BOURGIN-YANG VERSIONS OF THE BORSUK-ULAM THEOREM FOR p-TORAL GROUPS

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ABSTRACT. Let V and W be orthogonal representations of G with $V^G = W^G = \{0\}$. Let $S(V)$ be the sphere of V and $f : S(V) \to W$ be a G-equivariant mapping. We give an estimate for the dimension of the set $Z_f = f^{-1}\{0\}$ in terms of $\dim V$ and $\dim W$, if G is the torus $\mathbb{T}^k$, or the p-torus $\mathbb{Z}_p^k$. This extends the classical Bourgin-Yang theorem onto this class of groups. Finally, we show that for any p-toral group G and a G-map $f : S(V) \to W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$.

1. INTRODUCTION

Let G be a compact Lie group and let V, W be two orthogonal representations of G such that $V^G = W^G = \{0\}$ for the sets of fixed points of G. Let $f : S(V) \to W$ be a G-equivariant mapping. By Z_f, we denote the set $Z_f := \{v \in S(V) \mid f(v) = 0\}$.

The problem of estimating the covering dimension of the set Z_f was considered firstly by C. T. Yang [21, 22] and (independently) D. G. Bourgin [6] for the case $G = \mathbb{Z}_2$. Specifically, they proved that for a $\mathbb{Z}_2$-equivariant mapping f from the unit sphere $S(\mathbb{R}^m)$ in $\mathbb{R}^m$ into $\mathbb{R}^n$, where the Euclidean spaces are considered as representations of $\mathbb{Z}_2$ with the antipodal action,

$$\dim Z_f \geq m - n - 1,$$

where dim is the covering dimension. Consequently, it is a strengthening of the classical Borsuk-Ulam theorem, which is its direct consequence.

In [10], Dold extended the Bourgin-Yang problem to a fibre-wise setting, giving an estimate for the set $Z_f = f^{-1}\{0\}$, where $\pi : E \to B$ and $\pi' : E' \to B$ are vector bundles and $f : S(E) \subset E \to E'$ is a $\mathbb{Z}_2$-map, which preserve fibres ($\pi' \circ f = \pi$). In [11] and [19] this problem was considered for the case of the cyclic group $G = \mathbb{Z}_p$ (p prime), and in [16] for bundles $E \to B$ whose fibre has the same cohomology (mod p) of a product of spheres. All these results gave the Bourgin-Yang theorem for $G = \mathbb{Z}_p$, with p prime, if we take B as a single point.

Recently, in [14] the authors considered the Bourgin-Yang problem for the case that G is a cyclic group of a prime power order, $G = \mathbb{Z}_{p^k}$, $k \geq 1$. Based on the result of [2] it was proved [14, Theorem 1.1] that if V, W are two orthogonal representations of the

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cyclic group $\mathbb{Z}_{p^k}$ and $f : S(V) \to W$ an equivariant map then the covering dimension $\dim(Z_f) = \dim(Z_f/G) \geq \phi(V,W)$, where ϕ is a function depending on $\dim V$, $\dim W$ and the orders of the orbits of actions on $S(V)$ and $S(W)$ (cf. [14, Theorems 3.6 and 3.9]). In particular, if $\dim W < \dim V/p^{k-1}$, then $\phi(V,W) \geq 0$, which means that there is no G-equivariant map from $S(V)$ into $S(W)$. Up to our knowledge the above are all known classes of groups for which the Bourgin-Yang theorem has been shown.

It is worth pointing out that the classical Bourgin-Yang problem studied here is similar but different than the Bourgin-Yang, or correspondingly Borsuk-Ulam problem for coincidence points along an orbit of the action. The latter studied for $G = \mathbb{Z}_{p^k}$ by Munkholm and for $G = \mathbb{Z}_p^k$ by Volovikov in several papers (cf. [17], [18] and [20] with references there). The resulted outcomes are called the Borsuk-Ulam, or respectively *Bourgin-Yang type* theorem. It studies dimension of the set $A(f) = \{x \in X : \mid f(x) = f(gx), \text{ for all } g \in G\}$ for a map (not equivariant in general) $f : X \to Y$ between two G-spaces X and Y. There are relations between the estimations of dimensions derived for these two distinct problems but hopeless to get each of them from the second directly and we will not discuss it.

