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ZETA-DETERMINANTS OF STURM–LIOUVILLE OPERATORS WITH QUADRATIC POTENTIALS AT INFINITY

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Abstract. We consider Sturm–Liouville operators on a half line $[a,\infty)$, $a>0$, with potentials that are growing at most quadratically at infinity. Such operators arise naturally in the analysis of hyperbolic manifolds, or more generally manifolds with cusps. We establish existence and a formula for the associated zeta-determinant in terms of the Wronski-determinant of a fundamental system of solutions adapted to the boundary conditions. Despite being the natural objects in the context of hyperbolic geometry, spectral geometry of such operators has only recently been studied in the context of analytic torsion.

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1. Introduction and formulation of the main results

In this paper we will investigate the zeta-determinant of Sturm–Liouville operators with potentials that are growing quadratically at infinity. More precisely, we consider operators of the form

$$H=-\frac{d}{dx}\Big(x^2\frac{d}{dx}\Big)+x^2\mu^2-\frac{1}{4}+V(x)=:D_\mu+V(x) \tag{1.1}$$

on the interval $[a,\infty)$, $a>0$, with $\mu>0$ and only minimal regularity assumptions on the potential V. Ignoring the potential V for a moment, such operators are also referred to as totally characteristic operators and have been studied by Melrose and Mendoza in [MeMe83]. However the relation to our analysis here

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is only formal, since [MeMe83] studies the operators of totally characteristic type near $x = 0$, while here we are interested in the behavior of such operators as x approaches infinity.

Our motivation for looking at zeta-determinants of such operators arises from geometry of hyperbolic manifolds or more generally manifolds with cusps. Spectral geometry of such manifolds has been initiated by Müller in his paper [Mül83]. A recent work by the third named author [Ver14] discusses analytic torsion on such spaces and in particular strongly relies on computations of zeta-determinants of such operators.

With the present paper we intend to initiate further discussion of such operators, parallel to developments in the setting of regular singular operators, which in turn are motivated by the geometry of spaces with isolated conical singularities. Analysis of such spaces has been initiated by Cheeger in his seminal papers [Che79], [Che83], and corresponding zeta-determinants have been considered by the second named author in [Les98]. These results have been employed in various studies of analytic torsion on conical singularities by the first named author jointly with Spreafico [HaSp11], [HaSp16] as well the third author jointly with Müller [MüVe14]. We expect a similar impact of our discussion here in the setting of manifolds with cusps.

Before stating the main result of our paper, let us briefly recall the formula for the ζ–determinant of a second order Sturm–Liouville operator on a finite interval with separated boundary conditions, *cf.* [BFK95]. Let

$$H_0 = -\frac{d}{dx}\Big(p(x)\frac{d}{dx}\cdot\Big) + V(x) \tag{1.2}$$

be a differential operator on the finite interval $[a, b]$. Here, p, V are smooth functions and $p(x) > 0$ for all $x \in [a, b]$. We impose separated boundary conditions at a, b of the form

$$R_c f := \sin\theta_c \cdot f'(c) + \cos\theta_c \cdot f(c), \quad 0 \le \theta_c < \pi, c \in \{a, b\}. \tag{1.3}$$

A solution of the homogeneous equation $H_0 g = 0$ is called *normalized* at c if $R_c g = 0$ and (we set $\operatorname{sgn}(a) = 1, \operatorname{sgn}(b) = -1$)

$$\begin{aligned} g'(c) &= \operatorname{sgn} \cdot p(c)^{-3/4}, && \text{if } \theta_c = 0 \text{ (Dirichlet)}\,, \\ g(c) &= p(c)^{-1/4}, && \text{if } \theta_c > 0 \text{ (generalized Neumann)}\,. \end{aligned} \tag{1.4}$$

One might wonder where this normalization comes from. For a regular operator of the form Eq. (1.2) there is a coordinate transformation $y(x) := \int_a^x p(x')^{-\frac{1}{2}} dx'$, which uniformly transforms the operator into a Sturm–Liouville operator of the form $-\partial_y^2 + V$. The known normalization for the latter operator, *cf.* [Les98], is equivalent to Eq. (1.4) under the transformation.

With this notation the following Theorem, which is a special case of a more general result due to Burghelea, Friedlander and Kappeler, holds.

Theorem 1.1 ([BFK95]). *Let φ, ψ be a fundamental system of solutions of the differential equation $H_0 g = 0$ with $R_a\varphi = 0, R_b\psi = 0$ being both normalized in the sense of*

Eq. (1.4). *Then the realization* $H_0 = H_0(R_a, R_b)$ *of* H_0 *with respect to the boundary conditions* R_a, R_b *is self-adjoint and discrete. Its* ζ*–function has a meromorphic continuation to the complex plane with simple poles.* $0 \in \mathbb{C}$ *is not a pole and moreover the* ζ*–regularized determinant is given by*

$$\det_\zeta(H_0(R_a, R_b)) = 2 \cdot p \cdot W(\psi, \varphi) = 2 \cdot p \cdot (\psi \cdot \varphi' - \psi' \cdot \varphi). \tag{1.5}$$

Note that the Wronskian $p \cdot W(\psi, \varphi)$ *is constant.*

This result has been generalized to regular singular operators by the second and third named authors, *cf.* [Les98, LeVe11].

In this paper we prove the analogue of BFK's Theorem for operators of the form Eq. (1.1). To the best of our knowledge this is the first example of a singular operator on an unbounded interval for which such a formula for the ζ–determinant is proven. We will use a capital letters to represent the multiplicative operator by function, i. e., the multiplication operator by x is denoted by X.

Theorem 1.2. *Fix any* $\nu \geq 0$ *and suppose the potential* V *in the differential expression* Eq. (1.1) *satisfies* $V \in X^\gamma L^1[a, \infty)$ *for a fixed* $\gamma < 2$*. We impose boundary conditions at* $x = a$ *of the form* Eq. (1.3)*. The operator* H *is in the limit point case at infinity and hence essentially self-adjoint on the core domain* $\{f \in C_0^\infty[a,\infty) \mid R_a f = 0\}$*. By abuse of notation we denote by* $H = H(R_a)$ *this self-adjoint realization. Choose a fundamental system of solutions* ϕ, ψ *of* $(H + \nu^2)f = 0$*, where* $R_a\phi = 0$ *satisfies the boundary conditions at the left end point and* $\psi \in L^2[a, \infty)$ *is square integrable. We normalize* ϕ *as above in* Eq. (1.4) *and* ψ *by*

$$\lim_{x \to \infty} \psi(x) \sqrt{x}\, K_\nu(\mu x)^{-1} = 1. \tag{1.6}$$

Here K_ν *is the modified Bessel function of the second kind of order* ν*.*

Then $H(R_a) + \nu^2$ *is self-adjoint with a discrete spectrum. Furthermore, its* ζ*–function admits a meromorphic continuation into a half plane* $\{z \in \mathbb{C} \mid \operatorname{Re} z > r\}$ *for some* $r < 0$ *with* 0 *being a regular point. Therefore, its zeta-regularized determinant is well–defined.*

Furthermore, we have the explicit formula

$$\det_\zeta(H(R_a) + \nu^2) = \sqrt{\frac{2}{\pi}} \cdot a^2 \cdot W(\psi, \phi)(a). \tag{1.7}$$

Note that when comparing with Eq. (1.5) we have $p(x) = x^2$.

1.1. Outline of proof and further results.

1.1.1. *The model operator.* The operator H is treated as a perturbation of the model *cusp* operator parametrized by μ

$$D_\mu = -\frac{d}{dx}\left(x^2 \frac{d}{dx}\right) + x^2\mu^2 - \frac{1}{4}. \tag{1.8}$$

We observe that if $\mu = 0$ then D_0 has a continuous spectrum, therefore we consider $\mu > 0$. A fundamental system of solutions to the differential equation $(D_\mu + z^2)f = 0$ is explicitly given in terms of the modified Bessel functions I_z, K_z by

$$x^{-1/2} \cdot K_z(\mu x), \quad x^{-1/2} \cdot I_z(\mu x). \tag{1.9}$$

Spectral problems and the analysis of the resolvent therefore ultimately reduce to questions about the modified Bessel functions and their asymptotic behavior. Besides the fairly standard asymptotics for large arguments and fixed order resp. large order and fixed arguments we will also need less standard *uniform* asymptotics for large order. The necessary facts about Bessel functions are compiled in Section 3.

1.1.2. *The resolvent expansion, and meromorphic continuation of the zeta-function.* The first step is to analyze the asymptotic expansion of the resolvent trace.

Theorem 1.3. *Let $V \in X^\gamma L^1[a,\infty)$ as in Theorem 1.2 and suppose that a fixed (Dirichlet or generalized Neumann) boundary condition for H at a is given. Then the resolvent $(H(R_a)+z^2)^{-1}$ is trace class and there is an asymptotic expansion*

$$\operatorname{Tr}(H(R_a)+z^2)^{-1} = b_0\cdot z^{-1}\cdot \log z + a_0\cdot z^{-1} + a_1\cdot z^{-2} + O(z^{-2-\delta}), \quad \text{as } z\to\infty \tag{1.10}$$

for some $\delta > 0$. The constants b_0, a_0, a_1 do not depend on the potential and z. Explicitly $a_0 = \frac{1}{2}\log\frac{2}{\mu a}$, , $b_0 = \frac{1}{2}$; for Dirichlet boundary conditions at a we have $a_1 = \frac{1}{4}$ while for generalized Neumann conditions we have $a_1 = -\frac{1}{4}$.

For the model operator D_μ ($V = 0$) there is a full asymptotic expansion

$$\operatorname{Tr}(D_\mu + z^2)^{-1} = \sum_{k=0}^{\infty} a_k(\mu) z^{-1-k} + \sum_{k=0}^{\infty} b_{2k}(\mu) z^{-1-2k}\log z, \quad \text{as } z\to\infty. \tag{1.11}$$

The full asymptotic expansion for the model operator is due to the third named author [Ver14, Sec. 4], however without explicitly specifying the first few coefficients.

The symbol $⨍$ will denote the Hadamard partie finie integral which we will briefly review in Section 2.1. The well-known formula, *cf.* [LeTo98, (2.30)],

$$\zeta_H(s) := \sum_{\lambda\in \operatorname{spec} D} \lambda^{-s} = \frac{\sin \pi s}{\pi}\cdot ⨍_0^\infty x^{-s}\operatorname{Tr}(H+x)^{-1}dx \tag{1.12}$$

relates the zeta-function of $H = H(R_a)$ to the resolvent trace and the asymptotic expansion Eq. (1.10) implies that $\zeta_H(s)$ has a meromorphic continuation to $\operatorname{Re} s > -\delta$ with 0 being a regular point. Therefore one has

$$\log\det\nolimits_\zeta H := -\zeta_H'(0) = -⨍_0^\infty \operatorname{Tr}(H+z)^{-1}dz = -2⨍_0^\infty z\cdot \operatorname{Tr}(H+z^2)^{-1}dz. \tag{1.13}$$

A consequence of the expansion Eq. (1.10) is that

$$\log\det\nolimits_\zeta(H+z^2) = 2\cdot b_0\cdot z\cdot\log z + 2\cdot(a_0-b_0)\cdot z + 2\cdot a_1\cdot \log z + O(z^{-\delta}\log z). \tag{1.14}$$

Note that there is no constant term, thus $\operatorname{LIM}_{z\to\infty} \log\det_\zeta(H+z^2) = 0$ (Lemma 2.2). LIM (regularized) limit is a short hand for the constant term in the asymptotic expansion, *cf.* Sec. 2.1.

1.1.3. *Weyl eigenvalue asymptotics.* In this subsection we discuss the asymptotic behavior of the eigenvalue counting function $N(\lambda)$ for the cusp operator H.

Note that the presence of $z^{-1}\log z$ as the leading term in the resolvent trace asymptotics in Theorem 1.3 above, distinguishes our case significantly from similar discussions of regular-singular Sturm–Liouville operators over a finite interval in [Les98] and [LeVe11]. There the singular potential in the Sturm–Liouville operator leads to logarithms in the resolvent trace asymptotics as well, however in contrast to our case, the logarithm does not appear in the leading term.

The logarithmic leading term $z^{-1}\log z$ is obviously a new phenomenon of our non-compact setting and has an important consequence for the Weyl asymptotics of the cusp operator $H(R_a)$. Indeed, by the resolvent trace expansion, one concludes that

$$\zeta_H(s) - \frac{1}{\Gamma(s)}\left(\frac{-c_0}{(s-\frac{1}{2})} + \frac{c_1}{(s-\frac{1}{2})^2}\right) \tag{1.15}$$

is continuous for $\mathrm{Re}(s) \geq \frac{1}{2}$, where the constants c_0 and c_1 are determined explicitly by the coefficients in the resolvent trace asymptotics and in particular $c_1 = -\frac{b_0(\mu)}{2\Gamma(\frac{1}{2})}$. Now, by a Tauberian argument, *cf.* Shubin [Shu01, Problem 14.1 pp. 127] and Aramaki [Ara83], one concludes for the eigenvalue counting function

$$N(\lambda) \sim \frac{\sqrt{\lambda}\log(\lambda)}{2\Gamma(1/2)^2}, \quad \lambda \to \infty. \tag{1.16}$$

Note that it is by no means straightforward to conclude a similar expansion for the eigenvalue counting function of the Laplace–Beltrami operator on cusps, which can be written as a direct sum of the cusp operators H. This is due to the non-uniform behaviour of the resolvent trace expansion in Theorem 1.3 as μ goes to infinity. A similar question has been studied in the joint work of the second and third author [LeVe15].

