AVERAGING THEORY FOR DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS

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ABSTRACT. We develop the averaging theory of first and second order for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems with the minimal conditions of differentiability. We also provide two applications.

1. Introduction and statement of the main results

1.1. **Introduction.** In these last years a big interest has appeared for studying discontinuous differential systems, that is differential equations with discontinuous right–hand sides. This interest has been stimulated by discontinuous phenomena in control systems [2], impact and friction mechanics [7], nonlinear oscillations [1, 20], economics [12, 13], and biology [3, 15], and it has become certainly one of the common frontiers between Mathematics, Physics and Engineering. For more details see Teixeira [22]. A recent review appears in [24].

One of the main problem in the qualitative theory of differential systems is the study of their periodic solutions. A good tool to study the periodic solutions is the averaging theory, see for instance the books of Sanders, Verhulst, and Murdock [21] and Verhulst [23]. We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behaviour of nonlinear smooth dynamical systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [10]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [6] in the 1930s and Bogoliubov [5] in 1945.

The classical results in the averaging theory require, at least, that the systems are of class $\mathcal{C}^2$. Nevertheless, Buica and Llibre in [9], using mainly topological tools as the Brouwer degree theory, extended the averaging theory up to order 3 for studying periodic orbits of continuous Lipschitz differential systems. Their results were generalized for any order in [16, 17]. Recently, the theory of regularization was used, by Llibre, Novaes and Teixeira in [18], to developed the averaging theory of first order for studying periodic orbits of discontinuous piecewise differential systems with two systems (pieces).

Here we develop the averaging theory of first and second order for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems (pieces). We generalize the results established in [9, 18] considering minimal conditions of differentiability. Furthermore, we use this theory to study the planar

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linear centers perturbed by discontinuous piecewise differential systems when the set of discontinuity is composed by rays passing through the origin and when the set of discontinuity is a parabola.

1.2. **Preliminaries.** In what follows we define the necessary elements for the statements of our main results.

Let D be an open subset of $\mathbb{R}^d$ and $\mathbb{S}^1 = \mathbb{R}/T$ for some period $T > 0$. We consider a finite set of ODE's

$$x'(t) = f^n(t,x), \quad (t,x) \in I \times D \quad \text{for} \quad n = 1, 2, \ldots, M, \tag{1}$$

where $f^n : \mathbb{S}^1 \times D \to \mathbb{R}^d$ is a continuous function. Here the prime denotes derivative with respect to the time t. Let (S_n) be a finite sequence of open disjoints subset of $\mathbb{S}^1 \times D$ for $n = 1, 2, \ldots, M$. We suppose that the boundaries of each S_n are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$. Furthermore the union of all boundaries, denoted by Σ, and all S_n together cover $\mathbb{S}^1 \times D$. So we define a *M–Discontinuous Piecewise Differential System* (*M–DPDS*) as

$$x'(t) = f(t,x) = \begin{cases} f^1(t,x), & (t,x) \in \overline{S}_1, \\ f^2(t,x), & (t,x) \in \overline{S}_2, \\ & \vdots \\ f^M(t,x), & (t,x) \in \overline{S}_M, \end{cases} \tag{2}$$

where $\overline{S}_k$ denotes the closure of S_k in D. When the context is clear we shall refer to the systems of kind (2) only by *DPDS*. Later on in this paper it will be assumed that the functions f^n are Lipschitz in the second variable for $n = 1, 2, \ldots, M$. However the theory described in the following is developed without these assumptions.

Let A be a subset of $\mathbb{S}^1 \times D$ and let $\chi_A(t,x)$ be the *characteristic function* defined as

$$\chi_A(t,x) = \begin{cases} 1 & \text{if} \quad (t,x) \in A, \\ 0 & \text{if} \quad (t,x) \notin A. \end{cases}$$

So system (2) can be written as

$$x'(t) = f(t,x) = \sum_{n=1}^{M} \chi_{\overline{S}_n}(t,x) f^n(t,x), \quad (t,x) \in \mathbb{S}^1 \times D. \tag{3}$$

We stress that systems (2) and (3) does not coincides in Σ. Indeed system (2) is multivalued in Σ whereas system (3) is single valued in Σ. Nevertheless the Filippov's convention for the solutions of these systems (see [11]) passing through a point $(t,x) \in \Sigma$ does not depend on the value $f(t,x)$. So the solutions of systems (2) and (3), in the sense of Filippov, are the same.

We say that a point $p \in \Sigma$ is a *generic point of discontinuity* if there exists a neighborhood $G_p \subset \mathbb{S}^1 \times D$ of p such that $\mathcal{S}_p = G_p \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface in $\mathbb{S}^1 \times D$ with $k \geq 1$. In this case we can always assume that $\mathcal{S}_p$ splits $G_p \backslash \mathcal{S}_p$ in two disconnected regions, namely G_p^+ and G_p^-, and that the vector fields $f_p^+ = f|_{G_p^+}$ and $f_p^- = f|_{G_p^-}$ are continuous. We define $l(p)$ as the segment connecting the vectors $f_p^+(p)$ and $f_p^-(p)$ when they have the same origin p (see Figures 1 and 2).

An embedded hypersurface $\mathcal{S} \subset \Sigma$ can be decomposed as the union of the closure of its *crossing* $\Sigma^c(\mathcal{S})$ (see Figure 1) and *sliding* $\Sigma^s(\mathcal{S})$ (see Figure 2) regions, which are defined as

$$\Sigma^c(\mathcal{S}) = \{p \in \mathcal{S} : \, l(p) \cap T_p\mathcal{S} = \emptyset\} \quad \text{and} \quad \Sigma^s(\mathcal{S}) = \{p \in \mathcal{S} : \, l(p) \cap T_p\mathcal{S} \neq \emptyset\},$$

where as usual $T_p\mathcal{S}$ denotes the tangent space of $\mathcal{S}$ at the point p. If $\mathcal{S} = h^{-1}(0)$ for some $\mathcal{C}^r$ $(r \geq 1)$ function $h : \mathbb{S}^1 \times D \to \mathbb{R}$ which has 0 as a regular value, then the above definitions become

$$\begin{aligned} \Sigma^c(\mathcal{S}) &= \big\{p \in \mathcal{S} : \langle \nabla h(p), (1, f^+(p))\rangle \langle \nabla h(p), (1, f^-(p))\rangle > 0\big\} \quad \text{and} \\ \Sigma^s(\mathcal{S}) &= \big\{p \in \mathcal{S} : \langle \nabla h(p), (1, f^+(p))\rangle \langle \nabla h(p), (1, f^-(p))\rangle < 0\big\}. \end{aligned} \tag{4}$$

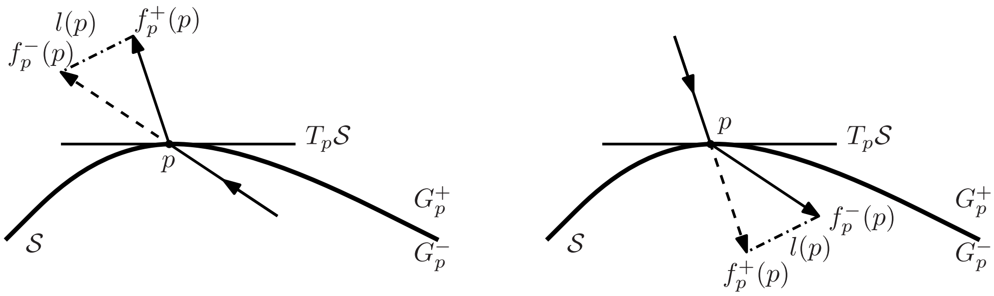

FIGURE 1. Crossing region of $\mathcal{S}$: $\Sigma^c(\mathcal{S})$.

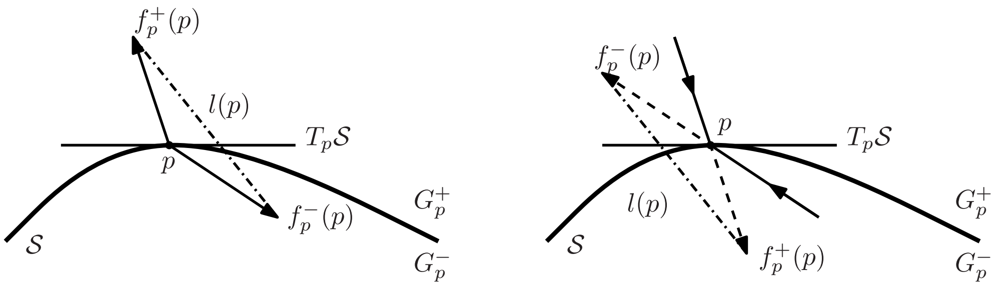

FIGURE 2. Sliding region of $\mathcal{S}$: $\Sigma^s(\mathcal{S})$.

The *crossing region* Σ^c is defined as the generic points of discontinuity p such that $p \in \Sigma^c(\mathcal{S}_p)$. Analogously, we define the *sliding region* Σ^s.

Let $\varphi_{f^n}(t, q)$ be the solution of system (1) passing through the point $q \in S_n$ at time $t = 0$, that is $\varphi_{f^n}(0, q) = q$. The local solution $\varphi_f(t, q)$ of system (3) passing through a point $p \in \Sigma^c$ at time $t = 0$ is given by the Filippov convention, that is for $p \in \Sigma^c$ such that $l(p) \subset G_p^+$ and taking the origin of time at p, the trajectory through p is defined as $\varphi_f(t, p) = \varphi_{f_p^-}(t, p)$ for $t \in I_p \cap \{t < 0\}$, and $\varphi_f(t, p) = \varphi_{f_p^+}(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $l(p) \subset G_p^-$ the definition is the same reversing the time. Here I_p is an open interval having the 0 in its interior.

The following proposition gives a condition for the existence and uniqueness of solutions of system (3).

Proposition 1. *For every point $p \in \Sigma^c$ there is a solution passing either from G_p^- into G_p^+, or from G_p^+ into G_p^-, and uniqueness in not violated.*

For a proof of Proposition 1 see Corollary 1 of Section 10 of chapter 2 of [11].

Assuming that the functions $f^n(t,x)$ are Lipschitz in the variable x for $n = 1, 2, \ldots, N$, Proposition 1 implies the uniqueness of the solutions reaching the set of discontinuity only at points of Σ^c.