In this paper we study the problem of estimation the cohomological dimension of the set $Z_f = \{x \in S(V) : f(x) = 0\}$ where $f : S(V) \to W$ is an equivariant map, and V, W are orthogonal representations of the group G such that $W^G = V^G = \{0\}$, where $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$.

First in Section 2, we prove in Theorem 2.1 that

$$\text{coh.dim} Z_f \geq \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1.$$

It gives an answer to the classical Bourgin-Yang problem for this class of groups.

As an accompanying result we give a sufficient condition on V and W for the existence of equivariant map $f : S(V) \to S(W)$ which together with earlier known classical necessary condition completes the Borsuk-Ulam theorem for p-torus (Theorem 2.4).

Finally, in Section 3 we discuss the Bourgin-Yang problem for an action of a p-toral group, i.e. a group G of the form $1 \to \mathbb{T}^k \to G \to \mathcal{P} \to 1$, where $\mathcal{P}$ is a p-group. Theorem 3.2 states that for any G-map $f : S(V) \to W$, of two orthogonal representations of a p-toral group with $\dim V = \infty$ and $\dim W < \infty$, $V^G = W^G = \{0\}$ we have $\dim Z_f = \infty$. However here the argument is more complicated and uses the mentioned result of [5] based on the G. Carlsson theorem on the G. Segal conjecture and theorem of Laitinen on the completion of Burnside ring of a p-group in the p-adic topology. In particular it is purely infinite-dimensional, i.e. does not give any estimate if V is of finite dimension. Moreover, combining it with the old result of T. Bartsch ([3], and [4]) one get a characterization of p-toral groups as the unique class of groups with this property (Theorem 3.2).

We should say that the Borsuk-Ulam theorem has many interesting applications to the discrete mathematics (see [15] for details). Also one can deduce a generalization of the Tverberg theorem (cf. [20]), or equipartition theorems as in [12]. On the other hand there are many mini-max invariants of a G-spaces which computations are based on the Borsuk-Ulam theorem. They are used in several nonlinear variational problems with symmetry to estimate from below the number of solutions (cf. [4] for a thorough exposition). Up to our knowledge the Bourgin-Yang theorem does not have so spectacular applications yet, but there is an expectation for results giving estimates of the dimension of the set of solutions of some problems where the Borsuk-Ulam theorem gives only the existence of them.

Throughout the paper $\dim X$ stands for the covering dimension of a space X and coh.dim$X$ stands for the cohomological dimension of a space X, i.e.,

$$\text{coh.dim}X = \max\{n \mid \check{H}^n(X) \neq 0\}$$

where $\check{H}^n(-)$ denotes the Čech cohomology with coefficients $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$, depending on whether $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$. Since we are working with Čech cohomology theory, we have coh.dim$X \leq \dim X$. Also, $H_*(-)$, $H^*(-)$ ($\tilde{H}_*(-)$, $\tilde{H}^*(-)$) denote the (reduced) singular (co)homology with coefficients $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$, depending on whether $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$.

Next, let us consider the important Borsuk-Ulam type theorem proved by Assadi in [1, page 23] (for p-torus) and Clapp and Puppe in [8, Theorem 6.4].

Theorem 1.1. *Let G be a p-torus or a torus. Let X and Y be G-spaces with fixed-points-free actions; moreover, in the case of a torus action assume additionally that Y has finitely many orbit types. Suppose that $\tilde{H}_j(X) = \tilde{H}^j(X) = 0$ for $j < n$, Y is compact or paracompact and finite-dimensional, and $H_j(Y) = H^j(Y) = 0$ for $j \geq n$. Then there exists no G-equivariant map of X into Y.*

We recall that for $G = \mathbb{Z}_p^k$, with p prime odd, and $G = \mathbb{T}^k$ every nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure, thus V and W admit it too. Denote $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$, and correspondingly $d(W) = \dim_{\mathbb{C}} W = \frac{1}{2}\dim_{\mathbb{R}} W$. If $G = \mathbb{Z}_2^k$ and V,W are orthogonal representations of G, then denote $d(V) = \dim_{\mathbb{R}} V$, and respectively $d(W) = \dim_{\mathbb{R}} W$.

