A CORRECTION TO THE PAPER "INJECTIVE MAPPINGS AND SOLVABLE VECTOR FIELDS OF EUCLIDEAN SPACES"

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ABSTRACT. We construct a counterexample that disproves the claim of Theorem 0.2 of the paper "Injective mappings and solvable vector fields of Euclidean spaces", which appeared in Topology Appl. **136** (2004), 261–274.

1. INTRODUCTION

Given a C^∞ local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$, it has been an interest of research along the years to find additional condition ensuring that F is globally injective. In $\mathbb{R}^2$, the non-injective local diffeomorphism $(x, y) \mapsto (e^x \cos y, e^x \sin y)$ shows that, in fact, more hypotheses are needed. It is not sufficient, for instance, to assume that F is a polynomial map, since in [8] it was constructed a non injective polynomial map in $\mathbb{R}^2$ with non-zero Jacobian determinant, disproving the so called *real Jacobian conjecture*. Yet in the polynomial case, if the Jacobian determinant is a non zero *constant*, then the injectivity of F is the famous *Jacobian conjecture* in $\mathbb{R}^n$, which is up to now an open problem. We address the reader to [9] for the very general result that F is a global diffeomorphism if and only if it is proper. We also mention a spectral condition in $\mathbb{R}^2$ given in [5] and, in a more general frame, in [4]. For the polynomial case and the Jacobian conjecture, we indicate [7].

In [10] and [11] an approach using vector fields was given. To explain it, we introduce some definitions and notations.

We say that a vector field $\mathcal{X} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is *globally solvable* if it is a surjective operator.

Letting $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ map, we define n vector fields $\mathcal{V}_i : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, $i = 1, \ldots, n$, as follows:

$$\mathcal{V}_i(\varphi) = \det D(F_1, \ldots, F_{i-1}, \varphi, F_{i+1}, \ldots, F_n),$$

for each $\varphi \in C^\infty(\mathbb{R}^n)$.

In $\mathbb{R}^2$, the following is true: if $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism and $\mathcal{V}_1$ or $\mathcal{V}_2$ is globally solvable, then F is injective. Indeed, if for instance $\mathcal{V}_1$ is globally solvable, then all the level sets of F_2 are connected, see [1] or [2] for proofs of this result. Then the injectivity of F follows, see Proposition 9 bellow.

We could think that a similar result remains true in higher dimensions. In this paper we give a counterexample for a generalization of this result, disproving the extension given in [10], and consequently in [11]. Precisely, we prove the following.

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Theorem 1. *There exists a C^∞ non-injective local diffeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathcal{V}_1$ and $\mathcal{V}_2$ are globally solvable.*

Theorem 0.2 of [10] claims that if a C^∞ local diffeomorphism $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is such that Ω is *F-convex*, then F is injective. From the Definition 0.1 of [10], if $\mathbb{R}^3$ is *$\mathcal{V}_i$-convex for supports*, for $i = 1, 2$, then $\mathbb{R}^3$ is F-convex. According to Definition 0.0 of [10] and comments after it, $\mathbb{R}^3$ is $\mathcal{V}_i$-convex for supports if and only if $\mathcal{V}_i$ is globally solvable. Therefore our Theorem 1 disproves Theorem 0.2 of [10].

Moreover, we mention here the approach given in [12] and [13] (for dimension 2) and in [6] (for dimension n) in the case of F being a *polynomial* map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $\mathcal{V}_i$, $i = 1, \ldots, n$, being as above (but now complex vector fields) from E_n to E_n, where E_n is the space of the entire functions. It was proved that if $\mathcal{V}_i$ is surjective for $i = 1, \ldots, n-1$, then F is a polynomial automorphism. Our example given in Theorem 1 also shows that we can not expect a similar result for more general maps defined in $\mathbb{R}^n$.

The main tool to prove Theorem 1 is a relation between global solvability of the vector field $\mathcal{V}_i$ and connectedness of some suitable intersections of level sets of F_j, $j \neq i$. It is our Theorem 2 below. Actually we will use the sufficient condition given in Theorem 2 to prove the global solvability of $\mathcal{V}_1$ and $\mathcal{V}_2$ for the map F of Theorem 1.

