A characterization of singular packing subspaces with an application to limit-periodic operators

Silas L. Carvalho and César R. de Oliveira

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Abstract

A new characterization of the singular packing subspaces of general bounded self-adjoint operators is presented, which is used to show that the set of operators whose spectral measures have upper packing dimension equal to one is a G_δ (in suitable metric spaces). As an application, it is proven that, generically (in space of continuous sampling functions), spectral measures of the limit-periodic Schrödinger operators have upper packing dimensions equal to one. Consequently, in a generic set, these operators are quasiballistic.

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Running head: Singular packing subspaces

1 Introduction and results

A study of packing continuity and singularity of bounded self-adjoint operators is performed. Our main goals here are to present a dynamical characterization of singular α-packing subspaces (Theorem 3.9), and use this result (along with the well known equivalence between strong convergence and strong dynamical convergence of operators) to prove, for some metric space of self-adjoint operators, that the set of operators whose spectral measures have upper packing dimension equal to 1 is a G_δ set (Theorem 4.2). The dynamical characterization will be obtained in Section 3 through the notion of uniformly α-Hölder singular measures and Lemma 3.4 (which is a "singular version" of Strichartz's Theorem [15]).

To put our work into perspective, we mention that, although important, it is not always easy to present dynamical characterizations of spectral subspaces of a self-adjoint operator T. For a vector ξ, denote the so-called return probability at time t by $p_\xi(t) := \left|\langle \xi, e^{-itT}\xi\rangle\right|^2$, clearly a dynamical quantity. It is well known (see, for instance, Theorem 13.5.5 in [7]) that the absolutely continuous subspace of T is the closure of the vectors ξ so that $p_\xi \in \mathrm{L}^1(\mathbb{R})$. By using Strichartz's Theorem, recalled in Section 3, Last (Theorem 5.3 in [12]) has presented a dynamical characterization of the α-Hausdorff continuous subspace $\mathcal{H}^T_{\alpha\mathrm{Hc}}$ of T (the definition is similar to $\mathcal{H}^T_{\alpha\mathrm{Pc}}$ in Proposition 2.5) in terms of averages $\langle p_\xi\rangle(t) := \frac{1}{t}\int_0^t p_\xi(s)\,\mathrm{d}s$, that is, given $0 < \alpha < 1$, then for all $\varepsilon > 0$,

$$\text{closure}\left\{\xi \mid \sup_t t^{\alpha+\varepsilon}\langle p_\xi\rangle(t) < \infty\right\} \subset \mathcal{H}^T_{\alpha\mathrm{Hc}} \subset \text{closure}\left\{\xi \mid \sup_t t^{\alpha}\langle p_\xi\rangle(t) < \infty\right\}.$$

Here we have a parallel of this result (Theorem 3.9) for the α-packing singular subspace; for this, we have introduced an appropriate quantity in equation (3.1) and the concept of UαHS measures (see Definition 3.1).

We apply our general packing results to a class of limit-periodic operators. These are discrete one-dimensional ergodic Schrödinger operators, denoted by $H^\kappa_{g,\tau}$, acting in $l^2(\mathbb{Z})$, whose action is given by

$$(H^\kappa_{f,\sigma}\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n(\kappa)\psi_n\ , \tag{1.1}$$

with

$$V_n(\kappa) = g(\tau^n(\kappa))\,; \tag{1.2}$$

here, κ belongs to a Cantor group Ω, $\tau : \Omega \to \Omega$ is a minimal translation on Ω and $g : \Omega \to \mathbb{R}$ a continuous sampling function, i.e., $g \in \mathrm{C}(\Omega, \mathbb{R})$ with the norm of uniform convergence. For more details, see [1].

For each $\kappa \in \Omega$, let X_κ be the set of limit-periodic operators $H^\kappa_{g,\tau}$ given by (1.1) and (1.2), with metric

$$d(H^\kappa_{g,\tau}, H^\kappa_{g',\tau}) = \|g - g'\|_\infty\ . \tag{1.3}$$

We shall prove the following

Theorem 1.1 *For each $\kappa \in \Omega$, the set $C^\kappa_{1\mathrm{uPd}} := \{T \in X_\kappa \mid \sigma(T)$ is purely 1-upper packing dimensional$\}$ is generic in X_κ.*

We stress that in a recent work [5], it was proven, again for some metric spaces of self-adjoint operators, that the set of operators whose spectral measures have upper correlation dimension equal to 1 is a G_δ. Since every Borel and finite measure on $\mathbb{R}$ whose upper correlation dimension is one has upper packing dimension equal to 1, one can conclude that if the hypotheses in Theorem 4.1 in [5] are fulfilled, then Theorem 4.2 follows. However, the hypotheses in Theorem 4.2 below are weaker than in Theorem 4.1 in [5], and therefore easier to meet. In particular, we were not able to apply the method discussed in [5] to the class (1.1) of limit periodic operators, since the estimation of upper correlation dimension seems to be far from trivial in this case.

The notorious Wonderland theorem [14] gives sufficient conditions for a set of self-adjoint operators, whose spectrum is purely singular continuous, to be generic. Theorem 4.2 is another step towards a better comprehension of the typical (in a topological sense) behavior of the spectral measures of a self-adjoint operator, since it generalizes Wonderland theorem in the sense that every application of Wonderland theorem discussed in [14] for bounded operators can be extended to results about generic sets of operators whose spectral measures have upper packing dimension equal to 1 (on top of singular continuous spectrum).