2. Bourgin-Yang theorem for p-torus and torus

The next result is the classical version of the Bourgin-Yang theorem for p-torus and torus.

Theorem 2.1. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$ such that $V^G = W^G = \{0\}$. If $f : S(V) \to W$ be a G-equivariant map, then*

$$\text{coh.dim}Z_f \geq \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1\,.$$

In particular, if $\dim_{\mathbb{R}} W < \dim_{\mathbb{R}} V$, then there is no G-equivariant map from $S(V)$ into $S(W)$.

Proof. Denote $m = \dim_{\mathbb{R}} V$ and $n = \dim_{\mathbb{R}} W$ and suppose

$$\text{coh.dim}Z_f < m - n - 1.$$

Then,

$$\check{H}^i(Z_f) = 0,\ \text{for any } i > m - n - 2.$$

By using Poincaré-Alexander-Lefschetz duality and the long exact sequence of the pair $(SV, SV \setminus Z_f)$, we conclude

$$0 = \check{H}^i(Z_f) = H_{m-1-i}(SV, SV \setminus Z_f) = \tilde{H}_{m-i-2}(SV \setminus Z_f),\ \text{for } j = m - i - 2 < n,\ \text{i.e.,}$$

$$\tilde{H}_j(SV \setminus Z_f) = 0,\ \text{for } j < n.$$

On the other hand, we have

$$H_j(W \setminus \{0\}) = H_j(SW) = 0,\ \text{for } j \geq n.$$

However,

$$f : SV \setminus Z_f \to W \setminus \{0\}$$

is a G-equivariant map, which contradicts Theorem 1.1.

In particular, if $\dim_{\mathbb{R}} V > \dim_{\mathbb{R}} W$, for a G-map $f : S(V) \to S(W) \subset W$ it implies that coh.dim$Z_f \geq 0$ and, consequently, $Z_f \neq \emptyset$, which gives a contradiction. □

As a consequence we get the following corollary.

Corollary 2.2. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$ with $p > 2$, or $G = \mathbb{T}^k$, such that $V^G = W^G = \{0\}$. If $f : S(V) \to W$ is a G-equivariant map and $\dim_{\mathbb{R}} V > \dim_{\mathbb{R}} W$ then*

$$\text{coh.dim} Z_f \geq 1\,.$$

Proof. Indeed, since every nontrivial orthogonal representation of $G = \mathbb{Z}_p^k$ with $p > 2$, or $G = \mathbb{T}^k$, has a complex structure, the integral number $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1 = 2(d(V) - d(W)) - 1$ is positive and odd. □

Now, for the group $G = \mathbb{Z}_p^k$ and $H \subset G$ an isotropy group, considering $f^H = f_{|S(V^H)} : S(V^H) \to W^H$ and the set

$$Z_f^H := \{v \in S(V^H) \mid f^H(v) = 0\} = Z_f \cap S(V^H)$$

we have the following results.

Corollary 2.3. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$, with $V^G = W^G = \{0\}$, and let $f : S(V) \to W$ be an equivariant map. Then, for every isotropy group $H \subset G$ of the action on $S(V)$, we have $f(S(V^H)) \subset W^H$, and for the cohomological dimension*

$$\text{coh.dim}(Z_f^H) \geq \dim_{\mathbb{R}} V^H - \dim_{\mathbb{R}} W^H - 1.$$

Proof. Let H be an isotropy subgroup in $S(V)$ and $f^H : S(V^H) \to W^H$ the restriction of f to $S(V^H)$. Note that V^H is a sub-representation of G, the action of G on $S(V^H)$ factorizes through $K = G/H$. Here $H \simeq \mathbb{Z}_p^{k'}$ and consequently, $G/H \simeq \mathbb{Z}_p^{\bar{k}}$ with $\bar{k} = k - k'$. Moreover, $f : S(V^H) \to W^H$ is a K-equivariant map and therefore, the result follows from Theorem 2.1 applied to the group $\mathbb{Z}_p^{\bar{k}}$. □