1.1.4. *Variation formula.* The next step is to establish a variation formula. Let us state it informally first: let V_t be a (sufficiently nice) one parameter family of potentials (satisfying the overall assumptions of Theorem 1.2 and depending differentiably on t) and denote by ϕ_t, ψ_t be a normalized fundamental system of solutions of the differential equation $H_t f = -D_\mu f + V_t f = 0$. Then

$$\partial_t \log \det\nolimits_\zeta H_t = \partial_t \log\big(p \cdot W(\psi_t, \phi_t)\big). \tag{1.17}$$

This variation formula goes back to Lemit and Smilansky [LeSm77] for the situation of Theorem 1.1. For $V_t = t^2$ being the resolvent parameter of the model operator it is due to the third named author [Ver14, Sec. 5]. In Section 4 we will present an expanded version which includes some important details. To the general case we investigate the dependence of the asymptotic behavior of a fundamental system of solutions at infinity on the parameter t and we proved a Bôcher Theorem for $H + \nu^2$ in Section 5. Hence we analyze the asymptotic expansion of the resolvent trace of the perturbed operator and prove the Theorem 1.3 in Section 6. We prove the general case in Section 7.

We apply Eq. (1.17) to $V_z = V + z^2$ and obtain in view of Eq. (1.14)

$$\det_\zeta(H + z_0^2) = p \cdot W(\psi_{z_0}, \phi_{z_0}) \cdot \exp\Big(-\operatorname*{LIM}_{z\to\infty} \log\big(p \cdot W(\psi_z, \phi_z)\big)\Big). \tag{1.18}$$

For general one dimensional elliptic differential operators on a finite interval this formula was established in [LeTo98, Thm. 3.3].

It remains to compute the constant $\operatorname*{LIM}_{z\to\infty} \log\big(p \cdot W(\psi_z, \phi_z)\big)$. The variation formula Eq. (1.17) shows that this constant is independent of the potential V. Therefore, it suffices to compute it for the model operator. In [BFK95] and [Les98] this is done by proving another formula for the variation of generalized Neumann conditions and then finally by computing explicit examples. Namely, on a finite interval the ζ–determinant for $-\frac{d^2}{dx^2}$ (with Dirichlet or Neumann boundary conditions) can explicitly be expressed in terms of the Riemann ζ–function for which the derivative at 0 is known (Lerch's formula). The case of a regular singular operator on a finite interval can also be reduced to this case; alternatively one can take advantage of the fact that the spectrum of the Jacobi differential operator is explicitly known [Les98].

For our model operator D_μ we need to employ a different strategy as we do not know the spectrum of any self-adjoint realization of Eq. (1.1) for any parameter value μ. However, for each boundary condition the normalized fundamental system ϕ_ν, ψ_ν can explicitly be expressed in terms of the modified Bessel functions. Consequently, the asymptotic behavior of $\log\big(p \cdot W(\psi_\nu, \phi_\nu)\big)$ can be studied with the help of the known asymptotics of the modified Bessel functions.

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2. Generalities: Regularized limits, ζ–determinants, and Wronskians of Sturm–Liouville operators

For the convenience of the reader and to fix some notation we collect here some general facts on regularized limits, zeta-determinants and Wronskians of Sturm–Liouville operators, *cf.* also [Les98], [LeTo98], [LeVe11, Sec. 1], [LeVe15, Sec. 1] and the references therein.

2.1. Regularized limits and integrals.

Let $f : (0, \infty) \to \mathbb{C}$ be a function with a (partial) asymptotic expansion

$$f(x) \sim \sum_{j=1}^{N-1} \sum_{k=0}^{M_j} a_{jk} x^{\alpha_j} \log^k(x) + \sum_{k=0}^{M_0} a_{0k} \log^k(x) + f_N(x), \qquad x \geq x_0 > 0, \tag{2.1}$$

where $\alpha_j \in \mathbb{C}$ are ordered with decreasing real part and the remainder $f_N(x) = o(1)$ (Landau notation) as $x \to \infty$. Then we define its *regularized limit* as $x \to \infty$

by

$$\operatorname*{LIM}_{x\to\infty} f(x) := a_{00}. \tag{2.2}$$

If f has an expansion of the form Eq. (2.1) as $x \to 0$ then the regularized limit as $x \to 0$ is defined accordingly.

If f is locally integrable and the remainder $f_N \in L^1[1,\infty)$ even integrable, the integral $\int_1^R f(x)dx$ also admits an asymptotic expansion of the form Eq. (2.1) and one defines the *regularized integral* as

$$⨍_1^\infty f(x)dx := \operatorname*{LIM}_{R\to\infty} \int_1^R f(x)dx. \tag{2.3}$$

Similarly, $⨍_0^1 f(x)dx := \operatorname*{LIM}_{\varepsilon\to 0} \int_\varepsilon^1 f(x)dx$, if this regularized limit exists.

$⨍$ is a linear functional extending the ordinary integral. However, it has some pathologies. E. g. the formula for changing variables $x \mapsto \lambda \cdot x$ in the integral has correction terms, [LeVe15, Lemma 1.1]. Relevant for us will be the behavior under translations. Namely, assuming that f is locally integrable with remainder $f_N \in L^1[1,\infty)$, consider for $x > 0$

$$\begin{aligned} ⨍_0^\infty f(x+t)dt &= \operatorname*{LIM}_{R\to\infty} \int_0^R f(x+t)dt \\ &= \operatorname*{LIM}_{R\to\infty} \Big(\int_x^R f(t)dt + \int_R^{R+x} f(t)dt \Big) \\ &= ⨍_x^\infty f(t)dt + \operatorname*{LIM}_{R\to\infty} \int_R^{R+x} f(t)dt. \end{aligned} \tag{2.4}$$

In general $\operatorname{LIM}_{R\to\infty} \int_R^{R+x} f(t)dt \neq 0$, *cf.* the discussion after Lemma 2.2 in [LeTo98]. However it vanishes whenever there are no terms of the form $x^\alpha \log^k x$ with $\alpha \in \mathbb{Z}_+ \subset \mathbb{C}\setminus\{0\}$ in the expansion Eq. (2.1). For later reference we record

Lemma 2.1. *Let* $f : (0,\infty) \to \mathbb{C}$ *be locally integrable with an asymptotic expansion as in* Eq. (2.1) *where* $\alpha_j \notin \mathbb{Z}_+$ *and* $f_N \in L^1[1,\infty)$. *Then for all* $x > 0$

$$⨍_0^\infty f(x+t)dt = ⨍_x^\infty f(t)dt, \tag{2.5}$$

in particular $\operatorname*{LIM}_{x\to\infty} ⨍_0^\infty f(x+t)dt = 0$.

Proof. The last claim is a consequence of the identity Eq. (2.5), as

$$\operatorname*{LIM}_{x\to\infty} ⨍_x^\infty f(t)dt = ⨍_1^\infty f(t)dt - \operatorname*{LIM}_{x\to\infty} \int_1^x f(t)dt = 0. \tag{2.6}$$

The identity Eq. (2.5) follows from the very definition of the regularized integral, regardless of the values of the exponents α_j. □

2.2. **Zeta-regularized determinants.** Let $H > 0$ be a self-adjoint positive operator acting on some Hilbert space. We assume that the resolvent of H is trace class, and that for $z \geq 0$ we have

$$\operatorname{Tr}(H+z)^{-1} = \sum_{-1-\delta<\operatorname{Re}\alpha<0} z^{\alpha} \cdot P_{\alpha}(\log z) + O(z^{-1-\delta}), \quad \text{as } z \to \infty, \tag{2.7}$$

with polynomials $P_{\alpha}(t) \in \mathbb{C}[t]$, $P_{\alpha} = 0$ for all but finitely many α. Moreover, we assume that P_{-1} is of degree 0, that is there are no terms of the form $z^{-1} \cdot \log^k z$ with $k \geq 1$. For $1 < \operatorname{Re} s < 2$ the *zeta-function (ζ–function)* of H is given by, *cf.* [LeTo98, (2.30)],

$$\zeta_H(s) := \sum_{\lambda \in \operatorname{spec} H} \lambda^{-s} = \frac{\sin \pi s}{\pi} \cdot ⨍_0^{\infty} x^{-s} \operatorname{Tr}(H+x)^{-1} dx. \tag{2.8}$$

From the asymptotic expansion Eq. (2.7) one deduces that $\zeta_H(s)$ extends meromorphically to the half plane $\operatorname{Re} s > -\delta$, [LeTo98, Lemma 2.1]. The identity Eq. (2.8) persists except for the poles of the function $s \mapsto \frac{\pi}{\sin \pi s} \zeta_H(s)$. From the assumption that $\deg P_{-1} = 0$ in Eq. (2.7) it follows that ζ_H is regular at $s = 0$ and one puts

$$\log \det_{\zeta} H := -\zeta_H'(0) = -⨍_0^{\infty} \operatorname{Tr}(H+z)^{-1} dz = -2 ⨍_0^{\infty} z \cdot \operatorname{Tr}(H+z^2)^{-1} dz. \tag{2.9}$$

$\det_{\zeta} H$ is called the *zeta-determinant (ζ-determinant)* or *zeta-regularized determinant* of H. For non-invertible H one puts $\det_{\zeta} H = 0$. With this setting the function $z \mapsto \det_{\zeta}(H+z)$ is an entire holomorphic function with zeroes exactly at the eigenvalues of $-H$. The multiplicity of a zero z equals the algebraic multiplicity of the eigenvalue z.

If P_{-1} is a higher order polynomial then ζ_H has poles at 0. One still could define $-\log \det_{\zeta} H$ to be the coefficient of s in the Laurent expansion about 0 of $\zeta_H(s)$. However, in this case the relation $\zeta_H'(0) = ⨍_0^{\infty} \operatorname{Tr}(H+z)^{-1} dz$ would not hold any more.

Lemma 2.2. *Let H be a bounded below self-adjoint operator in some Hilbert space. Assume that the resolvent is trace class and that, as $z \to \infty$, the expansion* Eq. (2.7) *holds. Then, as $z \to \infty$ we have an asymptotic expansion*

$$\log \det_{\zeta}(H+z) = \sum_{-1-\delta<\operatorname{Re}\alpha<0} z^{\alpha+1} \cdot Q_{\alpha}(\log z) + O(z^{-\delta}) \tag{2.10}$$

with polynomials Q_{α} satisfying $Q_{\alpha}' = -(\alpha+1)Q_{\alpha} + P_{\alpha}$. Moreover, $Q_{-1}(\log x) = P_{-1}(0) \cdot \log x$, in particular $\operatorname{LIM}_{z\to\infty} \log \det_{\zeta}(H+z) = 0$.

Proof. This follows in a straightforward fashion from the definition of the regularized integral, the relation Eq. (2.9) and Lemma 2.1. For details, *cf.* [LeTo98, Lemma 2.2]. □

With regard to Lemma 2.1 we emphasize that under the assumptions of the previous Lemma we have for the zeta-determinant of $H+\nu^2$

$$\log \det_\zeta(H+\nu^2) = -\,\fint_{\nu^2}^{\infty} \operatorname{Tr}(H+z)^{-1} dz = -2\fint_{\nu}^{\infty} z\cdot \operatorname{Tr}(H+z^2)^{-1} dz. \tag{2.11}$$

The result

$$\operatorname*{LIM}_{z\to\infty} \log \det_\zeta(H+z) = 0 \tag{2.12}$$

contains the main result of [Fri89] as a special case. Namely, one has for $z \geq 0$ and invertible H

$$\det_F(I+zH^{-1}) = \frac{\det_\zeta(H+z)}{\det_\zeta H}, \tag{2.13}$$

where $\det_F$ denotes the Fredholm determinant. This follows immediately from the fact that the left hand side and the right hand side have the same z-derivatives and that they coincide at $z=0$. Therefore, we have as $z\to\infty$,

$$\begin{aligned} \log \det_F(I+zH^{-1}) &= \log\det_\zeta(H+z) - \log\det_\zeta H \\ &= \sum_{-1-\delta<\operatorname{Re}\alpha<0} z^{\alpha+1}\cdot Q_\alpha(\log z) - \log\det_\zeta H + O(z^{-\delta}). \end{aligned} \tag{2.14}$$

In particular,

$$\operatorname*{LIM}_{z\to\infty} \log\det_F(I+zH^{-1}) = -\log\det_\zeta H. \tag{2.15}$$

2.3. **Wronskians and their variation.** Let

$$H_0 = -\frac{d}{dx}\Big(p(x)\frac{d}{dx}\cdot\Big) + V_0(x), \tag{2.16}$$

be a differential operator on the interval (a,∞), $a>0$, with a positive continuous function $p\in C[a,\infty)$, $p(x)>0$ and locally integrable potential $V_0\in L^1_{loc}[a,\infty)$. Furthermore, we assume that a is a regular point and ∞ is in the limit point case for H_0. We fix a self-adjoint boundary condition $R_a f=0$ at a and assume that H_0 with this boundary condition is invertible.