The proof of Theorem 1.1 is presented in Section 5, since a previous preparation is required. Let us briefly discuss some dynamical consequences of Theorem 1.1 within the context of the unitary evolution group e^{-itT}, that is, the solution to the corresponding Schrödinger equation [7]. Regarding the dynamics generated by a self-adjoint operator T acting on $l^2(\mathbb{Z})$, the growth of the width of "quantum wave packets" is usually probed by the algebraic growth of the (time-averaged) q-moments, $q > 0$,

$$\langle M_T^q\rangle(t) := \sum_n |n|^q \, \frac{2}{t}\int_0^\infty e^{-2s/t}|\langle e^{-isT}\delta_0, \delta_n\rangle|^2 \, \mathrm{d}s \,,$$

of the position operator at time $t > 0$; such wave packets are represented here by the initial state δ_0. To describe this algebraic growth $\langle M_T^q\rangle(t) \sim t^{q\beta(q)}$ for large t, one usually considers the lower and the upper transport exponents, given respectively by

$$\beta_T^-(q) := \liminf_{t\to\infty} \frac{\ln\langle M_T^q\rangle(t)}{q\ln t}, \quad \beta_T^+(q) := \limsup_{t\to\infty} \frac{\ln\langle M_T^q\rangle(t)}{q\ln t}.$$

The following result, extracted from [9], gives basic properties of moments within the setting of bounded self-adjoint operators T acting on $l^2(\mathbb{Z})$, in particular, for bounded discrete Schrödinger operators (that is, operators whose action is given by (1.1) with bounded potentials (V_n)).

Proposition 1.2 *If T is a bounded self-adjoint operator on $l^2(\mathbb{Z})$, then*

1. *$\langle M_T^q\rangle(t)$ is well defined for all $q, t > 0$;*
2. *$\beta_T^\pm(q)$ are increasing functions of q;*
3. *$\beta_T^\pm(q) \in [0, 1]$, for all $q > 0$.*

In case $\beta_T^+(q) = 1$ (resp. $\beta_T^-(q) = 1$), for all $q > 0$, the corresponding dynamics is called *quasi-ballistic* (resp. *ballistic*). We shall make use of the general inequality (see Definition 2.2

for the description of the upper packing dimension $\dim_{\rm P}^+(\cdot)$)

$$\beta_T^+(q) \geq \dim_{\rm P}^+\left(\mu_{\delta_0}^T\right), \qquad \forall q > 0, \tag{1.4}$$

proven in [11]. It is worth mentioning that $\beta_T^-(q)$ is related to the upper Hausdorff dimension [10, 2], but we will not use them in this work. Following the discussion above and Theorem 1.1, one has

Corollary 1.3 *For every $\kappa \in \Omega$, the set $C_{\rm QB}^\kappa := \{T \in X_\kappa \mid \beta_T^+(q) = 1$ for every $q > 0\}$ is generic in X_κ.*

In Section 2, we recall suitable results on decompositions of Borel measures with respect to packing measures and dimensions, and how these components are related to pointwise scaling exponents and generalized dimensions. In Section 3, we present a singular version-like of Strichartz's Theorem [15, 12] for finite measures, which is used to give a dynamical characterization of packing singular subspaces of bounded self-adjoint operators in general separable Hilbert spaces (see Theorem 3.9). The very same arguments lead to a version for spectral measures restricted to any given subinterval of the real line.

In Section 4, we state and prove Theorem 4.2. In the last section, as previously stated, we present the proof of Theorem 1.1. It is important to emphasize that the results of Section 4 can be used to prove, for every general class of bounded operators (including classes of not necessarily Schrödinger-like operators) discussed in [14], existence of generic sets of operators whose spectral measures have upper packing dimensions equal to 1.

Some words about notation: $\mathcal{H}$ will always denote a complex separable Hilbert space. The spectrum of a self-adjoint operator T is denoted by $\sigma(T)$. μ will always indicate, unless explicitly stated, a finite nonnegative Borel measure on $\mathbb{R}$, and its restriction to a Borel set E will be denoted by $\mu_{;E}$; it is *singular* if μ and the Lebesgue measure are mutually singular; it is *supported* on a Borel set S if $\mu(\mathbb{R} \setminus S) = 0$. $\text{supp}(\mu)$ denotes the *support* of μ, that is, the complement of the largest open set B with $\mu(B) = 0$. $P^T(E)$ represents the spectral projection of T associated with the Borel set $E \subset \mathbb{R}$.

2 Basic tools

In this section, we recall important decompositions of Borel measures on $\mathbb{R}$ with respect to packing measures and dimensions, along with the corresponding spectral decompositions of self-adjoint operators. We also recollect how these decompositions are related to the upper and generalized dimensions. This discussion parallels the rather well-known corresponding Hausdorff properties.

2.1 Packing decompositions

Given a set $S \subset \mathbb{R}$ and $0 \le \alpha \le 1$, denote by $h^\alpha(S)$ its α-dimensional (exterior) Hausdorff measure and by $\dim_H(S)$ its Hausdorff dimension. Since packing measures and dimensions are not so familiar as the Hausdorff versions, we recall their definitions in what follows.

A δ-packing of an arbitrary set $S \subset \mathbb{R}$ is a countable disjoint collection $(\bar{B}(x_k; r_k))_{k\in\mathbb{N}}$ of closed intervals centered at $x_k \in S$ and radii $r_k \le \delta/2$, so with diameters at most of δ. Define $P^\alpha_\delta(S)$, $0 \le \alpha \le 1$, as

$$P^\alpha_\delta(S) = \sup\left\{ \sum_{k=1}^{\infty} (2r_k)^\alpha \mid (\bar{B}(x_k; r_k))_k \text{ is a } \delta\text{-packing of } S \right\};$$

that is, the supremum is taken over all δ-packings of S. Then, take the decreasing limit

$$P^\alpha_0(S) = \lim_{\delta \downarrow 0} P^\alpha_\delta(S)$$

as a pre-measure.

