A MEAN VALUE THEOREM FOR METRIC SPACES

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ABSTRACT. We present a form of the Mean Value Theorem (MVT) for a continuous function f between metric spaces, connecting it with the possibility to choose the $\varepsilon \mapsto \delta(\varepsilon)$ relation of f in a homeomorphic way. We also compare our formulation of the MVT with the classic one when the metric spaces are open subsets of Banach spaces. As a consequence, we derive a version of the Mean Value Propriety for measure spaces that also possesses a compatible metric structure.

Keywords metric space, mean value property, mean value theorem, homeomorphism, average of functions.
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1. INTRODUCTION

One of the most important results in Analysis is the Mean Value Theorem (MVT), which is used to prove many other significant results, from Several Complex Variables to Partial Differential Equations. It claims, in its original formulation (see Dieudonné, [5]) that if $I = [a, b] \subset \mathbb{R}$ is a closed real interval and $\mathcal{B}$ is a Banach space, then every pair of continuous functions $f : I \to \mathcal{B}$ and $\phi : I \to \mathbb{R}$ with derivatives in (a, b) satisfies the following implication

$$\|f'(\xi)\|_{\mathcal{B}} \leq \phi'(\xi) \text{ for all } \xi \in (a,b) \Longrightarrow \|f(b) - f(a)\|_{\mathcal{B}} \leq \phi(b) - \phi(a). \tag{1}$$

The most known consequence of the previous result happens when we fix some constant $M > 0$ and set $\phi(\xi) = M(\xi - a)$. This guarantees that

$$\|f'(\xi)\|_{\mathcal{B}} \leq M \text{ for all } \xi \in (a,b) \Longrightarrow \|f(b) - f(a)\|_{\mathcal{B}} \leq \phi(b) - \phi(a) = M(b-a). \tag{2}$$

There is also the classic form of the MVT for functions of several variables, which is directly obtained from the previous version. Consider $\mathcal{B}_1, \mathcal{B}_2$ Banach spaces and $U \subset \mathcal{B}_1$ an open subset. If $a, b \in U$ are such that the line segment $[a, b] \subset U$ and $f : U \to \mathcal{B}_2$ is a continuous function in $[a, b]$ which is differentiable in (a, b), then we achieve a similar conclusion of (2) by considering f composed with the curve

$$t \mapsto a + (b-a)t, \text{where } 0 \leq t \leq 1.$$

Inspired by Dieudonné, several authors generalized such theorems to more abstract spaces and/or to less regular functions. For instance, Clarke and Ledyaev in [1, 2] proposed the study of MVT in Hilbert spaces for lower semi-continuous functions in a new multidirectional sense. In a subsequent paper, Radulescu and Clarke (cf. [12]) proved that — in certain particular spaces — the locally Lipschitz functions also fulfill a variant of MVT.

Another distinct and celebrated approach was obtained by D. Preiss in [11]. He considered an Asplund space $\mathcal{B}$ and a Lipschitz continuous function $f : \mathcal{B} \to \mathbb{R}$. By denoting as $\mathcal{D}$ the set of

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points where f is differentiable, D. Preiss proved that $\mathcal{D}$ is dense in $\mathcal{B}$ and then concluded that for any $x, y \in \mathcal{B}$ the following MVT holds

$$|f(x) - f(y)| \leq L\|x - y\|_{\mathcal{B}}, \text{ where } L = \sup_{x \in \mathcal{D}} \|f'(x)\|_{\mathcal{L}(\mathcal{B},\mathbb{R})}.$$

An important formulation is due to M. Turinici, which proved (cf. [13]) a version resembling the original MVT in (1) to partially ordered metric spaces under some restrictive conditions on the metric. As we can see, there is a rich literature claiming different formulations of the MVT. On the other hand, there is a lack of discussion for a version of the theorem which is suitable for metric spaces.

In what follows, inspired by Turinici's paper and collected work about this topic, we propose a lot of results that discuss the viability of MVT for those general spaces. To do this, we must clarify what a suitable substitute of MVT means when no differential and/or vectorial framework is assumed.

To motivate our definition, let us examine the situation pictured in (2). Let $(x-r, x+r) \subset (a, b)$ and consider $\Psi : [0, r) \subset \mathbb{R}^+ \to \mathbb{R}^+$ given as

$$\Psi(d) = \sup_{\xi \in [x-d, x+d]} \|f'(\xi)\|_{\mathcal{B}}. \tag{3}$$

If we write $d_{\mathbb{R}}$ to indicate the Euclidian distance in $\mathbb{R}$, then it follows that the MVT in (2) is equivalent to

$$\|f(x) - f(y)\|_{\mathcal{B}} \leq \Psi(d_{\mathbb{R}}(x, y))\, d_{\mathbb{R}}(x, y) \text{ for all } y \in (x - r, x + r). \tag{4}$$

Note that the previous formulation is expressed in a metric fashion, with the aid of a certain function Ψ. Also, observe that extra proprieties of Ψ can be obtained from some regularity that f may possess. This discussion motivate us to consider the next definition.

Definition 1. *Let (M_1, d_1) and (M_2, d_2) be metric spaces. Consider a function $f : M_1 \to M_2$ and a point $x \in M_1$. We say that f satisfies the* Mean Value Inequality *(MVI) in the open ball $B_{M_1}(x, r)$ if there is a function $\Psi : [0, r) \subset \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$d_2(f(x), f(y)) \leq \Psi(d_1(x, y))\, d_1(x, y) \text{ for all } y \in B_{M_1}(x, r).$$

Also, the following properties must hold

(i) *Ψ is a continuous function;*
(ii) *The function $\mathcal{I} : [0, r) \subset \mathbb{R}^+ \to \mathbb{R}^+$ defined as*

$$\mathcal{I}(d) = \Psi(d)\, d$$

is a non decreasing homeomorphism over its image.

In this work we have two main objectives. First, establish a necessary and sufficient condition for the MVI in the open ball $B_{M_1}(x, r)$. Note that our applications will be defined in abstract metric spaces, not necessarily partially ordered ones (cf. [13]). To deal with this problem, we introduce a subclass of continuous applications that, roughly speaking, are the ones which we can find a $\varepsilon \mapsto \delta(\varepsilon)$ relation that possesses some regularity.

Our second goal is to show that the MVI in a Banach space is reduced to the supremum of the norm of the derivative in a open ball. We point out that in metric spaces there is no usual way to define directions, as opposed to what can be done in Banach spaces. Therefore, it is natural to expect that in our metric-only situation, the supremum is taken in the open ball, instead of the line segment.

To conclude, this paper is organized in the following sequence: in Section 2, we discuss the basic properties associated with sets that describes the regularity of a given function defined on a metric space. These sets are fundamental to the base structure of Sections 3 to 5, in which we investigate the $\varepsilon \mapsto \delta(\varepsilon)$ regularity of continuous functions. In Section 6, we establish a necessary and sufficient condition for the MVI. Further, in Section 7, we derive a version of the Mean Value Propriety, which is about computing and estimating average of functions, and how it relates to its counterpart in Harmonic Analysis.

2. Preliminary Ideas and Initial Concepts

Throughout this paper, (M_1, d_1) and (M_2, d_2) denote metric spaces, unless stated otherwise. For $i = 1, 2$, we write $B_{M_i}(x, r)$ to represent the open ball centered at $x \in M_i$ with radius $r > 0$. Also, let $\mathcal{F}(M_1, M_2)$ (or simply $\mathcal{F}$) be the set of all functions between M_1 and M_2. An element of $\mathcal{F} \times M_1 \times \mathbb{R}^+ \setminus \{0\}$ is called a *triplet.* We say that a positive real number δ is *suitable* for the (f, x, ε) triplet if $f(B_{M_1}(x, \delta)) \subset B_{M_2}(f(x), \varepsilon)$. In other words, $\delta > 0$ is suitable for the (f, x, ε) triplet if the following implication holds

$$y \in M_1 \text{ and } d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon.$$

With the previous definition, it is clear that $f : M_1 \to M_2$ is continuous at $x \in M_1$ if, and only if, for every $\varepsilon > 0$ there exists a suitable number for the (f, x, ε) triplet. Based in the notion introduced above, we now discuss properties of some new structures.