We say that an isotropy subgroup $H \subset G$ is maximal if it is maximal with respect to the inclusion. Note that if $G = \mathbb{Z}_p^k$ and the G-space is $S(V)$, $V^G = \{0\}$, then H is maximal if and only if it is p-subtorus of rank $k - 1$. Indeed, these subgroups are maximal and they appear as the isotropy subgroups, because for an irreducible representation $V_\alpha \subset V$ given by $\rho_\alpha : G \to \mathbb{Z}_p \subset \{z \in \mathbb{C} : |z| = 1\}$ and each point $x \in S(V_\alpha) \subset S(V)$ we have $G_x = \ker \rho_\alpha$.

Theorem 2.4. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$, with $V^G = W^G = \{0\}$. A necessary and sufficient condition for the existence of a $\mathbb{Z}_p^k$-equivariant map $f : S(V) \to S(W)$ is*

$$\dim_{\mathbb{R}} V^H \leq \dim_{\mathbb{R}} W^H$$

for every maximal isotropy subgroup H on $S(V)$.

Proof. If there is a G-map $f : S(V) \to S(W) \subset W$, then $Z_f = \emptyset$, which gives $\dim Z_f^H = -1$, for every H. By Corollary 2.3, we have $\dim_{\mathbb{R}} V^H \le \dim_{\mathbb{R}} W^H$, for every maximal isotropy subgroup H, which appears in the decomposition of V.

The converse was already proved in [13, Theorem 2.5], in another formulation. We present this proof. It is enough to show that for every maximal subgroup $H \subset G$, under the assumption $0 < \dim_{\mathbb{R}} V^H \le \dim_{\mathbb{R}} W^H$, there exists a $\mathbb{Z}_p$-map $f^H : S(V^H) \to S(W^H)$. Indeed, once more, using the fact that the action of G on $S(V^H)$ and $S(W^H)$ factorizes through $K = G/H \simeq \mathbb{Z}_p$, and any such a map f^H is G-equivariant, it is sufficient to take the joint of maps of the corresponding joints

$$f = \mathop{*}_{H} f^H : S(V) = S(\bigoplus_H V^H) = \mathop{*}_{H} S(V^H) \;\to\; \mathop{*}_{H} S(W^H) = S(\bigoplus_H W^H) = S(W).$$

We have $V^H = \oplus_{j=1}^{p-1} l_j\, V_j$, with $l_j \ge 0$ and $W^H = \oplus_{j=1}^{p-1} \tilde{l}_j\, V_j$, with $\tilde{l}_j \ge 0$, where each V_j is an irreducible representation of G. Since $d(V^H) = \sum_{j=1}^{p-1} l_j \le \sum_{j=1}^{p-1} \tilde{l}_j = d(W^H)$, it is enough to show that for every $1 \le j_1, j_2 \le p-1$, there exists a $\mathbb{Z}_p$-equivariant map from $S(V_{j_1}) \to S(V_{j_2})$. Let $1 \le j_1^{-1} \le p-1$ be the inverse of j_1 in $\mathbb{Z}_p^*$. It is easy to check that the map $S^1 \ni z \mapsto z^{j_1^{-1} j_2} \in S^1$ is the required $\mathbb{Z}_p$-map.

□

Now, we will consider the problem of existence of G-equivariant maps from the sphere $S(V)$ of an infinite dimensional representation V into the sphere $S(W)$ of a finite dimensional representation W, or the estimate of dimension of the set Z_f, for a G-equivariant map $f : S(V) \to W$.

Theorem 2.5. *Let V, W be an orthogonal representations of a p-torus $G = \mathbb{Z}_p^k$, p prime, or the torus $G = \mathbb{T}^k$, such that $V^G = \{0\} = W^G$. If $\dim V = \infty$ and $\dim W < \infty$, then for every G-equivariant map $f : S(V) \to W$ we have*

$$\dim\, Z_f \ge \text{coh.dim} Z_f \;=\; \infty\,.$$

In particular, there is no G-equivariant map $S(V) \to S(W)$ under this assumption.