Before stating the theorem, we introduce the notation $\mathscr{F}(f)$ to indicate the foliation of codimension 1 given by the connected components of the non-empty level sets of a a C^∞ submersion $f : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 2. *Let $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ map satisfying* $\det DF(x) \neq 0$ $\forall x \in \mathbb{R}^3$*, and $i \in \{1, 2, 3\}$. If for each $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j \neq k$, $L_j \in \mathscr{F}(F_j)$ and $c_k \in \mathbb{R}$, the set*

$$L_j \cap F_k^{-1}\{c_k\}$$

is connected, then $\mathcal{V}_i$ is globally solvable.

A version for $\mathbb{R}^n$ of Theorem 2 will soon appear in [2]. We remark that the converse of Theorem 2 is also true, see [2].

The paper is organized as follows. We begin section 2 explaining the general ideas for the construction of the counterexample given in Theorem 1. Then we give all the details of our construction culminating with the proof of Theorem 1, assuming Theorem 2. Finally, in section 3, we give some properties of the vector fields $\mathcal{V}_i$ and prove Theorem 2.

2. Hunting for a counterexample. The proof of Theorem 1

We search for a C^∞ map $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

- $\det DF(x) \neq 0$ for all $x \in \mathbb{R}^3$,
- F is not injective,
- $\mathcal{V}_1$ and $\mathcal{V}_2$ are globally solvable.

From our Theorem 2, to obtain the global solvability of $\mathcal{V}_1$ it is sufficient to have the connectedness of the sets

$$L_2 \cap F_3^{-1}\{c_3\}, \quad L_3 \cap F_2^{-1}\{c_2\},$$

for each $L_2 \in \mathscr{F}(F_2)$ and $L_3 \in \mathscr{F}(F_3)$, and for each $c_2, c_3 \in \mathbb{R}$. From the same result, the global solvability of $\mathcal{V}_2$ is attained by proving the connectedness of the sets

$$L_1 \cap F_3^{-1}\{c_3\}, \qquad L_3 \cap F_1^{-1}\{c_1\},$$

for each $L_1 \in \mathscr{F}(F_1)$ and $L_3 \in \mathscr{F}(F_3)$, and for each $c_1, c_3 \in \mathbb{R}$. As mentioned in the introduction section, the connectedness of the above sets are also necessary for the global solvability of $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, see [2].

Now since for the injectivity of F it is sufficient to have for $i \neq j$ that $F_i^{-1}\{c_i\} \cap F_j^{-1}\{c_j\}$ is connected for all $c_i, c_j \in \mathbb{R}$ (see Proposition 9 below), in order to satisfy the second condition above, it is necessary the existence of $a_1, a_2, b_1, b_3, c_2, c_3 \in \mathbb{R}$ such that the intersections

$$F_1^{-1}\{a_1\} \cap F_2^{-1}\{a_2\}, \qquad F_1^{-1}\{b_1\} \cap F_3^{-1}\{b_3\}, \qquad F_2^{-1}\{c_2\} \cap F_3^{-1}\{c_3\},$$

are disconnected.

In particular, in the map of Theorem 1, no component F_i can have all its level sets connected.

Our counterexample will have the following form

$$F(x,y,z) = \left((\arctan x + g_{h_1}(z))e^y, (\arctan x + g_{h_2}(z))e^y, (1-z^2)e^y\right), \tag{1}$$

with g_{h_1} and g_{h_2} C^∞ functions in $\mathbb{R}$ to be determined in this section.

Let $h > \pi/2$. Throughout the paper, $g_h : \mathbb{R} \to \mathbb{R}$ will stand for a C^∞ function satisfying the following conditions:

(a) $g_h([-1,1]) = [-h,h]$,
(b) $g_h'(z) > 0$ for each $z \in (-1,1)$,
(c) $g_h'''(z) < 0$ for each $z \in (-1,1)$,
(d) $g_h(z) = -\theta_h(z-1)^2 + h$ for each $z \geq 1$, for a suitable $\theta_h > 0$,
(e) $g_h(z)$ is odd for $|z| \geq 1$.

We explicitly construct a family of such functions in Example 6 bellow. Figure 1 shows how the graphics of g_h and its first three derivatives should look like.