Let ψ, ϕ be a fundamental system of solutions of the differential equation $H_0 f = 0$ with $R_a\phi=0$ and $\psi\in L^2[a,\infty)$. Then the *Wronskian*

$$p\cdot W(\psi,\phi) = p\cdot\Big(\psi\cdot\phi' - \psi'\cdot\phi\Big) \tag{2.17}$$

is constant. The Schwartz kernel (Green function) of H_0^{-1} is given by

$$G(x,y) = \frac{1}{p\cdot W(\psi,\phi)}\cdot\begin{cases}\phi(x)\cdot\psi(y), & x\leq y,\\ \psi(x)\cdot\phi(y), & y\leq x.\end{cases} \tag{2.18}$$

Suppose now that V_0 depends differentiably on a parameter t. Assume that ψ_t and ϕ_t are solutions as above depending differentiably on t. Denote the differentiation by t by a dot decorator, e.g. $\partial_t\phi =: \dot\phi$ and differentiation by x by a $'$ decorator, e.g. $\partial_x\phi =: \phi'$. Differentiate the differential equation $-(p\cdot\psi')' + V_0\cdot\psi = 0$ by t to obtain

$$-\big(p\cdot\dot\psi'\big)' + V_0\cdot\dot\psi(x) = -\dot V_0\cdot\psi, \tag{2.19}$$

and similarly for ϕ. Hence

$$\dot{V}_0(x) \cdot G(x,x) = \dot{V}_0(x) \cdot \phi(x) \cdot \psi(x), \tag{2.20}$$

thus

$$\begin{aligned} \dot{V}_0 \cdot \phi \cdot \psi &= (\dot{V}_0 \cdot \phi) \cdot \psi = \Big(\partial_x(p \cdot \partial_x \dot{\phi}) - V_0 \dot{\phi}\Big) \cdot \psi \\ &= \psi \cdot \partial_x(p \cdot \partial_x \dot{\phi}) - \dot{\phi} \cdot \partial_x(p \cdot \partial_x \psi) \\ &= \frac{d}{dx}\Big(p \cdot (\partial_x \dot{\phi}) \cdot \psi - p \cdot \dot{\phi} \cdot \partial_x \psi\Big) = \frac{d}{dx}\Big(p \cdot W(\psi, \dot{\phi})\Big). \end{aligned} \tag{2.21}$$

Thus if the operator $\dot{V}_0 H_0^{-1}$ is trace class, then

$$\begin{aligned} p \cdot W(\psi, \phi) \cdot \operatorname{Tr}(\dot{V}_0 H_0^{-1}) &= \int_a^\infty \dot{V}_0 \cdot \phi \cdot \psi \\ &= p(x) \cdot W(\psi, \dot{\phi})(x)\big|_{x=a}^{x=\infty} = p(x) \cdot W(\phi, \dot{\psi})(x)\big|_{x=a}^{x=\infty}, \end{aligned} \tag{2.22}$$

where the second equation follows by exchanging ϕ and ψ in the calculation. Note that, the trace class property plus the regularity at a imply the existence of the limit $\lim_{x\to\infty} p(x) W(\psi, \dot{\phi})(x)$. Furthermore,

$$\partial_t\Big(p \cdot W(\psi, \phi)\Big) = p \cdot \Big(W(\dot{\psi}, \phi) + W(\psi, \dot{\phi})\Big). \tag{2.23}$$

By Eq. (2.17) the Wronskian $p \cdot W(\psi, \phi)$ is a constant function in x. Moreover, if at the regular end ϕ is normalized, then $\dot{\phi}(a) = \frac{d}{dx}\dot{\phi}(a) = 0$, hence $W(\dot{\phi}, \psi)(a) = 0$. Thus altogether we have proved

Proposition 2.3. *Let* H_0 *be the differential operator* Eq. (2.16) *and assume that* $(V_{0,t})_t$ *depends differentiably on a parameter* t*. Furthermore, let* ϕ_t, ψ_t *be a fundamental system of solutions such that* ϕ_t *is normalized at* a *and* $\psi_t \in L^2[a, \infty)$*; assume that* ϕ_t, ψ_t *depend differentiably on* t*. Then*

$$p(a) \cdot W(\phi, \psi)(a) = \partial_t\big(p \cdot W(\phi, \psi)\big) = -\partial_t\big(p \cdot W(\psi, \phi)\big). \tag{2.24}$$

Furthermore, if $\dot{V}_0 H_0^{-1}$ *is trace class and if*

$$\lim_{x\to\infty} p(x) W(\phi, \dot{\psi})(x) = 0$$

then

$$\begin{aligned} \operatorname{Tr}(\dot{V}_0 H_0^{-1}) &= \frac{1}{p \cdot W(\psi, \phi)}\, p \cdot W(\phi, \dot{\psi})\big|_a^\infty \\ &= \frac{1}{p \cdot W(\psi, \phi)} \partial_t\Big(p \cdot W(\psi, \phi)\Big) \\ &= \partial_t \log\Big(p \cdot W(\psi, \phi)\Big). \end{aligned} \tag{2.25}$$

2.4. **Perturbative solutions, Bôcher's Theorem.** Let H_0 be as in Eq. (2.16). We do not impose any boundary condition in this subsection. Let ϕ, ψ be any fundamental system of solutions of the differential equation $H_0 f = 0$. Without loss of generality, we may assume their Wronskian equals 1, *i.e.*, $p \cdot W(\psi, \phi) = 1$. The solution formula for the inhomogeneous equation $H_0 u = v$ then reads

$$\begin{aligned} u(x) = c_1 \cdot \psi(x) + c_2 \cdot \phi(x) - \psi(x) \cdot \int_x^\infty \phi(y) \cdot v(y) dy \\ + \phi(x) \cdot \int_x^\infty \psi(y) \cdot v(y) dx, \end{aligned} \tag{2.26}$$

if, for all $x \in (a, \infty)$,

$$\int_x^\infty |\psi(y) v(y)| dy < \infty, \quad \text{and} \quad \int_x^\infty |\phi(y) v(y)| dx < \infty. \tag{2.27}$$

This formula may be used to find a fundamental system of solutions with prescribed asymptotics for perturbations $H = H_0 + V$ of H_0. Here we just present the general pattern. We will apply this to our concrete model operator in section 5 below. For a solution of $Hf = 0$ we make the perturbative Ansatz $h_1(x) = \psi(x)(1 + f_1(x))$. This leads to the non-homogeneous equation

$$H_0\big(\psi \cdot f_1\big) = -V \cdot \psi \cdot \big(1 + f_1\big). \tag{2.28}$$

We denote by L the integral operator with kernel

$$L(x, y) = p(y) \cdot \psi^2(y) \cdot \Big(\frac{\phi(x)}{\psi(x)} - \frac{\phi(y)}{\psi(y)}\Big). \tag{2.29}$$

Then writing $V =: p \cdot W$, Eq. (2.26) implies

$$f_1(x) = \int_x^\infty L(x, y) \cdot W(y) \cdot (1 + f_1(y)) dy. \tag{2.30}$$

Note that, since

$$\left(\frac{\phi}{\psi}\right)' = \frac{1}{p \cdot \psi^2}, \tag{2.31}$$

we find

$$(Lf)'(x) = -\frac{1}{p(x) \cdot \psi^2(x)} \int_x^\infty p(y) \cdot \psi^2(y) \cdot f(y) dy. \tag{2.32}$$

Now consider the following assumptions:

$$\sup_{a \le x \le y \le \infty} |L(x, y)| < \infty, \tag{2.33}$$

$$\lim_{x \to \infty} \psi(x) = 0, \tag{2.34}$$

$$\frac{\psi(x)}{\psi'(x)} = O(1) \text{ at } \infty, \tag{2.35}$$

$$\sup_{a \le x \le y \le \infty} \frac{p(y) \cdot \psi(y)}{p(x) \cdot \psi(x)} < \infty, \tag{2.36}$$

$$\int_1^x \frac{\phi(y)}{\psi(y)} dy = O\left(\frac{\phi(x)}{\psi(x)}\right) \text{ at } \infty, \text{ and } \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \infty. \tag{2.37}$$

Denote by $C_b^k[a,\infty)$ the Banach space of k-times continuously differentiable functions with bounded derivatives up to order k. I. e.

$$\|f\|_{C_b^k} := \sum_{j=0}^{k} \sup_{a\le x<\infty} |f^{(k)}(x)|. \tag{2.38}$$

Furthermore, the space $X^{-\gamma}C_b^k[a,\infty)$ is a Banach space with norm $\|f\|_{C_b^k,\gamma} := \|X^\gamma f\|_{C_b^k}$. We write $C_b[a,\infty) := C_b^0[a,\infty)$ for the Banach space of bounded continuous functions. We also write $C_\bullet^k[a,\infty) \subset C_b^k[a,\infty)$ for the subspace of bounded k times continuously differentiable functions which converge to zero at infinity along with their derivatives.

Lemma 2.4. *Let W be a function in* $L^1(a,\infty)$*. Then for each* $\gamma \ge 0$ *the Volterra operator* LW *maps* $X^{-\gamma}C_b[a,\infty)$ *continuously into* $X^\gamma C_\bullet^1[a,\infty)$*. Moreover, as an operator in* $X^{-\gamma}C_b[a,\infty)$ *it has spectral radius zero. Finally, the map*

$$L^1[a,\infty) \ni W \mapsto LW \in \mathcal{L}(X^{-\gamma}C_b[a,\infty)) \tag{2.39}$$

is continuous from $L^1[a,\infty)$ *into the bounded linear operators on* $X^{-\gamma}C_b[a,\infty)$*, where for* $f \in X^{-\gamma}C_b[a,\infty)$

$$LWf(x) = \int_x^\infty L(x,y)W(y)f(y)dy. \tag{2.40}$$

Proof. Clearly by Eq. (2.33),

$$\begin{aligned} |(LWf)(x)| &\le \int_x^\infty |W(y)|\cdot|f(y)|dy \\ &\le x^{-\gamma}\cdot\|f\|_\gamma\cdot\int_x^\infty |W(y)|dy \end{aligned} \tag{2.41}$$

and inductively

$$\left|((LW)^n f)(x)\right| \le x^{-\gamma}\cdot\frac{\|f\|_\gamma}{n!}\cdot\left(\int_x^\infty |W(y)|dy\right)^n. \tag{2.42}$$

This proves that LW maps $X^{-\gamma}C_b[a,\infty)$ continuously into $C_{\bullet,\gamma}[a,\infty)$ and that, as an operator in $X^\gamma C_b[a,\infty)$ it has spectral radius zero. It also implies the last sentence of the Lemma.

Furthermore, from Eq. (2.32) we infer

$$\begin{aligned} |(LWf)'(x)| &= \left|\frac{1}{p(x)\cdot\psi^2(x)}\int_x^\infty p(y)\cdot\psi^2(y)\cdot W(y)\cdot f(y)dy\right| \\ &\le C_1 x^{-\gamma}\cdot\|f\|_\gamma\cdot\int_x^\infty |W(y)|dy \\ &\le C_2 x^{-\gamma}\|f\|_\gamma, \end{aligned} \tag{2.43}$$

by Eq. (2.36). □

Theorem 2.5. *Suppose that there exists* $x_0 \in [a,\infty)$ *such that* $\psi(x) \ne 0$ *for* $x \ge x_0$*. Under the assumptions* Eq. (2.33), (2.34), (2.35), (2.36) *and* (2.37) *the perturbed operator*

$$H = H_0 + V = H_0 + p\cdot W \tag{2.44}$$

has a fundamental system of solutions h_1, h_2 *of the form*

$$h_1(x) = \psi(x) \cdot g_1(x), \qquad h_2(x) = \phi(x) \cdot g_2(x), \tag{2.45}$$

with $g_j \in C_b[a,\infty)$, $\lim_{x\to\infty} g_j(x) = 1$, $j = 1, 2$. *Furthermore,*

$$h_1'(x) = \psi'(x) \cdot \tilde{g}_1(x), \qquad h_2'(x) = \phi'(x) \cdot \tilde{g}_2(x), \tag{2.46}$$

with $\tilde{g}_j \in C_b[a,\infty)$, $\lim_{x\to\infty} \tilde{g}_j(x) = 1$, $j = 1, 2$ *and* $p(x)W(h_1, h_2)(x) = 1$.

Proof. We denote the constant function equal to one by a bold number one, $\mathbf{1}(x) = 1$ for all x. By equation Eq. (2.30) and the properties of Lemma 2.4 we conclude that

$$f_1(x) = (I - LW)^{-1}(LW\mathbf{1})(x) \tag{2.47}$$

is in $C^1_\bullet[a,\infty)$, hence

$$h_1(x) = \psi(x) \cdot (1 + f_1(x)) \tag{2.48}$$

has the claimed properties. Note that

$$h_1'(x) = \psi'(x) \cdot \big(1 + f_1(x) + \frac{\psi(x)}{\psi'(x)} \cdot f_1'(x)\big), \tag{2.49}$$

and by assumption Eq. (2.35), we conclude that $\tilde{g}_1 = 1 + f_1 + \frac{\psi}{\psi'} \cdot f_1' \in C_b[a,\infty)$ and

$$\lim_{x\to\infty} \tilde{g}_1(x) = 1. \tag{2.50}$$

For the second solution one finds

$$h_2(x) = c(x) \cdot h_1(x), \tag{2.51}$$

with

$$c(x) = \int_{x_0}^{x} p(y)^{-1} \cdot h_1(y)^{-2} dy. \tag{2.52}$$

From Eq. (2.31) we infer by integrating by parts

$$\begin{aligned} c(x) &= \int_{x_0}^{x} \frac{1}{p(y)\cdot\psi(y)^2}(1 + f_1(y))^{-2} dy \\ &= \frac{\phi(y)}{\psi(y)}(1 + f_1(y))^{-2}\Big|_{x_0}^{x} + 2\int_{x_0}^{x} \frac{\phi(y)}{\psi(y)} \cdot \frac{g_1'(y)}{g_1(y)^3} dy. \end{aligned} \tag{2.53}$$

Thus,

$$\begin{aligned} g_2(x) =& c(x)\frac{h_1(x)}{\phi(x)} \\ =& \frac{1}{g_1(x)} - \frac{\psi(x)}{\phi(x)} \cdot \frac{\psi(x_0)}{\phi(x_0)} \cdot \frac{g_1(x)}{g_1(x_0)^2} + 2g_1(x) \cdot \frac{\psi(x)}{\phi(x)} \int_{x_0}^{x} \frac{\phi(y)}{\psi(y)} \cdot \frac{g_1'(y)}{g_1(y)^3} dy. \end{aligned} \tag{2.54}$$

By assumption Eq. (2.37),

$$\Big|\frac{\psi(x)}{\phi(x)} \int_{x_0}^{x} \frac{\phi(y)}{\psi(y)} dy\Big| \le C, \tag{2.55}$$

and since $\lim_{x\to\infty} g_1'(x) = 0$ by Lemma 2.4, we obtain $\lim_{x\to\infty} g_2(x) = 1$.