1.3. **Statements of the main results.** We consider the following DPDS.

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{5}$$

with

$$F_i(t,x) = \sum_{j=1}^{M} \chi_{\overline{S}_j}(t,x) F_i^j(t,x), \quad \text{for } i = 1,2, \text{ and}$$

$$R(t,x,\varepsilon) = \sum_{j=1}^{M} \chi_{\overline{S_j}}(t,x) R^j(t,x),$$

where $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ for $i = 1,2$ and $j = 1, 2, \ldots, M$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^d$. For $i = 1,2$ we denote

$$D_x F_i(t,z) = \sum_{j=1}^{M} \chi_{\overline{S}_j}(t,z) D_x F_i^j(t,z). \tag{6}$$

In order to state our main results we define the averaged functions $f_1, f_2 : D \to \mathbb{R}^d$ as

$$f_1(z) = \int_0^T F_1(t,z)dt, \quad \text{and} \tag{7}$$

$$f_2(z) = \int_0^T \Big(D_x F_1(t,z) y_1(t,z) + F_2(t,z) \Big) dt, \tag{8}$$

where

$$y_1(t,z) = \int_0^t F_1(s,z)ds.$$

Moreover we state the next condition which is common for our main results.

(HC) There exists an open bounded set $C \subset D$ such that for each $z \in \overline{C}$ the curve $\{(t,z) : t \in \mathbb{S}^1\}$ reaches transversely the set Σ and only at generic points of discontinuity.

The principal consequence of assumption (HC) is the following:

Proposition 2. *The assumption (HC) implies that, for $|\varepsilon| \neq 0$ sufficiently small, every solution of* (5) *starting in $\overline{C}$ reaches the set of discontinuity Σ only at its crossing region.*

Proposition 2 is proved in Section 3.

Our main results on the periodic orbits of DPDS (5) are given in the next two theorems. Their proofs use the Brouwer degree theory for finite dimensional spaces (see the appendix for a definition of the Brouwer degree $d_B(f, V, 0)$).

Theorem A (First order averaging theorem for DPDS)**.** *In addition to the crossing hypothesis (HC) assume the following conditions.*

$(Ha1)$ *For $i = 1,2$ and $j = 1, 2, \ldots, M$, the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x, and T–periodic with respect to the time t. Furthermore, for $j = 1, 2, \ldots, M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.*

($Ha2$) *For* $a^* \in C$ *with* $f_1(a^*) = 0$, *there exist a neighborhood* $U \subset C$ *of* a^* *such that* $f_1(z) \neq 0$ *for all* $z \in \overline{U}\backslash\{a^*\}$ *and* $d_B(f_1, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ *sufficiently small, there exists a* T*–periodic solution* $x(t, \varepsilon)$ *of system* (5) *such that* $x(0, \varepsilon) \to a^*$ *as* $\varepsilon \to 0$.

Theorem A is proved in Section 3.

Theorem B (Second order averaging theorem for DPDS)**.** *Suppose that* $f_1(z) \equiv 0$. *In addition to the crossing hypothesis* (HC) *assume the following conditions.*

($Hb1$) *For* $j = 1, 2, \ldots, M$, *the functions* $F_1^j(t, \cdot)$ *are of class* $\mathcal{C}^1$ *for all* $t \in \mathbb{R}$; *for* $j = 1, 2, \ldots, M$, *the functions* $D_x F_1^j$, F_2^j *and* R *are locally Lipschitz with respect to* x. *Furthermore, for* $j = 1, 2, \ldots, M$, *the boundaries of* S_j *are piecewise* $\mathcal{C}^k$ *embedded hypersurfaces with* $k \geq 1$.

($Hb2$) *If* $(t, z) \in \Sigma$ *then* $\big(0, y_1(t, z)\big) \in T_{(t,z)}\Sigma$.

($Hb3$) *For* $a^* \in C$ *with* $f_2(a^*) = 0$, *there exist a neighborhood* $U \subset C$ *of* a^* *such that* $f_2(z) \neq 0$ *for all* $z \in \overline{U}\backslash\{a^*\}$ *and* $d_B(f_2, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ *sufficiently small, there exists a* T*–periodic solution* $x(t, \varepsilon)$ *of system* (5) *such that* $x(0, \varepsilon) \to a^*$ *as* $\varepsilon \to 0$.

Theorem B is also proved in Section 3.

We remark that when f_1 (resp. f_2) is a $\mathcal{C}^1$ function the assumption "there exists $a^* \in V$ such that $f_1(a^*) = 0$ (resp. $f_2(a^*) = 0$) and $\det(f_1'(a^*)) \neq 0$ (resp. $\det(f_2'(a^*)) \neq 0$)" is a sufficient condition to guarantees the validity of the hypothesis ($Ha2$) (resp. ($Hb3$)).

1.4. **Discontinuous perturbation of planar linear centers.** In this subsection we show how to use the Theorems A and B for studying the linear centers perturbed by DPDS having the set of discontinuity composed by rays passing through the origin of coordinates. In other words we shall show that the hypothesis of crossing (HC) and the hypothesis ($Hb2$) of Theorem B always hold for such systems after a change of variables and a time–rescaling.

Let M be a positive integer greater than 1, let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M) \in \mathbb{T}^M$ (M–Torus) be a M–tuple of angles such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_M < 2\pi$ and let $\mathcal{X} = (X_1, X_2, \cdots, X_M)$ be a M–tuple of locally Lipschitz vector fields defined on an open neighborhood $D \subset \mathbb{R}^2$ of the origin.

We define the set of discontinuity $\Sigma = \bigcup_{i=1}^M L_i$, where L_i for $i = 1, 2, \ldots, M$, is the intersection between the ray starting at the origin and passing through the point $(\cos\alpha_i, \sin\alpha_i)$ with the set D. We note that the set Σ splits the set $D\backslash\Sigma \subset \mathbb{R}^2$ in M disjoint open sectors. We denote the sector delimited by L_i and L_{i+1} by C_i for $i = 1, 2, \ldots, M$.

Now let $Z_{\mathcal{X},\alpha}(x, y)$ be the DPDS defined in D as

$$Z_{\mathcal{X},\alpha}(x, y) = X_i(x, y) \quad \text{if} \quad (x, y) \in C_i.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two M–tuples of vector fields. We shall study the following DPDS.

$$(\dot{x}\,,\,\dot{y}) = (y\,,\,-x) + \varepsilon Z_{\mathcal{X},\alpha}(x, y) + \varepsilon^2 Z_{\mathcal{Y},\alpha}(x, y). \tag{9}$$

Here the dot denotes derivative with respect to the time t.

Using the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (9) becomes equivalent to

$$\left(\dot{\theta}\,,\,\dot{r}\right) = (-1, 0) + \varepsilon A(\theta, r) + \varepsilon^2 B(\theta, r), \tag{10}$$

where A and B are DPDS with the set of discontinuity $\widetilde{\Sigma}$ being the union of the rays $\{(\alpha_i, r) : r > 0\}$ for $i = 1, 2, \ldots, M$. Moreover $A(\theta, r) = A_i(\theta, r)$ and $B(\theta, r) = B_i(\theta, r)$ if $\alpha_i \le \theta \le \alpha_{i+1}$ for $i = 1, 2, \ldots, M$, where $\alpha_{M+1} = \alpha_1$, and

$$A_i(\theta, r) = \Bigg(\frac{1}{r} \left(X_i^2 (r\cos\theta, r\sin\theta)\cos\theta - X_i^1 (r\cos\theta, r\sin\theta)\sin\theta \right),$$
$$X_i^1 (r\cos\theta, r\sin\theta)\cos\theta + X_i^2 (r\cos\theta, r\sin\theta)\sin\theta \Bigg)$$

and

$$B_i(\theta, r) = \Bigg(\frac{1}{r} \left(Y_i^2 (r\cos\theta, r\sin\theta)\cos\theta - Y_i^1 (r\cos\theta, r\sin\theta)\sin\theta \right),$$
$$Y_i^1 (r\cos\theta, r\sin\theta)\cos\theta + Y_i^2 (r\cos\theta, r\sin\theta)\sin\theta \Bigg).$$

Here $X_i = (X_i^1, X_i^2)$ and $Y_i = (Y_i^1, Y_i^2)$ for $i = 1, 2, \ldots, M$.

Taking θ as the new time system (10) writes

$$\frac{dr}{d\theta} = \frac{\dot r}{\dot\theta} = \frac{\varepsilon A_i^2(\theta, r) + \varepsilon^2 B_i^2(\theta, r)}{-1 + \varepsilon A_i^1(\theta, r) + \varepsilon^2 B_i^1(\theta, r)},$$

for $\alpha_i \le \theta \le \alpha_{i+1}$. Here $A_i = (A_i^1, A_i^2)$ and $B_i = (B_i^1, B_i^2)$ for $i = 1, 2, \ldots, M$. So system (10) and consequently system (9) become equivalent to

$$r' = \mathcal{R}(\theta, r, \varepsilon), \tag{11}$$

where, for $i = 1, 2, \ldots, M$, $\mathcal{R}(\theta, r, \varepsilon) = \mathcal{R}_i(\theta, r, \varepsilon)$ if $\alpha_i \le \theta \le \alpha_{i+1}$, and

$$\begin{aligned}\mathcal{R}_i(\theta, r, \varepsilon) = \; & -\varepsilon \left(X_i^1 (r\cos\theta, r\sin\theta)\cos\theta + X_i^2 (r\cos\theta, r\sin\theta)\sin\theta \right) \\ & -\varepsilon^2 \Big(\frac{1}{r} \left(X_i^2 (r\cos\theta, r\sin\theta)\cos\theta - X_i^1 (r\cos\theta, r\sin\theta)\sin\theta \right) \\ & \cdot \left(X_i^1 (r\cos\theta, r\sin\theta)\cos\theta + X_i^2 (r\cos\theta, r\sin\theta)\sin\theta \right) \\ & + \left(Y_i^1 (r\cos\theta, r\sin\theta)\cos\theta + Y_i^2 (r\cos\theta, r\sin\theta)\sin\theta \right) \Big) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

Now the prime denotes derivative with respect to the time θ.

Proposition 3. *The hypotheses* (HC) *and* $(Hb2)$ *hold for system* (11).

