SEMI-GLOBAL SOLVABILITY WITH LOSS OF ONE DERIVATIVE OF PARTIAL DIFFERENTIAL OPERATORS ON SURFACES

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Abstract. In this work we prove that for linear partial differential operators in two variables the well known Hörmander's semi-global solvability theorem [Hör2] can be improved by showing that solutions can in fact be taken with a sharp loss of one derivative.

Introduction

The semi-global solvability for a pseudo-differential operator defined on a smooth manifold and satisfying condition $(\mathcal{P})$ was established by L. Hörmander in [Hör2] (cf. also [Hör3]). In more precise terms, let Ω be a smooth manifold and let P be a (properly supported) classical pseudo-differential operator on Ω whose principal symbol p is a homogeneous functions of order m. Assume that P has simple real characteristics, which means that $d_\xi p(x,\xi) \neq 0$ in $T^*\Omega\backslash 0$ and also that p satisfies the well known Nirenberg-Treves solvability condition $(\mathcal{P})$ (cf. [Hör3] where equivalent formulations of this condition are given). The semi-global solvability result for P mentioned above can be stated in the following way: if $K \subset \Omega$ is a compact subset for which every characteristic point for P over K lies on a compact semi-bicharacteristic interval with no characteristic endpoint over K, then the equation $Pu = f$, $f \in C^\infty(\Omega)$, can be smoothly solved in a full neighborhood of K if f is orthogonal to the finite dimensional space $\mathcal{N}(K) \doteq \{u \in \mathcal{E}'(K) : P^*u = 0\} \subset C_c^\infty(K)$. Here by a semi-bicharacteristic we mean an integral curve of the Hamilton field of $\mathsf{Re}\, qp$ on which $\mathsf{Re}\, qp$ vanishes, where $q \neq 0$ is smooth and positively homogeneous on $T^*\Omega\backslash 0$.

This statement regarding the existence of smooth solutions, which even in the case of local solvability (i.e., when the compact set K reduces to a point) was an open question until the appearance of [Hör2], is a consequence of a precise statement involving Sobolev spaces [1]: if K is a compact subset of Ω satisfying the non-trapping condition just described then the following holds: if $s \in \mathbb{R}$ and if $f \in H^s_{\mathrm{loc}}(\Omega)$ is orthogonal to $\mathcal{N}(K)$ then for every $t < s + m - 1$ there is $u \in H^t_{\mathrm{loc}}(\Omega)$ solving the equation $Pu = f$ in a neighborhood of K.

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[1] In this work we shall denote by $H^s_{\mathrm{loc}}(\Omega)$ the local Sobolev space of order $s \in \mathbb{R}$ in Ω. Also if $K \subset \Omega$ is compact we shall set $H^s(K) = H^s_{\mathrm{loc}}(\Omega) \cap \mathcal{E}'(K)$. Finally we shall denote by $H^s_c(\Omega)$ the union of the spaces $H^s(K)$ when K runs over the set of all compact subsets of Ω.

However, for linear partial differential operators (LPDO's) with simple real characteristics satisfying condition $(\mathcal{P})$, it is known that local solutions of the equation $Pu = f \in H^s$ can be taken in the smaller space H^{s+m-1} ([BF], [NT]). This result is sharp for general LPDO's with simple real characteristics and is usually termed as (local) solvability with loss of one derivative.

It is then natural to ask whether in the semi-global case, when K is a general compact set rather than a point, a similar result can be achieved, i.e., whether solutions near K can always be found with a regularity loss not higher than one derivative.

In other words, can Hörmander's solvability result be improved, at least for LPDO's with simple real characteristics satisfying condition $(\mathcal{P})$, by providing solutions near K in the space $H^{s+m-1}_{\mathrm{loc}}(\Omega)$?

This is the question we address in the present article and prove that the answer is yes when Ω is two dimensional.

We start our exposition by presenting a very detailed study for first order operators in two variables. Write $P = L + c$, where L is a nowhere vanishing complex vector field in two variables and c is a smooth function. In this case the characterization of the non-trapping condition for a given compact set K is described in terms of the Sussmann's orbits of L: no closed one dimensional orbit of L is contained in K (such condition is similar to the one obtained in [DH, Theorem 7.1.5]). According to one of our main results (Theorem 1.1 below) given such K and $f \in H^s_{\mathrm{loc}}(\Omega)$ we can find a solution $u \in H^s_{\mathrm{loc}}(\Omega)$ to the equation $Pu = f$ near K (notice that in this case we have $\mathcal{N}(K) = 0$, a consequence of the property of uniqueness in the Cauchy problem, which is valid for P).