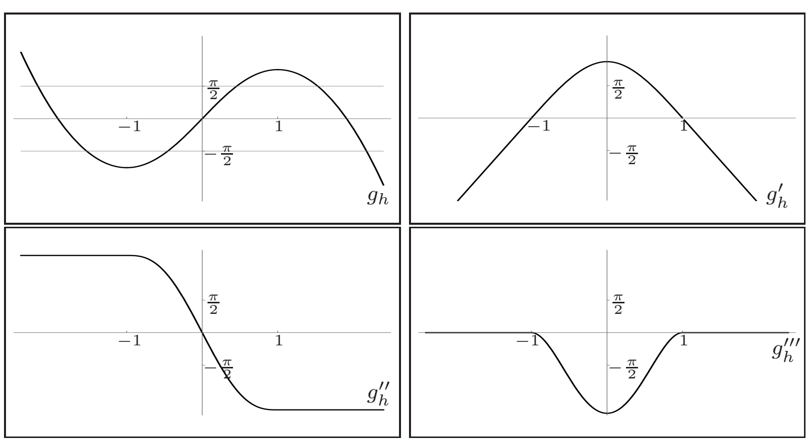

FIGURE 1. A function g_h and its first three derivatives.

In Lemma 4 below we discuss the connectedness of intersections of the type $L_i \cap F_j^{-1}\{c_j\}$ as explained in the beginning of this section. We first prove a technical result.

Lemma 3. *Let $h > \pi/2$, $\alpha \in \mathbb{R}$ and $k_\alpha : \mathbb{R} \to \mathbb{R}$ be defined by*

$$k_\alpha(z) = \alpha(1 - z^2) - g_h(z).$$

Let also $I_1 = (-\infty, -1)$, $I_2 = (-1, 1)$ and $I_3 = (1, \infty)$. Then the sets

$$k_\alpha^{-1}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap I_j, \ \ j = 1, 2, 3,$$

are connected. In particular, for each $c_3 \neq 0$ the sets

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid e^{-y} = \frac{1 - z^2}{c_3}, \ \arctan x = k_\alpha(z), \ \ z \in I_j\right\}, \ \ j = 1, 2, 3, \tag{2}$$

are connected.

Proof. For $z \in I_1$, we have from the properties of g_h that

$$k_\alpha(z) = \alpha(1 - z^2) - \theta_h(-z - 1)^2 + h = -(\alpha + \theta_h)z^2 - 2\theta_h z + \alpha - \theta_h + h.$$

We observe that if $\alpha + \theta_h = 0$, then k_α is injective and hence $k_\alpha^{-1}(-\pi/2, \pi/2) \cap I_1$ is connected. If $\alpha + \theta_h \neq 0$, then the quadratic function k_α has the critical point $z_m = -\theta_h/(\alpha+\theta_h)$. If $z_m \geq -1$, then k_α is injective in I_1, and the lemma is proven in this case. On the other hand, if $z_m < -1$, then $\alpha + \theta_h > 0$ (recall that $\theta_h > 0$) and thus z_m is a maximum point of k_α. Therefore, since $k_\alpha(-1) = h > \pi/2$, it follows that

$$k_\alpha^{-1}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap I_1 = k_\alpha^{-1}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap (-\infty, z_m),$$

which gives the lemma in this case, because k_α is injective (indeed increasing) in $(-\infty, z_m)$.

Since $k_\alpha(z) = -k_{-\alpha}(-z)$ for $z > 1$, it follows from the first part of the proof that $k_\alpha^{-1}(-\pi/2, \pi/2) \cap I_3$ is also connected.

Now for $z \in I_2$, if $\alpha = 0$, since $g_h'(z) > 0$, it follows that $k_\alpha = -g_h$ is injective in I_2, and hence the lemma follows in this case. If $\alpha < 0$, we have that $k_\alpha'(-1) = 2\alpha < 0$ and $k_\alpha'(1) = -2\alpha > 0$, because $g_h'(\pm 1) = 0$. Then since $k_\alpha'''(z) = -g_h'''(z) > 0$, it follows that there exists exactly one $z_1 \in (-1, 1)$ such that $k_\alpha'(z) < 0$ in $(-1, z_1)$, $k_\alpha'(z_1) = 0$ and $k_\alpha'(z) > 0$ in $(z_1, 1)$. In particular, k_α is decreasing in $(-1, z_1)$ and increasing in $(z_1, 1)$. Hence, since $k_\alpha(1) = -h$, it follows that $k_\alpha(z) < -h < -\pi/2$ for all $z \in (z_1, 1)$. Therefore

$$k_\alpha^{-1}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap I_2 = k_\alpha^{-1}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap (-1, z_1),$$

which proves the lemma in this case, because k_α is decreasing in $(-1, z_1)$. The case $\alpha > 0$ is entirely analogous.