Proof. The assumption (HC) holds because the set of discontinuity of system (11) is the union of the rays $\widetilde{\Sigma}_i = \{(\alpha_i, r) : r > 0\}$ for $i = 1, 2, \ldots, M$. Let $h_i(\theta, r) = \theta - \alpha_i$, so $\widetilde{\Sigma}_i = h_i^{-1}(0)$. Hence $\big(s, y_1(\alpha_i, r)\big) \in T_{(\alpha_i, r)}\widetilde{\Sigma}$ if and only if $0 = \big\langle (1, 0), \big(s, y_1(\alpha_i, r)\big)\big\rangle = \big\langle \nabla h_i(\alpha_i, r), \big(s, y_1(\alpha_i, r)\big)\big\rangle = s$. Therefore $\big(0, y_1(\theta, r)\big) \in T_{(\theta, r)}\widetilde{\Sigma}$ for every $(\theta, r) \in \widetilde{\Sigma}$. □

1.5. **Example 1.** In the following example we solve a problem of type (9).

Consider $\alpha = (0, \pi/2, \pi, 3\pi/2, 2\pi) \in \mathbb{T}^4$. Thus $L_1 = \{(x, 0) : x > 0\}$, $L_2 = \{(0, y) : y > 0\}$, $L_3 = \{(x, 0) : x < 0\}$, and $L_4 = \{(0, y) : y < 0\}$. Then for $i = 1, ..., 4$ we have that C_i is the first, second, third and fourth quadrants, respectively.

In this example we study the maximum number of limit cycles given by the averaging theory of first and second order for DPDS, which can bifurcate from the periodic orbits of the linear center $\dot x = y$, $\dot y = -x$, perturbed inside the following class of linear DPDS:

$$\dot X = Y_i(x, y) \quad \text{if} \quad (x, y) \in C_i,\ i = 1, ..., 4 \tag{12}$$

where

$$Y_i(x,y) = \begin{pmatrix} y + \varepsilon P_i^1(x,y) + \varepsilon^2 P_i^2(x,y) \\ -x + \varepsilon Q_i^1(x,y) + \varepsilon^2 Q_i^2(x,y) \end{pmatrix},$$

with $P_i^1(x,y) = a_{0i} + a_{1i}x + a_{2i}y$, $P_i^2(x,y) = c_{0i} + c_{1i}x + c_{2i}y$, $Q_i^1(x,y) = b_{0i} + b_{1i}x + b_{2i}y$, $Q_i^2(x,y) = d_{0i} + d_{1i}x + d_{2i}y$ and $|\varepsilon| \neq 0$ is a small parameter.

Let $\mathcal{A}$ denote the set of the following two conditions

$$\begin{aligned} &4a_{01} - 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) = 0 \quad \text{and} \\ &2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) + \\ &(a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi = 0. \end{aligned}$$

Our results on the limit cycles of system (12) are stated in the next two propositions.

Proposition 4. *For $|\varepsilon| \neq 0$ sufficiently small and using Theorem A system* (12) *has at most* 1 *limit cycle for any chosen of parameters for which the conditions of $\mathcal{A}$ do not hold. Moreover we can find parameters a_{ij}, b_{ij}, c_{ij}, and d_{ij} such that system* (12) *has exactly* 0 *or* 1 *limit cycle.*

Proposition 5. *For $|\varepsilon| \neq 0$ sufficiently small and using Theorem B system* (12) *has at most* 4 *limit cycles for any chosen of parameters for which the two conditions of $\mathcal{A}$ holds. Moreover we can find parameters a_{ij}, b_{ij}, c_{ij}, and d_{ij} such that system* (12) *has exactly* 0, 1, 2, 3 *or* 4 *limit cycles.*

Proposition 4 and 5 are proved in Section 3.

1.6. **Example 2.** In the following example we solve a problem which is not of type (9).

Let $h(x,y) = y - x^2$. The set $\Sigma = h^{-1}(0)$ is a regular manifold which splits the set $\mathbb{R}^2 \backslash \Sigma$ in two disjoint open regions. We consider the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{array}{l} x + \varepsilon P^1(x,y) + \varepsilon^2 P^2(x,y), \\ -y + \varepsilon Q^1(x,y) + \varepsilon^2 Q^2(x,y), \end{array} & \text{if} \quad h(x,y) \geq 0, \\ \begin{array}{l} x + \varepsilon R^1(x,y) + \varepsilon^2 R^2(x,y,) \\ -y + \varepsilon S^1(x,y) + \varepsilon^2 S^2(x,y), \end{array} & \text{if} \quad h(x,y) \leq 0, \end{cases} \tag{13}$$

where

$$\begin{aligned} P^i &= p_{00}^i + p_{10}^i x + p_{01}^i y + p_{20}^i x^2 + p_{11}^i xy + p_{02}^i y^2, \\ Q^i &= q_{00}^i + q_{10}^i x + q_{01}^i y + q_{20}^i x^2 + q_{11}^i xy + q_{02}^i y^2, \\ R^i &= r_{00}^i + r_{10}^i x + r_{01}^i y + r_{20}^i x^2 + r_{11}^i xy + r_{02}^i y^2, \\ S^i &= s_{00}^i + s_{10}^i x + s_{01}^i y + s_{20}^i x^2 + s_{11}^i xy + s_{02}^i y^2, \end{aligned}$$

for $i = 1, 2$.

Let $\mathcal{B}$ denote the set of conditions

$$\begin{aligned} &p_{00}^1 = p_{10}^1 = q_{00}^1 = q_{01}^1 = q_{02}^1 = s_{00}^1 = s_{02}^1 = 0, \\ &q_{10}^1 = -p_{01}^1 - 2p_{20}^1, \quad q_{11}^1 = -p_{02}^1 - 2p_{20}^1, \\ &q_{20}^1 = -p_{11}^1, \quad s_{01}^1 = -r_{10}^1, \quad \text{and} \\ &s_{20}^1 = 3r_{10}^1 - r_{11}^1. \end{aligned}$$

Our results on the limit cycles of system (13) are given in the next two propositions.

Proposition 6. *For* $|\varepsilon| \neq 0$ *sufficiently small and using Theorem A system* (13) *has at most* 4 *limit cycles for any chosen of parameters for which the conditions of* $\mathcal{B}$ *do not hold. Moreover we can find parameters* p^1_{ij}, q^1_{ij}, r^1_{ij}, *and* s^1_{ij} *such that system* (13) *has exactly* 0, 1, 2, 3 *or* 4 *limit cycles.*

Proposition 7. *For* $|\varepsilon| \neq 0$ *sufficiently small and using Theorem B system* (13) *has at most* 6 *limit cycles for any chosen of parameters for which the conditions of* $\mathcal{B}$ *hold. Moreover we can find parameters* p^1_{01}, p^1_{20}, p^1_{11}, p^1_{02}, s^1_{10}, r^1_{ij} p^2_{ij}, q^2_{ij}, r^2_{ij}, *and* s^2_{ij} *such that system* (13) *has exactly* 0, 1, 2, 3, 4, 5 *or* 6 *limit cycles.*

Proposition 6 and 7 are proved in Section 3.

2. Proofs of the main results

We start this section proving Proposition 2. Then we state some preliminary lemmas needed to prove our main results. After that, the remainder of this section consists of the proof of Theorems A and B. As usual μ denotes the *Lebesgue Measure.*

Proof of Proposition 2. For a fixed $z \in \overline{C}$ let $(t^i, z) \in \Sigma$ be a generic point of discontinuity. So there exists a neighborhood $G_{(t^i,z)}$ of (t^i, z) such that $\mathcal{S}_{(t^i,z)} = G_{(t^i,z)} \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface of $\mathbb{S}^1 \times \mathbb{R}^d$ with $k \geq 1$. It is well known that $\mathcal{S}_{(t^i,z)}$ can be locally described as the inverse image of a regular value of a $\mathcal{C}^k$ function, that is, there exists a $\mathcal{C}^k$ function $h_i : G_{(t^i,z)} \to \mathbb{R}$ such that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. Here $\widetilde{G}_{(t^i,z)}$ is an open subset such that $(t^i, z) \in \widetilde{G}_{(t^i,z)} \subseteq G_{(t^i,z)}$. For $(t, z) \in \widetilde{G}_{(t^i,z)}$ system (5) becomes

$$x' = \begin{cases} f^+_{(t^i,z)}(t,x,\varepsilon) = \varepsilon F_1^{j_{i+1}}(t,x) + \varepsilon^2 F_2^{j_{i+1}}(t,x) + \varepsilon^3 R^{j_{i+1}}(t,x,\varepsilon) & \text{if} \quad h_i(t,x) > 0, \\ f^-_{(t^i,z)}(t,x,\varepsilon) = \varepsilon F_1^{j_i}(t,x) + \varepsilon^2 F_2^{j_i}(t,x) + \varepsilon^3 R^{j_i}(t,x,\varepsilon) & \text{if} \quad h_i(t,x) < 0. \end{cases}$$

From hypothesis (HC) we know that $(\partial/\partial t)h_i(t^i, z)^2 > 0$. Hence

$$\left\langle \nabla h_i(t^i,z), f^+_{(t^i,z)}(t,x,\varepsilon) \right\rangle \left\langle \nabla h_i(t^i,z), f^-_{(t^i,z)}(t,x,\varepsilon) \right\rangle = \left(\frac{\partial h_i}{\partial t}(t^i,z) \right)^2 + \mathcal{O}(\varepsilon),$$

which is positive for $|\varepsilon| \neq 0$ sufficiently small. So from (4) we conclude this proof. □

Lemma 8. *The averaged functions* (7) *and* (8) *are continuous in* C.

Proof. Let $z_0 \in C$ and let V be a neighborhood of z_0 with a compact closure contained in C. For $z \in V$ we define the sets $A^i_z(t) = \{s \in [0,t] : (s,z) \in S_i\}$, and $A^0_z(t) = \{s \in [0,t] : (s,z) \in \Sigma\}$.