The starting point in our argument is a semi-global estimate for degenerate Cauchy-Riemann operators proved in [Hör3, Lemma 26.7.1]. From this result a corresponding semi-global estimate in Sobolev norms can be obtained for first order operators satisfying condition $(\mathcal{P})$, where now changes of sign are allowed (cf. Lemma 3.1 and Theorem 3.1). Besides the importance of these results for the proof of Theorem 1.1, a standard argument allows one to prove their microlocal counterparts (Theorem 3.2 and Corollary 3.1), which provide the key arguments for the proof of Hörmander's semi-global solvability with loss of one derivative in the case of a LPDO of arbitrary order with simple real characteristics in two variables that satisfies condition $(\mathcal{P})$ (Theorem 6.2).

As an application of the semi-global solvability results for first-order LPDO's we also discuss in Section 5 the global solvability in Sobolev spaces with loss of one derivative for first-order LPDO's and present a geometric characterization of those which are globally solvable in $H^s_{\mathrm{loc}}(\Omega)$ (Theorem 5.1).

Finally, we take the opportunity to express our gratitude to Prof. Nicolas Lerner who brought this question to our attention, discussed with us several of its aspects and showed constant interest in our research. We are also thankful to the anonymous referee whose comments helped us to improve the presentation of the paper.

1. Sobolev regularity for first-order operators

In this section we deal with a first-order operator on a connected, orientable, non compact, two-dimensional smooth manifold Ω which in a local coordinate patch $U \subset \Omega$ is expressed as

$$L + c = A(x,t)\frac{\partial}{\partial t} + B(x,t)\frac{\partial}{\partial x} + c(x,t)$$

with smooth complex coefficients such that

$$|A(x,t)| + |B(x,t)| > 0, \quad (x,t) \in U.$$

We always assume that

($\mathcal{P}$) $\mathsf{Re}\, L \wedge \mathsf{Im}\, L$ does not change sign on the two-dimensional orbits of L

which is well known to be necessary for local solvability. We recall some background on the geometry of condition ($\mathcal{P}$) for vector fields. Let us write $X = \mathsf{Re}\, L$, $Y = \mathsf{Im}\, L$. The orbits of the pair of real vector fields X and Y in the sense of Sussmann [Su] are called the orbits of L. Two points belong to same orbit if and only if they can be joined by a continuous piecewise differentiable curve, such that each piece is an integral curve of $\pm X$ or $\pm Y$. The orbits are connected submanifolds of Ω, tangent to X and Y. They can be used to characterize the vector fields that satisfy ($\mathcal{P}$) [Hou2, Thm 3.1]. For a more detailed discussion on the orbits of L and their role in the Nirenberg-Treves condition ($\mathcal{P}$) we refer to [Hou2] and [BCH, p. 177]. In particular, if L satisfies ($\mathcal{P}$), the orbits of L are submanifolds of dimension 1 or 2. In general, for a complex vector field without zeros, a finer qualitative description than the simple decomposition of Ω in orbits may be given. Consider the open subset Ω_0 of Ω where X and Y are linearly independent and set $\Omega_1 = \Omega \setminus \Omega_0$. Note that Ω_0 is the set of elliptic points of L in Ω. If $p \in \Omega_0$ the orbit $\mathcal{O}_p$ of p has dimension two, on the other hand, if If $p \in \Omega_1$, $\mathcal{O}_p$ may have dimension one or two. We now define an equivalence relation $\sim$ on Ω_1.

Definition 1.1. *Let $p_1, p_2 \in \Omega_1$. We say that $p_1 \sim p_2$ if and only if either $p_1 = p_2$ or else there exist a smooth curve $\gamma : [a,b] \to \Omega$ such that $\gamma(a) = p_1$, $\gamma(b) = p_2$ and* $\mathsf{Re}\, L$, $\mathsf{Im}\, L$ *are parallel to $\gamma'(s)$ along $\gamma(s)$. This is an equivalence relation and the equivalence class of p will be denoted by $[p]$.*

Note that this equivalence relation depends only on the vector bundle spanned by L, i.e., if L' is a nonvanishing multiple of L the equivalence relation induced by L and L' are identical. It is known ([Hou2]) that

Proposition 1.1. *Let $p \in \Omega_1$ and let $\mathcal{O}_p$ denote the orbit of L in Ω that passes through p. Then, $[p]$ is a subset of $\mathcal{O}_p$ homeomorphic to one of the following spaces:*

$$(i)\ \{0\}, \qquad (ii)\ I = [0,1], \qquad (iii)\ S^1 = \{e^{i\theta}\}, \qquad (iv)\ \mathbb{R}, \qquad (v)\ \mathbb{R}^+ = [0,\infty).$$

The inclusion $[p] \subset \mathcal{O}_p$ is an embedding if we give $[p]$ its natural differential structure of integral curve. If $\mathcal{O}$ is a generic orbit and $\dim \mathcal{O} = 2$, $\mathcal{O}$ does not contain any class $[p]$ homeomorphic to $\mathbb{R}$ or S^1 and for any $p \in \mathcal{O}$ there is a curve $\alpha : [0,1] \to \mathcal{O}$ such that $\alpha(0) = p$ and $\mathsf{Re}\, L$, $\mathsf{Im}\, L$ are linearly independent at $\alpha(1)$.