Definition 2. *Given the (f, x, ε) triplet, consider the set*

$$\Delta_{f,x}(\varepsilon) = \{\delta > 0 : \delta \text{ is suitable for the } (f, x, \varepsilon) \text{ triplet}\}.$$

The idea behind the previous set, is to capture all possible choices of $\delta > 0$ one can find while trying to prove that a certain function is continuous. Note that we are not excluding the possibility of $\Delta_{f,x}(\varepsilon) = \emptyset$, which would means that f is discontinuous at x. It is important to understand every possible scenario for the set of suitable numbers for a triplet.

Theorem 3. *Let (f, x, ε) be a triplet. One, and only one of the following alternatives occurs.*

(i) $\Delta_{f,x}(\varepsilon) = \emptyset$;
(ii) $\Delta_{f,x}(\varepsilon) = (0, \infty)$;
(iii) *There exists a certain $\delta > 0$ such that $\Delta_{f,x}(\varepsilon) = (0, \delta]$.*

Proof. Let us first prove that $\Delta_{f,x}$ is connected. We claim that if $\delta_2 \in \Delta_{f,x}(\varepsilon)$ and if $0 < \delta_1 \leq \delta_2$, then $\delta_1 \in \Delta_{f,x}(\varepsilon)$. Indeed, if $y \in M_1$ and $d_1(x, y) < \delta_1$, then $d_1(x, y) < \delta_2$. Since δ_2 is suitable for the (f, x, ε) triplet, we conclude that $d_2(f(x), f(y)) < \varepsilon$. This guarantees that δ_1 is also suitable for the (f, x, ε) triplet.

Now let us check that $\Delta_{f,x}$ is closed from the right side. More precisely, if $\{\delta_n\}_{n=1}^{\infty} \subset \Delta_{f,x}(\varepsilon)$ is an increasing sequence that converges to δ, then $\delta \in \Delta_{f,x}(\varepsilon)$. Fix $y \in M_1$ such that $d_1(x, y) < \delta$. Choose a natural number N such that $|\delta - \delta_N| < \delta - d_1(x, y)$. Since the sequence is increasing, we conclude that $d_1(x, y) < \delta_N$. But we already know that δ_N is suitable for the (f, x, ε) triplet, therefore we conclude that $d_2(f(x), f(y)) < \varepsilon$. Once $y \in M_1$ was an arbitrary choice, δ is also suitable for the (f, x, ε) triplet.

Now, we are able to conclude the argument. Three alternatives occur.

(i) $\Delta_{f,x}(\varepsilon) = \emptyset$. Then, we are done;

(ii) $\Delta_{f,x}(\varepsilon)$ is nonempty unbounded. Then, by the hereditary property above, $\Delta_{f,x}(\varepsilon) = (0,\infty)$;

(iii) $\Delta_{f,x}(\varepsilon)$ is nonempty bounded. Then, by the sequential argument we just developed, $\Delta_{f,x}(\varepsilon) = (0,\delta]$, where $\delta = \sup \Delta_{f,x}(\varepsilon) = (0,\delta]$.

□

To study more properties of $\Delta_{f,x}$, fix $x \in M_1$ and consider the family $\{\Delta_{f,x}(\varepsilon) : \varepsilon > 0\}$. Roughly speaking, we are going to show that the set of suitable numbers for the (f, x, ε) triplet does not became smaller as ε gets bigger.

Proposition 4. *Consider any function $f : M_1 \to M_2$. For each element $x \in M_1$, if $0 < \varepsilon_1 \leq \varepsilon_2$, then $\Delta_{f,x}(\varepsilon_1) \subset \Delta_{f,x}(\varepsilon_2)$.*

Proof. If $\Delta_{f,x}(\varepsilon_1) = \emptyset$, there is nothing to be done. On the other hand, suppose that there exists a value $\delta \in \Delta_{f,x}(\varepsilon_1)$. Hence, for $y \in M_1$ with $d_1(x, y) < \delta$, we have that $d_2(f(x), f(y)) < \varepsilon_1$. But $\varepsilon_1 \leq \varepsilon_2$, therefore for any $y \in M_1$ with $d_1(x, y) < \delta$, we have that $d_2(f(x), f(y)) < \varepsilon_2$. In other words, $\delta \in \Delta_{f,x}(\varepsilon_2)$. □

Definition 5. *Given $x \in M_1$ and a function $f : M_1 \to M_2$, define*

$$E_f(x) = \{\varepsilon > 0 : \Delta_{f,x}(\varepsilon) \textit{ is a non empty bounded set}\}.$$

Shortly, we may say that $E_f(x)$ captures all the possible values of $\varepsilon > 0$ such that there exists a maximum real number $\delta > 0$ that verifies the sentence

$$y \in M_1 \text{ and } d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon.$$

Working in the same way as we did in Theorem 3, we examine the full range of topological possibilities about the set $E_f(x)$. Note that, from now on, the continuity of the function f at x plays an important role. This allow us to state and prove the following result.

Theorem 6. *Let $f : M_1 \to M_2$ be a given function which is continuous at $x \in M_1$. Then one, and only one of the following alternatives occurs.*

(i) *If f is an unbounded, then $E_f(x) = (0,\infty)$;*

(ii) *If f is a constant function, then $E_f(x) = \emptyset$;*

(iii) *In the other cases, there exists some real number $\varepsilon > 0$ such that $E_f(x) = (0, \varepsilon]$ or $E_f(x) = (0, \varepsilon)$.*

Proof. Note that since f is continuous at $x \in M_1$, Proposition 4 guarantees that $E_f(x)$ is an interval. Now let us prove that f is a constant function if, and only if, $E_f(x) = \emptyset$. Indeed, if there exists $x \in M_1$ such that $E_f(x) = \emptyset$, then for any $\varepsilon > 0$ we have that $\Delta_{f,x}(\varepsilon) = (0,\infty)$. Thus, for any $\varepsilon, \delta > 0$ we must have the following

$$y \in M_1 \text{ and } d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon.$$

Therefore, for any $y \in M_1$ and any $\varepsilon > 0$ we have that $d_2(f(x), f(y)) < \varepsilon$; which implies that for any $y \in M_1$, $d_2(f(x), f(y)) = 0$. The other part of the claim is trivial.

To conclude, let us prove that $E_f(x)$ is a proper subset of $(0,\infty)$ if, and only if, f is bounded. Indeed, let f be bounded. In this case, there exists $\varepsilon > 0$ such that $f(M_1) \subset B_{M_2}(f(x), \varepsilon)$. In other words, for any $\delta > 0$

$$f(B_{M_1}(x, \delta)) \subset B_{M_2}(f(x), \varepsilon).$$

This last statement implies that any δ is suitable for the (f, x, ε) triplet and therefore $\Delta_{f,x}(\varepsilon) = (0, \infty)$, which means that $\varepsilon \notin E_f(x)$. If there exists $\varepsilon \in (0, \infty) \setminus E_f(x)$, by definition $\Delta_{f,x}(\varepsilon) = (0, \infty)$. Hence, for any $y \in M_1$ we have that $d_2(f(x), f(y)) < \varepsilon$, which implies that f is bounded. □

Example 7. *Note that all the situations listed in the last case can happen, as we can see in the following couple examples.*

(i) *Consider $M_1 = M_2 = \mathbb{R}$ and $d_1 = d_2$ the real Euclidean metric. If $f : M_1 \to M_2$ is the characteristic function of $[0, \infty)$, then*

$$\Delta_{f,x}(\varepsilon) = \begin{cases} (0, |x|], & \varepsilon \in (0, 1] \text{ and } x \neq 0 \\ (0, \infty), & \varepsilon > 1 \text{ and } x \neq 0 \\ \emptyset, & \varepsilon \in (0, 1] \text{ and } x = 0 \\ (0, \infty), & \varepsilon > 1 \text{ and } x = 0 \end{cases}$$

which implies that $E_f(0) = \emptyset$ and $E_f(x) = (0, 1]$ for $x \neq 0$;

(ii) *Let $M_1 = M_2 = [0, \infty)$ and $d_1 = d_2$ be the induced real Euclidean metric. If $f : M_1 \to M_2$ is defined as $f(x) = 1 - e^{-x}$, then*

$$\Delta_{f,0}(\varepsilon) = \begin{cases} \left(0, \ln \frac{1}{1-\varepsilon}\right], & \varepsilon \in (0, 1) \\ \\ (0, \infty), & \varepsilon \geq 1. \end{cases}$$

Therefore $E_f(0) = (0, 1)$.