From hypothesis (HC) we have that $\mu\left(A_z^0(t)\right)=0$ for every $t\in[0,T]$ and $\mathbf{z}\in\overline{C}$. So

$$
\begin{aligned}
\Delta(t,z,z_0) &= |y_1(t,z_0)-y_1(t,z)|\\
&= \left|\sum_{j=1}^{M}\int_{A_{z_0}^j(t)} F_1^j(s,z_0)ds-\sum_{j=1}^{M}\int_{A_z^j(t)} F_1^j(s,z)ds\right|\\
&\le \sum_{j=1}^{M}\left|\int_{A_{z_0}^j(t)} F_1^j(s,z_0)ds-\int_{A_z^j(t)} F_1^j(s,z)ds\right|\\
&\le \sum_{j=1}^{M}\int_{A_{z_0}^j(t)\cap A_z^j(t)}\left|F_1^j(s,z_0)-F_1^j(s,z)\right|ds+\sum_{j=1}^{M}\left|\int_{A_{z_0}^j(t)\backslash A_z^j(t)} F_1^j(s,z_0)-\int_{A_z^j(t)\backslash A_{z_0}^j(t)} F_1^j(s,z)ds\right|\\
&\le MTL|z_0-z|+\sum_{j=1}^{M}\left|\int_{A_{z_0}^j(t)\backslash A_z^j(t)} F_1^j(s,z_0)ds-\int_{A_z^j(t)\backslash A_{z_0}^j(t)} F_1^j(s,z)ds\right|\\
&\le MTL|z_0-z|+\sum_{j=1}^{M}L_{1,j}\left(\mu\left(A_{z_0}^j(t)\backslash A_z^j(t)\right)+\mu\left(A_z^j(t)\backslash A_{z_0}^j(t)\right)\right),
\end{aligned}
\tag{14}
$$

where L is maximum of the Lipschitz constants of the functions F_i^j for $j=1,2,\dots,M$, and $L_{1,j}=\max\{F_1^j(s,z):\,(s,z)\in[0,T]\times\overline{V}\}$ for $j=1,2,\dots,M$. We observe that $\mu\left(A_{z_0}^j(t)\backslash A_z^j(t)\right)\to 0$ and $\mu\left(A_z^j(t)\backslash A_{z_0}^j(t)\right)\to 0$, as $z\to z_0$ for every $t\in[0,T]$. Thus $\Delta(t,z,z_0)\to 0$, as $z\to z_0$ for every $t\in[0,T]$. So the function $y_1(t,z)$ is continuous in C for each $t\in[0,T]$. Since $f_1(z)=y_1(T,z)$, we conclude that the averaged function f_1 is continuous in C.

Repeating the computations (14), now for $\int_0^t F_2(s,z)ds$, we get that this function is continuous for $z\in C$. So to prove the continuity of the function f_2 it is sufficient to estimate the difference

$$
D(z_0,z)=\left|\int_0^T\Big(D_xF_1(t,z_0)y_1(t,z_0)-D_xF_1(t,z)y_1(t,z)\Big)dt\right|,
$$

for $z\in V$. Thus

$$
\begin{aligned}
D(z_0,z)\le\ & \int_0^T|D_xF_1(t,z_0)-D_xF_1(t,z)|\,|y_1(t,z_0)|dt+\int_0^T|D_xF_1(t,z)|\,|y_1(t,z_0)-y_1(t,z)|dt\\
\le\ & TY\int_0^T|D_xF_1(t,z_0)-D_xF_1(t,z)|\,dt+TL'\int_0^T|y_1(t,z_0)-y_1(t,z)|dt,
\end{aligned}
$$

where $Y=\max\{|y_1(s,z)|\,:\,(s,z)\in[0,T]\times\overline{V}\}$ and $L'=\max_{j=1}^{M}\{|D_xF_1^j(s,z)|\,:\,(s,z)\in[0,T]\times\overline{V}\}$. The function $y_1(t,z)$ is continuous in z. Hence repeating the computations (14), now for $D_xF_1(t,z)$, we conclude that $D(z_0,z)\to 0$ when $z\to z_0$, which implies the continuity of the averaged function f_2 in C. □

Let $g:(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^d$ be a function defined on a small interval $(-\varepsilon_0,\varepsilon_0)$. We say that

1. $g(\varepsilon)=\mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1>0$ and $k>0$ such that $|g(\varepsilon)|\le k|\varepsilon^\ell|$ for $-\varepsilon_1<\varepsilon<\varepsilon_1$.
2. $g(\varepsilon)=o(\varepsilon^\ell)$ for some positive integer ℓ if

$$
\lim_{\varepsilon\to 0}\frac{|g(\varepsilon)|}{\varepsilon^\ell}=0.
$$

Here $|\cdot|$ denotes the usual norm in the Euclidean space $\mathbb{R}^n$ for $n \geq 1$. The symbols $\mathcal{O}$ and o are called the *Landau's symbols* (see for instance [21]).

Lemma 9 (Fundamental lemma)**.** *Let $x(\cdot, z, \varepsilon) : [0, t_z) \to \mathbb{R}^n$ be the solution of system* (3) *with $x(0, z, \varepsilon) = z$. Then we have the following statements.*

(a) *Under the hypotheses of Theorem A $t_z \geq T$ and the equality $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \mathcal{O}(\varepsilon^2)$ holds.*

(b) *Under the hypotheses of Theorem B $t_z \geq T$ and the equality $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \varepsilon^2 \int_0^t (D_x F_1(s, z)\, y_1(s, z) + F_2(s, z))ds + \varepsilon\, o(\varepsilon)$ holds. Furthermore if for $j = 1, 2, \ldots, M$ the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 2$ then we have that $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \varepsilon^2 \int_0^t (D_x F_1(s, z) y_1(s, z) + F_2(s, z))ds + \mathcal{O}(\varepsilon^3)$.*

Proof. For each $z \in C$ the function $t \in [0, t_z) \mapsto x(t, z, \varepsilon)$ is continuous and piecewise differentiable. From hypothesis (HC), for $|\varepsilon| \neq 0$ sufficiently small, we can assume that

$$x(t,z,\varepsilon) = \begin{cases} x_1(t,z,\varepsilon) & \text{if} \quad 0 = \; t_\varepsilon^0 \leq t \leq t_\varepsilon^1, \\ x_2(t,z,\varepsilon) & \text{if} \quad t_\varepsilon^1 \leq t \leq t_\varepsilon^2, \\ \quad \vdots & \\ x_i(t,z,\varepsilon) & \text{if} \quad t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i, \\ \quad \vdots & \\ x_\kappa(t,z,\varepsilon) & \text{if} \quad t_\varepsilon^{\kappa-1} \leq t \leq t_\varepsilon^\kappa = t_z \leq T, \end{cases}$$

for which we have the following recurrence

$$x_1(0, z, \varepsilon) = z \quad \text{and} \quad x_i(t_\varepsilon^{i-1}, z, \varepsilon) = x_{i-1}(t_\varepsilon^{i-1}, z, \varepsilon), \tag{15}$$

for $i = 2, \ldots, \kappa$. Moreover each function $x_i(t, z, \varepsilon)$ satisfies the DPDS (5), that is, there exists a subsequence (j_i) for $i = 1, \ldots, \kappa$ such that

$$\frac{\partial}{\partial t} x_i(t, z, \varepsilon) = \varepsilon F_1^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^2 F_2^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^3 R^{j_i}(t, x_i(t, z, \varepsilon), \varepsilon), \tag{16}$$

In other words, for $i = 1, \ldots, \kappa$, the function $x_i(t, z, \varepsilon)$ is the solution of the *Cauchy Problem* defined by the differential equation (16) together with the initial condition (15).

We note that there exists $|\varepsilon_0| \neq 0$ sufficiently small such that, for each $z \in \overline{C}$, the solution $x_i(t, z, \varepsilon)$ of (16) is defined in $[0, T]$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $i = 1, 2, \ldots, \kappa$. Indeed, using the *Existence and Uniqueness Theorem* of solutions (see, for instance, Theorem 1.2.4 of [21]) we have that, for each $z \in C$, $x_i(t, z, \varepsilon)$ is defined for all $0 \leq t \leq \inf\left(T, d/M_i(\varepsilon)\right)$, where

$$M_i(\varepsilon) \geq \left| \varepsilon F_1^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^2 F_2^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^3 R^{j_i}(t, x_i(t, z, \varepsilon), \varepsilon) \right|$$

for all $t \in [0, T]$, for each x with $|x - z| \leq d$ and for every $z \in \overline{C}$. When ε is sufficiently small we can take $d/M_i(\varepsilon)$ sufficiently large in order that $\inf\left(T, d/M_i(\varepsilon)\right) = T$ for all $z \in \overline{C}$. So for any $z \in \overline{C}$ we have that the solution $x(t, z, \varepsilon)$ of system (3) is also defined for every $t \in [0, T]$.

From the continuity of the solution $x(t, z, \varepsilon)$ and by compactness of the set $[0, T] \times \overline{C} \times [-\varepsilon_0, \varepsilon_0]$, there exits a compact subset K of D such that $x(t, z, \varepsilon) \in K$ for all $t \in [0, T]$, $z \in \overline{C}$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Now, by the piecewise continuity of the function R, $|R(s, x(s, z, \varepsilon), \varepsilon)| \leq \max\{|R(t, x, \varepsilon)|, (t, x, \varepsilon) \in [0, T] \times K \times [-\varepsilon_1, \varepsilon_1]\} = N$. Then

$$\left| \int_0^t R(s, x(s, z, \varepsilon), \varepsilon) ds \right| \leq \int_0^T |R(s, x(s, z, \varepsilon), \varepsilon)|\, ds = TN,$$

which implies that

$$\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds = \mathcal{O}(1).$$

Now for a given $t\in(0,T)$ there exists $\bar\kappa\in\{1,2,\dots,\kappa-1\}$ such that $t\in[t_\varepsilon^{\bar\kappa-1},t_\varepsilon^{\bar\kappa})$ and

$$\begin{aligned} x(t,z,\varepsilon) = & \quad x_{\bar\kappa}(t,z,\varepsilon)\\ = & \quad x_{\bar\kappa-1}(t_\varepsilon^{\bar\kappa-1},z,\varepsilon)+\varepsilon\int_{t_\varepsilon^{\bar\kappa-1}}^t F_1(s,x(s,z,\varepsilon))ds+\varepsilon^2\int_{t_\varepsilon^{\bar\kappa-1}}^t F_2(s,x(s,z,\varepsilon))ds+\mathcal{O}(\varepsilon^3). \end{aligned}$$

