,2l+1,2l+1}(R) &= \int_0^{2\pi}\frac{\cos^{2l+1}(\psi)}{(1-R\sin(\psi))^{2l+1}}\,d\psi \\ &= \int_0^{2\pi}\frac{\cos^{2l}(\psi)\cos(\psi)}{(1-R\sin(\psi))^{2l+1}}\,d\psi \\ &= \int_0^{1}\frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} \\ &\quad -\int_{-1}^{1}\frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} \\ &\quad +\int_{-1}^{0}\frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} \\ &= 0. \end{aligned}$$

□

Lemma 2 *$J_{0,2l,2l}(R) = \lambda\dfrac{u_{l-1}(Z)}{Z^{2l-1}(1+Z)^l}$, for all l, where $Z = \sqrt{1-R^2}$, λ is some constant and u_j is a j-degree polynomial.*

Proof. If we write $z = \tan(\psi/2)$ then $\cos(\psi) = \dfrac{1-z^2}{1+z^2}$, $\sin(\psi) = \dfrac{2z}{1+z^2}$, $d\psi = \dfrac{2\,dz}{1+z^2}$; then we have to solve

$$\int_{-\infty}^{\infty}\frac{2(1-z^2)^{2l}}{(1+z^2)(z^2-2Rz+1)^{2l}}\,dz.$$

We proceed using an well-know application of the Residue Theorem [1].

Lemma 3 ([1], section 5.3) *If G is a rational function, an integral of the form $\int_{-\infty}^{\infty} G(x)\,dx$ converges if and only if the degree of the denominator of G is at least two units higher than the degree of the numerator, and if no poles lies on the real axis. In this case,*

$$\int_{-\infty}^{\infty} G(x)\,dx = 2\pi i\sum_j \mathrm{Res}_{w_j} G, \tag{21}$$

where $\mathrm{Res}_{w_j} G$ is the residue of G on the pole w_j, and the summation is done over all poles in the upper half plane.

Remark 2 Obviously, the complex product in (21) is a real number.

Let

$$G(w)=\frac{2(1-w^2)^{2l}}{(1+w^2)(w^2-2Rw+1)^{2l}}$$

where w is a complex variable.

The poles on the upper half plane are $w_1=i$ (simple pole) and $w_2=R+i\sqrt{1-R^2}$ (pole of order $2l$). We have to compute the residues of G over these poles.

Lemma 4 ([1]) *The point w_0 is a pole of order $m\geq 1$ for G if and only if $G(w)=\dfrac{\phi(w)}{(w-w_0)^m}$ for some analytic function ϕ. In this case, the residue of G over w_0 is given by*

$$Res_{w_0}G=\frac{\phi^{(m-1)}(w_0)}{(m-1)!},$$

where $\phi^{(m-1)}$ is the $(m-1)$-derivative of ϕ.

The residue over w_1 is easy to compute: we write

$$G_1(w)=(w-i)G(w)=\frac{2(1-w^2)^{2l}}{(w+i)(w^2-2Rw+1)^{2l}}$$

and then $Res_{w_1}G=G_1(w_1)=\dfrac{i(-1)^{l+1}}{R^{2l}}$.

Now let $\rho_1=R+i\sqrt{1-R^2}$ and $\rho_2=R-i\sqrt{1-R^2}$. Then

$$G(w)=\frac{1}{(w-\rho_1)^{2l}}\frac{2(1-w^2)^{2l}}{(1+w^2)(w-\rho_2)^{2l}}.$$

If we define $\phi(w)=\dfrac{2(1-w^2)^{2l}}{(1+w^2)(w-\rho_2)^{2l}}$, then to obtain the residue of G over w_1 we need to compute $\phi^{(2l-1)}(\rho_1)$.

For $l=1$, $\phi(w)=\dfrac{2(1-w^2)^2}{(1+w^2)(w-\rho_2)^2}$ and

$$\phi^{(1)}(\rho_1)=\frac{i\left(-i+iR^2-R\sqrt{1-R^2}\right)^2}{\left(iR-\sqrt{1-R^2}\right)^2R^2\left(1-R^2\right)^{3/2}}$$

Using the change $Z=\sqrt{1-R^2}$ in both residues we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{2(1-z^2)^2}{(1+z^2)(z^2-2Rz+1)^2}\,dz\\&=2\pi i\left(\frac{i}{1-Z^2}+\frac{-i\left(iZ+\sqrt{1-Z^2}\right)^2}{Z\left(i\sqrt{1-Z^2}-Z\right)^2(-1+Z^2)}\right)\\&=\frac{2\pi\left(-2Z^2+2i\sqrt{1-Z^2}Z+1\right)}{\left(i\sqrt{1-Z^2}-Z\right)^2(Z+1)Z}\\&=\frac{2\pi}{Z(Z+1)}\end{aligned}$$

where $Z=\sqrt{1-R^2}$. So the case $l=1$ is done. The case $l>1$ is similar. The specific degree in the statement is obtained when we simplify the sum of residues. □

Lemma 5 *$J_{0,2l+3,2l+1}(R)\equiv 0$, for all l.*

Lemma 6 *$J_{0,2,0}(R)=\pi$, $J_{0,4,2}(R)=\dfrac{3\pi}{(1+Z)^2}$,*

$J_{0,2l+2,2l}(R)=\dfrac{2l+1}{2^{l-1}}\dfrac{v_{l-2}(Z)}{(Z+1)^{l+1}Z^{2l-3}}$ for all $l\geq 2$, where $Z=\sqrt{1-R^2}$ and v_j is a j-degree polynomial.