Definition 2.1 The *α-packing (exterior) measure* $P^\alpha(S)$ of S is given by

$$P^\alpha(S) := \inf\left\{ \sum_{k=1}^{\infty} P^\alpha_0(S_k) \mid S \subset \bigcup_{k=1}^{\infty} S_k \right\}.$$

The *packing dimension* of the set S, $\dim_P(S)$, is defined as the infimum of all α such that $P^\alpha(S) = 0$, which coincides with the supremum of all α so that $P^\alpha(S) = \infty$.

It is possible to show [8] that the Hausdorff and packing dimensions are related by the inequality $\dim_H(S) \le \dim_P(S)$, and this inequality is in general strict. It is also important to mention that P^α and h^α are Borel (regular) measures and, for $0 \le \alpha < 1$, they are not σ-finite; furthermore, $P^0 \equiv h^0$ and $P^1 \equiv h^1$, and they are equivalent, respectively, to counting and Lebesgue measures.

Definition 2.2 The packing *upper dimension* of μ is defined as

$$\dim^+_P(\mu) := \inf\{\dim_P(S) \mid \mu(\mathbb{R} \setminus S) = 0,\ S \text{ a Borel subset of } \mathbb{R}\}.$$

The notions of packing measures and dimensions lead to concepts of continuity and singularity of Borel measures with respect to them.

Definition 2.3 Let $\alpha \in [0, 1]$. μ is called:

1. *α-packing continuous*, denoted αPc, if $\mu(S) = 0$ for every Borel set S with $\mathrm{P}^\alpha(S) = 0$.

2. *α-packing singular*, denoted αPs, if it is supported on some Borel set S with $\mathrm{P}^\alpha(S) = 0$.

3. *α-packing dimension continuous*, denoted αPdc, if $\mu(S) = 0$ for every Borel set S with $\dim_{\mathrm{P}}(S) < \alpha$.

4. *almost α-packing dimension singular*, denoted aαPds, if it is supported on some Borel set S with $\dim_{\mathrm{P}}(S) \le \alpha$.

5. 1-*packing dimensional*, denoted 1Pd, if $\mu(S) = 0$ for any Borel set S with $\dim_{\mathrm{P}}(S) < 1$.

Proposition 2.4 *Let μ as before.*

1. *Fix $\alpha \in (0, 1]$. If μ is αPs, then it is aαPds.*

2. *Fix $\alpha \in [0, 1)$. If μ is aαPds, then it is $(\alpha + \varepsilon)$Ps for every $0 < \varepsilon \le 1 - \alpha$.*

Proposition 2.5 *Let $T : \operatorname{dom} T \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator in the Hilbert space $\mathcal{H}$, and μ_ψ^T the spectral measure of T associated with the vector $\psi \in \mathcal{H}$. Given $\alpha \in (0, 1)$, the sets*

$$\mathcal{H}^T_{\alpha\mathrm{Pc}} := \{\psi \mid \mu_\psi^T \text{ is } \alpha\mathrm{Pc}\} \quad \text{and} \quad \mathcal{H}^T_{\alpha\mathrm{Ps}} := \{\psi \mid \mu_\psi^T \text{ is } \alpha\mathrm{Ps}\}$$

are closed and mutually orthogonal subspaces of $\mathcal{H}$, which are invariant under T, and $\mathcal{H} = \mathcal{H}^T_{\alpha\mathrm{Pc}} \oplus \mathcal{H}^T_{\alpha\mathrm{Ps}}$.

Proof. The proof follows closely the proofs of the corresponding statements involving Hausdorff versions in Theorem 5.1 in [12]. □

2.2 Generalized upper dimensions

Definition 2.6 The *pointwise upper scaling exponent* of μ at $x \in \mathbb{R}$ is defined as

$$d_\mu^+(x) := \limsup_{\varepsilon \to 0} \frac{\ln \mu(B(x; \varepsilon))}{\ln \varepsilon},$$

if, for every $\varepsilon > 0$, $\mu(B(x; \varepsilon)) > 0$, and $d_\mu^+(x) := +\infty$ otherwise.

The function $x \mapsto d_\mu^+(x)$ is measurable [8]. The following results relate packing singularity properties of nonnegative finite Borel measures on $\mathbb{R}$ with their upper pointwise scaling exponent and dimensions (see [11] for details).

Proposition 2.7 *Let μ as before.*

1. *If μ is αPs, then $\mu\text{-}\operatorname{ess.sup} d_\mu^+ \le \alpha \le 1$;*

2. μ *is* aαPds *if, and only if,* μ-$\operatorname{ess.sup} d_\mu^+ \leq \alpha \leq 1$.

Proposition 2.8 *Let E be a Borel subset of $\mathbb{R}$. Then,*

$$\dim_{\rm P}^+(\mu_{;E}) = \mu_{;E}\text{-}\operatorname{ess.sup} d_{\mu_{;E}}^+ \ .$$

Item 2. in Proposition 2.7 can be restated in the following way:

Corollary 2.9 *Let E be a Borel subset of $\mathbb{R}$. $\mu_{;E}$ is* aαPds *if, and only if,* $\dim_{\rm P}^+(\mu_{;E}) \leq \alpha$.

Now we recall the definition of generalized upper dimensions of positive Borel measures and how they are connected to the upper packing dimensions.

Definition 2.10 Let μ be a positive Borel measure on $\mathbb{R}$. The *upper generalized dimensions* of μ are defined, for $q \neq 1$, as

$$D_q^+(\mu) := \limsup_{\varepsilon\to 0} \frac{\ln\left[\int [\mu(B(x;\varepsilon))]^{q-1}\, {\rm d}\mu(x)\right]}{(q-1)\ln\varepsilon} \ ,$$

with integrals taken on $\operatorname{supp}\mu$.