Since

$$x_{i+1}(t_\varepsilon^{i+1},z,\varepsilon) \quad = x_i(t_\varepsilon^i,z,\varepsilon)+\varepsilon\int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}} F_1(t,x(t,z,\varepsilon))dt+\varepsilon^2\int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}} F_2(t,x(t,z,\varepsilon))dt+\mathcal{O}(\varepsilon^3),$$

proceeding by induction on i, we obtain

$$x(t,z,\varepsilon)=z+\varepsilon\int_0^t F_1(s,x(s,z,\varepsilon))ds+\varepsilon^2\int_0^t F_2(s,x(s,z,\varepsilon))ds+\mathcal{O}(\varepsilon^3). \tag{17}$$

Claim 1. *Statement (a) of Lemma 9 holds.*

For $i=1,2,\dots,\kappa$ and for $t_\varepsilon^{i-1}\le t\le t_\varepsilon^i$, $x_i(t,z,\varepsilon)=x(t,z,\varepsilon)$. Since $F_1^{j_i}$ is Lipschitz for $i=1,2,\dots,\kappa$ in the variable x, we have that

$$\begin{aligned} \left|F_1^{j_i}(t,x_i(t,z,\varepsilon))-F_1^{j_i}(t,z)\right| = & \quad \left|F_1^{j_i}(t,x(t,z,\varepsilon))-F_1^{j_i}(t,z)\right|\\ \le & \quad L_{j_i}|x(t,z,\varepsilon)-z|=\mathcal{O}(\varepsilon), \end{aligned}$$

for all $t_\varepsilon^{i-1}\le t<t_\varepsilon^i$, where L_{j_i} is the Lipschitz constant of the function $F_1^{j_i}$. It implies that

$$F_1^{j_i}(t,x_i(t,z,\varepsilon))=F_1^{j_i}(t,z)+\mathcal{O}(\varepsilon), \tag{18}$$

for $t_\varepsilon^{i-1}\le t<t_\varepsilon^i$ and for each $i=1,2,\dots,\kappa$.

Let $t^i=\lim_{\varepsilon\to0}t_\varepsilon^i$ for $i=1,2,\dots,\kappa-1$. Observing that, for $t^{i-1}\le t<t^i$, $F_1^{j_i}(s,z)=F_1(s,z)$ and using (18) we compute

$$\begin{aligned} \int_0^t F_1(s,x(s,z,\varepsilon))ds & = \left(\sum_{i=1}^{\bar\kappa-1}\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s,x_i(s,z,\varepsilon))ds\right)+\int_{t_\varepsilon^{\bar\kappa-1}}^t F_1^{j_{\bar\kappa}}(s,x_{\bar\kappa}(s,z,\varepsilon))ds\\ & = \left(\sum_{i=1}^{\bar\kappa-1}\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s,z)ds\right)+\int_{t_\varepsilon^{\bar\kappa-1}}^t F_1^{j_{\bar\kappa}}(s,z)ds+\mathcal{O}(\varepsilon)\\ & = \left(\sum_{i=1}^{\bar\kappa-1}\int_{t^{i-1}}^{t^i} F_1^{j_i}(s,z)ds\right)+\int_{t^{\bar\kappa-1}}^t F_1^{j_{\bar\kappa}}(s,z)ds+E_1(\varepsilon)+\mathcal{O}(\varepsilon)\\ & = \left(\sum_{i=1}^{\bar\kappa-1}\int_{t^{i-1}}^{t^i} F_1(s,z)ds\right)+\int_{t^{\bar\kappa-1}}^t F_1(s,z)ds+E_1(\varepsilon)+\mathcal{O}(\varepsilon)\\ & = \int_0^t F_1(s,z)ds+E_1(\varepsilon)+\mathcal{O}(\varepsilon), \end{aligned} \tag{19}$$

where

$$E_1(\varepsilon)= \quad \sum_{i=1}^{\bar\kappa-1}\left(\int_{t_\varepsilon^{i-1}}^{t^{i-1}} F_1^{j_i}(s,z)ds-\int_{t_\varepsilon^i}^{t^i} F_1^{j_i}(s,z)ds\right)+\int_{t_\varepsilon^{\bar\kappa-1}}^{t^{\bar\kappa-1}} F_1^{j_{\bar\kappa}}(s,z)ds.$$

It is easy to see that there exists a constant $\widetilde{E}$ such that

$$|E_1(\varepsilon)| \leq \widetilde{E} \sum_{i=0}^{\bar{\kappa}-1} |t^i - t^i_\varepsilon|. \tag{20}$$

We shall prove that $\tau^i : \varepsilon \mapsto t^i_\varepsilon$ is a $\mathcal{C}^k$ function with $k \geq 1$.

As in the proof of Proposition 2, for a generic point of discontinuity $(t^i, z) \in \Sigma$ with $z \in \overline{C}$, let $\widetilde{G}_{(t^i,z)}$ be a neighbourhood of (t^i, z) such that $\mathcal{S}_{(t^i,z)} = \widetilde{G}_{(t^i,z)} \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface of $\mathbb{S}^1 \times \mathbb{R}^d$ with $k \geq 1$, for which there exists a $\mathcal{C}^k$ function $h_i : \widetilde{G}_{(t^i,z)} \to \mathbb{R}$ such that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. We define $H_i(t, \varepsilon) = h_i(t, x_i(t, z, \varepsilon))$. So $H_i(t^i, 0) = 0$ and from hypothesis (HC)

$$\begin{aligned}
\frac{\partial}{\partial t} H_i(t^i, 0) &= \frac{\partial}{\partial t} h_i(t, x_i(t, z, \varepsilon))\Big|_{(t^i,0)} \\
&= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i, z, 0)) + \frac{\partial}{\partial z} h_i(t^i, x_i(t^i, z, 0)) \frac{\partial}{\partial t} x_i(t^i, z, 0) \\
&= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i, z, 0)) \neq 0,
\end{aligned}$$

because (16) implies $(\partial/\partial t) x_i(t, z, 0) = 0$. Hence from the Implicit Function Theorem, $\tau^i(\varepsilon)$ is a $\mathcal{C}^k$ function with $H(\tau^i(\varepsilon), \varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small and $\tau^i(0) = t^i$. So

$$\tau^i(\varepsilon) = t^i + (\tau^i)'(0)\varepsilon + o(\varepsilon) \tag{21}$$

for every $i = 1, 2, \ldots, \kappa - 1$, because $k \geq 1$. This implies that $E_1(\varepsilon) = \mathcal{O}(\varepsilon)$.

Going back to the equality (19) we have

$$\int_0^t F_1(s, x(s, z, \varepsilon)) = \int_0^t F_1(s, z) ds + \mathcal{O}(\varepsilon). \tag{22}$$

Hence from (17) and (22) we conclude that

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, z) ds + \mathcal{O}(\varepsilon^2).$$

Therefore the claim 1 is proved.

Claim 2. *Statement (b) of Lemma 9 holds.*

For $i = 1, 2, \ldots, \kappa$ and for $t^{i-1}_\varepsilon \leq t \leq t^i_\varepsilon$ we prove that

$$|F_1^{j_i}(t, x_i(t, z, \varepsilon)) - F_1^{j_i}(t, z) - \varepsilon D_x F_1^{j_i}(t, z) y_1(t, z)| = \mathcal{O}(\varepsilon^2). \tag{23}$$

To do this we define

$$G(\lambda) = F_1^{j_i}(t, \lambda x_i(t, z, \varepsilon) + (1 - \lambda) z).$$

Computing the derivative in λ we get

$$G'(\lambda) = D_x F_1^{j_i}(t, \lambda x_i(t, z, \varepsilon) + (1 - \lambda) z)(x_i(t, z, \varepsilon) - z).$$

So from the Fundamental Theorem of Calculus and observing that, for $t^{i-1}_\varepsilon \leq t \leq t^i_\varepsilon$, $x_i(t, z, \varepsilon) = x(t, z, \varepsilon)$ it follows that

$$G(1) - G(0) = \int_0^1 D_x F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda) z)(x(t, z, \varepsilon) - z) d\lambda.$$

Thus for $t_\varepsilon^{i-1}\le t\le t_\varepsilon^i$,

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\Big(F_1^{j_i}(t,x_i(t,z,\varepsilon))-F_1^{j_i}(t,z)-\varepsilon D_xF_1^{j_i}(t,z)y_1(t,z)\Big)=\\
&\frac{1}{\varepsilon^2}\Big(G(1)-G(0)-\varepsilon D_xF_1^{j_i}(t,z)y_1(t,z)\Big)=\\
&\frac{1}{\varepsilon}\int_0^1 D_xF_1^{j_i}(t,\lambda x(t,z,\varepsilon)+(1-\lambda)z)\frac{(x(t,z,\varepsilon)-z)}{\varepsilon}d\lambda-\frac{1}{\varepsilon}D_xF_1^{j_i}(t,z)y_1(t,z)=\\
&\frac{1}{\varepsilon}\left(\int_0^1 D_xF_1^{j_i}(t,\lambda x(t,z,\varepsilon)+(1-\lambda)z)d\lambda\right)\int_0^t F_1(s,x(s,z,\varepsilon))ds-\\
&\frac{1}{\varepsilon}D_xF_1^{j_i}(t,z)y_1(t,z)+\mathcal{O}(1)=\\
&\frac{1}{\varepsilon}\left(\int_0^1\Big[D_xF_1^{j_i}(t,\lambda x(t,z,\varepsilon)+(1-\lambda)z)-D_xF_1^{j_i}(t,z)\Big]d\lambda\right)\int_0^t F_1(s,x(s,z,\varepsilon))ds+\\
&\frac{1}{\varepsilon}D_xF_1^{j_i}(t,z)\left[\int_0^t F_1(s,x(s,z,\varepsilon))-F_1(s,z)ds\right]+\mathcal{O}(1).
\end{aligned}
$$

Let $B=\max\{|F_1(s,x(s,z,\varepsilon))|:\,(t,z)\in[0,T]\times\overline{C}\}$. Observing that $D_xF_1^{j_i}$ is locally Lipschitz in the second variable, and (from (22)) that $\int_0^t F_1(s,x(s,z,\varepsilon))-\int_0^t F_1(s,z)=\mathcal{O}(\varepsilon)$, it follows that

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^2}\Big(F_1^{j_i}(t,x_i(t,z,\varepsilon))-F_1^{j_i}(t,z)-\varepsilon D_xF_1^{j_i}(t,z)y_1(t,z)\Big)\right|\le\\
&\frac{1}{\varepsilon}\int_0^1\Big|D_xF_1^{j_i}(t,\lambda x(t,z,\varepsilon)+(1-\lambda)z)-D_xF_1^{j_i}(t,z)\Big|d\lambda\left|\int_0^t F_1(s,x(s,z,\varepsilon))ds\right|+\\
&\frac{1}{\varepsilon}\Big|D_xF_1^{j_i}(t,z)\Big|\left|\int_0^t F_1(s,x(s,z,\varepsilon))-F_1(s,z)ds\right|+\mathcal{O}(1)\le\\
&TL_iB\frac{|x(t,z,\varepsilon)-z|}{\varepsilon}+\mathcal{O}(1)=\mathcal{O}(1),
\end{aligned}
$$

where L_i is the Lipschitz constant of the function $D_xF_1^{j_i}$. Hence for $t_\varepsilon^{i-1}\le t\le t_\varepsilon^i$ and for every $i=1,2,\dots,\kappa$ the equality (23) holds, which implies that $F_1^{j_i}(t,x(t,z,\varepsilon))=F_1^{j_i}(t,z)+\varepsilon D_xF_1^{j_i}(t,z)y_1(t,z)+\mathcal{O}(\varepsilon^2)$.