Proof. For a given $d \in \mathbb{N}$, let us take a sub-representation $V(d) \subset V$ such that $\dim_{\mathbb{R}} V(d) \ge d$. Restricting the map $f : S(V) \to W$ to $S(V(d)) \subset S(V)$, we have an equivariant map $f_d : S(V(d)) \to W$ with $Z_{f_d} \subset Z_f$. By Theorem 2.1, $\dim Z_{f_d} \ge \text{coh.dim} Z_{f_d} \ge d - \dim_{\mathbb{R}} W - 1$. Now using the monotonicity of dimension we get

$$\dim Z_f \ge \text{coh.dim} Z_f \ge \lim_{d\to\infty} d - \dim W - 1 = \infty.$$

□

3. p-toral groups

In this section, we show that Theorem 2.5 can be extended on a larger class of groups called p-toral. The main result will be formulated analogous to [4, Theorem 3.1].

Definition 3.1. A compact Lie group G is called *p-toral* if it is of the form of an extension

$$1 \hookrightarrow \mathbb{T}^k \hookrightarrow G \to P \to 1,$$

where P is a finite p-group.

In this section, we will use the G-index of G-spaces defined by the Borel equivariant stable cohomotopy theory, i.e the theory $h_G^*(X, A) = \pi_s^*(X \times_G EG, A \times_G EG)$, where π_s^* denotes the stable cohomotopy theory. Following [4, 5.4)], as a family $\mathcal{B}$ of orbits defining value of this length index, we take $\mathcal{B} = \{G/H : H \subsetneq G\}$. Taking $I = h^*(pt)$, h_G^* and $\mathcal{B}$ as above, the value of the length index defined by the triple $\{\mathcal{B}, h_G^*, I\}$ at a pair of G-spaces (X, X') will be denote by $l(X, X')$.

Theorem 3.2 (Characterization of p-toral groups)**.**

a) *Let G be a p-toral group $1 \hookrightarrow \mathbb{T}^k \to G \to P \to 1$. Then, for the sphere $S(V)$ of an infinite-dimensional fixed point free G-Hilbert space (orthogonal representation) V and a finite dimensional orthogonal representation W of G, such that $W^G = \{0\}$, and a G-equivariant map $f : S(V) \to W$, we have*

$$\dim Z_f \;=\; l(Z_f) \;=\; \infty\,.$$

b) *If G is not p-toral, then there exist an infinite-dimensional fixed point free G-Hilbert space V, a finite dimensional representation W of G with $W^G = \{0\}$ and an equivariant map $f : S(V) \to W$ such that*

$$Z_f = \emptyset, \quad e.g. \quad \dim Z_f = -1 < \infty\,.$$

Proof. The part b) follows directly from [3, Theorem 2)]. It states that for any not p-toral group and every orthogonal Hilbert representation V, $V^G = \{0\}$, there exist an orthogonal representation W, with $W^G = \{0\}$ and $\dim W < \infty$, and a G-map $f : S(V) \to W \setminus \{0\}$. Therefore $Z_f = \emptyset$, which proves part b).

To show a) we adapt the arguments of [5] and [3] exposed in an extended form in [4, Chapter 5]. First, we have the following

Proposition 3.3 ([4, 5.11, 5.12])**.** *For a p-group P and a contractible P-space X, we have $l(X) = \infty$.*

Note that we do not require $X^P = \emptyset$. Recall that if V is an infinite-dimensional Hilbert space then $X = S(V)$ is a metric G-space which is contractible, because $S(V)$ is homeomorphic to $\overset{\circ}{D}(V)$. Consequently, it follows from Proposition 3.3 that $l(S(V)) = \infty$.

On the other hand, we have the following

Proposition 3.4 (cf. [4, 5.4])**.** *For every finite dimensional orthogonal representation W, with $W^G = \{0\}$, and any G-length index as above, we have $l(S(W)) < \infty$.*

Proof. Indeed, $S(W)$ is a compact G-space and each orbit of $S(W)$ can be mapped into some element (orbit) of $\mathcal{B}$. Now, the statement reduces to [4, Corollary 4.9 b)]. ☐

Note that the statement in the last propositions holds for every equivariant cohomology theory h^*.