We may fix a smooth positive density dV on Ω. The (formal) L^2-adjoint of L will be denoted by L^*; it is determined by the identity

$$\int Lu\,\bar{v}\,dV = \int u\,\overline{L^* v}\,dV, \quad u, v \in C_c^\infty(\Omega).$$

In local coordinates where $L = A\partial_t + B\partial_x$ and $dV = \rho\, dxdt$, we have $L^* = -\bar{L} - ((\rho\bar{A})_t + (\rho\bar{B})_x)/\rho$. In particular, L satisfies $(\mathcal{P})$ if and only if L^* satisfies $(\mathcal{P})$ which allows to interchange the roles of L and L^* in many arguments. We are interested in the semi-global solvability in Sobolev spaces with loss of one derivative of the equation $Lu = f$ on some open neighborhood of an arbitrary compact subset

$$K \subset \Omega.$$

Given a real number s, we want to find an open subset U of Ω containing K with the property that for any $f \in H^s_{\mathrm{loc}}(\Omega)$ there exists $u \in H^s_{\mathrm{loc}}(\Omega)$ such that the equation $(L + c)u = f$ holds on U. It is a standard fact that such solvability property is implied by a convenient Sobolev regularity property satisfied by the L^2-adjoint $(L + c)^* = L^* + \bar{c}$ as described in the next

Proposition 1.2. *Let $s \in \mathbb{R}$ and $K \subset\subset \Omega$. Assume that there is an open set U, $K \subset U \subset \Omega$ such that*

$$u \in \mathcal{E}'(\overline{U}) \cap H^{s-1}_{\mathrm{loc}} \text{ and } (L^* + \overline{c})u \in H^s_{\mathrm{loc}} \implies u \in H^s_{\mathrm{loc}}. \tag{1.1}$$

Then, if $f \in H^{-s}_{\mathrm{loc}}(\Omega)$, we may find $u \in H^{-s}_{\mathrm{loc}}(\Omega)$ such that

$$(L + c)u = f \quad \text{on } U.$$

For the sake of completeness, we recall briefly the well known argument that shows that (1.1) implies H^{-s}-solvability on U. Consider the space $\mathfrak{X}$ of distributions $u \in H^{s-1}_{\mathrm{loc}}(\Omega)$ supported in $\overline{U}$ such that $(L^* + \bar{c})u \in H^s_{\mathrm{loc}}(\Omega)$, equipped with the norm $\|u\|_{s-1} + \|L^* u\|_s$, $u \in \mathfrak{X}$. It follows from (1.1) that $\mathfrak{X} \subset H^s_{\mathrm{loc}}(\Omega)$ and by the closed graph theorem there is an estimate

$$\|u\|_s \le C_1(\|(L + c)^* u\|_s + \|u\|_{s-1}), \quad u \in \mathfrak{X}. \tag{1.2}$$

Next we show that this implies that for some $C > 0$ we must have

$$\|u\|_s \le C\|(L + c)^* u\|_s, \quad u \in \mathfrak{X}. \tag{1.3}$$

Otherwise, there would be a sequence $u_j \in \mathfrak{X}$, with $\|u_j\|_s = 1$ and $\|L^* u_j\|_s \to 0$. Passing through a subsequence we may assume that u_j is convergent in $H^{s-1}_{\text{loc}}(\Omega)$ to a distribution $u \in \mathfrak{X}$ that satisfies $(L+c)^* u = 0$. In view of (1.3), $C_1 \|u\|_{s-1} \geq 1$, so $u \neq 0$. On the other hand, $0 = (L+c)^* u \in H^s_{\text{loc}}(\Omega)$ for every $s \in \mathbb{R}$, so iterating (1.2) we conclude that $u \in C^\infty_c(\mathbb{R}^2)$. However, $(L+c)^*$ does not have nontrivial homogeneous solutions with compact support, in fact, condition $(\mathcal{P})$ implies that $(L+c)^*$ enjoy uniqueness in the noncharacteristic Cauchy problem for C^1 functions (see, e.g.,[ST]) and that contradiction proves (1.3). Since $C^\infty_c(U) \subset \mathfrak{X}$, we have in particular that

$$\|\phi\|_s \leq C \|(L+c)^* \phi\|_s, \quad \phi \in C^\infty_c(U),$$

which by a standard application of the Hahn-Banach theorem (cf. [Hör3], p.64) allows one to solve the equation $(L+c)u = f$ on U with $u \in H^{-s}_{\text{loc}}(\Omega)$ for any given $f \in H^{-s}_{\text{loc}}(\Omega)$. □