Until this moment, the constructions that we made do not seem to have a direct connection with the original function f. However, now that we have discussed these prerequisites, we are ready to define the function that is related to the continuity of f.

Definition 8. *Given $f : M_1 \to M_2$ and $x \in M_1$ such that $E_f(x) \neq \emptyset$, define $\Pi_{f,x} : E_f(x) \to (0, \infty)$ as*

$$\Pi_{f,x}(\varepsilon) = \max \Delta_{f,x}(\varepsilon).$$

Remark 9. *A few remarks are in order.*

(i) *Since the set $\Delta_{f,x}(\varepsilon)$ admits a maximum for every $\varepsilon \in E_f(x)$, the function Π_x is well defined;*

(ii) *Is a direct consequence, from the previous section, that Π_x is a non decreasing function that for every $\varepsilon \in E_f(x)$ provided the largest possible number such that $f(B_{M_1}(x, \Pi_{f,x}(\varepsilon))) \subset B_{M_2}(f(x), \varepsilon)$;*

(iii) *Even if, during the construction of the function $\Pi_{f,x}$, the original function f plays an important role, whenever there is no possibility of confusion, we omit f as an index, writing instead just Π_x.*

3. The Continuity of Π_x

We now focus our attention to the continuity properties of the function Π_x.

Lemma 10. *Let $f : M_1 \to M_2$ be a non constant and continuous function. Choose $x \in M_1$ and suppose that for any $r > 0$ the closure of $B_{M_1}(x, r)$ is compact in M_1. Under these conditions, for all $\varepsilon \in E_f(x)$ the following holds.*

(i) *If* $\{\varepsilon_n\}_{n=1}^{\infty} \subset E_f(x)$ *is an increasing sequence that converges to* ε*, then*

$$\lim_{n\to\infty} \max \Delta_{f,x}(\varepsilon_n) = \max \Delta_{f,x}(\varepsilon);$$

(ii) *If* $\{\varepsilon_n\}_{n=1}^{\infty} \subset E_f(x)$ *is a decreasing sequence that converges to* ε*, then*

$$\lim_{n\to\infty} \max \Delta_{f,x}(\varepsilon_n) = \max \Delta_{f,x}(\varepsilon).$$

Proof. We only verify the first part; the second one follows in a similar way. By Proposition 4, the map $\varepsilon \mapsto \max \Delta_{f,x}(\varepsilon)$ is non decreasing. Denote, for simplicity

$$\widetilde{\delta} = \lim_{n\to\infty} \max \Delta_{f,x}(\varepsilon_n) \qquad \text{and} \qquad \delta = \max \Delta_{f,x}(\varepsilon)$$

If $\widetilde{\delta} \neq \delta$, then the non decreasing property above gives $\widetilde{\delta} < \delta$. Consider the auxiliary sequence $\{\delta_n\}_{n=1}^{\infty}$ given by

$$\delta_n = \frac{\widetilde{\delta} + \delta}{2} + \left[\frac{\delta - \widetilde{\delta}}{2(n+1)}\right]$$

and observe that it satisfies the following properties

(i) $\delta_n \to (\widetilde{\delta} + \delta)/2$;
(ii) $\widetilde{\delta} < \delta_n < \delta$.

Since for each $n = \{1, 2, \ldots\}$ the number δ_n is strictly bigger then $\widetilde{\delta}$, we have that $\delta_n \notin \Delta_{f,x}(\varepsilon_n)$. Therefore, for each n, there exists $y_n \in M_1$ such that

$$d_1(y_n, x) < \delta_n \quad \text{and} \quad d_2(f(y_n), f(x)) \geq \varepsilon_n.$$

Since $y_n \in B_{M_1}(x, \delta)$, choosing a subsequence if necessary, we can assume that there exists $y \in \overline{B_{M_1}(x, \delta)}$ such that $y_n \to y$. Then, when $n \to \infty$, we obtain

$$d_1(y, x) \leq (\widetilde{\delta} + \delta)/2 \quad \text{and} \quad d_2(f(y), f(x)) \geq \varepsilon,$$

and conclude that $\delta \notin \Delta_{f,x}(\varepsilon)$, which is a contradiction. □

Theorem 11. *Let* $f : M_1 \to M_2$ *be a non constant and continuous function. Choose* $x \in M_1$ *and suppose that for any* $r > 0$ *the closure of* $B_{M_1}(x, r)$ *is compact in* M_1*. Under these conditions* Π_x *is a continuous function.*

Proof. Choose and fix $\varepsilon_0 \in E_f(x)$. Lets prove the equality

$$\lim_{\varepsilon\to\varepsilon_0^-} \Pi_x(\varepsilon) = \Pi_x(\varepsilon_0) = \lim_{t\to\varepsilon_0^+} \Pi_x(\varepsilon).$$

We shall begin proving the left limit. Let $\{\widetilde{\varepsilon}_n\}$ be sequence to the left of ε_0 such that $\widetilde{\varepsilon}_n \to \varepsilon_0$. It is not difficult to construct another increasing sequence $\{\varepsilon_n\}$ with the following proprieties

(i) $\varepsilon_n < \widetilde{\varepsilon}_n$, for each $n \in \mathbb{N}$;
(ii) $\varepsilon_n \to \varepsilon_0$, when $n \to \infty$.

Then we know that $\Delta_{f,x}(\varepsilon_n) \subset \Delta_{f,x}(\widetilde{\varepsilon}_n) \subset \Delta_{f,x}(\varepsilon_0)$ for all natural number n. Using the maximum function in this sequence of inclusions, we deduce

$$\max \Delta_{f,x}(\varepsilon_n) \leq \max \Delta_{f,x}(\widetilde{\varepsilon}_n) \leq \max \Delta_{f,x}(\varepsilon_0).$$

Finally, applying the limit when $n \to \infty$ on both sides and using Lemma 10 we obtain

$$\max \Delta_{f,x}(\varepsilon_0) \leq \lim_{n\to\infty} \max \Delta_{f,x}(\widetilde{\varepsilon}_n) \leq \max \Delta_{f,x}(\varepsilon_0).$$

Since $\{\widetilde{\varepsilon}_n\}$ was an arbitrary sequence, we conclude the proof of the left limit. The right one is obtained in a similar way and therefore we are done. □

The next step in our work is to extend the domain of Π_x. To this end, we recall a useful definition.

Definition 12. *A function $f : M_1 \to M_2$ is called* locally constant *at a point $x \in M_1$, if there is exists $r > 0$ such that $f|_{B_{M_1}(x,r)}$ is a constant function.*

Observe that locally constant functions are almost similar to the constant functions, when we compute its Π_x function. Hence, we shall exclude this class of functions in our future results.

Theorem 13. *Let $f : M_1 \to M_2$ be a continuous function. Choose $x \in M_1$ and suppose that f is not locally constant at x. In this case, 0 is an accumulation point of $E_f(x)$ and*

$$\lim_{\varepsilon \to 0^+} \Pi_x(\varepsilon) = 0.$$

Proof. Since f is continuous and non constant, it follows easily (Theorem 6) that 0 is an accumulation point of $E_f(x)$. In fact, $E_f(x)$ turns out to be an interval in this case.

Now, consider $\{\varepsilon_n\} \subset E_f(x)$ a decreasing sequence that converges to 0. By monotonicity, we have that $\Delta_{f,x}(\varepsilon_{n+1}) \subset \Delta_{f,x}(\varepsilon_n)$. If $\operatorname{diam} A$ is the diameter of the set A, then $\operatorname{diam} \Delta_{f,x}(\varepsilon_{n+1}) \leq \operatorname{diam} \Delta_{f,x}(\varepsilon_n)$. Since $\Pi_x(\varepsilon) = \max \Delta_{f,x}(\varepsilon) = \operatorname{diam} \Delta_{f,x}(\varepsilon)$, we observe that

$$\lim_{n \to \infty} \Pi_x(\varepsilon_n) = \operatorname{diam} \left(\bigcap_{n=1}^{\infty} \Delta_{f,x}(\varepsilon_n) \right).$$

If $\delta \in \cap_{n=1}^{\infty} \Delta_{f,x}(\varepsilon_n)$, then $f(B(x,\delta)) \subset B(f(x), \varepsilon_n)$ for any $n \in \mathbb{N}$. Since $\varepsilon_n \to 0$, we conclude that f is locally constant at x. Since by hypothesis this situation cannot happen, we obtain that $\cap_{n=1}^{\infty} \Delta_{f,x}(\varepsilon_n) = \emptyset$. Therefore $\lim_{n \to \infty} \Pi_x(\varepsilon_n) = 0$. Following the same final steps done in Theorem 11, we conclude the proof of this theorem. □

This last theorem allow us to continuously extend our definition of Π_x to $\varepsilon = 0$ if f is not locally constant at x. This extension will be useful in the next result, which connects the image of Π_x with the set of suitable values.