For all $q < 1 < s$, Proposition 4.1 in [3] gives

$$D_q^+(\mu) \geq \dim_{\rm P}^+(\mu) \geq D_s^+(\mu)\,. \tag{2.1}$$

This will be used in Section 3, particularly with $q = 1/2$.

3 Dynamical characterization of $\mathcal{H}^T_{\alpha\mathbf{Ps}}$

In Definition 3.1, we introduce a class of special measures for which a singular version-like of Strichartz's Theorem (Theorem 3.3) will be deduced (see Lemma 3.4). The arguments there will be used to prove the main result of this section, that is, Theorem 3.9. We assume, in what follows, that $0 \neq T$ represents a bounded self-adjoint operator on $\mathcal{H}$.

Definition 3.1 *Let $\alpha \in [0,1]$ and μ be a positive Borel measure on $\mathbb{R}$. We say that μ is uniformly α-Hölder singular (UαHS) if there exist positive constants C and r_0, with $r_0 < 1$, such that, for all $0 < r < r_0$ and for μ a.e. x, $\mu(B(x;r)) \geq Cr^\alpha$.*

Besides the application to limit-periodic operators ahead, the next results have interest on their own (as well as the results in Section 4). Next a description of α-packing singular measures in terms of UαHS ones.

Theorem 3.2 *Let μ be a positive Borel measure on $\mathbb{R}$ and $\alpha \in (0,1)$. If μ is αPs, then, for every $0 < \varepsilon \le 1-\alpha$ and every $\delta > 0$, there exist a Borel set $S_\delta = S \subset \mathbb{R}$ such that $\mu(S^c) < \delta$ and positive constants C and $r_0 < 1$ such that, for each $0 < r < r_0$ and for each $x \in S$, $\mu(B(x;r)) \ge Cr^\alpha$. Conversely, if, for every $\delta > 0$, there exist mutually singular Borel measures μ_1^δ and μ_2^δ such that $\mu = \mu_1^\delta + \mu_2^\delta$, with μ_1^δ UαHS and $\mu_2^\delta(\mathbb{R}) < \delta$, then, for every $0 < \varepsilon \le 1-\alpha$, μ is $(\alpha+\varepsilon)$Ps.*

Proof. Suppose that, for every $\delta > 0$, $\mu = \mu_1^\delta + \mu_2^\delta$, with μ_1^δ and μ_2^δ satisfying the properties in the statement of the theorem. We must show, for every $0 < \varepsilon \le 1-\alpha$, that μ is $(\alpha+\varepsilon)$Ps; by Propositions 2.4 and 2.7, this is equivalent to showing that $\mu\text{-}\operatorname{ess.sup} d_\mu^+ \le \alpha$.

Let us assume, nonetheless, that $\mu\text{-}\operatorname{ess.sup} d_\mu^+ > \alpha$. Thus, there is a Borel set, say B, of positive μ-measure such that $d_\mu^+(x) > \alpha$ for every $x \in B$. Fix $0 < \zeta < \mu(B)$. By hypothesis, there is a Borel set E (which may depend on ζ) such that μ can be decomposed as $\mu = \mu_1^\zeta + \mu_2^\zeta$, with $\mu_1^\zeta(\cdot) := \mu(E \cap \cdot)$ UαHS and $\mu_2^\zeta(\cdot) := \mu(E^c \cap \cdot)$, with $\mu_2^\zeta(\mathbb{R}) < \zeta$.

By Definition 3.1, there are constants $C > 0$ and $0 < r_0 < 1$ such that, for every $0 < r < r_0$ and every $x \in E \setminus D$ (D a set of zero μ_1^ζ-measure), $\mu_1^\zeta(B(x;r)) \ge Cr^\alpha$. Now, since $\ln \mu(\cdot) \ge \ln \mu_1^\zeta(\cdot)$,

$$d_\mu^+(x) = \limsup_{r\downarrow 0} \frac{\ln \mu(B(x;r))}{\ln r} \le \limsup_{r\downarrow 0} \frac{\ln \mu_1^\zeta(B(x;r))}{\ln r} \le \limsup_{r\downarrow 0} \frac{\ln C}{\ln r} + \alpha = \alpha,$$

and $d_\mu^+(x) \le \alpha$ for every $x \in E \setminus D$. But then, $(E \setminus D)^c \supset B$, which implies that $\zeta > \mu_2^\zeta(E^c \cup D) = \mu_2^\zeta(E^c \cup D) + \mu_1^\zeta(E^c \cup D) = \mu(E^c \cup D) \ge \mu(B)$, a contradiction with $\mu(B) > \zeta$. Thus, $\mu\text{-}\operatorname{ess.sup} d_\mu^+ \le \alpha$, and we are done.