The last statement of the lemma follows because the function $1 - z^2$ does not change sign in each interval I_j. Thus or a set in (2) is empty or x and y are continuous functions of z defined in the connected $k_\alpha^{-1}(-\pi/2, \pi/2) \cap I_j$, and hence the set is connected. □

Lemma 4. *Let $h > \pi/2$ and define $F_h, F_3 : \mathbb{R}^3 \to \mathbb{R}$ by*

$$F_h(x, y, z) = \big(\arctan x + g_h(z)\big)e^y, \qquad F_3(x, y, z) = \left(1 - z^2\right) e^y.$$

Then for each $i, j \in \{h, 3\}$, $i \neq j$, and for each $L_i \in \mathscr{F}(F_i)$ and $c_j \in \mathbb{R}$, the set $L_i \cap F_j^{-1}\{c_j\}$ is connected.

Proof. We begin calculating the level sets of F_h and F_3 and their connected components. In the boxes bellow, when we write for instance $\arctan x = c_h e^{-y} - g_h(z)$, $z < 1$, we mean the largest interval contained in $z < 1$ where this expression makes sense.

First the level sets of F_h, i.e. the sets $F_h^{-1}\{c_h\}$:

$c_h < 0$: two components:

$\arctan x = c_h e^{-y} - g_h(z)$, $z < 1$,

$\arctan x = c_h e^{-y} - g_h(z)$, $z > 1$.

$c_h = 0$: three components:

$\arctan x = -g_h(z)$, $z < -1$,

$\arctan x = -g_h(z)$, $-1 < z < 1$,

$\arctan x = -g_h(z)$, $z > 1$.

$c_h > 0$: two components:

$\arctan x = c_h e^{-y} - g_h(z)$, $z < -1$,

$\arctan x = c_h e^{-y} - g_h(z)$, $z > -1$.

Indeed, it is simple to calculate these connected components for the level set $F_h^{-1}\{0\}$. We detail here the calculation of the negative level sets, i.e. the sets $F_h^{-1}\{c_h\}$, for $c_h < 0$. For $c_h > 0$ the calculation is analogous. Observe that we have to search for the connected sets of $\mathbb{R}^3$ satisfying $\arctan x = c_h e^{-y} - g_h(z)$. These sets have the form $\{(x, y, z) \mid x = \tan(c_h e^{-y} - g_h(z)),\ (y, z) \in A\}$, where $A \subset \mathbb{R}^2$ is the greatest connected set such that $-\pi/2 < c_h e^{-y} - g_h(z) < \pi/2$, i.e, $g_h(z) - \pi/2 < c_h e^{-y} < g_h(z) + \pi/2$. Since $c_h < 0$, it follows in particular that $g_h(z) - \pi/2$ has to be negative. By the properties of g_h, it follows that this happens in two intervals, one contained in $z < 1$ and the other in $z > 1$. Then there are two connected regions of $\mathbb{R}^2$ where $(c_h e^{-y}, z)$ varies. Since $y \mapsto c_h e^{-y}$ is injective, it follows then that there are two connected sets A as above and the sets of the first box above are connected. See figure 2 for the two regions where varies z and $c_h e^{-y}$ for $c_h < 0$. For $c_h > 0$, similar regions will stay above the z-axis.

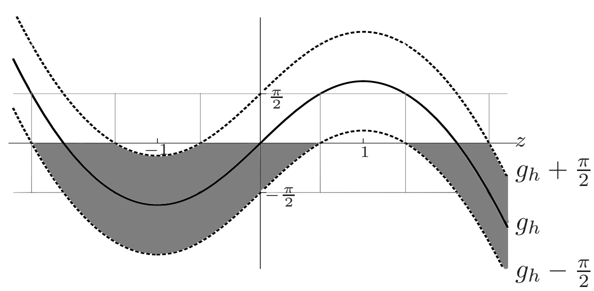

Figure 2. The connected regions where lives $(z, c_h e^{-y})$, for $c_h < 0$. Here $c_h e^{-y}$ varies vertically.