Theorem 14. *Let $f : M_1 \to M_2$ be a continuous function. Choose $x \in M_1$ such that f is not locally constant at x. Suppose that for any $r > 0$ the closure of $B_{M_1}(x, r)$ is compact in M_1. Under these conditions, for all $\varepsilon \in E_f(x)$ we have that $\Delta_{f,x}(\varepsilon) = \Pi_x(0, \varepsilon]$.*

Proof. If $\delta > 0$ is suitable for the (f, x, ε) triplet, then $\Pi_x(0) < \delta \leq \Pi_x(\varepsilon)$. Since Π_x is continuous, it must exist some $\varepsilon' \in (0, \varepsilon]$ such that $\Pi_x(\varepsilon') = \delta$. Conversely, if $\varepsilon' \in (0, \varepsilon]$, then $f(B_{M_1}(x, \Pi_x(\varepsilon'))) \subset B_{M_2}(f(x), \varepsilon')$ what guarantees that $f(B_{M_1}(x, \Pi_x(\varepsilon'))) \subset B_{M_2}(f(x), \varepsilon)$. This proves that $\Pi_x(\varepsilon')$ is suitable for the (f, x, ε) triplet. □

4. Π_x as a Homeomorphism

There is no special reason to expect that Π_x is a homeomorphism. For example, if $\mathbb{R}$ is considered with the discrete metric, than any given $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In the case where $f(x) = x$, it is quite easy to see that 1 is the maximal suitable value for the (f, x, ε) triplet. More precisely, we have that $\Pi_x(\varepsilon) = 1$ for all $\varepsilon \in (0, 1)$. Inspired by this example, we shall now investigate under which circumstances we can guarantee that Π_x is a homeomorphism. We start with some basic definition.

Definition 15. *Given a function $f : M_1 \to M_2$ and a point $x \in M_1$, we say that f is* adherent *at x if the following holds*

(i) *f is continuous at x;*

(ii) *If $y \in M_1$ and $\varepsilon \in E_f(x)$ are such that $d_1(x,y) = \Pi_x(\varepsilon)$, then $d_2(f(x), f(y)) \leq \varepsilon$.*

If $f : M_1 \to M_2$ is adherent at any $x \in M_1$, we simply say that f is adherent.

This last technical definition has the purpose of categorize a critical behavior of the function f. Recall that if f is continuous at x and $d_1(x,y) < \Pi_x(\varepsilon)$, then $d_2(f(x), f(y)) < \varepsilon$. Then, it is natural to ask what happens with f when $d_1(x,y) = \Pi_x(\varepsilon)$. In other words, suppose that there exists some point y such that $d_1(x,y) = \Pi_x(\varepsilon)$ and $d_2(f(x), f(y) \geq \varepsilon$. Intuitively speaking, if f is adherent at x, then we are requesting that the number $d_2(f(x), f(y))$ stays, at the worst possible scenario, equal to ε.

Example 16. *As outlined previously, consider $\mathbb{R}$ with the discrete metric and $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. Then we can verify that f is not adherent at any $x \in \mathbb{R}$. Indeed, if $\varepsilon = 1/2$ and y is such that $d_1(x,y) = \Pi(\varepsilon) = 1$, then y can be any real number, except x itself. Under these conditions, it is false that $d_2(x,y) = d_2(f(x), f(y)) \leq 1/2$ for all y such that $d_1(x,y) = 1$.*

The next theorem investigates the relation between Π_x and the definition of adherence at a point x.

Theorem 17. *Consider $f : M_1 \to M_2$ and $x \in M_1$. Assume that for any $r > 0$ the closure of $B_{M_1}(x,r)$ is compact in M_1. If f is continuous at x, then the following are equivalent*

(i) *f is adherent at x.*

(ii) *Π_x is injective.*

Proof. Suppose that f is not adherent at x. Under these conditions, there exist $y \in M_1$ and $\varepsilon \in E_f(x)$ such that $d_1(x,y) = \Pi_x(\varepsilon)$ and $d_2(f(x), f(y)) > \varepsilon > 0$. We observe that $\varepsilon' = d_2(f(x), f(y)) \in E_f(x)$. Indeed, since f is continuous at x and $\varepsilon' > 0$ we already know that $\Delta_{f,x}(\varepsilon')$ is a non empty set. Now, assume that $\Delta_{f,x}(\varepsilon')$ is not a bounded set. Using Theorem 3, we conclude that $\Delta_{f,x}(\varepsilon') = (0, \infty)$. Therewith, $0 < d_1(x,y) + 1 \in \Delta_{f,x}(\varepsilon')$, which implies that $d_1(x,y) + 1$ is suitable for the triplet (f, x, ε'). Therefore, if $\widetilde{y} \in M_1$ and

$$d_1(x, \widetilde{y}) < d_1(x,y) + 1, \text{ then } d_2(f(x), f(\widetilde{y})) < \varepsilon'.$$

Once we can suppose that $\widetilde{y} = y$, in the previous statement, we obtain that $\varepsilon' < \varepsilon'$; contradiction. In other words, $\Delta_{f,x}(\varepsilon')$ is a bounded set and $\varepsilon' = d_2(f(x), f(y)) \in E_f(x)$.

Now, since Π_x is a non decreasing function, we obtain that $\Pi_x(\varepsilon) \leq \Pi_x(\varepsilon')$. Assume, for an instant, that $\Pi_x(\varepsilon) < \Pi_x(\varepsilon')$. Using the fact that $d_1(x,y) = \Pi_x(\varepsilon)$, we deduce that $d_1(x,y) < \Pi_x(\varepsilon')$. Therefore $d_2(f(x), f(y)) < \varepsilon'$, which is a contradiction. Hence, $\Pi_x(\varepsilon) = \Pi_x(\varepsilon')$, which implies that Π_x is not injective.

Conversely, suppose that f is adherent at x. Consider ε_1 and ε_2 in $E_f(x)$ such that $\Pi_x(\varepsilon_1) = \Pi_x(\varepsilon_2)$. Without loss of generality, assume that $\varepsilon_1 \leq \varepsilon_2$. By Lemma 6, we already know that $[\varepsilon_1, \varepsilon_2] \subset (0, \varepsilon_2] \subset E_f(x)$.

Let $\delta = \Pi_x(\varepsilon_1) = \Pi_x(\varepsilon_2)$. Since Π_x is a non decreasing function and $[\varepsilon_1, \varepsilon_2] \subset E_f(x)$, we have that $\Pi_x(\varepsilon) = \delta$ for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$. But δ is the largest suitable number for the (f, x, ε) triplet, whenever ε lies in $[\varepsilon_1, \varepsilon_2]$. Writing it in another way: for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ and for any number $\delta + 1/n$, we have that $\delta + 1/n$ is not suitable for the (f, x, ε) triplet. Hence, for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ and for any natural number n, there is some $y_{n,\varepsilon} \in M_1$ such that $d_1(x, y_{n,\varepsilon}) < \delta + 1/n$ and $d_2(f(x), f(y_{n,\varepsilon})) \geq \varepsilon$.