Conversely, if μ is αPs, then, by Proposition 2.7, $\mu\text{-}\operatorname{ess.sup} d_\mu^+ \le \alpha$; that is,

$$\limsup_{r\downarrow 0} \frac{\ln \mu(B(x;r))}{\ln r} \le \alpha$$

for μ a.e. x. Since the sequence $(f_r(x))$ of measurable functions

$$f_r(x) := \sup_{r' \le r} \frac{\ln \mu(B(x;r'))}{\ln r'}$$

converges to $d_\mu^+(x)$, Egoroff's Theorem implies that given an arbitrary $\delta > 0$, there is a Borel set S such that $\mu(S^c) < \delta$ and $f_r(x)$ converges uniformly on S to $d_\mu^+(x)$, as $r \downarrow 0$. But then, given an arbitrary $0 < \varepsilon \le 1-\alpha$, there is a $0 < r_0 < 1$ such that, for every $0 < r < r_0$ and all $x \in S$, $\ln \mu(B(x;r))/\ln r < \alpha + \varepsilon$; that is, $\mu(B(x;r)) > r^{\alpha+\varepsilon}$, for all $x \in S$. □

Now we introduce another quantity that has proven useful. For a finite and positive Borel measure μ on $\mathbb{R}$ and every $t \in \mathbb{R}$, write

$$\Xi_\mu(t) := \int d\mu(x) \left(\int d\mu(y)\, e^{-(x-y)^2 t^2/4} \right)^{-1/2}. \qquad (3.1)$$

If the measure μ is a spectral measure μ_ψ^T, we denote $\Xi_\mu(t)$ by $\Xi_\psi^T(t)$.

Recall that μ is uniformly α-Hölder continuous [12] if there are positive and finite constants C and r_0, so that for each $0 < r < r_0$ and for μ a.e. x, $\mu(B(x;r)) \le Cr^\alpha$. The following result is known as Strichartz's Theorem (it is, in fact, an adapted version of the Theorem presented in [12] for $f \equiv 1 \in \mathrm{L}^2(\mathbb{R}; \mathrm{d}\mu)$; see also [15] for the original result).

Theorem 3.3 *Let μ be a finite uniformly α-Hölder continuous measure, and for each $s > 0$, denote*

$$\widehat{\mu}(s) := \int e^{-isx}\, \mathrm{d}\mu(x).$$

Then, there exist constants $\tilde{D}$ and $t_0 > 0$, depending only on μ, such that, for any $t > t_0$,

$$\frac{1}{t}\int_0^t |\widehat{\mu}(s)|^2\, \mathrm{d}s \le \tilde{D} t^{-\alpha}. \tag{3.2}$$

The proof of Theorem 3.3 in [12], after some preparation, essentially consists of showing that there exist constants $\tilde{D}$ and $t_0 > 0$ so that

$$\int \mathrm{d}\mu(x) \left(\int \mathrm{d}\mu(y)\, e^{-(x-y)^2 t^2/4} \right) \le \tilde{D}/t^\alpha, \qquad t > t_0. \tag{3.3}$$

In such proofs, in case $\mu = \mu_\psi^T$, the parameter "t" comes from the time evolution $e^{-itT}\psi$ and the left hand side of (3.2) coincides with the average return probability $\langle p_\psi \rangle(t)$, so one may look at $\Xi_\mu(t)$ as a dynamical quantity. Equation (3.3), related to Hausdorff continuity, was our main motivation to introduce $\Xi_\mu(t)$, and we have got the following singular version of this result; although simple, it will be very useful ahead.

Lemma 3.4 *Let μ be a finite positive Borel measure on $\mathbb{R}$ and UαHS for some $\alpha \in [0,1]$. Then, there exist finite constants $D > 0$ and $t_0 > 1$ such that, for every $t > t_0$,*

$$\Xi_\mu(t) \le \mu(\mathbb{R})\, D\, t^{\alpha/2}\,.$$

In case of spectral measures μ_ψ^T, one has $\Xi_\psi^T(t) \le \|\psi\|^2\, D\, t^{\alpha/2}$.

Proof. Since μ is UαHS, there are positive constants C and r_0, with $r_0 < 1$, such that, for every $0 < r < r_0$ and μ a.e. y, $\mu(B(y;r)) \ge Cr^\alpha$. Thus, by taking $t_0 \equiv 1/r_0$, it follows, for every $t > t_0$ and every $x \in \mathbb{R}$, that

$$\begin{aligned} \int \mathrm{d}\mu(y)\, e^{-(x-y)^2 t^2/4} &= \sum_{n\ge 0} \int_{n/t \le |x-y| < (n+1)/t} \mathrm{d}\mu(y)\, e^{-(x-y)^2 t^2/4} \\ &\ge 2Ct^{-\alpha} \sum_{n\ge 0} e^{-(n+1)^2/4}\,. \end{aligned} \tag{3.4}$$

Finally, by letting $D \equiv \left(2C \sum_{n\geq 0} e^{-(n+1)^2/4}\right)^{-1/2}$, we obtain

$$\Xi_\mu(t) = \int \mathrm{d}\mu(x) \left(\int \mathrm{d}\mu(y)\, e^{-(x-y)^2 t^2/4}\right)^{-1/2} \leq \mu(\mathbb{R})\, D\, t^{\alpha/2}\,.$$

In case of $\mu = \mu_\psi^T$, just recall that $\mu_\psi^T(\mathbb{R}) = \|\psi\|^2$. □

Proposition 3.5 *Let T be a bounded self-adjoint operator on $\mathcal{H}$ and $\alpha \in (0,1)$. Then, for every $0 < \varepsilon \leq 1-\alpha$,*

$$\mathcal{H}^T_{\alpha\mathrm{Ps}} \setminus \{0\} \subset \{\psi \mid \limsup_{t\to\infty} t^{-(\alpha+\varepsilon)/2}\, \Xi^T_\psi(t) < \infty\}.$$

Proof. Suppose that $\psi \in \mathcal{H}^T_{\alpha\mathrm{Ps}} \setminus \{0\}$, that is, that μ_ψ^T is positive and αPs. By Theorem 3.2, it follows, for every $0 < \varepsilon \leq 1-\alpha$ and every $\delta > 0$, that there exist $S \subset \mathbb{R}$ such that $\mu(S^c) < \delta$ and positive constants C and $r_0 < 1$ such that, for each $0 < r < r_0$ and for each $x \in S$, $\mu(B(x;r)) \geq Cr^\alpha$. Since $e^{-(x-y)^2t^2/4}$ is positive, one has, for every $x, t \in \mathbb{R}$,

$$0 < \int_S \mathrm{d}\mu^T_\psi(y)\, e^{-(x-y)^2t^2/4} \leq \int \mathrm{d}\mu^T_\psi(y)\, e^{-(x-y)^2t^2/4} < \infty.$$