The level sets of F_3 are very easy to calculate, as they do not depend on x:

$c_3 < 0$: two components:	$c_3 = 0$: two components:	$c_3 > 0$: one component:
$e^{-y} = (1 - z^2)/c_3$, $z < -1$, $e^{-y} = (1 - z^2)/c_3$, $z > 1$.	$z = -1$, $z = 1$.	$e^{-y} = (1 - z^2)/c_3$, $-1 < z < 1$.

Now we study the intersections $L_i \cap F_j^{-1}\{c_j\}$, for $i, j \in \{h, 3\}$, $i \neq j$.

We first consider $c_3 = 0$. When we intersect each of the connected components $z = -1$ and $z = 1$ of the level set $F_3 = 0$ with the level sets of F_h, we obtain empty or connected sets, because for each $c_h \neq 0$ the function $y \mapsto c_h e^{-y} + \mathit{constant}$ is injective.

On the other hand, since each connected component L_h of a level set of F_h does not intersect the planes $z = -1$ and $z = 1$ at the same time, and for each $c_h \neq 0$ the function $y \mapsto c_h e^{-y} + \mathit{constant}$ is injective, it follows that $L_h \cap F_3^{-1}\{0\}$ is connected.

Finally each of the other required intersections is the solution of one of the following systems of equations

$$e^{-y} = \frac{1 - z^2}{c_3}, \qquad \arctan x = c_h e^{-y} - g_h(z),$$

for z in only one of the intervals $I_1 = (-\infty, -1)$, $I_2 = (-1, 1)$ or $I_3 = (1, \infty)$. These solutions are the sets (2) with α of the form c_h/c_3. They are all connected from Lemma 3. □

Now we choose a class of pair of functions g_{h_1} and g_{h_2} in order to obtain maps F as in (1) such that the determinant of DF is nowhere zero.

Lemma 5. *Let $h_2 > h_1 > \pi/2$. If there exists $z_0 > 1$ such that $g_{h_1}(z_0) = g_{h_2}(z_0) = 0$ and $g'_{h_2}(z) > g'_{h_1}(z)$ in $(-1, 1)$, then the Jacobian determinant of the map defined in (1) is strictly negative in $\mathbb{R}^3$.*

In the family of functions constructed in Example 6 below there are pairs g_{h_1} and g_{h_2} satisfying these assumptions.

Proof of Lemma 5. From the formula of F in (1), a straightforward calculation shows that

$$\det DF(x, y, z) = \frac{e^{3y}}{1 + x^2} f(z),$$

with $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(z) = (1 - z^2)\left(g'_{h_1}(z) - g'_{h_2}(z)\right) + 2z\left(g_{h_1}(z) - g_{h_2}(z)\right).$$

Thus it is enough to prove that $f(z)$ is strictly negative in $\mathbb{R}$.

Since $\left(g_{h_2} - g_{h_1}\right)'(z) > 0$ in the interval $(-1, 1)$, it follows that $(1 - z^2)(g'_{h_1}(z) - g'_{h_2}(z)) < 0$ in $(-1, 1)$. From the same fact and since $\left(g_{h_2} - g_{h_1}\right)(0) = 0$, it follows that the function $\left(g_{h_2} - g_{h_1}\right)(z)$ is negative in $(-1, 0)$ and positive in $(0, 1)$. In particular $2z(g_{h_1}(z) - g_{h_2}(z)) < 0$ in $(-1, 0) \cup (0, 1)$. Therefore, since $f(0) = g'_{h_1}(0) - g'_{h_2}(0) < 0$ and $f(\pm 1) = h_1 - h_2 < 0$, it follows that $f(z) < 0$ in the interval $[-1, 1]$.

Thus it remains to prove that $f(z) < 0$ if $|z| > 1$. Since g_{h_i}, $i = 1, 2$, are odd functions in $|z| > 1$, it follows that f is an even function in $|z| > 1$, and hence it is enough to prove that $f(z) < 0$ for all $z \geq z_0$.