If L is a multiple of a real vector field, the solvability of the equation $(L+c)u = f$ on compact subsets of Ω is well understood and characterized by [DH, Theorem 6.4.1], in particular our results will give new information only if L has at least one elliptic point.

We now look at some geometric conditions concerning L and K that are relevant for the solvability of the equation $(L+c)u = f$. They will bear upon the equivalence classes $[p]$, $p \in K \cap \Omega_1$, described in Proposition 1.1. Suppose $[p]$ is of type (iv) and consider an embedding $j : \mathbb{R} \longrightarrow [p]$. We define two sets:

$$\omega(p) = \bigcap_{k=1}^{\infty} \overline{j[k,\infty)}, \qquad \alpha(p) = \bigcap_{k=1}^{\infty} \overline{j(-\infty,k]}.$$

If j' is another embedding and $j^{-1} \circ j'$ is increasing we get the same sets $\omega(p)$ and $\alpha(p)$ by using j' instead of j while if $j^{-1} \circ j'$ is decreasing then $\omega(p)$ and $\alpha(p)$ are interchanged. At any rate, the pair of sets $\{\alpha(p), \omega(p)\}$ does not depend on the particular embedding. Similarly, if $[p]$ is of type (v) and $j : \mathbb{R}^+ \longrightarrow [p]$ is an embedding, we define

$$\omega(p) = \bigcap_{k=1}^{\infty} \overline{j[k,\infty)}$$

and the definition is independent of the particular embedding j. Note that $\alpha(p)$ and $\omega(p)$ are closed connected sets, invariant under the equivalence relation $\sim$. A set $\Gamma \subset \Omega$ is said to be $\sim$-invariant (or simply invariant when there is no risk of confusion) if it is a union of equivalence classes: $\Gamma = \bigcup_{q \in \Gamma} [q]$. The following lemma will be needed in the proof of the main result.

Lemma 1.1. *Let L satisfy $(\mathcal{P})$ in Ω, let K be a compact subset of Ω and let $p \in K \cap \Omega_1$ be a point. If $[p]$ is of type (iv) and either $\emptyset \neq \alpha(p) \subset K$ or $\emptyset \neq \omega(p) \subset K$,* *then K contains a closed one-dimensional orbit of L. The same conclusion holds if $[p]$ is of type (v) and $\omega(p) \subset K$.*

PROOF: Assume that $[p]$ is of type (iv) or (v) and $\emptyset \neq \omega(p) \subset K$. Then $\omega(p)$ is compact, $\sim$-invariant and contains a minimal set $\mu \neq \emptyset$, i.e., $\mu \subset \omega(p) \subset \Omega_1$ is closed, invariant and contains no such proper subset. Let $q \in \mu$ and consider the orbit $\mathcal{O}_q$. If $\mathcal{O}_q$ is two-dimensional, then $[q]$ is neither of type (iii) nor (iv). In this case, we may find local coordinates (x,t) around $[q]$ and a nonvanishing multiple L' of L such that in the coordinate patch $|x| < a$, $|t| < T$, $L' = \partial_t + ib(x,t)\partial_x$ with $b(x,t)$ real, $q = (0,0)$, $[q] \subset \{0\} \times (-T,T)$ and $b(0,-T) \neq 0$. If $j : \mathbb{R}^+ \longrightarrow [p]$ is an embedding, note that for any $k = 1, 2, \ldots$, the intersection of $j[k,\infty)$ with the coordinate patch is a union of vertical segments $\{x_\ell\} \times (-T,T)$ such that $b(x_\ell,t) \equiv 0$, $-T < t < T$, and then $j[k,\infty)$ must remain at a fixed positive distance of q which implies that $q \notin \omega(p)$. This contradiction shows that $\mathcal{O}_q$ is a one-dimensional orbit, either closed or homeomorphic to $\mathbb{R}$. We shall now rule out the latter possibility. Note that $\overline{\mathcal{O}_q} = \mu$. If the interior of μ is nonempty, we look at $\mu \setminus \operatorname{Int}\mu$ and conclude by minimality that $\operatorname{Int}\mu = \mu$ is both open and closed which implies that $\mu = \Omega$ contradicting that μ is compact. Hence, μ is nowhere dense. At this point, the arguments in the proof of the generalized Poincaré-Bendixson-Schwartz theorem [Hou1, Theorem 1] apply to show that μ cannot be nowhere dense if $\mathcal{O}_q$ is homeomorphic to $\mathbb{R}$. It follows that $\mathcal{O}_q$ is a closed orbit of dimension one contained in K. The same arguments apply if $[p]$ is of type (iv) and $\emptyset \neq \alpha(p) \subset K$. $\square$