Thus, using the same reasoning presented in the proof of Lemma 3.4, one has

$$\Xi^T_\psi(t) \leq \int \mathrm{d}\mu^T_\psi(x) \left(\int_S \mathrm{d}\mu^T_\psi(y)\, e^{-(x-y)^2t^2/4}\right)^{-1/2} \leq \|\psi\|^2\, D\, t^{(\alpha+\varepsilon)/2},$$

for some finite D and large t; relation (3.4) and the identity $\mu^T_\psi(\mathbb{R}) = \|\psi\|^2$ were used in the last inequality. □

Lemma 3.6 *Let T be a bounded self-adjoint operator on $\mathcal{H}$ and $\alpha \in (0,1)$. Then, for each $0 < \varepsilon \leq \min\{\alpha, 1-\alpha\}$, one has*

$$\begin{aligned}\left\{\psi \mid \limsup_{t\to\infty} t^{-(\alpha-\varepsilon)/2}\, \Xi^T_\psi(t) < \infty\right\} \quad &\subset \quad \left\{\psi \mid D^+_{1/2}(\mu^T_\psi) \leq \alpha - \varepsilon/2\right\} \\ &\subset \quad \left\{\psi \mid \limsup_{t\to\infty} t^{-(\alpha+\varepsilon)/2}\, \Xi^T_\psi(t) < \infty\right\}.\end{aligned}$$

Proof. Since, by hypothesis, T is bounded, μ^T_ψ has compact support. Hence, the result is immediate from

$$\limsup_{t\to\infty} \frac{\ln \Xi^T_\psi(t)}{\ln t} = \frac{1}{2} D^+_{1/2}(\mu^T_\psi)\,, \tag{3.5}$$

proved in Lemma 4.3 in [4] (note the different notation for Ξ^T_ψ in [4]). □

Remark 3.7 The hypothesis that the operator T is bounded can be dropped as soon as one verifies (3.5). See [4] for a discussion about the validity of (3.5) in more general situations.

Proposition 3.8 *Let $\alpha \in (0,1)$. Then, for every $0 < \varepsilon \le \alpha$,*

$$\{\psi \mid \limsup_{t\to\infty} t^{-(\alpha-\varepsilon)/2}\, \Xi_\psi^T(t) < \infty\} \subset \mathcal{H}_{\alpha\mathrm{Ps}}^T \setminus \{0\}.$$

Proof. Fix $0 < \varepsilon \le \alpha$. If ψ is such that $\limsup_{t\to\infty} t^{-(\alpha-\varepsilon)/2}\, \Xi_\psi^T(t) < \infty$, one has, from Lemma 3.6, that $D_{1/2}^+(\mu_\psi^T) \le \alpha - \varepsilon/2$. By inequalities (2.1), $D_{1/2}^+(\mu_\psi^T) \ge \dim_{\mathrm{P}}^+(\mu_\psi^T)$, and therefore, $\dim_{\mathrm{P}}^+(\mu_\psi^T) \le \alpha - \varepsilon/2$; consequently, it follows from Corollary 2.9 that μ_ψ^T is a$(\alpha - \varepsilon/2)$Pds and so it is αPs, by Proposition 2.4. □

By combining Propositions 3.5 and 3.8, we obtain the following characterization of the α-packing singular subspace.

Theorem 3.9 *Let T be a bounded self-adjoint operator on $\mathcal{H}$. If $\alpha \in (0,1)$, then, for every $0 < \varepsilon \le \min\{\alpha, 1-\alpha\}$,*

$$\Big\{\psi \mid \limsup_{t\to\infty} t^{-(\alpha-\varepsilon)/2}\, \Xi_\psi^T(t) < \infty\Big\} \subset \mathcal{H}_{\alpha\mathrm{Ps}}^T \setminus \{0\} \subset \Big\{\psi \mid \limsup_{t\to\infty} t^{-(\alpha+\varepsilon)/2}\, \Xi_\psi^T(t) < \infty\Big\}.$$

4 Generic quasiballistic and upper packing sets

Let (X,d) be a complete metric space of bounded self-adjoint operators, acting on the infinite-dimensional separable Hilbert space $\mathcal{H}$, such that the metric d convergence implies strong convergence of operators. We denote its elements by T. In order to obtain the main results of this section (i.e., Propositions 4.3 and 4.5), we will prove, for each fixed vector $\psi \in \mathcal{H}$, that the set

$$C_{1\mathrm{uPd}}^{\psi;(a,b)} := \Big\{T \in X \mid \dim_{\mathrm{P}}^+(\mu_{\psi;(a,b)}^T) = 1\Big\}$$

is a G_δ set in X. Recall that here, $\mu_{\psi;(a,b)}^T$ denotes the restriction of μ_ψ^T to the open interval (a,b), $-\infty \le a < b \le +\infty$. If $(a,b) = \mathbb{R}$, we simply denote $C_{1\mathrm{uPd}}^{\psi;(a,b)}$ by $C_{1\mathrm{uPd}}^{\psi}$.