Furthermore, direct computation shows

$$p(x) \cdot W(h_1, h_2)(x) = 1. \tag{2.56}$$

□

2.5. **Comparison of zeta-determinants for Dirichlet and generalized Neumann boundary conditions.** In this subsection we will show that under very mild assumptions it is possible to compute the relative ζ–determinant of the operator H_0 in Eq. (2.16) with respect to two different boundary conditions at the left endpoint. Under the general assumptions of Sec. 2.3 we consider the Dirichlet boundary condition at a, $f(a) = 0$, and a generalized Neumann boundary condition at a, $R_a f = f'(a) + \alpha \cdot f(a)$. Furthermore, let $\phi = \phi_z, \psi = \phi_z$ be a fundamental system of solutions to the equation $(H + z^2)u = 0$ such that $\phi(a) = 0, \phi'(a) = p(a)^{-\frac{3}{4}}$ (that is, ϕ is normalized at a) and such that $\psi \in L^2[a, \infty)$. We suppress the z–dependence from the notation. Furthermore, we consider z such that $H + z^2$ with both Dirichlet and the generalized Neumann boundary condition are invertible. This is certainly the case if z is large enough. Consequently, $\psi(a) \neq 0 \neq R_a\psi$. Denote by H_D the self-adjoint extension of H with Dirichlet boundary condition and by H_α the self-adjoint extension of H with generalized boundary condition R_a.

For the normalized solution $\phi_\alpha = \phi_{\alpha,z}$ satisfying the generalized Neumann condition we make the Ansatz

$$\begin{aligned} \phi_\alpha &= \lambda_\alpha \cdot \phi + \mu_\alpha \cdot \psi \\ \phi_\alpha(a) &= p(a)^{-\frac{1}{4}}, \quad \phi'_\alpha(a) + \alpha \cdot \phi_\alpha(a) = 0 \end{aligned} \tag{2.57}$$

and find

$$\begin{aligned} \mu_\alpha &= \psi(a)^{-1} \cdot p(a)^{-\frac{1}{4}} \\ \lambda_\alpha &= -\mu_\alpha \cdot \frac{R_a\psi}{\phi'(a)} = -\sqrt{p(a)} \cdot \left(\alpha + \frac{\psi'(a)}{\psi(a)}\right). \end{aligned} \tag{2.58}$$

Note that μ_α is independent of α while $\partial_\alpha \lambda_\alpha = -\sqrt{p(a)}$.

Furthermore, we have for the Wronskians

$$p \cdot W(\psi, \phi) = p(a) \cdot \psi(a) \cdot \phi'(a) = \mu_\alpha^{-1}, \tag{2.59}$$

$$p \cdot W(\psi_\alpha, \phi_\alpha) = \lambda_\alpha \cdot p \cdot W(\psi, \phi) = \frac{\lambda_\alpha}{\mu_\alpha}. \tag{2.60}$$

Lemma 2.6. *For $z \geq 0$ such that $\psi(a) = \psi_z(a) \neq 0$ we have*

$$\int_a^\infty \psi_z(y)^2 dy = \frac{1}{2z \cdot \mu_{\alpha,z}^2} \frac{d}{dz} \lambda_{\alpha,z}. \tag{2.61}$$

Proof. We use the formula for the Green function Eq. (2.18) of the operator $H_\alpha + z^2$ and obtain for $z \geq 0$ such that $-z^2$ is in the resolvent set (and still suppressing the index z where appropriate)

$$\begin{aligned}\int_a^\infty \psi_z(y)^2 dy &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2 \int_a^\infty \Big(\frac{\psi_z(y)}{p\cdot W(\psi,\phi_\alpha)}\Big)^2 dy\\ &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2\cdot \phi_\alpha(a)^{-2}\int_a^\infty \Big(\frac{\psi_z(y)\phi_\alpha(a)}{p\cdot W(\psi,\phi_\alpha)}\Big)^2 dy\\ &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2\cdot \phi_\alpha(a)^{-2}\int_a^\infty \Big((H_\alpha+z^2)^{-1}(a,y)\Big)^2 dy\\ &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2\cdot \phi_\alpha(a)^{-2}\cdot\Big(-\frac{1}{2z}\frac{d}{dz}(H_\alpha+z^2)^{-1}(a,a)\Big)\\ &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2\cdot \phi_\alpha(a)^{-2}\cdot (H_\alpha+z^2)^{-2}(a,a)\\ &= \big(p\cdot W(\psi,\phi_\alpha)\big)^2\cdot \phi_\alpha(a)^{-2}\cdot\Big(-\frac{1}{2z}\frac{d}{dz}\frac{\phi_\alpha(a)\cdot\psi_z(a)}{p\cdot W(\psi_z,\phi_\alpha)}\Big).\end{aligned} \tag{2.62}$$

The claim now follows by noting that $\phi_\alpha(a) = p(a)^{-3/4}$ is independent of z and by plugging in the known values for $\psi_z(a) = p(a)^{-1/4}\cdot\mu_\alpha^{-1}$ and $p\cdot W(\psi,\phi_\alpha) = \lambda_\alpha/\mu_\alpha$. □

Finally, let us do the following, a priori formal, calculation for the relative ζ–determinant of H_α and H_D:

$$\begin{aligned}\log\frac{\det_\zeta H_\alpha}{\det_\zeta H_D} &= -2 ⨍_0^\infty z\,\mathrm{Tr}\Big((H_\alpha+z^2)^{-1}-(H_D+z^2)^{-1}\Big)dz\\ &= 2 ⨍_0^\infty z\,\mathrm{Tr}\Big((H_D+z^2)^{-1}-(H_\alpha+z^2)^{-1}\Big)dz\\ &= 2 ⨍_0^\infty z\int_a^\infty \frac{\psi\cdot\phi}{p\cdot W(\psi_z,\phi_z)}-\frac{\psi_z\cdot\phi_{\alpha,z}}{p\cdot W(\psi_z,\phi_{\alpha,z})}dz\\ &= ⨍_0^\infty \frac{2z}{p\cdot W(\psi_z,\phi_z)}\Big(-\frac{\mu_{\alpha,z}}{\lambda_{\alpha,z}}\Big)\int_a^\infty \psi(y)^2 dy\,dz\\ &= -⨍_0^\infty \frac{d}{dz}\log\lambda_{\alpha,z}dz\\ &= \log\lambda_{\alpha,z}|_{z=0} - \operatorname*{LIM}_{z\to\infty}\log\lambda_{\alpha,z}.\end{aligned} \tag{2.63}$$

This calculation is valid whenever the trace under the first integral has an asymptotic expansion as $z\to\infty$ as Eq. (2.7). Note that then the existence of the regularized limit $\mathrm{LIM}_{z\to\infty}\log\lambda_{\alpha,z}$ follows automatically. In concrete situations, as e.g. in Section 4 below, the computation of this regularized limit (ideally proving that it is 0) is a separate issue.

Instead of formulating a formal Theorem we record for later reference that if the computations of this subsection are valid and if $\mathrm{LIM}_{z\to\infty}\log\lambda_{\alpha,z} = 0$ then by Eq. (2.59), (2.60) and (2.63) the quotient $\det_\zeta H/pW(\psi,\phi)$ is *independent* of the boundary condition at a. Replacing H by $H+\nu^2$ (e.g. to ensure invertibility) we even conclude from this calculation that the quotient $\det_\zeta(H+\nu^2)/pW(\psi_\nu,\phi_\nu)$ is *independent* of the boundary condition at a.

3. Asymptotic expansions of modified Bessel functions

In this section we present the relevant asymptotic expansions for the modified Bessel functions of the first and second kind, which will be employed throughout this paper. We employ the standard references Abramowitz and Stegun [AbSt92], Gradshteyn and Ryzhik [GrRy15], Olver [Olv97] as well as Watson [Wat95]. We will also refer to Sidi and Hoggan [SiHo11] in Sec. 3.2.

We begin by recalling the definitions of modified Bessel functions. Modified Bessel functions of order $z \in \mathbb{R}$ are defined as a fundamental system of solutions $f \in C^\infty(0,\infty)$ of the following differential equation

$$f''(x) + \frac{1}{x} f'(x) - \left(1 + \frac{z^2}{x^2}\right) \cdot f(x) = 0. \tag{3.1}$$

A fundamental system of solutions to this second order differential equation is given in terms of the modified Bessel functions of first and second kind

$$I_z(x) := \frac{x^z}{2^z} \sum_{k=0}^{\infty} \frac{x^{2k}}{4^k \cdot k! \cdot \Gamma(z+k+1)}, \qquad K_z(x) := \frac{(I_{-z}(x) - I_z(x))}{2\pi \sin(z\pi)}, \tag{3.2}$$

where $K_z(x)$ is defined for $z \notin \mathbb{Z}$ and for $z \in \mathbb{Z}$ by the limit

$$K_z(x) := \lim_{t \to z} \frac{I_{-t}(x) - I_t(x)}{2\pi \sin(t\pi)}. \tag{3.3}$$

We gather some important properties of the modified Bessel functions in the following proposition. These properties are classical and can be inferred from the aforementioned references Abramowitz and Stegun [AbSt92], Gradshteyn and Ryzhik [GrRy15], as well as Olver [Olv97].

Proposition 3.1.

(i) *The Wronskian of the fundamental system is given by* $W(K_z, I_z)(x) = x^{-1}$.
(ii) *The modified Bessel functions are positive for* $x > 0$.
(iii) *For* z *fixed,* $K_z(x)$ *is decreasing and* $I_z(x)$ *is increasing.*
(iv) *For* x *fixed and* $z \in [0,\infty)$, $K_z(x)$ *is increasing and* $I_z(x)$ *is decreasing.*
(v) $I_z \in L^1[0,a]$, *but* $I_z \notin L^1[a,\infty)$, *for* $a > 0$.
(vi) $K_z \notin L^1(0,a]$, *but* $K_z \in L^1[a,\infty)$, *for* $a > 0$.

We now begin with studying asymptotic expansions of the modified Bessel functions. We distinguish between the following three cases: large argument x and fixed order z, fixed argument and large order, as well as uniform asymptotic expansion for larger order.

3.1. Asymptotics for large arguments and fixed order.

We begin with an analysis of the asymptotic behavior of the Bessel functions for fixed order $z \geq 0$ and large argument $x \to \infty$. We infer from [AbSt92, (9.7.1), (9.7.2)], see also [Wat95, p. 202,

§7.23 (1)-(2)], that the Bessel functions admit the following asymptotic expansions

$$I_z(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 + \sum_{k=1}^{\infty} (-1)^k A_k(z)\, x^{-k}\right), \quad x \to \infty, \tag{3.4}$$

$$K_z(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + \sum_{k=1}^{\infty} A_k(z)\, x^{-k}\right), \quad x \to \infty. \tag{3.5}$$

The coefficients of in the asymptotic expansions above are given explicitly by

$$A_k(z) = \frac{1}{8^k k!} \prod_{n=1}^{k} \left(4z^2 - (2n-1)^2\right). \tag{3.6}$$

3.2. **Asymptotics for fixed arguments and large order.** For the asymptotics of modified Bessel functions for large order we refer to Sidi and Hoggan [SiHo11]. Asymptotics of $I_z(x)$ also follows from the asymptotic expansion of the (unmodified) Bessel function [AbSt92, (9.3.1)]. Using the Stirling formula asymptotics for the Gamma function, see e.g. [AbSt92, (6.1.37)], we infer from [SiHo11] for $x > 0$ fixed

$$I_z(x) \sim \frac{1}{\sqrt{2\pi z}} \left(\frac{ex}{2z}\right)^z \left(1 + \sum_{j=1}^{\infty} \frac{B_j(x)}{z^j}\right), \quad z \to \infty, \tag{3.7}$$

$$K_z(x) \sim \sqrt{\frac{\pi}{2z}} \left(\frac{ex}{2z}\right)^{-z} \left(1 + \sum_{j=1}^{\infty} (-1)^j \frac{B_j(x)}{z^j}\right), \quad z \to \infty. \tag{3.8}$$

The coefficients B_j are polynomials in $(x/2)^2$ of degree $j \in \mathbb{N}$. At several instances we will use a particular consequence of these expansions.

$$\frac{K_{z+1}(x)}{K_z(x)} = \frac{2z}{x} + O(z^{-1}), \quad z \to \infty. \tag{3.9}$$

Similar expansions hold for the derivatives just using the standard recurrence relations of Bessel functions, *cf.* [AbSt92, (9.6.26)]

$$I_z'(x) = I_{z+1}(x) + \frac{z}{x} I_z(x), \quad K_z'(x) = -K_{z+1}(x) + \frac{z}{x} K_z(x). \tag{3.10}$$

Combining Eq. (3.9) and (3.10) we find

$$\frac{K_z'(x)}{K_z(x)} = -\frac{z}{x} + O(z^{-1}), \quad z \to \infty. \tag{3.11}$$

3.3. **Uniform asymptotic expansion for large order.** We now turn to uniform asymptotics of Bessel functions, when the order go to infinity. Following Olver [Olv97, p. 377 (7.16), (7.17)], see also [AbSt92, (9.7.7), (9.7.8)], we have for large $z > 0$ and uniformly in $x > 0$

$$I_z(zx) \sim \frac{e^{z\xi(x)}}{\sqrt{2\pi z}(1+x^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} \frac{U_k(x)}{z^k}\right), \quad z \to \infty. \tag{3.12}$$

Similarly, for the modified Bessel functions of second kind we have for large $z > 0$ and uniformly in $x > 0$ the following asymptotic expansions

$$K_z(zx) \sim \sqrt{\frac{2\pi}{z}} \frac{e^{-z\xi(x)}}{(1+x^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{U_k(x)}{z^k}\right), \quad z \to \infty. \tag{3.13}$$

In both cases we have introduced the following notation

$$\begin{aligned} \xi = \xi(x) &:= \sqrt{1+x^2} + \log \frac{x}{1+\sqrt{1+x^2}}, \\ p = p(x) &:= \frac{1}{\sqrt{1+x^2}}. \end{aligned} \tag{3.14}$$

The coefficients $U_k(x)$ in the asymptotic expansions above, are polynomial functions in p of degree $3k$.