Since g_{h_1} and g_{h_2} annihilate in $z_0 > 1$, we have from the definition of g_{h_i} that

$$z_0 = 1 + \sqrt{\frac{h_1}{\theta_{h_1}}} = 1 + \sqrt{\frac{h_2}{\theta_{h_2}}},$$

and thus, as $h_2 > h_1$, it follows that

$$\theta_{h_2} > \theta_{h_1}. \tag{3}$$

We also have that for $z > 1$

$$f'(z) = 2\big(2(\theta_{h_1} - \theta_{h_2})z - 2(\theta_{h_1} - \theta_{h_2}) + h_1 - h_2\big).$$

From (3) it follows that $f'(z)$ has negative slope and could only be zero in

$$z_1 = 1 - \frac{h_1 - h_2}{2\,(\theta_{h_1} - \theta_{h_2})},$$

which again by (3) is a number less than 1. Thus $f'(z) < 0$ for all $z > 1$ and since $f(1) < 0$, it follows that $f(z) < 0$ for all $z > 1$. □

The following example presents an explicit family of functions g_{h_i} satisfying the assumptions (a) to (e) above and the hypotheses of Lemma 5.

Example 6. Let $h_2 > h_1 > \pi/2$ and $\varphi_3 : \mathbb{R} \to \mathbb{R}$ be the C^∞ function defined by

$$\varphi_3(z) = \begin{cases} e^{\frac{1}{z^2-1}}, & |z| < 1 \\ 0, & |z| \geq 1. \end{cases}$$

Let further $\varphi_2, \varphi_1, \varphi : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$\varphi_2(z) = \int_0^z \varphi_3(s)ds, \qquad \varphi_1(z) = \int_{-1}^z \varphi_2(s)ds, \qquad \varphi(z) = \int_0^z \varphi_1(s)ds.$$

Now we consider $a_i < 0$ and $\theta_{h_i} > 0$ be defined by

$$h_i = a_i\varphi(1), \qquad 2\theta_{h_i} = -a_i\varphi_2(1), \quad i = 1, 2. \tag{4}$$

We finally define

$$g_{h_i}(z) = \begin{cases} \theta_{h_i}(z+1)^2 - h_i, & z \leq -1 \\ a_i\varphi(z), & -1 \leq z \leq 1 \\ -\theta_{h_i}(z-1)^2 + h_i, & z \geq 1 \end{cases}, \qquad i = 1, 2$$

Since from the definition of φ and from (4) we have that $a_i\varphi(\pm 1) = \pm h_i$, $a_i\varphi'(\pm 1) = a_i\varphi_1(\pm 1) = 0$, $a_i\varphi''(\pm 1) = a_i\varphi_2(\pm 1) = \mp 2\theta_{h_i}$ and $a_i\varphi^{(k)}(\pm 1) = \varphi_3^{(k-3)}(\pm 1) = 0$ for all $k \geq 3$, it follows that g_{h_i} is a C^∞ function, for $i = 1, 2$. Moreover, since in $(-1, 1)$ we have that $g'_{h_i}(z) = a_i\varphi_1(z)$ and $g'''_{h_i}(z) = a_i\varphi_3(z)$, it follows that g'_{h_i} is positive and g'''_{h_i} is negative in $(-1, 1)$. In particular, $g_{h_i}([-1, 1]) = [-h_i, h_i]$. Finally, it is clear that g_{h_i} is odd. Thus g_{h_i}, $i = 1, 2$, satisfy the properties (a) to (e) of the beginning of the section.

Now we verify the hypotheses of Lemma 5. From (4) it follows that $h_1/\theta_{h_1} = h_2/\theta_{h_2}$, which gives that the functions g_{h_1} and g_{h_2} annihilate in $z_0 = 1 + \sqrt{h_1/\theta_{h_1}} = 1 + \sqrt{h_2/\theta_{h_2}} > 1$. Now by construction, $\varphi_1(z) < 0$ for all $z \in (-1, 1)$. Then as $a_2 < a_1 < 0$ (from (4)), it follows that $a_2\varphi_1(z) > a_1\varphi_1(z)$ and thus $g'_{h_2}(z) > g'_{h_1}(z)$ in $(-1, 1)$, as required.