Lemma 4.1 *For each $0 \ne \psi \in \mathcal{H}$, one has*

$$C_{1\mathrm{uPd}}^{\psi} = \bigcap_{l=1}^{\infty} \bigcap_{k=1}^{\infty} A_{1-1/(2l)-1/(2k)}^{\psi}\,, \tag{4.1}$$

where, for every $\alpha > 0$,

$$A_\alpha^\psi := \bigcap_{n=0}^{\infty} \left\{T \in X \mid \text{ for each } m,\ \exists\, t > m \text{ with } t^{-\alpha/2}\, \Xi_\psi^T(t) > n\right\}.$$

Proof. Fix $\alpha \in (0,1)$ and let $\mathcal{H}_{\mathrm{a}\alpha\mathrm{Pds}}^T := \{\zeta \in \mathcal{H} \mid \mu_\zeta^T \text{ is a}\alpha\text{Pds}\}$; by Corollary 2.9,

$$\mathcal{H}_{\mathrm{a}\alpha\mathrm{Pds}}^T = \{\zeta \mid \dim_{\mathrm{P}}^+(\mu_\zeta^T) \le \alpha\}\,.$$

By Proposition 2.4, one has the inclusions $\mathcal{H}^T_{\alpha\mathrm{Ps}} \subset \mathcal{H}^T_{\mathrm{a}\alpha\mathrm{Pds}} \subset \mathcal{H}^T_{(\alpha+\varepsilon)\mathrm{Ps}}$; then, by Theorem 3.9, it follows that, for each $T \in X$ and each $0 < \varepsilon \le \min\{\alpha, (1-\alpha)/2\}$,

$$\big\{\zeta \mid \limsup_{t\to\infty} t^{-(\alpha-\varepsilon)/2}\, \Xi^T_\zeta(t) < \infty\big\} \subset \mathcal{H}^T_{\mathrm{a}\alpha\mathrm{Pds}} \setminus \{0\} \subset \big\{\zeta \mid \limsup_{t\to\infty} t^{-(\alpha+2\varepsilon)/2}\, \Xi^T_\zeta(t) < \infty\big\}.$$

Hence, for fixed $0 \ne \psi \in \mathcal{H}$ and $0 < \varepsilon \le \min\{\alpha, (1-\alpha)/2\}$, one has

$$\bigcap_{n=0}^{\infty}\bigcap_{m=0}^{\infty}\bigcup_{t>m}\Big\{T \in X \mid t^{-(\alpha+2\varepsilon)/2}\, \Xi^T_\psi(t) > n\Big\} \subset \big\{T \in X \mid \dim^+_{\mathrm{P}}(\mu^T_\psi) > \alpha\big\}$$

$$\subset \bigcap_{n=0}^{\infty}\bigcap_{m=0}^{\infty}\bigcup_{t>m}\Big\{T \in X \mid t^{-(\alpha-\varepsilon)/2}\, \Xi^T_\psi(t) > n\Big\};$$

that is,

$$A^\psi_{\alpha+2\varepsilon} \subset \{T \in X \mid \dim^+_{\mathrm{P}}(\mu^T_\psi) > \alpha\} \subset A^\psi_{\alpha-\varepsilon}\,.$$

Finally, by replacing α by $\alpha - 3\varepsilon$ and taking $\varepsilon = 1/(8k)$, $k \ge 1$, and $\alpha = 1 - 1/(2l)$, $l \ge 1$, one obtains

$$\bigcap_{l=1}^{\infty}\bigcap_{k=1}^{\infty} A^\psi_{1-1/(2l)-1/(8k)} \subset C^\psi_{1\mathrm{uPd}} \subset \bigcap_{l=1}^{\infty}\bigcap_{k=1}^{\infty} A^\psi_{1-1/(2l)-1/(2k)}\,,$$

since $C^\psi_{1\mathrm{uPd}} = \bigcap_{l=1}^\infty \bigcap_{k=1}^\infty \{T \in X \mid \dim^+_{\mathrm{P}}(\mu^T_\psi) > 1 - 1/(2l) - 3/(8k)\}$. □

We remark that Lemma 4.1 holds true for restrictions to intervals (a, b). The choice $(a, b) = \mathbb{R}$ was just for simplicity. However, we will keep the interval in Theorem 4.2 to deal with some subtleties there.

Theorem 4.2 *Let $-\infty \le a < b \le +\infty$ and $\psi \in \mathcal{H}$. Then, the set $C^{\psi;(a,b)}_{1\mathrm{uPd}}$ is a G_δ set in X.*

Proof. If, for every $T \in X$, $\mu^T_\psi((a, b) \cap \cdot) = 0$ (which is the case when, for every $T \in X$, $\mathrm{supp}(\mu^T_\psi) \cap (a, b) = \emptyset$), then $\dim^+_{\mathrm{P}}(\mu^T_{\psi;(a,b)}) = 0$ and $C^{\psi;(a,b)}_{1\mathrm{uPd}} = \emptyset$ is a G_δ set in X.

Otherwise, suppose that $\xi = \xi(T, \psi, (a, b)) := P^T((a, b))\psi \ne 0$ for some $T \in X$. Since, for bounded operators, strong convergence implies strong resolvent convergence, which in turn is equivalent to strong dynamical convergence (see Theorem 10.1.8 in [7]), it follows, for each $t \in \mathbb{R}$, that the mapping $X \ni T \mapsto \Xi^T_\psi(t)$ is continuous.

Now, let $\mathcal{M}_+(I)$ represent the set of positive finite Borel measures on the open interval I endowed with the vague topology; the continuity of the mapping $\mathcal{M}_+(\mathbb{R}) \ni \mu(\cdot) \mapsto \mu(I \cap \cdot) \in \mathcal{M}_+(I)$ (see [13] for details), combined with the continuity of $X \ni T \mapsto \Xi^T_\psi(t)$, implies that $X \ni T \mapsto \Xi^T_\xi(t)$ is also continuous.