We conclude with an asymptotic expansion for a product of Bessel functions. Using Cauchy product formula we infer from the expansions above

$$I_z(zx)K_z(zx) = \frac{1}{2z} \frac{1}{\sqrt{1+x^2}} \left(1 + \sum_{k=1}^{\infty} \frac{\widetilde{U}_{2k}(x)}{z^{2k}}\right), \quad z \to \infty, \tag{3.15}$$

where the coefficients $\widetilde{U}_{2k}(x)$ are polynomials in p of degree $6k$.

4. Variation formula and the determinant of the model operator

Fix $\mu > 0$ and consider the family of scalar model *cusp* operators

$$D_\mu = -\frac{d}{dx}\Big(x^2 \frac{d}{dx}\Big) + x^2\mu^2 - \frac{1}{4} : C_0^\infty(a,\infty) \to C_0^\infty(a,\infty). \tag{4.1}$$

Let $z \geq 0$. Then a fundamental system of solutions for the second order differential equation $(D_\mu + z^2)f = 0$ is given in terms of the modified Bessel functions by $x^{-1/2}I_z(\mu x), x^{-1/2}K_z(\mu x)$. By Eq. (3.4), $x^{-1/2}I_z(\mu x)$ does not lie in $L^2[a,\infty)$. Consequently, ∞ is in the limit point case for the operator $D_\mu + z^2$ and self-adjoint extensions are obtained by imposing boundary conditions of the form Eq. (1.3) at $x = a$.

Consider first the case of Dirichlet boundary conditions $R_a f = f(a)$. By abuse of notation we use $D_\mu := D_\mu(R_a)$. A normalized fundamental system, *cf.* Eq. (1.4) and Theorem 1.2, of solutions of the equation $(D_\mu + z^2)f = 0$ is then given by

$$\psi_{z,\mu}(x) = x^{-1/2}K_z(\mu x), \quad \phi_{z,\mu}(x) = x^{-1/2}\big(K_z(\mu a)\cdot I_z(\mu x) - I_z(\mu a)\cdot K_z(\mu x)\big). \tag{4.2}$$

Note that $\psi_{z,\mu} \in L^2[a,\infty)$. Furthermore,

$$\phi'_{z,\mu}(a) = a^{-1/2}\cdot \mu \cdot W(K_z, I_z)(\mu a) = a^{-3/2} = p(a)^{-3/4}, \tag{4.3}$$

hence $\phi_{z,\mu}$ has the correct normalization according to Eq. (1.4). The Wronskian of the modified Bessel functions satisfies $W(K_z, I_z)(x) = \frac{1}{x}$. Furthermore, we point out that $K_z(\mu x) > 0$ is nowhere vanishing for $x > 0$ by Proposition 3.1. In particular, $D_\mu(R_a) + z^2$ is invertible. The Wronskian of $\psi_{z,\mu}, \phi_{z,\mu}$ is given by

$$x^2 \cdot W(\psi_{z,\mu}, \phi_{z,\mu}) = x\cdot \mu \cdot K_z(\mu a)\dot{W}(K_z, I_z)(\mu x) = K_z(\mu a). \tag{4.4}$$

Consequently, the Green function G_z of $(D_\mu + z^2)^{-1}$ is obtained as in Eq. (2.18)

$$G_z(x,y) = \begin{cases} (xy)^{-1/2} \cdot \big(I_z(\mu x)\cdot K_z(\mu y) - \frac{I_z(\mu a)}{K_z(\mu a)} K_z(\mu x)\cdot K_z(\mu y)\big), & x \le y, \\ (xy)^{-1/2} \cdot \big(I_z(\mu y)\cdot K_z(\mu x) - \frac{I_z(\mu a)}{K_z(\mu a)} K_z(\mu y)\cdot K_z(\mu x)\big), & y \le x. \end{cases} \tag{4.5}$$

In particular we find for the Green function at the diagonal

$$G_z(x) \equiv G_z(x,x) = x^{-1}\Big(I_z(\mu x)\cdot K_z(\mu x) - \frac{I_z(\mu a)}{K_z(\mu a)} K_z^2(\mu x)\Big). \tag{4.6}$$

The Green function $G_z(x,y)$ is continuous on $[a,\infty)^2$ and by positivity of solutions $\phi_{z,\mu}$ and $\psi_{z,\mu}$, the kernel is non-negative and positive away from $x,y = a$. Moreover, $G_z(x) = O(x^{-2})$ when $x \to \infty$ by the asymptotic expansions Eq. (3.4) and (3.5) then G_z is integrable on $[a,\infty)$ along the diagonal. Consequently, by Mercer's theorem, as worked out e.g. by Reed and Simon [ReSi79, §XI.4, Lemma on p. 65] we conclude that the resolvent $(D_\mu + z^2)^{-1}$ is trace class and the trace of $(D_\mu + z^2)^{-1}$ is given by

$$\operatorname{Tr}\big(D_\mu + z^2\big)^{-1} = \int_a^\infty G_z(x)\,dx. \tag{4.7}$$

A similar argument holds in case of generalized Neumann boundary conditions $R_a = f'(a) + \alpha f(a)$. In that case the solution $\phi_{z,\mu,\alpha}$, satisfying the generalized Neumann boundary conditions is given by

$$\phi_{z,\mu,\alpha}(x) = c_{z,\mu}\cdot x^{-\frac{1}{2}} \cdot \left(I_z(\mu x) - \frac{(\alpha - \frac{1}{2a})\cdot I_z(\mu a) + \mu \cdot I_z'(\mu a)}{(\alpha - \frac{1}{2a})\cdot K_z(\mu a) + \mu\cdot K_z'(\mu a)} \cdot K_z(\mu x) \right). \tag{4.8}$$

The constant $c_{z,\mu}$ is determined by the normalization requirement Eq. (1.4), *i.e.*, $\phi_{z,\mu}(a) = p(a)^{-1/4} = a^{-1/2}$. We construct the Green function as before. But now G_z is positive only if either $\alpha \in \big[0, \frac{1}{2}\big]$ or if $\alpha > \frac{1}{2}$ and μ sufficiently large. In these two cases we can use Mercer's Theorem as before to obtain $\operatorname{Tr}(D_\mu(R_a)+z^2)^{-1}$ integrating the Green function G_z along the diagonal. If G_z is not necessarily positive, the trace class property of $(D_\mu(R_a) + z^2)^{-1}$ follows by a well–known comparison principle for elliptic operators, *cf.* e.g. [LMP12, Chapter 3] and [LeVe11].

Proposition 4.1. *Consider Dirichlet boundary conditions* $R_{a,1}(f) = f(a)$ *and generalized Neumann boundary conditions* $R_{a,2}(f) = f'(a) + \alpha f(a)$ *for the model operator* D_μ. *Let* $D_{\mu,1}$ *and* $D_{\mu,2}$ *denote the corresponding self-adjoint realizations in* $L^2[a,\infty)$ *with boundary conditions* $R_{a,1}$ *and* $R_{a,2}$, *respectively. Then*

$$\big\|(D_{\mu,1} + z^2)^{-1} - (D_{\mu,2} + z^2)^{-1}\big\|_{\mathrm{tr}} = O(z^{-2}\log z), \quad z \to \infty. \tag{4.9}$$

Proof. Fix any $\delta > \delta' > a$. We choose cut-off functions $\chi_1, \chi_2 \in C_0^\infty[a,\delta)$, as illustrated in Figure 1, such that they are identically one over $[a,\delta']$ and moreover,

- $\operatorname{supp}(\chi_1) \subset \operatorname{supp}(\chi_2)$,
- $\operatorname{supp}(\chi_1) \cap \operatorname{supp}(d\chi_2) = \emptyset$.

We write $\eta_1 := 1 - \chi_1$, $\eta_2 := 1 - \chi_2$. By construction $\eta_1\eta_2 = \eta_2$. We consider

$$R(z) := \eta_1\big[(D_{\mu,1} + z^2)^{-1} - (D_{\mu,2} + z^2)^{-1}\big]\eta_2. \tag{4.10}$$

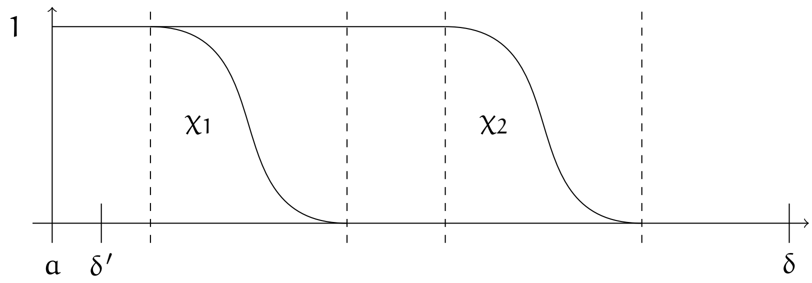

FIGURE 1. The cutoff functions χ_1 and χ_2.

$R(z)$ maps into the domain of both $D_{\mu,1}$ and $D_{\mu,2}$. On the support of η_1 the differential expressions $D_{\mu,1}$ and $D_{\mu,2}$ coincide and moreover $\eta_1\,\mathcal{D}(D_{\mu,1}) = \eta_1\,\mathcal{D}(D_{\mu,2})$. Thus we may compute

$$(D_{\mu,1}+z^2)R(z) = [D_{\mu,1},\eta_1]\big((D_{\mu,1}+z^2)^{-1} - (D_{\mu,2}+z^2)^{-1}\big). \tag{4.11}$$

Arguing similarly for $R(z)^*$ and taking adjoints one then finds

$$(D_{\mu,1}+z^2)R(z)(D_{\mu,2}+z^2) = [-\partial_x^2,\eta_1]\big((D_{\mu,1}+z^2)^{-1} - (D_{\mu,2}+z^2)^{-1}\big)[\partial_x^2,\eta_2], \tag{4.12}$$

where $[\cdot,\cdot]$ denotes the commutator of the corresponding operators and any function is viewed as a multiplication operator. Hence

$$R(z) = (D_{\mu,1}+z^2)^{-1}[-\partial_x^2,\eta_1]\big((D_{\mu,1}+z^2)^{-1} - (D_{\mu,2}+z^2)^{-1}\big)[\partial_x^2,\eta_2](D_{\mu,2}+z^2)^{-1}. \tag{4.13}$$

Since $(D_{\mu,1}+z^2)^{-1}$ is trace class by Mercer's theorem as explained above, and since the space of trace class operators forms an ideal in the space of bounded operators, we conclude that $R(z)$ is trace class as well and continue with the following estimate

$$\begin{aligned}\|R(z)\|_{tr} \le \|(D_{\mu,1}+z^2)^{-1}\|_{tr}\Big(\|[\partial_x^2,\eta_1](D_{\mu,1}+z^2)^{-1}\| + \|[\partial_x^2,\eta_1](D_{\mu,2}+z^2)^{-1}\|\Big)\cdot \\ \cdot\|[\partial_x^2,\eta_2](D_{\mu,2}+z^2)^{-1}\|.\end{aligned} \tag{4.14}$$

By Eq. (1.11), we have $\|(D_{\mu,1}+z^2)^{-1}\|_{tr} = O(z^{-1}\log z)$. Let f denote η_1 or η_2. Then $[\partial_x^2, f]$ is a first order differential operator whose coefficients are compactly supported in (a,δ), hence it maps the Sobolev space $H^1[a,\delta]$ continuously into $L^2_{comp}(a,\delta)$. Therefore we conclude for $j=1,2$ with a cut–off function $\chi \in C_0^\infty(a,\delta)$ with $\chi = 1$ in a neighborhood of $\operatorname{supp}([\partial_x^2, f])$,

$$\|[\partial_x^2,f](D_{\mu,j}+z^2)^{-1}\| \le \|[\partial_x^2,f]\|_{H^1\to L^2}\|\chi(D_{\mu,j}+z^2)^{-1}\|_{L^2\to H^1} = O(z^{-1}), \tag{4.15}$$

as $z\to\infty$. Hence for $j=1,2$ the operator norms $\|[\partial_x^2,\eta_1](D_{\mu,j}+z^2)^{-1}\|$ and $\|[\partial_x^2,\eta_2](D_{\mu,j}+z^2)^{-1}\|$ behave as $O(z^{-1})$ as $z\to\infty$. This proves

$$\Big\|\Big((D_{\mu,1}+z^2)^{-1} - (D_{\mu,2}+z^2)^{-1}\Big)\Big|_{L^2(\delta,\infty)}\Big\|_{tr} = O(z^{-2}\log z), \quad z\to\infty. \tag{4.16}$$

Note that the Schwartz integral kernel of $(D_{\mu,2}+z^2)^{-1}$ is smooth at the diagonal $[a,\infty)$ and hence by continuity is strictly positive (or strictly negative) over $[a,\delta]$ for $(\delta-a)>0$ sufficiently small. By Mercer's theorem, $(D_{\mu,2}+z^2)^{-1}$ is trace class in $L^2(a,\delta)$.