Lemma 7 *$J_{1,2l+1,2l+1}(R)\equiv 0$, for all l.*

Lemma 8 *$J_{1,2,2}(R)=\dfrac{2\pi(1-Z)}{RZ(1+Z)}$,*

$J_{1,2l,2l}(R)=\dfrac{\lambda_l\pi w_l(Z)}{R(1+Z)^lZ^{2l-1}}$, for all $l\geq 2$, where $Z=\sqrt{1-R^2}$ and w_j is a j-degree polynomial.

Now we apply Lemmas 1-8 to simplify (19).

Lemma 9 *The coefficients a_j,b_j of f_n,g_n with j odd don't contribute to (19).*

Proof. Just note that these coefficients are associated to the integrals $J_{0,2l+1,2l+1}$ or $J_{0,2l+3,2l+1}$ or $J_{1,2l+1,2l+1}$, and according to Lemmas (1), (5), (7), these integrals vanishes. □

By Lemma (9) we may consider f_n,g_n even degree polynomials; moreover, we may take these polynomials even functions, that is, without odd degree terms. So from now on we consider just this case.

Lemma 10 *Consider $n=2$ in (19). Then*

$$F(Z)=\pi\frac{(Z-1)^2((2a_2-2b_2)Z-a_2-b_2)}{\sqrt{1-Z^2}},$$

where $Z=\sqrt{1-R^2}$. The equation $F(Z)=0$ has exactly one solution for $Z\in(0,1)$ when $a_2\neq b_2$ and $0<\dfrac{a_2+b_2}{2a_2-2b_2}<1$; otherwise there is no solution.

Proof. Straightforward calculations. □

Lemma 11 *Consider $n=2m$ $(m>1)$ in (19). Then the numerator of $R\cdot F(Z)$ is given by*

$$\begin{aligned}\mathrm{numer}(R\cdot F(Z))=&\sum_{j=0}^{m}\alpha_{2j}(a,b)Z^{2j}\\&+\beta_{2m-3}(a,b)Z^{2m-3}\\&+\beta_{2m-1}(a,b)Z^{2m-1}\\&=(Z-1)^2\sum_{j=0}^{2m-2}\mu_j(a,b)Z^j,\end{aligned}$$

where $Z=\sqrt{1-R^2}$ and $\alpha_l(a,b)$, $\beta_s(a,b)$, $\mu_j(a,b)$, depend on a_j,b_j (recall that j is always even), while the denominator is a function without zeroes in the interval $(0,1)$. In particular, the equation $F(Z)=0$ have at most $2m-2$ solutions for $Z\in(0,1)$.

Proof of Theorem 1. Consider system (1) and the Melnikov function associated to it, as defined in (18). We must to find zeros of the function F in the interval $(0,1)$. It follows from Lemmas 9-10 that $F(Z)=0$ has at most one solution in $(0,1)$ if $n=2,3$. From Lemma 9-11 for $n \geq 4$, the number of zeros of F is $n-2$, if n is even and $n-3$, if n is odd.

From Theorem 2 the maximum number of limit cycles that bifurcates from the center (3) by a Liénard-like perturbation and applying the averaging theory is 1, if $n=2,3$, and $2\left[\dfrac{n-2}{2}\right]$, $n \geq 4$, where [.] denotes the integer part function. □

4 Examples

Example 1 (One limit cycle) Consider the system

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon a_2 y x^2, \end{cases} \tag{22}$$

where $\varepsilon > 0$ is a small parameter and a_2 is a real constant. Then

$$F(Z) = \pi (Z-1)^2 \left(\frac{2a_2 Z - a_2}{\sqrt{1-Z^2}} \right)$$

The equation $F(Z)=0$ has at most one solution. If we take, for instance, $a_2 = \frac{1}{2}$, the unique solution is $Z = \frac{1}{2}$. So we have just one limit cycle.

In the Figure 1 we show two trajectories of system (22) with the above mentioned choice of a_2 and $\varepsilon = 1/100$. As:
i) the outer trajectory pass through the point $(0;10)$ and is decreasing the radius;
ii) the inner trajectory pass through the point $(0;5)$ and is decreasing the radius,
then the limit cycle should be located between these trajectories (by the Poincar-Bendixson Theorem, see [2]).