Our main result concerning the solvability of $L + c$, where $c(x,t)$ is any smooth complex valued function, is

Theorem 1.1. *Let L satisfy $(\mathcal{P})$ in Ω and let K be a compact subset of Ω. Assume that K contains no closed one-dimensional orbit of L. There exists an open neighborhood $U \subset \Omega$ of K such that for every $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R})$ the equation*

$$Lu(x,t) + c(x,t)u(x,t) = f(x,t), \quad (x,t) \in U$$

is satisfied for some $u \in H^s_{\mathrm{loc}}(\Omega)$.

Note that if K satisfies the hypotheses of Theorem 1.1 with respect to L it also does with respect to L^* and so the roles of L and L^* may be interchanged.

Roughly speaking, the strategy to prove Theorem 1.1 is to find an open subset U, $K \subset U \subset \Omega$, such that the regularity hypothesis (1.1) in Proposition 1.2 (with s replaced by $-s$) is satisfied. To say that $u \in \mathcal{E}'(U)$ is in $H^s_{\mathrm{loc}}(\Omega)$ is equivalent to saying that the H^s-singular support of u, denoted by $SS_s u$, is empty. We recall that $p \in \Omega$ is not in $SS_s u$ if and only if there exist $\phi \in C^\infty_c(\Omega)$ such that $\phi(p) \neq 0$ and $\phi u \in H^s_{\mathrm{loc}}(\Omega)$. For every point $p \in K$ we will find an open set $U_p \subset \Omega$ containing p such that $(L+c)u \in H^s_{\mathrm{loc}}$ implies $SS_s u \cap U_p = \emptyset$ and taking $U = \bigcup_{p \in K} U_p$ we will see that (1.1) holds. The sets U_p will be chosen so that, after multiplication of $L+c$ by a nonvanishing factor and the introduction of a convenient change of coordinates, U_p becomes a rectangle $Q_\delta = (-\delta,\delta)\times(-T,T)$ and L has the simple canonical form $L = \partial_t - ib(x,t)\partial_x$ with $b(x,t)$ real. Once we are in this situation, the required Sobolev regularity will be proved by obtaining appropriate a priori estimates in Sobolev norms and the initial step will be to get them for the case $s=0$ which is the subject of the next section.

2. A priori L^2 estimates

We deal with a first-order operator of the form

$$L + c = \frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x} + c(x,t)$$

defined for all $(x,t)\in\mathbb{R}^2$. We always assume that

($\star$) $b(x,t)$ and $c(x,t)$ are smooth and bounded with bounded derivatives;

($\mathcal{P}$) $b(x,t)$ is real and the function $t\mapsto b(x,t)$ does not change sign for any $x\in\mathbb{R}$.

The L^2-adjoint of L is given by $L^* = -\bar{L} - ib_x$. We are interested in L^2 a priori estimates on a rectangle of the form

$$\|u\|_{L^2} \le C\|Lu\|_{L^2}, \quad u\in C_c^\infty(Q_\delta), \tag{2.1}$$

where $Q_\delta \doteq (-\delta,\delta)\times(-T,T)$, and also estimates like

$$\|u\|_{L^2} \le C\|(L+c)u\|_{L^2}, \quad u\in C_c^\infty(Q_\delta). \tag{2.2}$$

Estimates like this are well known when T is sufficiently small, however, here we want to obtain the estimates keeping T fixed but allowing δ to shrink. For T large and fixed, estimates of this type are proved in [Hör2, Lemma 4.2] under the additional hypothesis that $b(x,t)\ge 0$ on Q_δ and $b(0,t)\equiv 0$ on $(-T,T)$.

Theorem 2.1. *If L and $c(x,t)$ are as above then there are $K>0$ and $0<\delta$ so that*

$$\|u\|_{L^2} \le K\|(L+c)u\|_{L^2}, \quad u\in C_c^\infty(Q_\delta). \tag{2.3}$$

The proof of Theorem 2.1 is divided into two steps.

2.1. **Step 1.** If we assume that $b(x,t)\ge 0$ on Q_δ, (2.3) follows from [Hör3, Lemma 26.7.2]. Of course, a similar argument shows that (2.3) holds assuming instead that $b(x,t)\le 0$ on Q_δ.