The statement now follows from the fact that integrals of the Schwartz kernels for $(D_{\mu,1}+z^2)^{-1}$ and $(D_{\mu,2}+z^2)^{-1}$ along the diagonal in $[1,\delta]$ admit an asymptotic expansion of the form $\sum_{k=0}^{\infty} a_k(\mu) z^{-1-k}$, where the leading order term a_0 is independent of the boundary conditions. □

The next proposition is proved in [VER14] without specifying the leading coefficients in the asymptotic expansion. By keeping track of the coefficients in the asymptotic expansions of Sec. 3.3 we obtain the following more precise statement.

Proposition 4.2. *For any boundary condition of the form* Eq. (1.3) *at* a *the corresponding self-adjoint realization* $D_\mu(R_a)$ *of* D_μ *admits an asymptotic expansion of the resolvent trace*

$$\operatorname{Tr}(D_\mu(R_a)+z^2)^{-1} = \sum_{k=0}^{\infty} a_k(\mu)\cdot z^{-1-k} + \sum_{k=0}^{\infty} b_{2k}(\mu)\cdot z^{-1-2k}\cdot \log z, \quad \text{as} \quad z\to\infty, \tag{4.17}$$

where $a_0(\mu)=\frac{1}{2}\log\frac{2}{\mu a}$ *and* $b_0(\mu)=\frac{1}{2}$ *independent of the choice of boundary conditions at* a*. Moreover, for Dirichlet boundary conditions* $a_1(\mu)=\frac{1}{4}$*, while for generalized Neumann boundary conditions* $a_1(\mu)=-\frac{1}{4}$*.*

Note that the asymptotic expansion Eq. (4.17) does not admit terms of the form $z^{-2}\log^k(z), k\in\mathbb{N}$. The zeta-regularized determinant of $D_\mu(R_a)+\nu^2$, for any $\nu\geq 0$, is therefore defined according to Sec. 2.2.

The following variation formula is due to the third named author [VER14, Proposition 5.1 and Theorem 5.2]. We present it here with precise formulae for solutions and the normalizing constants. Theorem 7.1, for which we will give a complete proof, contains the following as a special case.

Theorem 4.3. *Fix a boundary condition* R_a *for the model operator* D_μ *at* a*. For* $\nu\geq 0$*, let* $\psi_{\nu,\mu}(x)=x^{-\frac{1}{2}}K_\nu(\mu x)$ *and* $\phi_{\nu,\mu}(x)$ *be a normalized fundamental system of solutions to* $(D_\mu+\nu^2)f=0$*, cf.* Eq. (1.3), (4.2), (4.8).

Assume that the kernel of $(D_\mu+\nu^2)$ *is trivial. Then the zeta-regularized determinant of* $(D_\mu+\nu^2)$ *is differentiable in* ν *and satisfies the following variational formula*

$$\frac{d}{d\nu}\log\det_\zeta(D_\mu+\nu^2) = \frac{d}{d\nu}\log\big(x^2\cdot W(\psi_{\nu,\mu}(x),\phi_{\nu,\mu}(x))\big). \tag{4.18}$$

Now we are already in a position to compute the zeta-regularized determinant for the model operator D_μ for any boundary condtion, *cf. e.g.*, [LETO98, Thm. 3.3].

Theorem 4.4. *Fix a boundary condition for the model operator* D_μ*. Then for the zeta-regularized determinant of* $D_\mu(R_a)+\nu^2$ *we have*

$$\det_\zeta(D_\mu(R_a)+\nu^2) = \sqrt{\frac{2}{\pi}}\cdot a^2\cdot W(\psi_\nu,\phi_{\nu,R_a})(a). \tag{4.19}$$

Here, ϕ_{ν,R_a}, ψ_ν is a normalized fundamental system of solutions to the equation $(D_\mu + \nu^2)f = 0$, $\psi_\nu(x) = x^{-1/2}K_\nu(\mu x)$ *and ϕ_{ν,R_a} is given in* Eq. (4.2) *(Dirichlet) resp.* Eq. (4.8) *(Neumann).*

Proof. From the previous Theorem and Lemma 2.2 we infer

$$\det_\zeta(D_\mu(R_a)+\nu^2) = a^2\cdot W(\psi_\nu, \phi_{\nu,R_a})\cdot \exp\Big(-\operatorname*{LIM}_{z\to\infty}\log\big(a^2\cdot W(\psi_z,\phi_{z,R_a})\big)\Big). \tag{4.20}$$

It therefore remains to compute the LIM in the exponential function on the right. Let us first look at the case of Dirichlet boundary conditions. Let $\phi_z = \phi_{z,R_a}$ for $R_a f = f(a)$. Then

$$a^2\cdot W(\psi_z,\phi_z)(a) = a^2\cdot\psi_z(a)\cdot\phi_z'(a) = K_z(\mu a), \tag{4.21}$$

since ϕ_z is normalized to $\phi_z'(a) = a^{-3/2}$. Using the asymptotic expansion Eq. (3.8), we obtain

$$\log K_z(\mu a) = \log\sqrt{\frac{\pi}{2}} - \frac{1}{2}\log z - z\log\Big(\frac{e\mu a}{2z}\Big) + \log\Big(1+O(z^{-1})\Big), \text{ as } z\to\infty, \tag{4.22}$$

hence $\operatorname*{LIM}_{z\to\infty}\log\big(K_z(\mu a)\big) = \log\sqrt{\frac{\pi}{2}}$ and the result follows.

Next consider a generalized Neumann boundary condition $R_a f = f'(a) + \alpha f(a)$ and denote by $\phi_{z,\alpha}$ the corresponding normalized solution satisfying $R_a\phi_{z,\alpha} = 0$. Then by Eq. (2.57), (2.58)

$$\begin{aligned}\frac{a^2\cdot W(\psi_z,\phi_{z,\alpha})}{a^2\cdot W(\psi_z,\phi_z)} &= \lambda_\alpha = -a\cdot\frac{R_a\psi_z}{\psi_z(a)} = -a^{\frac{1}{2}}\cdot\frac{(\alpha-\frac{1}{2a})\,K_z(\mu a) + \mu\,K_z'(\mu a)}{a^{-\frac{1}{2}}K_z(\mu a)}\\ &= -(\alpha a - \frac{1}{2}) - \mu a\cdot\frac{K_z'(\mu a)}{K_z(\mu a)}\\ &= z + O(1) = z\cdot\big(1+O(z^{-1})\big), \quad \text{as } z\to\infty.\end{aligned} \tag{4.23}$$

The last line follows from Eq. (3.11). Taking log on both sides yields

$$\log\Big(\frac{a^2\cdot W(\psi_z,\phi_{z,\alpha})}{a^2\cdot W(\psi_z,\phi_z)}\Big) = \log z + O(z^{-1}), \text{ as } z\to\infty, \tag{4.24}$$

consequently the regularized limit for the Neumann boundary condition equals that for the Dirichlet boundary condition, *i.e.*,

$$\operatorname*{LIM}_{z\to\infty}\log\big(a^2\cdot W(\psi_z,\phi_{z,\alpha})\big) = \operatorname*{LIM}_{z\to\infty}\log\big(a^2\cdot W(\psi_z,\phi_z)\big) = \log\sqrt{\frac{\pi}{2}}. \tag{4.25}$$

□

5. Bôcher theorem for operators with quadratic potentials at infinity

In this section we will prove a version of the Bôcher's Theorem for perturbations of the model cusp operator Eq. (4.1) and we analyse the Wronskian's behavior at infinity of a perturbed fundamental system of solution.

Recall that $\psi_z(x) = x^{-\frac{1}{2}}K_z(\mu x)$, $\phi_z(x) = x^{-\frac{1}{2}}I_z(\mu x)$ is a fundamental system of solutions of the equation $(D_\mu + z^2)f = 0$ with Wronskian

$$x^2\cdot W(\psi_z,\phi_z)(x) = 1. \tag{5.1}$$

In the notation of Section 2 we have $p(x) = x^2$. We will specialise the result of Section 2.4 to Eq. (4.1). For this we need to verify the conditions Eq. (2.33) - (2.37).

Lemma 5.1. *For the operator* Eq. (4.1) *we have, cf.* Eq. (2.29)*,*

$$L(x,y) := y \cdot K_z^2(\mu y)\left[\frac{I_z(\mu x)}{K_z(\mu x)} - \frac{I_z(\mu y)}{K_z(\mu y)}\right], \tag{5.2}$$

for $a \leq x \leq y < \infty$*. Furthermore,*

$$\sup_{a\leq x\leq y<\infty} |L(x,y)| \leq C(\mu). \tag{5.3}$$

Proof. This follows from the asymptotic expansion Eq. (3.4) and (3.5). Namely, choose y_0 such that for $y \geq y_0$,

$$K_z(\mu y) \leq 2 \cdot \sqrt{\frac{\pi}{2\mu y}} e^{-\mu y}, \qquad I_z(\mu y) \leq 2 \cdot \frac{1}{\sqrt{2\pi\mu y}} e^{\mu y}. \tag{5.4}$$

Since $x \mapsto \frac{I_z(\mu x)}{K_z(\mu x)}$ is an increasing function we then have for all $a \leq x \leq y$ and $y \geq y_0$

$$\big|L(x,y)\big| = yK_z^2(\mu y)\left[\frac{I_z(\mu y)}{K_z(\mu y)} - \frac{I_z(\mu x)}{K_z(\mu x)}\right] \leq yK_z(\mu y)I_z(\mu y) \leq \frac{2}{\mu}. \tag{5.5}$$

Since L is certainly continuous, it is bounded on the compact set $a \leq x \leq y \leq y_0$ and the claim follows. □

Theorem 5.2. *Let*

$$H = D_\mu + X^2 \cdot W, \tag{5.6}$$

with $W \in L^1[a,\infty)$ *and fix* $z \geq 0$*. Then the differential equation* $(H + z^2)f = 0$ *has a fundamental system of solutions* h_1, h_2*, such that*

$$h_1(x) = \psi_z(x) \cdot g_1(x), \qquad h_2(x) = \phi_z(x) \cdot g_2(x), \tag{5.7}$$

with $g_j \in C_b[a,\infty)$ *and* $\lim_{x\to\infty} g_j(x) = 1$, $j = 1, 2$*. Furthermore,*

$$h_1'(x) = \psi_z'(x) \cdot \tilde{g}_1(x), \qquad h_2'(x) = \phi_z'(x) \cdot \tilde{g}_2(x), \tag{5.8}$$

where $\tilde{g}_j \in C_b[a,\infty)$, $\lim_{x\to\infty} \tilde{g}_j(x) = 1$, $j = 1, 2$, *and* $x^2 \cdot W(h_1, h_2)(x) = 1$.

Proof. We just need to verify the assumptions Eq. (2.34) - (2.37). In view of the asymptotic expansion Eq. (3.5) it is easy to see that ψ satisfies Eq. (2.34) - (2.36). Assumption Eq. (2.37) follows from the asymptotic expansion Eq. (3.4) and (3.5). Namely, by these expansions

$$\frac{I_z(\mu x)}{K_z(\mu x)} = O\big(e^{2\mu x}\big), \quad x \to \infty, \tag{5.9}$$

thus the assumption follows observing that quotient $\frac{\phi_z(x)}{\psi_z(x)}$ is $\frac{I_z(\mu x)}{K_z(\mu x)}$. □

5.1. **Asymptotics of Wronskians for the perturbed operator.** Consider a family of functions $W_t \in L^1[a,\infty)$, depending on a real parameter t such that $t \mapsto W_t$ is differentiable as a map into $L^1[a,\infty)$. We apply Theorem 5.2 to study the fundamental system of solutions and their Wronskians for the perturbed operator

$$H_t + z^2 := (D_\mu + z^2) + X^2 W_t. \tag{5.10}$$

As a notational convenience we take $t_0 = 0$ as base point. By Theorem 5.2 the equation $(H_t + z^2)f = 0$ has two solutions. Let $h_{1,t}(x)$ be the solution of $(H_t + z^2)h_{1,t} = 0$ with

$$h_{1,t}(x) \sim x^{-\frac{1}{2}} K_z(\mu x) = O(x^{-1}e^{-\mu x}), \quad x \to \infty. \tag{5.11}$$

Note that by Eq. (5.11) the solution $h_{1,t}$ is unique as the space of solutions in the limit point case has dimension one.

Lemma 5.3. *The solution $h_{1,t}(x)$ is differentiable in t with the estimates $\dot{h}_{1,t}(x) = o(x^{-1}e^{-\mu x})$, $\partial_x \dot{h}_{1,t}(x) = o(x^{-1}e^{-\mu x})$ as $x \to \infty$ and the o-constants are locally uniform in t, i.e., $h_{1,t}(x) - h_{1,0}(x) = o(x^{-1}e^{-\mu x})$ as $x \to \infty$.*

Proof. The o-behavior as $x \to \infty$ follows directly from Lemma 2.4. In fact,

$$\begin{aligned} \dot{h}_{1,t} &= \psi_z \cdot \dot{f}_{1,t} \\ \partial_x \dot{h}_{1,t} &= \psi_z' \cdot \dot{f}_{1,t} + \psi \cdot \partial_x \dot{f}_1, \end{aligned} \tag{5.12}$$

where by equation Eq. (2.47),

$$\dot{f}_{1,t} = (I - LW_t)^{-1}(L\dot{W}_t \mathbf{1}) + (I - LW_t)^{-1} L\dot{W}_t (I - LW_t)^{-1}(\mathbf{1}), \tag{5.13}$$

and

$$\partial_x \dot{f}_{1,t} = (I - LW_t)^{-1}[L\dot{W}_t \mathbf{1}]' + (I - LW_t)^{-1}[L\dot{W}_t f_{1,t}]'. \tag{5.14}$$

Then $\dot{f}_{1,t}$ and $\partial_x \dot{f}_{1,t}$ are in $C_\bullet[a,\infty)$.