An essential step in our proof is the following result.

Lemma 3.5. *Let G be a finite group. If X is a finite dimensional metric G-space, with $X^G = \emptyset$, then $l(X) < \infty$.*

Proof. Since X is finite dimensional, so is X/G. Since G is finite, there is only a finite number of orbit types on X. Now, by the Mostow theorem (cf. [7, Theorem 10.1]), there exist a finite dimensional orthogonal representation V of G and a G-embedding $\iota : X \to V$ of X into V. Since $X^G = \emptyset$, we have $\iota(X) \cap V^G = \emptyset$. Consequently, composing ι with $p_0^\perp : V \to V_\perp^G$, the orthogonal projection of V onto the orthogonal complement of V^G, we get an equivariant map $\phi : X \to V_\perp^G \setminus \{0\}$. Now, composing ϕ with the retraction $V_\perp^G \setminus \{0\} \to S(V_\perp^G)$ we obtain a G-equivariant map $\psi : X \to S(V_\perp^G)$. Therefore, it follows from the monotonicity property of the length index l (cf. [4, Theorem 4.7]) and Proposition 3.4 that $l(X) \le l(S(V_\perp^G)) < \infty$. □

Proof of Theorem 3.2 a). Suppose first that G is finite, i.e. $k = 0$, and $G = P$ is a finite p-group and let $f : S(V) \to W$ be a P-equivariant map. Since Z_f is closed G-invariant subspace of $S(V)$, by continuity property of the length index (cf. [4, 4.7 Continuity]), there exists an open P-invariant neighborhood $Z_f \subset \mathcal{U}$ such that $l(Z_f) = l(\mathcal{U})$.

Denote $\mathcal{V} = S(V) \setminus Z_f$ which is a P-invariant open subset of $S(V)$. Note that $f : \mathcal{V} \to W \setminus \{0\}$ is an equivariant map, and it follows from the monotonicity property of the index that $l(\mathcal{V}) \le l(W \setminus \{0\})$. Also, by Proposition 3.3, we have $l(S(V)) = \infty$, and $l(S(W)) < \infty$, by Proposition 3.4 and the assumption $W^G = \{0\}$.

On the other hand $W \setminus \{0\}$ is P-equivariantly homotopy equivalent to $S(W)$ and, consequently, has the same index. Using the subadditivity property of the index, we get

$$\infty = l(S(V)) \le l(\mathcal{U}) + l(\mathcal{V}) \le l(Z_f) + l(S(W)).$$

Since $l(S(W)) < \infty$, we conclude that $l(Z_f) = \infty$. Note that $Z_f^G = \emptyset$, because $S(V)^G = \emptyset$. If $\dim Z_f < \infty$, then $l(Z_f) < \infty$ by Lemma 3.5, which is a contradiction. Thus, $\dim Z_f = \infty$.

Now, assume that G is an extension $1 \to \mathbb{T}^k \hookrightarrow G \to P \to 1$, with $k \ge 1$. We distinguish two cases:

$$\text{either } \dim V^{\mathbb{T}^k} = \infty\,, \text{ or } \dim V^{\mathbb{T}^k} < \infty\,.$$

First suppose that $\dim V^{\mathbb{T}^k} = \infty$. Note that $V^{\mathbb{T}^k}$ has a natural action of $P = G/\mathbb{T}^k$, with the fixed point set $V^G = (V^{\mathbb{T}^k})^P = \{0\}$. Moreover, the restriction $f_{|S(V)^{\mathbb{T}^k}}$ maps $S(V)^{\mathbb{T}^k}$ P-equivariantly into $W^{\mathbb{T}^k} \subset W$. Applying the previous case for $G = P$, and for the triple $(V^{\mathbb{T}^k}, W^{\mathbb{T}^k}, f_{|S(V)^{\mathbb{T}^k}})$ we conclude that $\dim Z_f \ge \dim Z_{f^{\mathbb{T}^k}} = \infty$.