2.2. **Step 2.** We now consider the general case. It follows from ($\mathcal{P}$) that we may decompose $\mathbb{R}$ as a finite or at most countable union

$$\mathbb{R} = \bigcup_j I_j \cup \mathcal{N}$$

where

$$\mathcal{N} = \{x\in\mathbb{R} : \quad b(x,t)=0,\ \forall t\in[-T,T]\}$$

and each I_j is an open interval such that $b(x,t)$ does not change sign on $I_j \times (-T,T)$ and is maximal for this property. In the language of orbits, $I_j \times (-T,T)$ is a two dimensional orbit of L in Q_δ while $\{x_0\} \times (-T,T)$ is a one dimensional orbit of L in Q_δ for any $x_0 \in \mathcal{N}$. If $I_j = (\alpha_j, \beta_j)$ it follows that $\alpha_j, \beta_j \in \mathcal{N}$. Let $\chi(x)$ denote the characteristic function of $\mathcal{N}$. Since $b(x,t) = 0$ on $\mathcal{N} \times (-T,T)$ it is easy to prove that $[(L+c), \chi] = 0$. Thus, to prove (1.3) for some $u \in C_c^\infty(\mathbb{R} \times (-T,T))$, it is enough to write $u = \chi u + (1-\chi)u$ and prove separately the inequalities

$$\|\chi u\|_{L^2} \leq K\|(L+c)\chi u\|_{L^2}$$
$$\|(1-\chi)u\|_{L^2} \leq K\|(1-\chi)(L+c)u\|_{L^2}.$$

For $x \in \mathcal{N}$, $L + c = \partial_t + c$ so the first inequality is easy to prove and we focus on the second one. We may write $(1-\chi)u = \sum_j \chi_j u$ with χ_j denoting the characteristic function of I_j and since the intervals I_j are disjoint we have $\|(1-\chi)u\|_{L^2} = \sum_j \|\chi_j u\|_{L^2}$ and $\|(1-\chi)Lu\|_{L^2} = \sum_j \|\chi_j Lu\|_{L^2}$, which further reduces the problem to show the inequality

$$\|\chi_j u\|_{L^2} \leq K\|\chi_j(L+c)u\|_{L^2} = K\|(L+c)\chi_j u\|_{L^2}$$

with K independent of j provided $u \in C_c^\infty(Q_\delta)$ and the width $2\delta > 0$ of Q_δ is sufficiently small. We now set $\tilde{I}_j = I_j \cap (-\delta, \delta)$ and call $\tilde{\chi}_j$ its characteristic function. Let $\psi_k(x) \in C_c^\infty(I_j)$ a sequence of that converges pointwise to $\tilde{\chi}_j(x)$ as $k \to \infty$, and satisfies $\|\psi_k\| \leq 1$, $|\psi_k'(x)| \leq C/\mathrm{dist}\,(x, \mathcal{N})$. It is clear that $\psi_k u \to \tilde{\chi}_j u$ in $L^2(\mathbb{R}^2)$ and using the fact that $b(x,t)$ vanishes on $\mathcal{N} \times (-T,T))$ it also follows that $(L+c)\psi_k u \to (L+c)\tilde{\chi}_j u$ in $L^2(\mathbb{R}^2)$ as $k \to \infty$. Therefore, we need only show that

$$\|\psi_k u\|_{L^2} \leq K\|(L+c)\psi_k u\|_{L^2}.$$

However, $\psi_k u \in C_c^\infty(\tilde{I}_j \times (-T,T)) = C_c^\infty(Q_j)$ where Q_j is a cube with width $< 2\delta$ on which $b(x,t)$ does not take opposite signs, so the desired estimate follows from Step 1 by taking $\delta > 0$ small enough. $\square$

Remark 2.1: Inspection of the proof of [Hör3, Lemma 26.7.2] shows that two basic properties were used in the proof of Theorem 2.1 concerning the zero order term $c(x,t)$ of $L + c$:

(i) the operator $u(x,t) \mapsto c(x,t)u(x,t)$ is bounded in $L^2(\mathbb{R}^2)$;
(ii) the operator $u(x,t) \mapsto [h(D_x), c]u(x,t) = \big(h(D_x)c(x,t) - c(x,t)h(D_x)\big)u(x,t)$ is bounded in $L^2(\mathbb{R}^2)$.