The last part follows as the o-constants are locally independent of t and

$$\dot{h}_{1,0}(x) = \lim_{t \to 0} \frac{h_{1,t}(x) - h_{1,0}(x)}{t}. \tag{5.15}$$

□

Now we will choose the second solution $h_{2,t}(x)$. By equation Eq. (2.47) there exists $x_0 \in [a,\infty)$ such that $h_{1,0}(x) \neq 0$ for $x \geq x_0$. Then for t in a neighborhood of 0 we have $h_{1,t}(x) \neq 0$ for all $x \geq x_0$. Thus

$$\begin{aligned} (xh_{1,t}(x))^{-2} - (xh_{1,0}(x))^{-2} &= \frac{(h_{1,0}(x) + h_{1,t}(x))(h_{1,0}(x) - h_{1,t}(x))}{x^2 h_{1,0}^2(x) h_{1,t}^2(x)} \\ &= o(e^{2\mu x}), \qquad x \to \infty. \end{aligned} \tag{5.16}$$

Define

$$h_{2,0}(x) = h_{1,0}(x) \int_{x_0}^{x} (yh_{1,0}(y))^{-2} dy, \tag{5.17}$$

and

$$h_{2,t}(x) = h_{1,t}(x) \int_{x_0}^{x} (yh_{1,t}(y))^{-2} - (yh_{1,0}(y))^{-2} dy - \frac{h_{1,t}(x)}{h_{1,0}(x)} h_{2,0}(x). \tag{5.18}$$

Note that, if $f(x) = o(e^{cx})$, with $c > 0$ then for $x \to \infty$

$$\int_a^x f(x)dx = o(e^{cx}), \tag{5.19}$$

as well.

Lemma 5.4. *Let* $h_{1,t}, h_{2,t}$ *be a fundamental system of solutions for* Eq. (5.10), *which satisfy* Eq. (5.7) *and* (5.8). *Then* $h_{2,t}$ *is differentiable in* t *and we have*

$$\begin{aligned} h_{2,t}(x) &= h_{2,0}(x) + O(x^{-1}e^{\mu x}) = O(x^{-1}e^{\mu x}),\\ \dot{h}_{2,t}(x) &= o(x^{-1}e^{\mu x}),\\ \partial_x \dot{h}_{2,t}(x) &= o(x^{-1}e^{\mu x}). \end{aligned} \tag{5.20}$$

Proof. Since

$$(x\, h_{1,t}(x))^{-2} - (x\, h_{1,0}(x))^{-2} = o(e^{2x}), \tag{5.21}$$

the integral

$$\int_{R_0}^x (y\, h_{1,t}(y))^{-2} - (y\, h_{1,0}(y))^{-2} dy = o(e^{2x}). \tag{5.22}$$

This implies that $h_{2,t}(x) = O(x^{-1}e^{\mu x})$. The orders of $\dot{h}_{2,t}(x)$ and $\partial_x \dot{h}_{2,t}(x)$ follows using last result and the previous lemma. □

Now we are ready to estimate the behavior of the following Wronskians as $x \to \infty$:

Corollary 5.5. *Let* $h_{1,t}, h_{2,t}$ *be a fundamental system of solutions for* Eq. (5.10), *which satisfy* Eq. (5.7) *and* (5.8). *Then as* $x \to \infty$,

$$\begin{aligned} x^2 \cdot W(h_{1,t}, \dot{h}_{1,t})(x) &= o(e^{-2\mu x});\\ x^2 \cdot W(h_{2,t}, \dot{h}_{1,t})(x) &= o(1);\\ x^2 \cdot W(h_{1,t}, \dot{h}_{2,t})(x) &= o(1);\\ x^2 \cdot W(h_{2,t}, \dot{h}_{2,t})(x) &= o(e^{2\mu x}). \end{aligned} \tag{5.23}$$

In particular,

$$\lim_{x\to\infty} x^2 W(h_{1,t}, \dot{h}_{1,t})(x) = \lim_{x\to\infty} x^2 W(h_{2,t}, \dot{h}_{1,t})(x) = 0. \tag{5.24}$$

6. Regularized determinant of the perturbed operator

In this section we establish a partial asymptotic expansion for the resolvent trace of the perturbed operator $H + \nu^2$, which allows the definition of its zeta-regularized determinant. In the case of the model operator we have the full asymptotic expansion of the trace of $(D_\mu + z^2)^{-1}$ when $z \to \infty$. The Green function Eq. (4.5) and the uniform asymptotic expansion of the modified Bessel function Eq. (3.12) and (3.13) are the main ingredients to obtain that result. Same argument does not apply to the perturbed case, however using a Neumann series argument we can still derive a partial asymptotic expansion for the resolvent trace.

The results in this section are independent of the boundary conditions at $x = a$, hence for simplicity we use Dirichlet boundary conditions $R_a f = 0$ and by abuse of notation $D_\mu := D_\mu(R_a)$.

Lemma 6.1. *For fixed z, μ and real numbers α, β with $\alpha+\beta \le 2$ the operator $X^\alpha(D_\mu + z^2)^{-1}X^\beta$ is a bounded operator in the Hilbert space $L^2[a,\infty)$.*

Proof. We apply Schur's test [HaSu78, Thm. 5.2] to the kernel function of the operator. Recall from Section 4 that the kernel of $(D_\mu + z^2)^{-1}$ is given by $G_z(x,y)$ Eq. (4.5). During the proof C denotes a generic constant depending on z, μ, α, β, γ, a but not on x, y; it may change from line to line.

From Eq. (3.4) and (3.5) we conclude that

$$\big|\psi_z(x)\big| \le C\cdot\frac{e^{-\mu x}}{x}, \qquad \big|\varphi_z(x)\big| \le C\cdot\frac{e^{\mu x}}{x}. \tag{6.1}$$

Furthermore, we need the inequalities

$$\int_a^x e^{\mu y}y^\gamma dy \le C\cdot e^{\mu x}x^\gamma, \qquad \int_x^\infty e^{-\mu y}y^\gamma dy \le C\cdot e^{-\mu x}x^\gamma. \tag{6.2}$$

We find

$$\begin{aligned}\int_a^\infty x^\alpha\cdot|G_z(x,y)|\cdot y^\beta dy &\le C\cdot e^{-\mu x}x^{\alpha-1}\int_a^x e^{\mu y}y^{\beta-1}dy\\ &\qquad + C\cdot e^{\mu x}x^{\alpha-1}\int_x^\infty e^{-\mu y}y^{\beta-1}dy\\ &\le C\cdot x^{\alpha+\beta-2}\le C,\end{aligned} \tag{6.3}$$

and reversing the roles of α, β,

$$\int_a^\infty x^\alpha\cdot|G_z(x,y)|\cdot y^\beta \mathbf{dx} \le C. \tag{6.4}$$

With these inequalities the claim follows from Schur's test. □

Proposition 6.2. *Fix μ and let $-z^2$ be in the resolvent set of D_μ. For $\delta > 0$ the operator $X^{\frac12-\delta}\cdot(D_\mu+z^2)^{-1/2}$ is of Hilbert-Schmidt class resp. $X^{\frac12-\delta}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac12-\delta}$ is of trace class.*

Moreover, for real numbers α, β with $\alpha+\beta < \frac32$, the operator $X^\alpha\cdot(D_\mu+z^2)^{-1}\cdot X^\beta$ is a Hilbert-Schmidt operator.

Proof. Since for any two z_1, z_2 with $-z_1^2$, $-z_2^2$ in the resolvent set the operator $(D_\mu + z_1^2)^{-1}\cdot(D_\mu+z_2^2)$ is bounded, it suffices to prove the claim for $z \ge 0$. Then

$$\begin{aligned}\|X^{\frac12-\delta}\cdot(D_\mu+z^2)^{-1}\|_{HS} &= \operatorname{Tr}\big(X^{\frac12-\delta}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac12-\delta}\big)\\ &= \int_a^\infty x^{1-2\delta}\cdot G_z(x,x)dx \le C\int_a^\infty x^{-1-2\delta}dx < \infty,\end{aligned} \tag{6.5}$$

since $\delta > 0$. Here we have used that $G_z(x,x) = O(x^{-2})$ as $x \to \infty$ by Eq. (3.4) and (3.5).

For the second part, pick $\delta > 0$ such that $2(\alpha+\beta)+2\delta < 3$. Let $k(x,y)$ be the Schwartz kernel of $(D_\mu+z^2)^{-1}X^{2\alpha}(D_\mu+z^2)^{-1}$. From the proof of the first part we infer that $X^{\frac{1}{2}-\delta}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac{1}{2}-\delta}$ is trace class and from Lemma 6.1 we infer that $X^{2\beta+\delta-\frac{1}{2}}\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha+\delta-\frac{1}{2}}$ is bounded. Consequently, the product of these two operators, $X^{2\beta+\delta-\frac{1}{2}}\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac{1}{2}-\delta}$ is trace class and Mercer's Theorem implies that

$$\begin{aligned}\operatorname{Tr}\big(X^{2\beta+\delta-\frac{1}{2}}\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac{1}{2}-\delta}\big)&\\ =\int_a^\infty x^{2\beta+\delta-\frac{1}{2}}\cdot k(x,x)\cdot x^{\frac{1}{2}-\delta}dx&\\ =\int_a^\infty x^\beta\cdot k(x,x)\cdot x^\beta dx.&\end{aligned}\tag{6.6}$$

On the other hand the operator $X^\beta\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^\beta$ is non-negative. Hence from Mercer's Theorem in the version of Reed and Simon [ReSi79, §XI.4, Lemma on p. 65] we infer that indeed

$$\begin{aligned}\int_a^\infty x^\beta\cdot k(x,x)\cdot x^\beta dx&=\operatorname{Tr}\big(X^\beta\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^\beta\big)\\&=\|X^\alpha\cdot(D_\mu+z^2)^{-1}\cdot X^\beta\|_{HS}^2.\end{aligned}\tag{6.7}$$

Since we know that the left hand side is finite we reach the conclusion. □

The argument using the kernel and applying Mercer's Theorem twice was necessary to justify the manipulation

$$\begin{aligned}\operatorname{Tr}\big(X^{2\beta+\delta-\frac{1}{2}}\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^{\frac{1}{2}-\delta}\big)&\\ =\operatorname{Tr}\big(X^\beta\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^\beta\big).&\end{aligned}\tag{6.8}$$

This rearrangement is not trivial since a priori $X^\beta\cdot(D_\mu+z^2)^{-1}\cdot X^{2\alpha}\cdot(D_\mu+z^2)^{-1}\cdot X^\beta$ need not be trace class.

Lemma 6.3. *For $W\in L^1[a,\infty)$ and $\varepsilon\ge 0$ we have*

$$\|X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-\frac{1}{2}}\|_{HS}=O(z^{-\frac{\min(1,\varepsilon)}{2}}),\quad z\to\infty.\tag{6.9}$$

Proof. For $\varepsilon>1$ we estimate,

$$\|X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-\frac{1}{2}}\|_{HS}\le\||W|^{\frac{1}{2}}(D_\mu+z^2)^{-\frac{1}{2}}\|_{HS},\tag{6.10}$$

hence it suffices to prove the Lemma for $0\le\varepsilon\le 1$.