Observing that for $t^{i-1} \le s < t^i$, $F_1^{j_i}(s,z) = F_1(s,z)$ we compute

$$
\begin{aligned}
\int_0^t F_1(s,x(s,z,\varepsilon))ds = & \sum_{i=1}^{\bar\kappa-1}\left(\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s,x_i(s,z,\varepsilon))ds\right) + \int_{t_\varepsilon^{\bar\kappa-1}}^t F_1^{j_{\bar\kappa}}(s,x_{\bar\kappa}(s,z,\varepsilon))ds\\
= & \sum_{i=1}^{\bar\kappa-1}\left(\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} \left[F_1^{j_i}(s,z)+\varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)\right]ds\right)\\
& + \int_{t_\varepsilon^{\bar\kappa-1}}^t \left[F_1^{j_{\bar\kappa}}(s,z)+\varepsilon D_x F_1^{j_{\bar\kappa}}(s,z)y_1(s,z)\right]ds + \mathcal{O}(\varepsilon^2)\\
= & \sum_{i=1}^{\bar\kappa-1}\left(\int_{t^{i-1}}^{t^i} \left[F_1^{j_i}(s,z)+\varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)\right]ds\right)\\
& + \int_{t^{\bar\kappa-1}}^t \left[F_1^{j_{\bar\kappa}}(s,z)+\varepsilon D_x F_1^{j_{\bar\kappa}}(s,z)y_1(s,z)\right]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2)\\
= & \sum_{i=1}^{\bar\kappa-1}\left(\int_{t^{i-1}}^{t^i} \left[F_1(s,z)+\varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)\right]ds\right)\\
& + \int_{t^{\bar\kappa-1}}^t \left[F_1(s,z)+\varepsilon D_x F_1^{j_{\bar\kappa}}(s,z)y_1(s,z)\right]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2)\\
= & \int_0^t \left[F_1(s,z)+\varepsilon D_x F_1(s,z)y_1(s,z)\right]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{24}
$$

The last equality comes from (6). Here

$$
\begin{aligned}
E_2(\varepsilon) = & \sum_{i=1}^{\bar\kappa-1}\left(\int_{t_\varepsilon^{i-1}}^{t^{i-1}} \left[F_1^{j_i}(s,z)+\varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)\right]ds\right.\\
& \left. - \int_{t_\varepsilon^{i}}^{t^i} \left[F_1^{j_i}(s,z)+\varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)\right]ds\right)\\
& + \int_{t_\varepsilon^{\bar\kappa-1}}^{t^{\bar\kappa-1}} \left[F_1^{j_{\bar\kappa}}(s,z)+\varepsilon D_x F_1^{j_{\bar\kappa}}(s,z)y_1(s,z)\right]ds.
\end{aligned}
$$

It is easy to see that there exists a constant $\widehat{E}$ such that

$$
|E_2(\varepsilon)| \le \widehat{E}\sum_{i=0}^{\bar\kappa-1}|t^i - t_\varepsilon^i|. \tag{25}
$$

From statement (a) the function $\varepsilon \mapsto x(\tau^i(\varepsilon),z,\varepsilon))$ is differentiable at $\varepsilon = 0$. Moreover $y_1(t,z) = (\partial/\partial\varepsilon)x(t,z,0)$. Since for $|\varepsilon| \neq 0$ sufficiently small $h_i(\tau^i(\varepsilon), x(\tau^i(\varepsilon),z,\varepsilon)) = 0$, so

$$
\begin{aligned}
0 = & \frac{\partial}{\partial\varepsilon}h(\tau^i(\varepsilon),x(\tau^i(\varepsilon),z,\varepsilon))\Big|_{\varepsilon=0}\\
= & \frac{\partial}{\partial t}h(t^i,z)(\tau^i)'(0) + \frac{\partial}{\partial z}h(t^i,z)\left(\frac{\partial}{\partial t}x(t^i,z,0)(\tau^i)'(0) + \frac{\partial}{\partial\varepsilon}x(t^i,z,\varepsilon)\Big|_{\varepsilon=0}\right)\\
= & \left\langle \nabla h(t^i,z), \Big((\tau^i)'(0), y_1(t^i,z)\Big)\right\rangle,
\end{aligned}
$$

So $((\tau^i)'(0), y_1(t^i,z)) \in T_{(t^i,z)}\Sigma$.

Now we shall prove that $(\tau^i)'(0) = 0$ and $E_2(\varepsilon) = o(\varepsilon)$. If $(\tau^i)'(0) \neq 0$, we get that $(s, y_1(t^i,z)) \in T_{(t^i,z)}\Sigma$ for every $s \in \mathbb{R}$, because from hypothesis $(Hb2)$, $(0, y_1(t^i,z)) \in T_{(t^i,z)}\Sigma$. Thus

$$0 = \Big\langle \nabla h(t^i,z), \Big(s, y_1(t^i,z)\Big)\Big\rangle = \frac{\partial}{\partial t}h(t^i,z)\, s + \frac{\partial}{\partial z}h(t^i,z)y_1(t^i,z),$$

for every $s \in \mathbb{R}$. Computing the derivative in s of the last equality it follows that $(\partial h/\partial t)(t^i,z) = 0$ contradicting then the hypothesis (HC). Hence we conclude that $(\tau^i)'(0) = 0$. Moreover from (25) and (21) we obtain that $E_2(\varepsilon) = o(\varepsilon)$.

Going back to the equality (24) we have

$$\int_0^t F_1(s, x(s,z,\varepsilon))ds = \int_0^t F_1(s,z)ds - \varepsilon\int_0^t D_xF_1(s,z)y_1(s,z)ds + o(\varepsilon). \tag{26}$$

Analogously to the proof of statement (a) and using that $E_2(\varepsilon) = o(\varepsilon) \subset \mathcal{O}(\varepsilon)$ we can show that

$$\int_0^t F_2(s, x(s,z,\varepsilon))ds = \int_0^t F_2(s,z)ds + \mathcal{O}(\varepsilon). \tag{27}$$

So from (17), (26) and (27) we get

$$x(t,z,\varepsilon) = z + \varepsilon\int_0^t F_1(s,z)ds - \varepsilon^2\int_0^t \big[D_xF_1(s,z)y_1(s,z) + F_2(s,z)\big]ds + \varepsilon o(\varepsilon). \tag{28}$$

To conclude the proof of statement (b) we assume that for $j = 1,2,\ldots,M$ the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 2$. From (HC) and following the steps of the proof of Claim 1 we can find a $\mathcal{C}^k$ function $h_i : G_{(t^i,z)} \to \mathbb{R}$, now with $k \geq 2$, such that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. Again, $\widetilde{G}_{(t^i,z)}$ is an open subset such that $(t^i,z) \in \widetilde{G}_{(t^i,z)} \subseteq G_{(t^i,z)}$. Applying the Inverse Function Theorem we conclude that $\tau^i(\varepsilon)$ is a $\mathcal{C}^2$ function. So

$$\tau^i(\varepsilon) = t^i + (\tau^i)'(0)\varepsilon + \mathcal{O}(\varepsilon^2).$$

which implies that $E_2(\varepsilon) = \mathcal{O}(\varepsilon^2)$. From here, analogously to (28), we obtain that

$$x(t,z,\varepsilon) = z + \varepsilon\int_0^t F_1(s,z)ds - \varepsilon^2\int_0^t \big[D_xF_1(s,z)y_1(s,z) + F_2(s,z)\big]ds + \mathcal{O}(\varepsilon^3).$$

It concludes this proof. □

Lemma 10. *Let U be a bounded open set of $\mathbb{R}^n$ and let $f : \overline{U} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ be a continuous function. We assume that $f(x,0) \neq 0$ for all $x \in \partial U$. Then for $|\varepsilon| \neq 0$ sufficiently small $d(f(x,\varepsilon), U, 0)$ is well defined and $d(f(x,\varepsilon), U, 0) = d(f(x,0), U, 0)$ for $|\varepsilon| \neq 0$ sufficiently small.*

Proof. For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}$ we consider the continuous homotopy

$$f_t(x,\varepsilon) = f(x,0) + t\left(f(x,\varepsilon) - f(x,0)\right).$$

Suppose that there exist sequences $(\varepsilon_i) \subset [-\varepsilon_0, \varepsilon_0]$, $(x_i) \subset \partial V$ and $(t_i) \in [0,1]$ with $\varepsilon_i \to 0$ when $\varepsilon \to \infty$ such that $f_{t_i}(x_i, \varepsilon_i) = 0$, that is $0 \in f_{t_i}(\partial V, \varepsilon_i)$. Since the sets ∂V and $[0,1]$ are compacts, there exists convergent subsequences $(x_{i_j}) \subset \partial V$ and $(t_{i_j}) \in [0,1]$, namely $x_{i_j} \to \bar{x} \in \partial V$ and $t_{i_j} \to \bar{t} \in [0,1]$ when $j \to \infty$. So $t_{i_j} f(x_{i_j}, 0) - f(x_{i_j}, 0) = t_{i_j} f(x_{i_j}, \varepsilon_{i_j})$. Passing the limit we conclude that $f(\bar{x}, 0) = 0$, contradicting then the hypotheses. So it must exists $\widetilde{\varepsilon} \in [0, \varepsilon_0]$ such that $0 \notin f_t(\partial V, \varepsilon)$ for every $\varepsilon \in [-\widetilde{\varepsilon}, \widetilde{\varepsilon}]$. From statement (iii) of Theorem 12 (see the Appendix) we conclude that $d(f(x,\varepsilon), V, 0) = d(f(x,0), V, 0)$ for every $\varepsilon \in [-\widetilde{\varepsilon}, \widetilde{\varepsilon}]$. □