Here, the operator $h(D_x)$ is the pseudo-differential operator with symbol $h(\xi) \in C^\infty(\mathbb{R})$ decreasing, equal to 1 on $(-\infty, -2)$ and 0 on $(-1, \infty)$. These two properties, that are essential in the proof of Step 1, are shared by a pseudo-differential operator of order zero $C(x,t,D_x,D_t)$. Hence, Step 1 of Theorem 2.1 remains valid for $L + C(x,t,D_x,D_t)$. However, in the proof of Step 2, we also used that $u(x,t) \mapsto c(x,t)u(x,t)$ commutes with multiplication by χ_j, a property that $C(x,t,D_x,D_t)$

does not satisfy in general. Furthermore, the commutators $[C(x,t,D_x,D_t),\chi_j]$ might have no uniform bounds as $j\to\infty$.

3. From L^2 estimates to Sobolev estimates

Our objective is to extend to Sobolev norms the L^2 estimates obtained for the operator $L+c(x,t)$ considered in Theorem 2.1 and assumed to satisfy $(\star)$ and $(\mathcal{P})$. We will work under the additional hypothesis that

$$b(0,t)\equiv 0,\quad |t|\le T'. \tag{3.1}$$

Set $Q=(-\delta_0,\delta_0)\times(-T',T')$. In view of Theorem 2.1 applied to $L+c_1(x,t)$ taking $\delta_0>0$ fixed but small enough we may assume that a priori estimate

$$\|v\|_0\le C\|Lv+c_1(x,t)v\|_0,\quad v\in C_c^\infty(Q), \tag{3.2}$$

holds for some $C>0$. The precise choice of $c_1(x,t)$ will be specified later. Fix any $0<T<T'$ and set, as before, $Q_\delta=(-\delta,\delta)\times(-T,T)$, where $0<\delta<\delta_0$. The main step is to obtain estimates for Sobolev norms of the form

$$\|u\|_s\le C_s(\|Lu+c(x,t)u\|_s+\|u\|_{s-1}),\quad u\in C_c^\infty(Q_\delta). \tag{3.3}$$

Set $\lambda^s(\xi,\tau)=(1+\xi^2+\tau^2)^{s/2}$ and let Λ^s be pseudo-differential operator with (total) symbol $\lambda^s(\xi,\tau)$, so $\|u\|_s=\|\Lambda^s u\|_0$, $u\in\mathcal{S}(\mathbb{R}^2)$. Pick up a function $\alpha(x,t)\in C_c^\infty(Q)$ which is identically equal to 1 on neighborhood of $\overline{Q}_\delta$ and for any $s\in\mathbb{R}$ denote by E_s the pseudo-differential operator with symbol $\alpha(x,t)\lambda^s(\xi,\tau)$. The symbol of $R_s\doteq\Lambda^s-E_s$ vanishes identically in a neighborhood of $\overline{Q}_\delta$ whence we see that it is a regularizing operator when acting on distributions supported on $\overline{Q}_\delta$ and

$$\|E_su-\Lambda^su\|_r=\|R_su\|_r\le K_{s,r,s'}\|u\|_{s'},\quad u\in C_c^\infty(Q_\delta), \tag{3.4}$$

for all $s,s',r\in\mathbb{R}$. In particular, if we replace $\|u\|_s$ by $\|E_s\|_0$ in any estimate, we introduce an error that may majorized by a norm of arbitrary lesser order. Keeping in mind (3.2), and observing that $v=E_su\in C_c^\infty(Q)$ if $u\in C_c^\infty(Q_\delta)$, we have for any $u\in C_c^\infty(Q_\delta)$

$$\begin{aligned}\|u\|_s&\le\|E_su\|_0+K_s\|u\|_{s-1}\le C\|LE_su+c_1E_su\|_0+K_s\|u\|_{s-1}\\&\le C\|E_s\big(Lu+E_{-s}[L,E_s]+c_1\big)u\|_0+K_s\|u\|_{s-1}\\&\le C\|Lu+E_{-s}[L,E_s]u+c_1u\|_s+K_s\|u\|_{s-1}.\end{aligned} \tag{3.5}$$

The homogeneous principal symbol of E_s on Q_δ where $\alpha\equiv 1$ is given by μ^s with

$$\mu(\xi,\tau)=(\xi^2+\tau^2)^{1/2}$$

so the principal symbol for $E_{-s}[L,E_s]$ is

$$-\mu^{-s}\{b(x,t)\xi,\mu^s\}=-s\xi\mu^{-1}\{b(x,t),\mu\}=-s\,\frac{b_x(x,t)\xi^2+b_t(x,t)\xi\tau}{\mu^2}$$

and we may write

$$\|u\|_s \leq C\|Lu + Du + c_1 u\|_s + K_s\|u\|_{s-1}, \quad u \in C_c^\infty(Q_\delta), \tag{3.6}$$