Example 2 (Two limit cycles) Consider the system

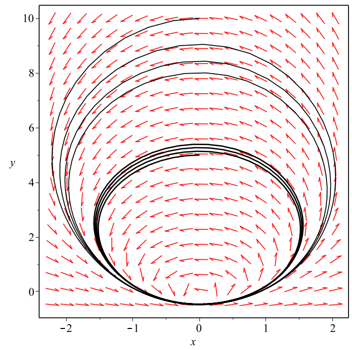
$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon \left(y(a_2 x^2 + a_4 x^4) + (b_2 x^2 + b_4 x^4) \right), \end{cases} \tag{23}$$

where $\varepsilon > 0$ is a small parameter and a_2, a_4, b_2, b_4 are real constants. Then

$$F(Z) = \pi (Z-1)^2 \left(\frac{(-3a_4 + 2a_2 + 3b_4 - 2b_2) Z^2}{Z\sqrt{1-Z^2}} + \frac{(5a_4 - a_2 - b_2) Z - 2a_4}{Z\sqrt{1-Z^2}} \right)$$

The equation $F(Z)=0$ has at most two solutions for $Z \in (0,1)$. In particular, if

Fig. 3 Some trajectories of system (22), with initial conditions $(0;5)$ and $(0;10)$.



$\Delta = {a_4}^2 - 6\,a_4 b_4 + 6\,a_4 a_2 - 26\,a_4 b_2 + 9\,{b_4}^2 + 6\,b_4 a_2 + 6\,b_4 b_2 + {a_2}^2 + 2\,a_2 b_2 + {b_2}^2 > 0$,
$\Gamma = -5\,a_4 + 3\,b_4 + a_2 + b_2$ and
$\zeta = 2(3\,b_4 - 2\,b_2 + 2\,a_2 - 3\,a_4) \neq 0$,
the solutions are $Z_{\pm} = \dfrac{\Gamma \pm \sqrt{\Delta}}{\zeta}$. For instance, choosing $a_2 = \frac{87}{50}$, $a_4 = \frac{7}{100}$, $b_2 = \frac{1}{2}$ and $b_4 = -\frac{169}{300}$ we obtain two solutions.

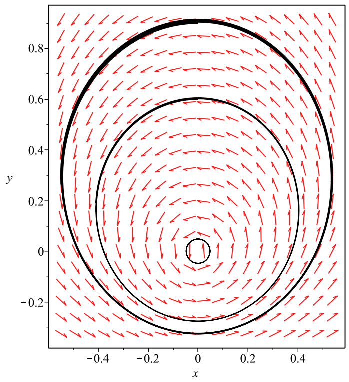
In the Figure 2 we show three trajectories of system (23) with the above mentioned choice of coefficients and $\varepsilon = 1/100$. As:
i) the outer trajectory pass through the point $(0;0.9)$ and is decreasing the radius;
ii) the middle trajectory pass through the point $(0;0.6)$ and is increasing the radius and;
iii) the inner trajectory pass through the point $(0;0.05)$ and is decreasing the radius,
then the two limit cycles should be located between these trajectories (by the Poincar-Bendixson Theorem, see [2]).

The estimative in the last lemma that the equation $F(Z)=0$ have at most $2m-2$ solutions is just based on the degree of $\dfrac{R \cdot F(Z)}{\pi(Z-1)^2}$ and is not sharp.

Example 3 (At most three limit cycles) Consider the system

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy + \varepsilon \left(y(a_2 x^2) \right. \\ \quad \left. + (b_2 x^2 + b_4 x^4 + b_6 x^6) \right), \end{cases} \tag{24}$$

Fig. 4 Some trajectories of system (23), with initial conditions (0; 0.9), (0; 0.6) and (0; 0.05).



where $\varepsilon > 0$ is a small parameter and $a_2, a_4, a_6, b_2, b_4, b_6$ are real constants. Then

$$\begin{aligned} G_6(Z) &= \frac{R \cdot F(Z)}{\pi(Z-1)^2} \\ &= (12b_4 - 15b_6 + 15a_6 - 12a_4 - 8b_2 + 8a_2)\, Z^4 \\ &+ (30b_6 - 12b_4 - 4b_2 + 20a_4 - 4a_2 - 38a_6)\, Z^3 \\ &+ (-15b_6 - 8a_4 + 29a_6)\, Z^2 - 4a_6 Z - 2a_6 \end{aligned}$$

The degree of polynomial G_6 is 4, but we cannot choose coefficients such that this polynomial has 4 roots in the interval $(0, 1)$. In this case we have just 3 roots.

5 Conclusions

In this paper we present an estimative for a number of limit cycles of one quadratic non-linear center in a Liénard-like perturbation. The exact number of these limit cycles is unknown but we have the following conjecture about this number using the averaging theory.

Remark 3 (Conjecture) Consider system (4) with f_n, g_n polynomials of degree n. We conjecture that the maximum number of limit cycles (solutions of $F(Z) = 0$ for $Z \in (0, 1)$) is $n - 3$ when n is odd and $n - 4$ when n is even.

As far as we know an estimate for the number of limit cycles to each one of the other 3 families of quadratic isochronous center in Table (1) is unknown except to the case $n = 2$, see [8].

We note that the difficult in proving general quotas is common in papers dealing with perturbation of non-linear centers. Note that in the proof of Theorem 1 (in special in the proof of Lemma 2) the degree of u_{l-1} was easy obtained, but its exact dependence on the coefficients is hard to determine. Similar problems are found in [6].

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