We finally give the

Proof of Theorem 1. Let $h_2 > h_1 > \pi/2$ and $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by (1), with g_{h_i}, $i = 1, 2$, satisfying the assumptions of Lemma 5. *We claim that*

(i) $\det DF(x, y, z) < 0$ *for all* $(x, y, z) \in \mathbb{R}^3$,
(ii) F *is not injective.*
(iii) $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are globally solvable,*

Thus to complete the proof, we only need to prove the claim.

Statement (i) follows from Lemma 5.

Now since g_{h_i} annihilates in $\pm z_0$ for $i = 1, 2$, it follows that $F(x, y, -z_0) = F(x, y, z_0)$, proving (ii).

Finally, we observe that for $i = 1, 2$, $F_i = F_{h_i}$ in the notation of Lemma 4. Therefore, from this lemma the intersections $L_2 \cap F_3^{-1}\{c_3\}$ and $L_3 \cap F_2^{-1}\{c_2\}$ are connected for all $L_j \in \mathscr{F}(F_j)$ and $c_k \in \mathbb{R}$, $j, k \in \{2, 3\}$. From Theorem 2 it thus follows that $\mathcal{V}_1$ is globally solvable. Analogously, Lemma 4 gives that $L_1 \cap F_3^{-1}\{c_3\}$ and $L_3 \cap F_1^{-1}\{c_1\}$ are connected for all $L_j \in \mathscr{F}(F_j)$ and $c_k \in \mathbb{R}$, $j, k \in \{1, 3\}$, which from Theorem 2 proves that $\mathcal{V}_2$ is globally solvable. Therefore, (ii) is proved. □

3. Global solvability of the vector fields $\mathcal{V}_i$ and the proof of Theorem 2

We begin this section recalling some results we shall use in the proof of Theorem 2. We first recall the following geometrical characterization of global solvability of vector fields. This result is part of Theorem 6.4.2 of [3].

Theorem 7. *Let M be a smooth manifold and $\mathcal{X} : C^\infty(M) \to C^\infty(M)$ be a vector field. Then $\mathcal{X}$ is globally solvable if and only if*

(1) *No integral curve of $\mathcal{X}$ is contained in a compact subset of M and*
(2) *For all compact $K \subset M$, there exists a compact $K' \subset M$ such that every compact interval on an integral curve of $\mathcal{X}$ with end points in K is contained in K'.*

Note that item (1) of Theorem 7 shows that on compact manifolds there are no globally solvable vector fields.

The following is a study of the integral curves of $\mathcal{V}_i$.

Lemma 8. *Let $F = (F_1, \dots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ map such that* $\det DF$ *is nowhere zero. Then for each $i \in \{1, \dots, n\}$,*

(i) *The integral curves of $\mathcal{V}_i$ are the non empty connected components of*

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{n} F_j^{-1}\{c_j\}, \; c_j \in \mathbb{R}. \tag{5}$$

(ii) *F_i is strictly monotone along the integral curves of $\mathcal{V}_i$.*

Proof. By definition, we have that $\mathcal{V}_i(F_j) = \delta_{ij} \det DF$, where δ_{ij} stands for the Kronecker delta. Thus for an integral curve $\gamma(t)$ of $\mathcal{V}_i$, we have that

$$(F_j \circ \gamma)'(t) = \delta_{ij} \det DF(\gamma(t)), \tag{6}$$

$j = 1, 2, \dots, n$. This shows that F_i is strictly monotone along $\gamma(t)$, proving (ii). In particular, $\mathcal{V}_i$ has no singular points. The equality in (6) also proves that the image of γ is contained in a connected component of an intersection in (5), that we denote by Γ. From the Implicit Function Theorem, Γ is a 1-dimensional connected manifold. Since $\mathcal{V}_i$ has no singular points, it follows that the image of γ coincides with Γ. This proves (i). □

As a direct application, we have the following result.