Without loss of generality, assume that there exists $\widetilde{y_\varepsilon}$ such that $\lim_{n\to\infty} y_{n,\varepsilon} = \widetilde{y_\varepsilon}$. Using a limit argument, we have that for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, there is some $\widetilde{y_\varepsilon} \in M_1$ such that $d_1(x, \widetilde{y_\varepsilon}) \leq \delta$ and $d_2(f(x), f(\widetilde{y_\varepsilon})) \geq \varepsilon$. If we suppose, for an instant, that $d_1(x, \widetilde{y_\varepsilon}) < \delta = \Pi_x(\varepsilon)$, a directly consequence would be that $d_2(f(x), f(\widetilde{y_\varepsilon})) < \varepsilon$, which is a contradiction. Because of that, for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, there is some $\widetilde{y_\varepsilon} \in M_1$ such that $d_1(x, \widetilde{y_\varepsilon}) = \delta$ and $d_2(f(x), f(\widetilde{y_\varepsilon})) \geq \varepsilon$. Since f is adherent at x, we conclude that

$$\forall\, \varepsilon \in [\varepsilon_1, \varepsilon_2],\ \exists\, \widetilde{y_\varepsilon} \in M_1 \text{ such that } d_1(x, \widetilde{y_\varepsilon}) = \Pi_x(\varepsilon) \text{ and } d_2(f(x), f(\widetilde{y_\varepsilon})) = \varepsilon.$$

In other words, there exists $\widetilde{y_{\varepsilon_2}} \in M_1$ such that $d_1(x, \widetilde{y_{\varepsilon_2}}) = \Pi_x(\varepsilon_2)$ and $d_2(f(x), f(\widetilde{y_{\varepsilon_2}})) = \varepsilon_2$. Since f is adherent at x and $d_1(x, \widetilde{y_{\varepsilon_2}}) = \Pi_x(\varepsilon_1)$, we conclude that $\varepsilon_2 = d_2(f(x), f(\widetilde{y_{\varepsilon_2}})) \leq \varepsilon_1$ what implies that $\varepsilon_1 = \varepsilon_2$. Therefore, Π_x is an injective function. □

Remark 18. *Since any continuous and injective function $g : I \to \mathbb{R}$, defined on any interval $I \subset \mathbb{R}$, is a homeomorphism (from I to $g(I)$), this last theorem guarantees that function Π_x is a homeomorphism. In other words, it guarantees that the dependence $\varepsilon \mapsto \delta(\varepsilon)$ from the continuity of a function in a metric space can be described by a homeomorphism.*

Next theorem specifies more clearly which functions can be expected to be adherent. From now on, if (M_1, d_1) denotes a metric space, we write $B_{M_1}[x, r]$ to denote the *closed ball* $B_{M_1}[x, r] = \{y \in M_1 : d_1(x, y) \leq r\}$.

Theorem 19. *Let $f : M_1 \to M_2$ be continuous at $x \in M_1$. Suppose that M_1 and M_2 are metric spaces such that the closure of the ball $B_{M_i}(x, r)$ is the closed ball $B_{M_i}[x, r]$. Then f is adherent at x.*

Proof. Let $\varepsilon \in E_f(x)$. Notice that $f(\overline{B_{M_1}(x, \Pi_x(\varepsilon))}) \subset \overline{B_{M_2}(f(x), \varepsilon)}$. Indeed, let $p \in f(\overline{B_{M_1}(x, \Pi_x(\varepsilon))})$. Then, there exists $z \in \overline{B_{M_1}(x, \Pi_x(\varepsilon))}$ such that $f(z) = p$. On the other hand, z is the limit of some sequence (z_n) of elements in $B_{M_1}(x, \Pi_x(\varepsilon))$. Since f is a continuous function, we must have that $p = \lim f(z_n)$. In other words, $p \in \overline{f(B_{M_1}(x, \Pi_x(\varepsilon))})$. But f is a continuous map, therefore $f(B_{M_1}(x, \Pi_x(\varepsilon)) \subset B_{M_2}(f(x), \varepsilon)$ what guarantees that $p \in \overline{B_{M_2}(f(x), \varepsilon)}$.

Now, using the hypothesis about the closure of the balls, we conclude that $f(B_{M_1}[x, \Pi_x(\varepsilon)]) \subset B_{M_2}[f(x), \varepsilon]$. At last, suppose that there exists $y \in M_1$ such that $d_1(x, y) = \Pi_x(\varepsilon)$. It follows that $y \in B_{M_1}[x, \Pi_x(\varepsilon)]$, what guarantees that $f(y) \in B_{M_2}[f(x), \varepsilon]$. This concludes the proof. □

We finish this section with an important theorem, that put together some previous results and discussions.

Theorem 20 (Π_x as a Homeomorphism)**.** *Suppose that $f : M_1 \to M_2$ is any given function, $x \in M_1$ and assume the following*

(i) *f is continuous;*
(ii) *f is adherent at x;*
(iii) *f is not locally constant at x;*
(iv) *For any $r > 0$ the closure of $B_{M_1}(x, r)$ is compact in M_1.*

Then there exists an interval I_x and a function $\Pi_x : I_x \to J_x \subset \mathbb{R}^+$ such that

(i) $0 \in I_x \subset \mathbb{R}^+$*;*
(ii) *Π_x is a monotonic increasing homeomorphism;*
(iii) $f(B_{M_1}(x, \delta)) \subset B_{M_2}(f(x), \varepsilon) \iff \delta \leq \Pi_x(\varepsilon)$, *for all $\varepsilon \in I_x$;*
(iv) $\Pi_x(0) = 0$.

5. Enclosed Functions

At this point, we observe that for a certain class of functions (see Theorem 20) we have already managed to construct a homeomorphism that is related to the continuity properties of f.

Definition 21. *Given a function $f : M_1 \to M_2$ and a point $x \in M_1$, we say that f is* strongly enclosed *at x if f satisfies the conclusions (i), (ii), (iii) and (iv) of Theorem 20. Furthermore, the Π_x map is called the* strong continuity function *of f at x. We can also say that f is* enclosed *at x if f satisfies all the previous conclusions, with the possible exception of the implication $f(B_{M_1}(x,\delta)) \subset B_{M_2}(f(x),\varepsilon) \Longrightarrow \delta \leq \Pi_x(\varepsilon)$. In this case, the Π_x map is just called a* continuity function *of f at x. If the function is strongly enclosed (or enclosed) at all points of the domain, we say that the function is strongly enclosed (or enclosed).*

Roughly speaking, we may not have a maximum suitable number for enclosed functions, while the opposite occurs with the strongly enclosed functions. In general, to guarantee that a function is strongly enclosed, we require some compactness hypothesis (see Theorem 20). On the other hand, all locally Lipschitz functions are enclosed.

It is interesting to note that the strong continuity function associated with a strong enclosed function f is unique, where the uniqueness here is considered in the sense of germs at 0. Note that this does not happen in the enclosed case. We are now interested in finding other classes of function which are enclosed or strongly enclosed.

Definition 22. *Given a function $f : M_1 \to M_2$ and a point $x \in M_1$, we are going to say that f satisfies the* Lagrange Propriety *at x if the following two conditions hold*

(i) *There exists a C^1 function $\Gamma : (-\zeta,\zeta) \subset \mathbb{R} \to \mathbb{R}$ such that*

$$\Gamma(r) = \sup_{y \in B_{M_1}[x,r]\setminus\{x\}} \frac{d_2(f(x),f(y))}{d_1(x,y)}$$

for all $r \in (0,\zeta)$;

(ii) *For all $r \in (0,\zeta)$, there is an element $y_r \in B_{M_1}[x,r] \setminus \{x\}$ such that*

$$\sup_{y \in B_{M_1}[x,r]\setminus\{x\}} \frac{d_2(f(x),f(y))}{d_1(x,y)} = \frac{d_2(f(x),f(y_r))}{d_1(x,y_r)}.$$

Example 23. *Consider $M_1 = M_2 = \mathbb{R}$ and $d_1 = d_2$ the real Euclidean metric. Assume that $f : M_1 \to M_2$ is given by $f(x) = 1 - e^{-x}$. Since $(1-e^{-y})/y$ is a positive decreasing function, we have that*

$$\Gamma(r) = \sup_{y \in B_{\mathbb{R}}[0,r]\setminus\{0\}} \frac{|1-e^{-y}|}{|y|} = \frac{1-e^{r}}{-r} = \frac{d_2(f(0),f(-r))}{d_1(0,-r)}.$$

In other words, f satisfies the Lagrange Propriety at 0 with $\Gamma(r) = \frac{e^r-1}{r}$.

The truly importance of the Lagrange class is uncovered by the following result, which connects the Γ function with the strong enclosed definition.