Proof of Theorem A. Let f be the function such that $\varepsilon f(z,\varepsilon) = x(T,z,\varepsilon) - z$. This function is well defined because, from statement (a) of Lemma 9, the solution $x(t,z,\varepsilon)$ is defined for all $t \in [0,T]$. Moreover f is continuous on C. Also from statement (a) of Lemma 9 we have that

$$f(z,\varepsilon) = f_1(z) + \mathcal{O}(\varepsilon),$$

where the function f_1 is the one defined in (7), which, from Lemma 8, is continuous. Clearly, $x(t,z,\varepsilon)$ is a T–periodic solution if and only if $f(z,\varepsilon) = 0$. However from Lemma 10 and hypothesis $(Ha2)$ we have, for $|\varepsilon| \neq 0$ sufficiently small, that

$$d_B\left(f_1(z), U, 0\right) = d_B\left(f(z,\varepsilon), U, 0\right) \neq 0.$$

Hence, by item (i) of Theorem 12 (see the Appendix), $0 \in f(U,\varepsilon)$ for $|\varepsilon| \neq 0$ sufficiently small, that is, there exists $a_\varepsilon \in U$ such that $f(a_\varepsilon,\varepsilon) = 0$. Therefore, for $|\varepsilon| \neq 0$ sufficiently small, $x(t,a_\varepsilon,\varepsilon)$ is a periodic solution of (3). We can choose a_ε such that $a_\varepsilon \to a^*$ when $\varepsilon \to 0$, because $f(z,\varepsilon) \neq 0$ in $U \setminus \{a^*\}$. It completes this proof. □

Proof of Theorem B. Let f be the function such that $\varepsilon^2 f(z,\varepsilon) = x(T,z,\varepsilon) - z$. From statement (b) of Lemma 9 we have that

$$f(z,\varepsilon) = f_2(z) + \frac{o(\varepsilon)}{\varepsilon},$$

where the function f_2 is the one defined in (8), which, from Lemma 8, is continuous. Since $o(\varepsilon)/\varepsilon \to 0$ when $\varepsilon \to 0$ the proof follows similarly to the proof of Theorem A. □

3. Proofs of Propositions 4, 5, 6 and 7

Proof of Proposition 4. The linear DPDS (12) in polar coordinates (r,θ) becomes

$$\begin{aligned}
\dot{r} = \;& \varepsilon\left(a_{0i}\cos\theta + a_{1i}r\cos^2\theta + b_{0i}\sin\theta + a_{2i}r\cos\theta\sin\theta + b_{1i}r\cos\theta\sin\theta + b_{2i}r\sin^2\theta\right) + \\
& \varepsilon^2\left(c_{0i}\cos\theta + c_{1i}r\cos^2\theta + c_{2i}r\cos\theta\sin\theta + d_{1i}r\cos\theta\sin\theta + d_{2i}r\sin^2\theta + d_{0i}\sin\theta\right), \\
\dot{\theta} = \;& -1 - \frac{\varepsilon}{r}\left(-b_{0i}\cos\theta - b_{1i}r\cos^2\theta + a_{0i}\sin\theta + a_{1i}r\cos\theta\sin\theta - b_{2i}r\cos\theta\sin\theta + a_{2i}r\sin^2\theta\right) - \\
& \frac{\varepsilon^2}{r}\left(-d_{0i}\cos\theta - d_{1i}r\cos^2\theta + c_{0i}\sin\theta + c_{1i}r\cos\theta\sin\theta - d_{2i}r\cos\theta\sin\theta + c_{2i}r\sin^2\theta\right),
\end{aligned}$$

with $i = 1$ if $0 < \theta < \pi/2$, $i = 2$ if $\pi/2 < \theta < \pi$, $i = 3$ if $\pi < \theta < 3\pi/2$, and $i = 4$ if $3\pi/2 < \theta < 2\pi$. Taking the angle θ as the new independent variable the DPDS (12) writes

$$\dot{r} = \varepsilon F_{1i} + \varepsilon^2 F_{2i} + \mathcal{O}(\varepsilon^3), \tag{29}$$

where

$$\begin{aligned}
F_{1i} = \;& -r(a_{0i}\cos\theta + a_{1i}r\cos^2\theta + b_{0i}\sin\theta + a_{2i}r\cos\theta\sin\theta + b_{1i}r\cos\theta\sin\theta + b_{2i}r\sin^2\theta), \\
F_{2i} = \;& \frac{1}{r}(-b_{1i}r\cos^2\theta - b_{0i}\cos\theta + a_{1i}r\sin\theta\cos\theta - b_{2i}r\sin\theta\cos\theta + a_{2i}r\sin^2\theta + a_{0i}\sin\theta) \\
& (a_{1i}r\cos^2\theta + a_{0i}\cos\theta + a_{2i}r\sin\theta\cos\theta + b_{1i}r\sin\theta\cos\theta + b_{2i}r\sin^2\theta + b_{0i}\sin\theta) \\
& -(c_{1i}r\cos^2\theta + c_{0i}\cos\theta + c_{2i}r\sin\theta\cos\theta + d_{1i}r\sin\theta\cos\theta + d_{2i}r\sin^2\theta + d_{0i}\sin\theta).
\end{aligned}$$

From Proposition 4 the assumptions of Theorem A hold for the DPDS (29). Computing the averaged function f_1 we obtain

$$\begin{aligned}
f_1(r) = \;& \frac{1}{4}r(-4a_{01} + 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) \\
& -(2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) \\
& +(a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi)r).
\end{aligned}$$

Clearly f_1 has at most 1 zero. Moreover we can choose coefficients a_{ij}, in such a way that f_1 has a simple positive zero. Hence this proposition is proved. □

Proof of Proposition 5. We choose coefficients a_{ij}, such that the conditions contained in $\mathcal{A}$ hold. Then $f_1(r) \equiv 0$. Again from Proposition 3 the assumptions of Theorem B hold for the DPDS (29). Using some algebraic manipulator as Mathematica or Maple we obtain

$$f_2(r) = k_0 + k_1 r + k_2 r^2 + k_3 r^3 + k_4 r^4, \tag{30}$$

where k_i, $i = 0, ..., 4$, depends on the coefficients a_{ij}, $i = 0, 1$, $j = 1, ..., 4$ and can be taken freely. The function (30) is a polynomial in the variable r of degree 4. So, clearly, it has at most 4 zeros. Moreover we can choose coefficients a_{ij}, $i = 0, 1$, such that (30) has 0, 1, 2, 3 or 4 simple zeros. So this proposition is proved. □

In order to prove the Propositions 6 and 7 we have to introduce the concept of ECT–systems.

Let I be a proper real interval of $\mathbb{R}$. We say that an ordered set of complex–valued functions $F = (f_0, f_1, \dots, f_n)$ defined on I is an *Extended Chebyshev* system or ET–system on I if and only if any nontrivial linear combination of the functions of F has at most n zeros counting multiplicities. We say that F is an *Extended Complete Chebyshev* system or an ECT–system on I if and only if for any k, $0 \le k \le n$, $(f_0, f_1, \dots, f_k)$ is an ET–system. For more details, see the book of Karlin and Studden [14].

In order to prove that F is a ECT–system on I is sufficient and necessary to show that $W(f_0, f_1, \dots, f_k)(t) \neq 0$ on I for $0 \le k \le n$. Here $W(f_0, f_1, \dots, f_k)(t)$ denotes the Wronskians of the functions $(f_0, f_1, \dots, f_k)$ with respect to t. We recall the definition of the Wronskian.

$$W(f_0, f_1, \dots, f_k)(t) = \begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f_0'(t) & f_1'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{vmatrix}.$$

We also recall the *Descartes Theorem* about the number of zeros of a real polynomial (for a proof see for instance either the pages 82 and 83 of [4], or the appendix of [19]).

Descartes Theorem *Consider the real polynomial* $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + \cdots + a_{i_r} x^{i_r}$ *with* $0 \le i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j} a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots.*

Now consider the functions

$$
\begin{aligned}
&g_1(u) = 1,\\
&g_2(u) = u^2,\\
&g_3(u) = u^4,\\
&g_4(u) = u\left(2+u^2\right)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right),\\
&g_5^1(u) = u\left(2+u^2\right),\\
&g_5^2(u) = u\left(2+u^2\right)\left(\pi - \arccos\left(\frac{u}{\sqrt{2+u^2}}\right)\right),\\
&g_6(u) = \sqrt{2}u^6 - u\left(8-4u^4-u^6\right)\left(\frac{\pi}{2} + \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \quad \text{and}\\
&g_7(u) = -\sqrt{2}u^6 - \frac{3\pi u^3\left(2+u^2\right)^2}{2} - u\left(8-4u^4-u^6\right)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right).
\end{aligned}
\tag{31}
$$

We define the sets of functions $G^1 = \{g_1, g_2, g_3, g_4, g_5^1\}$ and $G^2 = \{g_1, g_2, g_3, g_4, g_5^2, g_6, g_7\}$.