Proposition 9. *Let $F = (F_1, \dots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ local diffeomorphism. If there is $i \in \{1, \dots, n\}$ such that all the intersections in (5) are connected, then F is injective.*

Proof. Let $a, b \in \mathbb{R}^n$ with $a \neq b$. If $F_j(a) = F_j(b)\ \forall j \neq i$, then by statement (i) of Lemma 8, a and b are in a same integral curve of $\mathcal{V}_i$. Thus by statement (ii) of the same lemma it follows that $F_i(a) \neq F_i(b)$. Hence $F(a) \neq F(b)$. □

Before proceeding to the proof of Theorem 2, we remark that the assumption $\det DF(x) \neq 0$ for all $x \in \mathbb{R}^3$ guarantees that for $i \in \{1, 2, 3\}$, each leaf of the foliation $\mathscr{F}(F_i)$ is a local transversal to the foliations $\mathscr{F}(F_j)$, $j \neq i$ and to the flow of the vector field $\mathcal{V}_i$.

Proof of Theorem 2. We assume without loss of generality that $i = 1$.

We suppose on the contrary that $\mathcal{V}_1$ is not globally solvable. From Lemma 8 and Theorem 7 it follows that there exists a compact set $K \subset \mathbb{R}^3$ such that for each $n \in \mathbb{N}$ there exists an integral curve γ_n of $\mathcal{V}_1$ and t_n, s_n in the interval of solution of γ_n, $t_n > s_n > 0$, such that $\gamma_n(0), \gamma_n(t_n) \in K$ and

$$|\gamma_n(s_n)| > n. \tag{7}$$

Without loss of generality, we assume that $\gamma_n(0) \to a$ and $\gamma_n(t_n) \to b$ as $n \to \infty$, for suitable $a, b \in K$. We take γ_a and γ_b the integral curves of $\mathcal{V}_1$ through a and b respectively. We *claim that* $\gamma_a \cap \gamma_b = \emptyset$.

To prove the claim we first prove that $a \neq b$. Indeed, if $a = b$, let L_1 be the connected component of $F_1^{-1}\{F_1(a)\}$ containing a. Since L_1 is locally transversal to the flow of $\mathcal{V}_1$, we can take a bounded neighborhood U of a such that each interval of an integral curve of $\mathcal{V}_1$ crossing U intersects L_1. From (7), we can take n big enough such that $\gamma_n(0), \gamma_n(t_n) \in U$ and $\gamma_n(s_n) \notin U$. Thus γ_n intersects L_1 twice, a contradiction with the fact that F_1 is increasing along γ_n, see Lemma 8. Now if $a \neq b$ are in the same integral curve γ of $\mathcal{V}_1$, we use the Flow Box Theorem to construct a compact tubular neighborhood T along the interval of the curve γ from a to b. Using again the fact that the connected components of the level sets of F_1 are local transversals to the flow of $\mathcal{V}_1$, we can assume that each curve in T enters T by crossing the connected component of the level set of F_1 containing a and leaves T by crossing the level set of F_1 containing b. Thus since F_1 is increasing along the integral curves of $\mathcal{V}_1$, if an integral curve of $\mathcal{V}_1$ passes through T, it does it just once. From (7), there exists n_0 such that $\gamma_n(s_n) \notin T$ for each $n \geq n_0$. This is a contradiction, because for n big enough, the interval of curve γ_n from 0 to t_n must be contained in T. This finishes the proof of the claim.

We take $c_2, c_3 \in \mathbb{R}$ such that $\gamma_a, \gamma_b \subset F_2^{-1}\{c_2\} \cap F_3^{-1}\{c_3\}$, see Lemma 8. Since from hypothesis any leaf of $\mathscr{F}(F_3)$ intersected with $F_2^{-1}\{c_2\}$ is a connected set, it follows from the above claim that γ_a and γ_b are in distinct connected components of $F_3^{-1}\{c_3\}$. We denote these components by L_3^a and L_3^b, respectively. We take open neighborhoods N_a and N_b of a and b, respectively, such that all the leaves of the foliation $\mathscr{F}(F_2)$ crossing N_a intersect L_3^a and all the leaves of $\mathscr{F}(F_2)$ crossing N_b intersect L_3^b (this is possible because L_3^a and L_3^b are local transversals to the foliation

$\mathscr{F}(F_2)$). We then take n_0 big enough in order that $\gamma_{n_0}(0) \in N_a$ and $\gamma_{n_0}(t_{n_0}) \in N_b$ and consider $L_2 \in \mathscr{F}(F_2)$ containing γ_{n_0}. This leaf L_2 intersects L_3^a and L_3^b and thus $L_2 \cap F_3^{-1}\{c_3\}$ is disconnected, a contradiction with the hypothesis. □

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