Theorem 24. *Suppose that $f : M_1 \to M_2$ is any given function that satisfies the Lagrange Propriety at $x \in M_1$. If $\Gamma(0) \neq 0$ and $\Gamma'(0) > 0$, then f is strongly enclosed at x. Furthermore, the strong continuity function of f is a C^k diffeomorphism, provided that Γ is a C^k application.*

Proof. Let $\Delta : (-\zeta,\zeta) \to \mathbb{R}$ be given by $\Delta(t) = t\,\Gamma(t)$. Since Δ is a C^k application and $\Delta'(0)$ is an isomorphism, we can use the Inverse Function Theorem and assume that $\Delta|_{(-\zeta_1,\zeta_1)} : (-\zeta_1,\zeta_1) \to (-\zeta_2,\zeta_2)$ is a C^k diffeomorphism.

Let Δ^{-1} be the inverse map of $\Delta|_{(-\zeta_1,\zeta_1)}$. Then, $\Delta^{-1}(\varepsilon)\Gamma(\Delta^{-1}(\varepsilon)) = \varepsilon$, which implies that

$$\Delta^{-1}(\varepsilon) = \frac{\varepsilon}{\sup\limits_{y\in B_{M_1}[x,\Delta^{-1}(\varepsilon)]\setminus\{x\}} \dfrac{d_2(f(x),f(y))}{d_1(x,y)}}.$$

We claim that $\Delta^{-1}(\varepsilon)$ is suitable for (f,x,ε) triplet. Indeed, if $d_1(x,y) < \Delta^{-1}(\varepsilon)$, then

$$d_2(f(x),f(y)) \le \sup_{y\in B_{M_1}[x,\Delta^{-1}(\varepsilon)]\setminus\{x\}} \frac{d_2(f(x),f(y))}{d_1(x,y)}\, d_1(x,y).$$

This implies that

$$d_2(f(x),f(y)) < \sup_{y\in B_{M_1}[x,\Delta^{-1}(\varepsilon)]\setminus\{x\}} \frac{d_2(f(x),f(y))}{d_1(x,y)}\, \Delta^{-1}(\varepsilon) = \varepsilon.$$

If ε is small enough, let us prove that $\delta = \Delta^{-1}(\varepsilon)$ is the maximum possible suitable number for (f,x,ε) triplet. Indeed, if $\delta = \Delta^{-1}(\varepsilon)$, then $\Delta(\delta) = \varepsilon$, which implies that $\delta\,\Gamma(\delta) = \varepsilon$. Remember that for each δ, there exists an element y_δ such that

$$\Gamma(\delta) = \frac{d_2(f(x),f(y_\delta))}{d_1(x,y_\delta)}.$$

Hence, $\delta\, d_2(f(x),f(y_\delta)) = \varepsilon\, d_1(x,y_\delta)$. It is now clear that $d_2(f(x),f(y_\delta)) = \varepsilon$ if, and only if, $d(x,y_\delta) = \delta$. Since $y_\delta \in B_{M_1}[x,\delta]\setminus\{x\}$, we also conclude that $d_2(f(x),f(y_\delta)) < \varepsilon$ if, and only if, $d(x,y_\delta) < \delta$.

Now suppose that δ is not the maximum possible suitable number for (f,x,ε) triplet. Therefore, we can find some $E > 0$ such that

$$d_1(x,y) < E + \delta \Longrightarrow d_2(f(x),f(x)) < \varepsilon.$$

Since $y_\delta \in B_{M_1}[x,\delta]\setminus\{x\}$, we have that

$$d_1(x,y_\delta) < E + \delta \Longrightarrow d_2(f(x),f(y_\delta)) < \varepsilon.$$

Making $E \to 0$, we then conclude that

$$d_1(x,y_\delta) \le \delta \Longrightarrow d_2(f(x),f(y_\delta)) < \varepsilon.$$

Therefore, as proved above,

$$d_1(x,y_\delta) \le \delta \Longrightarrow d_2(f(x),f(y_\delta)) < \varepsilon \Longrightarrow d(x,y_\delta) < \delta.$$

Because of the last inequality, we have that $y_\delta \in B_{M_1}[x,\delta]\setminus\{x\}$ actually satisfies $d(x,y_\delta) < \delta$. Since $\Gamma'(0) > 0$ and Γ is a C^1 function, we know that Γ' is non negative on a certain interval starting at the origin. Considering that ε is small enough, we have that

$$\Gamma(d_1(x,y_\delta)) < \Gamma(\delta) = \frac{d_2(f(x),f(y_\delta)}{d_1(x,y_\delta)}.$$

Note that the last inequality is a contradiction, since by definition

$$\Gamma(d_1(x,y_\delta)) \ge \frac{d_2(f(x),f(y_\delta)}{d_1(x,y_\delta)}.$$

Therefore $\delta = \Delta^{-1}(\varepsilon)$ is the maximum suitable number for the (f,x,ε) triplet. In other words, $\Pi_x = \Delta^{-1}$ is the strongly enclosed function associated to f. Since the function $\Delta|_{(-\zeta_1,\zeta_1)} : (-\zeta_1,\zeta_1) \to (-\zeta_2,\zeta_2)$ is a C^k diffeomorphism, the proof is finished. $\square$

Corollary 25. *Assuming the last theorem hypothesis, we also have that*

$$\Pi_x(\varepsilon) = \frac{\varepsilon}{\displaystyle\sup_{y\in B_{M_1}[x,\xi(\varepsilon)]\setminus\{x\}} \frac{d_2(f(x),f(y))}{d_1(x,y)}} = \frac{\varepsilon}{\Gamma(\xi(\varepsilon))},$$

where ξ *satisfies the following conditions*

$$\begin{cases} \xi(0) = 0 \\ \xi'\left(\varepsilon\,\Gamma(\varepsilon)\right) = \dfrac{1}{\Gamma(\varepsilon)+\varepsilon\Gamma'(\varepsilon)} \end{cases}$$

Proof. We just have to note that $\xi = \Pi_x = \Delta^{-1}$. The rest of the conclusion follows from the Chain Rule. ∎

Example 26. *If* $M_1 = M_2 = [0,\infty)$ *and* $d_1 = d_2$ *are the real Euclidean metric, we know that if function* $f : M_1 \to M_2$ *is given by* $f(x) = 1 - e^{-x}$*, then*

$$\Pi_0(\varepsilon) = \ln\frac{1}{1-\varepsilon} \approx \varepsilon + \frac{\varepsilon^2}{2} + O(\varepsilon^3).$$

Now let us consider a slightly different situation. Suppose that $M_1 = M_2 = \mathbb{R}$ *and* $d_1 = d_2$ *are the real Euclidean metric. Suppose that* $f : M_1 \to M_2$ *is given by* $f(x) = 1 - e^{-x}$*. Since our domain is an open set, we can use the last theorem to find an expression for* $\Pi_0(\varepsilon)$*.*

We also know that f *satisfies the Lagrange Propriety with* $\Gamma(r) = \frac{1-e^r}{-r}$*. Under these conditions,* ξ *satisfies the following system*

$$\begin{cases} \xi(0) = 0 \\ \xi'\left(\varepsilon\,\Gamma(\varepsilon)\right) = \dfrac{1}{\Gamma(\varepsilon)+\varepsilon\Gamma'(\varepsilon)} \end{cases}$$

for sufficiently small ε*. Therefore* $\xi(\varepsilon) = \ln(1+\varepsilon)$ *and*

$$\Pi_0(\varepsilon) = \ln(1+\varepsilon) \approx \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3).$$

It is noteworthy that a simple domain change can alter the Π continuity function expression. This phenomenon should not be surprising, since there is a connection between the $\varepsilon \mapsto \delta(\varepsilon)$ relation of f and its domain.

Corollary 27. *Suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are Banach spaces. In addition to the last theorem hypothesis, suppose that* M_1 *is an open set of* $\mathcal{B}_1$ *and also assume that* $M_2 \subset \mathcal{B}_2$*. If* f *is differentiable and* M *is the maximum value of function* $t \mapsto \|f'(t)\|_{\mathcal{L}(B_1,B_2)}$*, then*

$$\frac{\varepsilon}{M} \leq \Pi_x(\varepsilon) \leq \frac{\varepsilon}{\Gamma(0)}$$

for all sufficiently small ε*. If* $t \mapsto \|f'(t)\|_{\mathcal{L}(B_1,B_2)}$ *does not reach a maximum value, then the first inequality is reduced to* $0 \leq \Pi_x(\varepsilon)$*.*

6. The Mean Value Inequality

Now we are ready to present our main theorem.