Lemma 11. *The sets of functions G^1 and G^2 are ECT-systems on the interval $(0,\infty)$.*

Proof. To prove the statement we compute the Wronskians $W_1(u) = g_1(u)$, $W_2(u) = W(g_1, g_2)(u)$, $W_3(u) = W(g_1, g_2, g_3)(u)$, $W_4(u) = W(g_1, g_2, g_3, g_4)(u)$, $W_5^1(u) = W(g_1, g_2, g_3, g_4, g_5^1)(u)$, $W_5^2(u) = W(g_1, g_2, g_3, g_4, g_5^2)(u)$, $W_6(u) = W(g_1, g_2, g_3, g_4, g_5^2, g_6)(u)$, and $W_6(u) = W(g_1, g_2, g_3, g_4, g_5^2, g_6, g_7)$ (u). So

$$
\begin{aligned}
W_1(u) =& \quad 1,\\
W_2(u) =& \quad 2u,\\
W_3(u) =& \quad 16u^3,\\
W_4(u) =& \quad \frac{16u}{\left(2+u^2\right)^2}\left(P_1(u) + P_2(u)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right)\right),\\
W_5^1(u) =& \quad -\frac{6144\sqrt{2}u^3}{\left(2+u^2\right)^3},\\
W_5^2(u) =& \quad -\frac{6144\sqrt{2}\pi u^3}{\left(2+u^2\right)^3},\\
W_6(u) =& \quad \frac{-12\,288\pi u}{\left(2+u^2\right)^6}\left(Q_1(u) + Q_2(u)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \quad \text{and}\\
W_7(u) =& \quad -\frac{14\,495\,514\,624\pi^2 u^3\left(11+u^2\right)}{\left(2+u^2\right)^8},
\end{aligned}
$$

where

$$
\begin{aligned}
P_1(u) = &\quad \sqrt{2}u\left(12+4u^2+3u^4\right),\\
P_2(u) = &\quad 3\left(2-u^2\right)\left(2+u^2\right)^2,\\
Q_1(u) = &\quad 144\sqrt{2}\pi-288u-336\sqrt{2}\pi u_7^2 20u^3+1\,656\sqrt{2}\pi u^4+14\,584u^5+5\,760\sqrt{2}\pi u^6\\
&\quad +14\,700u^7+4\,305\sqrt{2}\pi u^8+3\,780u^9+945\sqrt{2}\pi u^{10}, \quad \text{and}\\
Q_2(u) = &\quad 6\sqrt{2}\left(2+u^2\right)^2\left(12-40u^2+175u^4+315u^6\right).
\end{aligned}
$$

Clearly $W_1(u) \neq 0$, $W_2(u) \neq 0$, $W_3(u) \neq 0$, $W_5^1(u) \neq 0$, $W_5^2(u) \neq 0$ and $W_7(u) \neq 0$ for $u > 0$.

To see that the function $W_4(u)$ does not vanish for any $u > 0$ we shall prove that

$$\widetilde{P}(u) = P_1(u) + P_2(u)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is an increasing function. Computing its derivative we have

$$P'(u) = 6u\left(\sqrt{2}u\left(2+3u^2\right)+\left(3u^4+4u^2-4\right)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right).$$

It is easy to see that $\left(3u^4+4u^2-4\right)$ is increasing. So $\widetilde{P}'(u)$ is also a increasing function for $u > 0$, because it is sums and products of increasing functions. Since $\widetilde{P}'(0) = 0$ it follows that $\widetilde{P}'(u) > 0$ for every $u > 0$. This implies that $\widetilde{P}(u)$ is an increasing function for $u > 0$. Again, since $\widetilde{P}(0) = 0$ it follows that $\widetilde{P}(u) > 0$ for every $u > 0$. Thus $W_4(u) \neq 0$ for $u > 0$.

To see that the function $W_6(u)$ does not vanish for any $u > 0$ we shall prove that

$$\widetilde{Q}(u) = Q_1(u) + Q_2(u)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is a positive function for $u > 0$. From Descartes Theorem the polynomials Q_1 and Q_2 have at most 2 zeros, and 1 minimum or maximum. Numerically we find $u_1 \approx 0.247$ and $u_2 \approx 0.269$ as the minimums for Q_1 and Q_2 respectively. So $\widetilde{Q}(u)$ is an increasing function for $u > \max\{u_1, u_2\}$. Finally it is easy to see that $\widetilde{Q}(u) > 0$ for $0 < u \leq \max\{u_1, u_2\}$. Thus $W_6(u) \neq 0$ for $u > 0$. Hence this lemma is proved. □

Proof of Proposition 6. Consider system (13). Proceeding with the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and taking θ as the new time, system (13) becomes equivalent

$$r' = \begin{cases} A(\theta, r) & \text{if} \quad r\sin^2\theta + \sin\theta - r > 0,\\ B(\theta, r) & \text{if} \quad r\sin^2\theta + \sin\theta - r < 0, \end{cases} \tag{32}$$

where

$$\begin{aligned} A(\theta, r) = \quad & -p_{20}^1 r^2 \cos^3\theta - r\cos^2\theta \left(p_{10}^1 + \left(p_{11}^1 + q_{20}^1\right) r \sin\theta\right) \\ & - \cos\theta \left(p_{00}^1 + r\sin\theta \left(p_{01}^1 + q_{10}^1 + \left(p_{02}^1 + q_1^{11}\right) r\sin\theta\right)\right) \\ & - \sin\theta \left(q_{00}^1 + r\sin\theta \left(q_{01}^1 + q_{02}^1 r \sin\theta\right)\right), \end{aligned}$$

$$\begin{aligned} B(\theta, r) = \quad & -r_{20}^1 r^2 \cos^3\theta - r\cos^2\theta \left(r_{10}^1 + \left(r_{11}^1 + s_{20}^1\right) r \sin\theta\right) \\ & - \cos\theta \left(r_{00}^1 + r\sin\theta \left(r_{01}^1 + s_{10}^1 + \left(r_{02}^1 + s_1^{11}\right) r\sin\theta\right)\right) \\ & - \sin\theta \left(s_{00}^1 + r\sin\theta \left(s_{01}^1 + s_{02}^1 r \sin\theta\right)\right). \end{aligned}$$

Clearly hypothesis $(Ha1)$ holds for system (32). Given

$$\theta_1(r) = \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right) \quad \text{and} \quad \theta_2(r) = \pi - \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right),$$

we have that for $r > 0$, $r\sin^2\theta + \sin\theta - r > 0$ if and only $0 \le \theta < \theta_1(r)$ and $\theta_2(r) < \theta \le 2\pi$; and $r\sin^2\theta + \sin\theta - r < 0$ if and only if $\theta_1(r) < \theta < \theta_2(r)$. Let $\widetilde{h}(\theta, r) = r\sin^2\theta + \sin\theta - r$, thus the set of discontinuity of system (32) is given by $\widetilde{\Sigma} = \widetilde{h}^{-1}(0) = \{(\theta_1(r), r) : r > 0\} \cup \{(\theta_2(r), r) : r > 0\}$. Since

$$\left\langle \nabla\widetilde{h}(\theta_1(r), r), \left(1, A(\theta_1(r), r)\right)\right\rangle \left\langle \nabla\widetilde{h}(\theta_1(r), r), \left(1, B(\theta_1(r), r)\right)\right\rangle = \frac{\left(1+4r^2\right)\left(-1+\sqrt{1+4r^2}\right)}{2r^2},$$

$$\left\langle \nabla\widetilde{h}(\theta_2(r), r), \left(1, A(\theta_2(r), r)\right)\right\rangle \left\langle \nabla\widetilde{h}(\theta_2(r), r), \left(1, B(\theta_2(r), r)\right)\right\rangle = \frac{\left(1+4r^2\right)\left(-1+\sqrt{1+4r^2}\right)}{2r^2},$$

we conclude that $\widetilde{\Sigma}$ has only crossing regions. So hypothesis (HC) holds for system (32).

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_1(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5^1(u),$$

where

$$\begin{aligned} k_1 &= 24\sqrt{2}\left(q_{00}^1 - s_{00}^1\right), \\ k_2 &= 2\sqrt{2}\left(-3p_{10}^1 + 2p_{11}^1 + 3q_{01}^1 + 4q_{02}^1 + 2q_{20}^1 + 3r_{10}^1 - 2r_{11}^1 - 3s_{01}^1 - 4s_{02}^1 - 2s_{20}^1\right), \\ k_3 &= 6\sqrt{2}\left(q_{02}^1 - s_{02}^1\right), \\ k_4 &= -6\left(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1\right), \\ k_5 &= -3\left(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1\right). \end{aligned}$$

So from Lemma 11 and Theorem A the proof follows. □

Proof of Proposition 7. In order to apply Theorem B to system (32) we have to guarantee that $f_1(u) \equiv 0$. By the linearity of the set of functions G^1, $f_1(u) \equiv 0$ if and only if $k_i = 0$ for $i = 1, 2, \ldots, 5$. Thus assuming that $k_i = 0$ for $i = 1, 2, \ldots, 5$, it is easy to see, using some algebraic manipulator as Mathematica or Maple, that the statement $\left\langle \nabla h(\theta_1(r), r), \left(s, y_1(\theta_1(r), t)\right)\right\rangle = 0$ implies $s = 0$ holds if and only if the conditions $\mathcal{B}$ holds. So assuming conditions $\mathcal{B}$ the hypothesis $(Hb2)$ holds.

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_2(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5^2(u) + k_6 g_6(u) + k_6 g_6(u).$$

Hence from Lemma 11 and Theorem B the proof follows. □

Appendix: Basic results on the Brouwer degree

In this appendix we present the existence and uniqueness result from the degree theory in finite dimensional spaces. We follow the Browder's paper [8], where are formalized the properties of the classical Brouwer degree. We also present some results that we shall need for proving the main results of this paper.

Theorem 12. *Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets V of X, consider continuous mappings $f : \overline{V} \to Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each such triple (f, V, y_0), there corresponds an integer $d(f, V, y_0)$ having the following three properties.*

(i) *If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y, then for every bounded open set V and $y_0 \in V$, we have*

$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(ii) (*Additivity*) *If $f : \overline{V} \to Y$ is a continuous map with V a bounded open set in X, and V_1 and V_2 are a pair of disjoint open subsets of V such that*

$$y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2)),$$

then,

$$d\left(f_0, V, y_0\right) = d\left(f_0, V_1, y_0\right) + d\left(f_0, V_1, y_0\right).$$

(iii) (*Invariance under homotopy*) *Let V be a bounded open set in X, and consider a continuous homotopy $\{f_t : 0 \leq t \leq 1\}$ of maps of $\overline{V}$ in to Y. Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d(f_t, V, y_t)$ is constant in t on $[0, 1]$.*

Theorem 13. *The degree function $d(f, V, y_0)$ is uniquely determined by the conditions of Theorem 12.*

For the proofs of Theorems 12 and 13 see [8].

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