Observe that if there exist $T \in X$, $\xi \ne 0$, $t, x \in \mathbb{R}$ such that $\int \mathrm{d}\mu^T_\xi(y)\, e^{-(x-y)^2 t^2/4} > 0$, then the continuity of $X \ni T \mapsto \int \mathrm{d}\mu^T_\xi(y)\, e^{-(x-y)^2 t^2/4}$ implies that $\Xi^W_\xi(t)$ exists for every W in some neighbourhood of T.

Therefore, one has, for every $n \geq 0$ and every $k, l \geq 1$, that

$$\{T \in X \mid t^{-1/2+1/(4l)+1/(4k)}\, \Xi_\xi^T(t) > n\}$$

is an open subset of X. Now, relation (4.1), that is,

$$C_{\rm 1uPd}^{\psi;(a,b)} = \bigcap_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcup_{t>m} \{T \in X \mid t^{-1/2+1/(4l)+1/(4k)}\, \Xi_\xi^T(t) > n\},$$

completes the proof. □

Corollary 4.3 *Let $-\infty \leq a < b \leq +\infty$, $\psi \in \mathcal{H}$, and denote by $(\mu_\psi^T)_{\rm 1Pc}$ the 1-packing continuous component of μ_ψ^T. Suppose that $C_{\rm 1uPc}^{\psi;(a,b)} = \{T \in X \mid (\mu_\psi^T)_{\rm 1Pc}((a,b)) \neq 0\}$ is dense in X. Then, the set $C_{\rm 1uPd}^{\psi;(a,b)}$ is generic in X.*

Proof. Since, by Theorem 4.2, $C_{\rm 1uPd}^{\psi;(a,b)}$ is a G_δ set in X, we just need to show that $C_{\rm 1uPd}^{\psi;(a,b)}$ is dense. Suppose, then, that $(\mu_\psi^T)_{\rm 1Pc}((a,b)) > 0$; thus, by Definitions 2.2 and 2.3, $\dim_{\rm P}^+(\mu_{\psi;(a,b)}^T) = 1$, and therefore, $C_{\rm 1uPc}^{\psi;(a,b)} \subset C_{\rm 1uPd}^{\psi;(a,b)}$. But now, since $C_{\rm 1uPc}^{\psi;(a,b)}$ is dense, it follows that $C_{\rm 1uPd}^{\psi;(a,b)}$ is also dense. □

Remark 4.4 A well-known fact about discrete Schrödinger operators in $l^2(\mathbb{Z})$, with action (1.1) and general real potentials (V_n), is the presence of a common set of cyclic vectors $\{\delta_{-1}, \delta_0\}$. When the elements of the space X are of this type, the results stated in Corollary 4.3 can be strengthened. Namely, if for $\zeta \in \{\delta_{-1}, \delta_0\}$ the spectral measure $\mu_{\zeta;(a,b)}^T$ is 1Pd, then $\mu_{\psi;(a,b)}^T$ is 1Pd for every vector $\psi \neq 0$ (since $P_{\rm 1Pd}^T((a,b)) = P^T((a,b))$ in this case), which implies that $\{T \in X \mid \dim_{\rm P}(E) = 1 \text{ for some } E \subset \sigma(T) \cap (a,b)\}$ is a G_δ set.

Write $C_{\rm 1uPd} := \{T \in X_{\lambda,\nu} \mid \dim_{\rm P}^+(\mu_{\delta_0}^T) = \dim_{\rm P}^+(\mu_{\delta_{-1}}^T) = 1\}$. The inclusion $C_{\rm 1uPd} \subset C_{\rm QB}$ (see the definition of the latter in the Introduction; this inclusion results from Proposition 1.2 and the second inequality in (1.4)), together with Corollary 4.3, lead us to the following

Proposition 4.5 *Suppose that the hypotheses of Corollary 4.3 are satisfied for $\psi = \delta_0$ and $(a,b) = \mathbb{R}$. Then, $C_{\rm QB}$ is generic in X.*

5 Proof of Theorem 1.1

We need the following

Theorem 5.1 (Theorem 1.1 in [6]) *Suppose that Ω is a Cantor group and that $\tau : \Omega \to \Omega$ is a minimal translation. Then, for a dense set of $g \in {\rm C}(\Omega, \mathbb{R})$ and every $\kappa \in \Omega$, the spectrum of $H_{g,\tau}^\kappa$ is purely absolutely continuous.*

Proof. (Theorem 1.1) Fix $\kappa \in \Omega$, $\tau : \Omega \to \Omega$ a minimal translation of the Cantor group Ω, and consider in $\mathrm{C}(\Omega, \mathbb{R})$.

Since, by Theorem 5.1, $C^{\kappa}_{\mathrm{ac}} := \{T \in X_\kappa \mid \sigma(T) \text{ is purely absolutely continuous}\}$ is dense in X_κ, it follows that $C^{\kappa}_{\mathrm{1uPd}} \supset C^{\kappa}_{\mathrm{ac}}$ is also dense in X_κ. Thus, by Corollary 4.3 and Remark 4.4, we conclude that $C^{\kappa}_{\mathrm{1uPd}}$ is generic in X_κ.

The second assertion in the statement of the theorem follows from the inclusion $C^{\kappa}_{\mathrm{1uPd}} \subset C^{\kappa}_{\mathrm{QB}}$ and Proposition 4.5. □

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Email: `silas@mat.ufmg.br`, Departamento de Matemática, UFMG, Belo Horizonte, MG, 30161-970 Brazil

Email: `oliveira@dm.ufscar.br`, Departamento de Matemática, UFSCar, São Carlos, SP, 13560-970 Brazil