Theorem 28 (Mean Value Inequality for Metric Spaces)**.** *Given a function* $f : M_1 \to M_2$ *and a point* $x \in M_1$*, the following are equivalent*

(i) f *is enclosed in* x *and* Π_x^{-1} *is differentiable at* 0^+*;*

(ii) f *satisfies the MVI (see Definition 1) for metric spaces in some* $B_{M_1}(x,R)$*.*

In addition, suppose that M_1 and M_2 are open sets contained in Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If f is differentiable and strongly enclosed at x, then there exists $R > 0$ such that

$$\Psi(d_1(x,y)) \leq \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$$

for any $y \in B_{M_1}(x,R)$.

Proof. Suppose that f is enclosed at x and consider $I_x = D(\Pi_x)$. Define $\Psi : J_x \to \mathbb{R}^+$ by

$$\Psi(d) = \begin{cases} \dfrac{\Pi_x^{-1}(d)}{d}, & \text{if } d \in J_x \setminus \{0\} \\ (\Pi_x^{-1})'(0^+), & \text{if } d = 0, \end{cases}$$

where $J_x = \Pi_x(I_x)$.

Observe that Ψ is continuous and that the function $d \mapsto \Psi(d)d$ is a non decreasing homeomorphism. Now choose $R > 0$ with the following two properties holding.

(i) $R < (1/2)\sup J_x$;
(ii) For any $y \in B_{M_1}(x,R)$, we have $d_2(f(x), f(y) < \sup I_x$.

Let $y \in B_{M_1}(x,R)$. If $\varepsilon \in (0,R)$, then the value $d_1(x,y)+\varepsilon$ is not suitable for the $(f, x, d_2(f(x), f(y)))$ triplet. Therefore we obtain the following inequality

$$\Pi_x(d_2(f(x), f(y))) < d_1(x,y) + \varepsilon.$$

Applying Π_x^{-1} on both sides and taking the limit as $\varepsilon \to 0$, we achieve

$$d_2(f(x), f(y)) \leq \Pi_x^{-1}(d_1(x,y)) = \Psi(d_1(x,y))\, d_1(x,y).$$

Conversely, let us suppose that f satisfies the MVI. Then, there exists a function $\Psi : [0,r) \subset \mathbb{R}^+ \to \mathbb{R}^+$ with the properties described on Definition 1. Set Π_x as the inverse of the function $d \mapsto \Psi(d)d$. Hence, if $d_1(x,y) < \delta$ and $\delta \leq \Pi_x(\varepsilon)$, MVI guarantees that

$$d_2(f(x), f(y)) \leq \Psi(d_1(x,y))\, d_1(x,y) = \Pi_x^{-1}(d_1(x,y)).$$

Since $d_1(x,y) < \Pi_x(\varepsilon)$ and Π_x is strictly increasing, we obtain that $\Pi_x^{-1}(d_1(x,y)) < \Pi_x^{-1}(\Pi_x(\varepsilon)) = \varepsilon$, i.e., $d_2(f(x), f(y)) < \varepsilon$.

Finally, to prove the last statement, assume that M_1 and M_2 are open sets contained in Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively and f is differentiable. Choose $R > 0$ such that for any $y \in B_{M_1}(x,R)$

$$d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} < \sup I_x/2.$$

Now let us prove that for any $y \in B_{M_1}(x,R)$, the value $d_1(x,y)$ is suitable for the (f,x,r) triplet, where the value $r = d_1(x,y)\sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$. Indeed, if $d_1(x,\widetilde{y}) < d_1(x,y)$, we have that

$$\begin{aligned} d_2(f(x), f(\widetilde{y})) &\leq d_1(x,\widetilde{y}) \sup\nolimits_{B_{M_1}[x,d_1(x,\widetilde{y})]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \\ &\leq d_1(x,\widetilde{y}) \sup\nolimits_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \\ &< d_1(x,y) \sup\nolimits_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}. \end{aligned}$$

Therefore, since f is strongly enclosed at x, $d_1(x,y) \leq \Pi_x\left(d_1(x,y)\sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\right)$ what guarantees

$$\Pi_x^{-1}(d_1(x,y)) \leq d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)},$$

which finally implies that

$$\Psi(d_1(x,y)) \leq \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)},$$

and the proof is complete. □

Notice that the differentiability requirement of Π_x^{-1} at 0 is the natural substitute for f being differentiable on its domain. Also note that, since Ψ is a continuous function and $\sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$ may not be continuous, there is no hope to expect anything better than the inequality provided by the last theorem.

On the other hand, if f is continuously differentiable we have that $y \mapsto \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$ defines a continuous function. Moreover, it provide us more precise information about the function Ψ, as we can verify in the following result.

Theorem 29. *In addition to the last theorem hypothesis, suppose that M_1 and M_2 are open sets contained in Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If f is continuously differentiable and if there is a direction $v \in \mathcal{B}_1$ such that $\|f'(x)\cdot v\|_{\mathcal{B}_2} = \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \neq 0$, then there exists $R > 0$ such that*

$$\Psi(d_1(x,y)) = \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$$

for all $y \in B_{M_1}(x,R)$.

Proof. Choose $R > 0$ with the following two properties

(i) $d_1(x,y)\sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} < \sup I_x$, for any $y \in B_{M_1}(x,R)$;
(ii) $R < (1/2)\sup J_x$.

where I_x and J_x where defined on Theorem 28.

Let $y \in B_{M_1}(x,R)$. Suppose that for any $\varepsilon \in (0,R)$, the positive value $d_1(x,y)+\varepsilon$ is not suitable for the (f,x,r) triplet, where $r = d_1(x,y)\sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$. Under these circumstances, we obtain

$$\Pi_x\left(d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\right) < d_1(x,y)+\varepsilon,$$

which implies

$$\Pi_x\left(d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\right) \leq d_1(x,y).$$

Applying the inverse of Π_x, we have

$$d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \leq \Pi_x^{-1}\left(d_1(x,y)\right),$$

which implies that $\Psi(d_1(x,y)) \geq \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$.

To complete this proof we show that for any $\varepsilon > 0$, the value $d_1(x,y)+\varepsilon$ is not suitable for the triplet (f,x,r). Since $y \mapsto \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(B_1,B_2)}$ is a continuous application, than for all $\xi > 0$ we can find $\zeta > 0$ such that if $d_1(x,y) < \zeta$, then $\|f'(x)\|^{-1}_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} < \xi + 1$.

In other words, if $d_1(x,y) < \zeta$ then

$$\frac{d_1(x,y)\ \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}}{\|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}} < d_1(x,y)\,\xi + d_1(x,y) < \zeta\,\xi + d_1(x,y).$$

Let $M > 1$ and set $\xi = \varepsilon/M$. Without loss of generality, we assume that $\zeta < 1$. Hence, if $d_1(x,y) < \zeta$, then

$$\frac{d_1(x,y)\ \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}}{\|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}} < d_1(x,y) + \frac{\varepsilon}{M}.$$

Choose λ_y such that

$$\frac{d_1(x,y)\ \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}}{\|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}} < \lambda_y < d_1(x,y) + \frac{\varepsilon}{M}.$$

Since there is a direction $v \in \mathcal{B}_1$ such that $\|f'(x)\cdot v\|_{\mathcal{B}_2} = \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$, define $h = \lambda v$ and $\widetilde{y} = x + h$. Then $d_1(x,\widetilde{y}) = \lambda_y < d_1(x,y) + \frac{\varepsilon}{M} < d_1(x,y) + \varepsilon$.

On the other hand,

$$\|f'(x)\cdot h\|_{\mathcal{B}_2} = \lambda \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} > d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}.$$

Choosing a suitable M and a sufficiently small value $R > 0$, if $y \in B_{M_1}(x,R)$ then $\|f(x+h) - f(x) - f'(x)\cdot h\|_{\mathcal{B}_2}$ is small enough, so $\|f(\widetilde{y}) - f(x)\|_{\mathcal{B}_2} \geq d_1(x,y)\ \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$ even if $d_1(x,\widetilde{y}) < d_1(x,y) + \varepsilon$. Therewith, $d_1(x,y) + \varepsilon$ is not suitable for the desired triplet if $y \in B_{M_1}(x,R)$, which concludes the proof. □

Corollary 30. *Suppose that M_1 and M_2 are open sets contained in Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, $f : M_1 \to M_2$ is continuously differentiable and f is enclosed at x. If there is a direction $v \in \mathcal{B}_1$ such that $\|f'(x)\cdot v\|_{\mathcal{B}_1} = \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \neq 0$, then, there exists $R > 0$ such that*

$$\Pi_x\left(d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(t)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\right) = d_1(x,y)$$

for all $y \in B_{M_1}(x,R)$.