Now assume that $\dim V^{\mathbb{T}^k} < \infty$. First observe that $V^{\mathbb{T}^k}$ is a $N(\mathbb{T}^k)$ invariant subspace of V, where $N(H)$ is the normalizer of H in G. But $\mathbb{T}^k$ is a normal subgroup as the component of the identity, thus $V^{\mathbb{T}^k}$ is a sub-representation of V. Let $V' = V_\perp^{\mathbb{T}^k}$ be the orthogonal complement of $V^{\mathbb{T}^k}$. By our assumption $\dim V' = \infty$.

Following a standard argument also used in [4, Proof of Theorem 3.1a]), we claim that for any p-toral group G and for every compact G-space A, with $A^G = \emptyset$, there exists a finite p-group P of G, which acts on A without fixed points: $A^P = \emptyset$. To see this, observe that G can be approximated by finite p-groups. More precisely, for any natural number s consider the set $P^s := \{g \in G : g^{p^s} = e\}$. If p^s is a multiple of the order of G/G_0, G_0 the component of the identity, then P^s is a subgroup of G, according to known results (see [4, 5.4] for references). P^s is obviously a finite p-group which is an extension of $G_0 \cap P^s$ by G/G_0, i.e. has the form $1 \to P_s \hookrightarrow P^s \to P$ with $P_s = P^s \cap (G_0 = \mathbb{T}^k)$. Moreover, it is clear that $A^{P^s} = \emptyset$, if s is big enough, because A is a compact fixed point free G-space, i.e. has only a finite number of isotropy orbit types.

Now, take $A = S(W)$ and P^s as above. Then, $f : S(V') \to W$ is a P^s-equivariant map and $W^{P^s} = \{0\}$. Consider $\tilde{V} = (V')^{P^s}$. If $\dim \tilde{V} = \infty$, then $\dim Z_{f|S(\tilde{V})} = \dim S(\tilde{V}) = \infty$, because $f(S(V)^{P^s}) \subset W^{P^s} = \{0\}$. Consequently, assume that $\dim \tilde{V} < \infty$. Once more, $\tilde{V}$ is a P^s-subrepresentation and we can take the orthogonal complement V'' of it in V'. By our assumption and the choice of V'', we have $(S(V''))^{P^s} = \emptyset$ and $\dim V'' = \infty$.

In this way, we reduced the assumption to the already studied case of a finite p-group. Applying it to $f : S(V'') \to W$, we have $\dim Z_f \geq \dim Z_{f_{|S(V'')}} = \infty$, which completes the proof of Theorem 3.2. □

Remark 3.6. One can easily note that our proof of Theorem 3.2 a) is more complicated than the corresponding Borsuk-Ulam theorem presented in [5] and [4]. It is caused by the fact that we need the assumption $S(V)^{P^s} = \emptyset$, which is not necessary in the study of the Borsuk-Ulam problem. Indeed, if $S(V)^{P^s} \neq \emptyset$, then there is no P^s-map from $S(V)$ into $S(W)$, because $S(W)^{P^s} = \emptyset$. The mentioned assumption is necessary to know that $Z_f^{P^s} = \emptyset$, which is necessary to apply Lemma 3.5.

Remark 3.7. Note that the torus $G = \mathbb{T}^k$ and the p-torus $\mathbb{Z}_p^k$ are toral, with $p = 1$ or $k = 0$, respectively. Then, the above argument can be applied to any prime p. In this way, we obtain another proof of Theorem 2.5. On the other hand, our proof of Theorem 2.5 is completely elementary. Contrary to it, the proof of Theorem 3.2 uses Proposition 3.3. The latter is the result of [5] and is based on a deep topological result namely the Segal conjecture, proved by G. Carlsson.

Corollary 3.8. *The statement of Theorem 3.2 holds, if we replace* $\dim Z_f$ *by* $\dim Z_f/G$ *in its statement.*

Proof. The statement of corollary follows directly from the main result of [9], since for a compact Lie group G and a metric G-space X, we have $\dim X - \dim X/G \leq \dim G$. □

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