The same argument as in the proof of Proposition 6.2 shows that

$$X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-1}|W|^{\frac{1}{2}}X^{1-\frac{\varepsilon}{2}}\tag{6.11}$$

is trace class, hence $X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-\frac{1}{2}}$ is a Hilbert-Schmidt operator. We have

$$\begin{aligned}\|X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-\frac{1}{2}}\|_{HS}^2&=\operatorname{Tr}\left(X^{1-\frac{\varepsilon}{2}}|W|^{\frac{1}{2}}(D_\mu+z^2)^{-1}|W|^{\frac{1}{2}}X^{1-\frac{\varepsilon}{2}}\right)\\&=\int_a^\infty x^{2-\varepsilon}|W(x)|G_z(x,x)dx\\&=\frac{z}{\mu}\int_{\frac{\mu a}{z}}^\infty\left(\frac{xz}{\mu}\right)^{2-\varepsilon}\left|W\left(\frac{xz}{\mu}\right)\right|G_z\left(\frac{xz}{\mu},\frac{xz}{\mu}\right)dx.\end{aligned}\tag{6.12}$$

Before we estimate the integral note that, for all $x \geq 0$,

$$\frac{x^{1-\varepsilon}}{(1+x^2)^{\frac{1}{2}}} \leq 1. \tag{6.13}$$

According to Eq. (4.6) we split the integral into a sum and estimate each summand separately. For the first integral, we use the asymptotic expansion Eq. (3.15),

$$\begin{aligned}\frac{z^{2-\varepsilon}}{\mu^{2-\varepsilon}} \int_{\frac{\mu a}{z}}^{\infty} x^{1-\varepsilon} \left| W\left(\frac{xz}{\mu}\right)\right| I_z(xz)K_z(xz)dx &\leq C_1 \frac{z^{1-\varepsilon}}{\mu^{2-\varepsilon}} \int_{\frac{\mu a}{z}}^{\infty} \frac{x^{1-\varepsilon}}{(1+x^2)^{\frac{1}{2}}} \left| W\left(\frac{xz}{\mu}\right)\right| dx \\ &\leq C_2 z^{-\varepsilon}.\end{aligned} \tag{6.14}$$

For the second integral we use the asymptotic expansions Eq. (3.7), (3.8) and (3.13), and obtain

$$\frac{z^{2-\varepsilon}}{\mu^{2-\varepsilon}} \frac{I_z(\mu)}{K_z(\mu)} \int_{\frac{\mu a}{z}}^{\infty} x^{1-\varepsilon} \left| W\left(\frac{xz}{\mu}\right)\right| K_z^2(xz)dx \leq C_3 \frac{z^{1-\varepsilon}}{\mu^{2-\varepsilon}} \int_{\frac{\mu a}{z}}^{\infty} \left| W\left(\frac{xz}{\mu}\right)\right| dx \leq C_4 z^{-\varepsilon}. \tag{6.15}$$

□

Theorem 6.4. *Let $W \in L^1[a,\infty)$ and $\varepsilon > 0$. Let R_a be a boundary condition (Dirichlet or generalized Neumann) at a. By slight abuse of notation let $D_\mu := D_\mu(R_a)$. Then the resolvent, $(D_\mu + X^{2-\varepsilon}W + z^2)^{-1}$, is trace class and*

$$\|(D_\mu + X^{2-\varepsilon}W + z^2)^{-1} - (D_\mu + z^2)^{-1}\|_{tr} = O(z^{-2-\min(1,\varepsilon)}), \qquad \text{as } z \to \infty. \tag{6.16}$$

Consequently,

$$\begin{aligned}\mathrm{Tr}(D_\mu + X^{2-\varepsilon}W + z^2)^{-1} &= \mathrm{Tr}(D_\mu + z^2)^{-1} + O(z^{-2-\min(1,\varepsilon)}) \\ &= b_0 \cdot z^{-1} \cdot \log z + a_0 \cdot z^{-1} + a_1 \cdot z^{-2} + O(z^{-2-\min(1,\varepsilon)}), \quad \text{as } z \to \infty,\end{aligned} \tag{6.17}$$

where the constants b_0, a_0, a_1 are those of Prop. 4.2. Consequently, the zeta-regularized determinant of the operator $D_\mu + X^{2-\varepsilon}W + \nu^2$ is well-defined as explained in the Introduction Sec. 1 for any $\nu \geq 0$.

Proof. We apply the Neumann series, a priori formally,

$$\begin{aligned}&(D_\mu + X^{2-\varepsilon}W + z^2)^{-1} - (D_\mu + z^2)^{-1} \\ &= \sum_{n=1}^{\infty} (-1)^n (D_\mu + z^2)^{-\frac{1}{2}} \left[(D_\mu + z^2)^{-\frac{1}{2}} X^{1-\frac{\varepsilon}{2}} W X^{1-\frac{\varepsilon}{2}} (D_\mu + z^2)^{-\frac{1}{2}}\right]^n (D_\mu + z^2)^{-\frac{1}{2}} \\ &= \sum_{n=1}^{\infty} (-1)^n E_n(z).\end{aligned} \tag{6.18}$$

We estimate each summand using Lemma 6.3,

$$\begin{aligned}\|E_n(z)\|_{tr} &\le \|(D_\mu+z^2)^{-\frac12}\|_{L^2}^2\|(D_\mu+z)^{-\frac12}X^{1-\frac{\varepsilon}{2}}|W|X^{1-\frac{\varepsilon}{2}}(D_\mu+z^2)^{-\frac12}\|_{tr}^n\\ &\le \|(D_\mu+z^2)^{-1}\|_{L^2}\|(D_\mu+z^2)^{-\frac12}X^{1-\frac{\varepsilon}{2}}|W|^{\frac12}\|_{HS}^n\||W|^{\frac12}X^{1-\frac{\varepsilon}{2}}(D_\mu+z^2)^{-\frac12}\|_{HS}^n\\ &\le Cz^{-2}\|(D_\mu+z^2)^{-\frac12}X^{1-\frac{\varepsilon}{2}}|W|^{\frac12}\|_{HS}^n\||W|^{\frac12}X^{1-\frac{\varepsilon}{2}}(D_\mu+z^2)^{-\frac12}\|_{HS}^n\\ &\le C\cdot z^{-2}\cdot\left(z^{-\min(1,\varepsilon)}\right)^n.\end{aligned} \tag{6.19}$$

This shows that for z large enough the Neumann series indeed converges in the trace norm and that Eq. (6.16) holds. □

7. Variation formula and the determinant of the perturbed operator

In this section we prove our main result, Theorem 1.2. We generalized the Theorem 4.3 in next theorem.

Theorem 7.1. *Let W_t be a differentiable family of functions in $L^1[a,\infty)$ and $\varepsilon>0$. As a notational convenience we take $t_0=0$. Fix $\nu\ge 0$ and consider the perturbed operator*

$$H_t := D_\mu + X^{2-\varepsilon}W_t. \tag{7.1}$$

Furthermore, let R_a be a boundary condition at a. Let ψ_t, ϕ_t be a fundamental system of solutions to the equation $(H_t+\nu^2)u=0$ where ϕ_t is normalized in the sense of Eq. (1.4) *and ψ_t satisfies*

$$\lim_{x\to\infty}\psi_t(x)\sqrt{x}\,K_\nu(\mu x)^{-1}=1. \tag{7.2}$$

Assume that H_{t_0} is invertible. Then we have for the variation of the zeta-regularized determinant

$$\frac{d}{dt}\log\det_\zeta(H_t+\nu^2)\big|_{t=0}=\frac{d}{dt}\log\big(x^2\cdot W(\psi_t,\phi_t)(x)\big)\big|_{t=0}. \tag{7.3}$$

This theorem contains Theorem 4.3 as a special case; just put $W_t=t^2X^{-3/2}$, $t=z$, $\nu=0$, and $\varepsilon=1/2$.

Proof. This theorem is a consequence of Proposition 2.3, Theorem 5.2, Corollary 5.5, Lemma 6.3 and Theorem 6.4. Firstly, by Theorem 6.4 the zeta-determinant is defined for $H_t+\nu^2$ for any t and $\nu\ge0$, and the difference,

$$\log\det_\zeta(H_t+\nu^2)-\log\det_\zeta(H_0+\nu^2)=-2\int_\nu^\infty z\Big(\mathrm{Tr}(H_t+z^2)^{-1}-\mathrm{Tr}(H_0+z^2)^{-1}\Big)dz, \tag{7.4}$$

is well-defined by Eq. (6.16). Moreover, Theorem 6.4 shows that the map $t\mapsto(H_t+z^2)^{-1}$ is differentiable as a map into the space of trace class operators and hence

$$\begin{aligned}\frac{d}{dt}z\,\mathrm{Tr}(H_t+z^2)^{-1}\big|_{t=0}&=z\,\mathrm{Tr}\big((H_t+z^2)^{-1}(\partial_tW_t)(H_t+z^2)^{-1}\big)\big|_{t=0}\\ &=-\frac12\frac{d}{dz}\mathrm{Tr}\Big((\partial_tW_t)(H_t+z^2)^{-1}\Big)\big|_{t=0},\end{aligned} \tag{7.5}$$

and follows from the proof of Lemma 6.3 that the latter $O(z^{-2-\min(1,\varepsilon)})$ as $t \to \infty$ is locally uniformly in t. Therefore by Dominated Convergence, we may differentiate under the integral and find

$$\frac{d}{dt} \log \det_\zeta(H_t + \nu^2)\big|_{t=0} = \operatorname{Tr}((\partial_t W_t)(H_t + \nu^2)^{-1})\big|_{t=0}. \tag{7.6}$$

By Theorem 5.2, there exists a fundamental system of solutions to $(H_t + \nu^2)u = 0$ given by $h_{1,t}$ and $h_{2,t}$, such that $h_{1,t}$ satisfy Eq. (7.2). Consider ϕ_t a linear combination of $h_{1,t}$ and $h_{2,t}$ normalized as Eq. (1.4) and $\psi_t = h_{1,t}$. Now the result follows from Corollary 5.5 and Proposition 2.3. □

Proof of Theorem 1.2. We conclude the section with a proof of our main result, Theorem 1.2. We conclude by Theorem 7.1 for any $\gamma < 2$, potential $V \in X^\gamma L^1[a,\infty)$ and boundary conditions R_a at $x = a$

$$\det_\zeta(H(R_a) + \nu^2) = c_0(a,\mu) \cdot a^2 \cdot W(\psi,\phi)(a). \tag{7.7}$$

where the coefficient does not depend on V. The same argument as in Theorem 4.4 shows that $c_0(a,\mu)$ does not depend on the boundary condition. In particular the equality holds in the special case of a trivial potential $V \equiv 0$ and hence by Theorem 4.4

$$c_0(a,\mu) = \frac{\det_\zeta\left(H(R_a) + \nu^2\right)}{a^2 \cdot W(\psi,\phi)(a)} = \sqrt{\frac{2}{\pi}}. \tag{7.8}$$

This completes the proof[1]. □

References

[AbSt92] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, Inc., New York, 1992, Reprint of the 1972 edition. MR 1225604

[Ara83] J. Aramaki, *Complex powers of a class of pseudodifferential operators and their applications*, Hokkaido Math. J. **12** (1983), no. 2, 199–225. MR 707571

[BFK95] D. Burghelea, L. Friedlander, and T. Kappeler, *On the determinant of elliptic boundary value problems on a line segment*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3027–3038. MR 1301012 (95m:58131)

[Che79] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 5, 2103–2106. MR 530173 (80k:58098)

[Che83] ———, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657 (1984). MR 730920 (85d:58083)

[Fri89] L. Friedlander, *The asymptotics of the determinant function for a class of operators*, Proc. Amer. Math. Soc. **107** (1989), no. 1, 169–178. MR 975642

[GrRy15] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, eighth ed., Elsevier/Academic Press, Amsterdam, 2015, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010]. MR 3307944

[HaSp11] L. Hartmann and M. Spreafico, *The analytic torsion of a cone over an odd dimensional manifold*, J. Geom. Phys. **61** (2011), no. 3, 624–657. `arXiv:1001.4755 [math.DG]`, MR 2763625

[1] A posteiori $c(a,\mu)$ does not depend on a and μ either.

[HaSp16] ———, *On the Cheeger–Müller theorem for an even-dimensional cone*, St. Petersburg Math. J. **27** (2016), no. 1, 137–154. MR 3443271

[HaSu78] P. R. Halmos and V. S. Sunder, *Bounded integral operators on L^2 spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 96, Springer-Verlag, Berlin-New York, 1978. MR 517709

[Les98] M. Lesch, *Determinants of regular singular Sturm-Liouville operators*, Math. Nachr. **194** (1998), 139–170. arXiv:math/9902114 [math.DG], MR 1653090 (99j:58220)

[LeSm77] S. Levit and U. Smilansky, *A theorem on infinite products of eigenvalues of Sturm-Liouville type operators*, Proc. Amer. Math. Soc. **65** (1977), no. 2, 299–302. MR 0457836

[LeTo98] M. Lesch and J. Tolksdorf, *On the determinant of one-dimensional elliptic boundary value problems*, Comm. Math. Phys. **193** (1998), no. 3, 643–660. arXiv:dg-ga/9707022, MR 1624851 (2000b:58060)

[LeVe11] M. Lesch and B. Vertman, *Regular singular Sturm–Liouville operators and their zeta-determinants*, J. Funct. Anal. **261** (2011), no. 2, 408–450. arXiv:1005.0368 [math.SP], MR 2793118

[LeVe15] ———, *Regularizing infinite sums of zeta-determinants*, Math. Ann. **361** (2015), no. 3-4, 835–862. arXiv:1306.0780 [math.SP], MR 3319550

[LMP12] M. Lesch, H. Moscovici, and M. J. Pflaum, *Connes-Chern character for manifolds with boundary and eta cochains*, Mem. Amer. Math. Soc. **220** (2012), no. 1036, viii+92. arXiv:0912.0194 [math.OA], MR 3025890

[MeMe83] R. Melrose and G. Mendoza, *Elliptic operators of totally characteristic type*, MSRI preprint, 1983.

[Mül83] W. Müller, *Spectral theory for Riemannian manifolds with cusps and a related trace formula*, Math. Nachr. **111** (1983), 197–288. MR 725778

[MüVe14] W. Müller and B. Vertman, *The metric anomaly of analytic torsion on manifolds with conical singularities*, Comm. Partial Differential Equations **39** (2014), no. 1, 146–191. arXiv:1004.2067 [math.SP], MR 3169782

[Olv97] F. W. J. Olver, *Asymptotics and special functions*, AKP Classics, A K Peters, Ltd., Wellesley, MA, 1997, Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)]. MR 1429619

[ReSi179] M. Reed and B. Simon, *Methods of modern mathematical physics. III*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979, Scattering theory. MR 529429

[Shu01] M. A. Shubin, *Pseudodifferential operators and spectral theory*, second ed., Springer-Verlag, Berlin, 2001, Translated from the 1978 Russian original by Stig I. Andersson. MR 1852334

[SiHo11] A. Sidi and P. E. Hoggan, *Asymptotics of modified Bessel functions of high order*, Int. J. Pure Appl. Math. **71** (2011), no. 3, 481–498. MR 2883984

[Ver14] B. Vertman, *Cheeger–Müller theorem on manifolds with cusps*, arXiv:1411.0615.

[Wat95] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition. MR 1349110

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