Corollary 31. *Suppose that M_1 and M_2 are open sets contained in Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, $f : M_1 \to M_2$ is continuously differentiable and f is enclosed at x. If there is a direction $v \in \mathcal{B}_1$ such that $\|f'(x)\cdot v\|_{\mathcal{B}_1} = \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \neq 0$, then there exists $R > 0$ and a function $\Sigma : B_{M_1}(x,R) \to \mathbb{R}^+$ defined as*

$$\Sigma(y) = d_1(x,y) \sup_{B_{M_1}[x,d_1(x,y)]} \|f'(x)\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$$

that fulfills

(i) *If y and $\widetilde{y}$ are equidistant points from x, then $\Sigma(y) = \Sigma(\widetilde{y})$;*
(ii) *If $\Sigma(y) = \Sigma(\widetilde{y})$, then y and $\widetilde{y}$ are equidistant points from x.*

7. The Mean Value Propriety

The study of averages of analytic (or harmonic) functions comprises a huge literature (see for instance [6, 7, 10, 9]), and important results are proven by this theory. Our final intention in this paper is to relate such averages with our techniques, and prove some interesting properties when the averaging functional is evaluated on enclosed functions.

Let $(X, \mathcal{A}, \mu)$ be a measure space, f an integrable function, A a positive measure set and λ a scalar. We write $\mathcal{M}_\lambda f_A$ to indicate the *shifted average* of f over A. In other words,

$$\mathcal{M}_\lambda f_A = \frac{1}{\mu(A)} \int_A f - \lambda \, d\mu = \frac{1}{|A|} \int_A f - \lambda \, d\mu.$$

Observe that if $\mathcal{M}f_A$ denotes the standard average over A, then we have that

$$\mathcal{M}_\lambda f_A = \mathcal{M}f_A - \lambda.$$

Suppose that (X,d) is also a metric space and that $\mathcal{A}$ is the σ-algebra generated by the Borelian sets. It is a recurring task to determinate the maximum ball in which f stays bellow its average on certain fixed set. Let us start with some preliminary results.

Lemma 32. *If $B(x,r_0)$ is a ball in X centered at x with radius r_0 and $f : X \longrightarrow \mathbb{C}$ is an enclosed function at the same point x, then $|\mathcal{M}_{f(x)}f_{B(x,r_0)}| \leq \Pi_x^{-1}(r_0)$ for all r_0 in the domain of the Π_x^{-1} function.*

Proof. Since r_0 is in the domain of the Π_x^{-1} function, we have that if $d(x,t) < r_0$ then $|f(x)-f(t)| < \Pi_x^{-1}(r_0)$. Therefore,

$$\left| \int_{B(x,r_0)} f(t) - f(x)\, d\mu(t) \right| \leq \int_{B(x,r_0)} |f(t) - f(x)|\, d\mu(t) \leq \int_{B(x,r_0)} \Pi_x^{-1}(r_0)\, d\mu(t).$$

□

Therefore, we connected the shifted average of a function f with the inverse of its continuity function. If we work with the average itself, we can just write

$$|\mathcal{M}f_{B(x,r_0)} - f(x)| \leq \Pi_x^{-1}(r_0).$$

It is now clear that Π_x^{-1} actually works as an upper bound for the difference between the image of the function in its center point and its average.

Now suppose that $f : X \longrightarrow \mathbb{C}$ is a strongly enclosed function at some point x and take some fixed number $r_0 > 0$. Additionally, suppose that $f(x) = 0$. Under these conditions, determining the maximum ball $B(x,r)$ in which f stays bellow its average in $B(x,r_0)$ is equivalent to determine the maximum $r > 0$ such that if $d(x,t) < r$, then $|f(t)| < |\mathcal{M}f_{B(x,r_0)}|$.

If $|\mathcal{M}f_{B(x,r_0)}|$ lies in the domain of Π_x, then $r = \Pi_x\left(|\mathcal{M}f_{B(x,r_0)}|\right)$ is the solution to the problem. Since $f(x) = 0$, we have that

$$|\mathcal{M}f_{B(x,r_0)}| = |\mathcal{M}_{f(x)}f_{B(x,r_0)}| \leq \Pi_x^{-1}(r_0).$$

Using monotonicity arguments, we now obtain that $r = \Pi_x\left(|\mathcal{M}f_{B(x,r_0)}|\right) \leq r_0$. In other words, if the average is taken on a ball of radius r_0, then the maximum ball that keeps f under its average has radius less or equal to r_0. Precisely, we have proved the

Theorem 33. *Let $f : X \longrightarrow \mathbb{C}$ be a enclosed function at some point x and take a number $r_0 > 0$. Additionally, suppose that $f(x) = 0$ and that f is an unbounded function. Under these circumstances, there exists an $r = r(r_0, x, f) \geq 0$ such that*

(i) *If $z \in B(x,r)$, then $|f(z)| < |\mathcal{M}f_{B(x,r_0)}|$;*
(ii) *If f is strongly enclosed at x, then r is the maximum radius satisfying the previous statement;*
(iii) *If $r_0 \in \Pi_x(0,\infty)$, then $r \leq r_0$.*

In addition to that, if $f \in L^1(X)$, then $r_0 \longmapsto r(r_0)$ is a continuous application.

Proof. We are almost done. First, note that since f is an unbounded function, we have that $E_f(x) = (0,\infty)$. Also, using the Dominated Convergence Theorem, we have that if $f \in L^1(X)$, then $r_0 \longmapsto \mathcal{M}f_{B(x,r_0)}$ is a continuous application. □

We can also state the following version of the theorem, which is a trivial consequence of the previous one.

Theorem 34 (The Mean-Value Property for Enclosed Functions)**.** *Let $f : X \longrightarrow \mathbb{C}$ be a enclosed function at some point x and take a number $r_0 > 0$. Additionally, suppose that f is an unbounded function. Under these circumstances, there exists an $r = r(r_0, x, f) \geq 0$ such that*

(i) *If $z \in B(x, r)$, then $|f(z) - f(x)| < |\mathcal{M}f_{B(x,r_0)} - f(x)|$;*
(ii) *If f is strongly enclosed at x, then r is the maximum radius satisfying the previous statement;*
(iii) *If $r_0 \in \Pi_x(0, \infty)$, then $r \leq r_0$.*

In addition to that, if $f \in L^1(X)$, then $r_0 \longmapsto r(r_0)$ is a continuous application.

Some remarks are in order. First of all, the last theorem can be seen as a version of the Mean Value Property. In harmonic functions, one can expect that the image of the center of the ball is given by the average value of the function in the interior of the ball. In this version, for enclosed functions, we have shown that the image of any point which lies in a certain set is strictly small than the average taken on some larger ball.

Also note that we call it a version — not a generalization — because of the following: suppose that f is a harmonic function. Under the assumptions of the last theorem, we have that $|\mathcal{M}f_{B(x,r_0)} - f(x)| = 0$ and $r = \Pi_x(0) = 0$. This is a natural and expected phenomenon, since harmonic functions behave much more nicely with respect to averages than enclosed functions.

To conclude, observe that the unboundedness of the function should not pose a threat to the usability of the last theorem. Let us consider the following two cases.

First, suppose that we have $\Pi_x(0, \infty) = (0, \infty)$ and $r_0 \leq R$, for some $R > 0$. In this case, if there exists an unbounded function F such that $F|_{B(x,R)} = f$, then we can obtain the same conclusions.

On the other hand, if we do not have any control over $\Pi_x(0, \infty)$, but R is small enough, then we can still claim the same conclusions.

Corollary 35. *Let $f : X \longrightarrow \mathbb{C}$ be a enclosed function at some point x and take a number $r_0 > 0$. Additionally, suppose that $f(x) = 0$ and that f is an unbounded function. Under these circumstances, there exists an $r = r(r_0, x, f) \geq 0$ such that $|\mathcal{M}f_{B(x,r)}| \leq \mathcal{M}|f|_{B(x,r)} \leq |\mathcal{M}f_{B(x,r_0)}|$.*

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