where D is the pseudo-differential operator with symbol

$$d(x,t,\xi,\tau) = -s\,\frac{b_x(x,t)\xi^2 + b_t(x,t)\xi\tau}{\mu^2}.$$

Given $\rho > 0$ small, consider smooth functions $\chi_1(\xi,\tau)$, $\chi_2(\xi,\tau)$, positively homogeneous of degree zero for $\xi^2 + \tau^2 \geq 1$ such that $\chi_1 + \chi_2 \equiv 1$ for $\xi^2 + \tau^2 \geq 1$ and

$$\operatorname{supp}\chi_2 \subset \Gamma = \{(\xi,\tau): \quad |\tau| \leq \rho|\xi|\}, \quad \chi_2(\xi,\tau) = 1,\ |\xi| \geq 1,\ |\tau| \leq \rho|\xi|/2.$$

We consider pseudo-differential operators J_k, $k = 1,2$, with symbols $\alpha(x,t)\chi_k(\xi,\tau)$, $k = 1,2$, and set $u_k = J_k u$, $u \in C_c^\infty(Q_\delta)$, $k = 1,2$. We have $u = u_1 + u_2 + Ru$ with a regularizing remainder Ru. Since $L + c$ is microlocally elliptic for $\tau \neq 0$, and the symbol of J_1 is supported on the cone $Q_\delta \times \{|\tau| \geq \rho|\xi|/2\}$ for $|\xi| \geq 1$, we have the elliptic estimate

$$\begin{aligned}\|J_1 u\|_s &\leq C_s\left(\|(L+c)J_1 u\|_{s-1} + \|u\|_{s-1}\right)\\ &\leq C_s'\big(\|(L+c)u\|_{s-1} + \|[(L+c), J_1]u\|_{s-1} + \|u\|_{s-1}\big)\end{aligned}$$

that implies

$$\|u_1\|_s \leq C_s\big(\|(L+c)u\|_s + \|u\|_{s-1}\big), \quad u \in C_c^\infty(Q_\delta). \tag{3.7}$$

On the other hand, to estimate u_2 we invoke (3.6) to get

$$\|u_2\|_s \leq C\|Lu_2 + Du_2 + c_1 u_2\|_s + K_s\|u\|_{s-1}, \quad u \in C_c^\infty(Q_\delta). \tag{3.8}$$

We now declare that $c_1 = sb_x + c$, in which case the symbol of $D + c_1(x,t)$ is

$$\begin{aligned}d(x,t,\xi,\tau) + c_1(x,t) &= -s\,\frac{b_x(x,t)\xi^2 + b_t(x,t)\xi\tau}{\mu^2} + sb_x(x,t) + c(x,t)\\ &= -s\,\frac{b_t(x,t)\xi\tau - b_x(x,t)\tau^2}{\mu^2} + c(x,t) \doteq d_1(x,t,\xi,\tau) + c(x,t),\end{aligned}$$

where the last equality defines $d_1(x,t,\xi,\tau)$. If D_1 is now the pseudo-differential operator with symbol d_1 we note that the principal symbol of $D_1 J_2$, namely

$$\alpha(x,t)d_1(x,t,\xi,\tau)\chi_2(x,t),$$

can be made as small as we wish by taking ρ small because $|\tau|/\mu \leq \rho$ on the support of χ_2. Thus, (3.8) yields

$$\|u_2\|_s \leq C\|Lu_2 + cu_2\|_s + O(\rho)\|u\|_s + K_s\|u\|_{s-1}, \quad u \in C_c^\infty(Q_\delta),$$

which implies

$$\|u_2\|_s \leq C_s\|Lu + cu\|_s + C\|[L, J_2]u\|_s + O(\rho)\|u\|_s + K_s\|u\|_{s-1}, \quad u \in C_c^\infty(Q_\delta).$$

The principal symbol of $[L, J_2] = [-ib\partial_x, J_2]$ on Q_δ is

$$-\{b\xi, \chi_2\} = -\xi\big(b_x\partial_\xi\chi_2 + b_t\partial_\tau\chi_2\big)$$

which, for $|\xi| \geq 1$, vanishes when $x = 0$ and $\tau = 0$ because $b_t(0,t) \equiv 0$, $|t| \leq T'$, and $\partial_\xi\chi_2(\xi, 0) \equiv 0$, $|\xi| \geq 1$. Thus, $[L, J_2]$ is bounded in $H^s_c(Q_\delta)$ with operator norm $\leq C_s(\rho + \delta)$ and the previous inequality implies

$$\|u_2\|_s \leq C_s\|Lu + cu\|_s + C'_s(\rho + \delta)\|u\|_s + K_s\|u\|_{s-1}, \quad u \in C^\infty_c(Q_\delta). \tag{3.9}$$

Therefore, adding (3.7) and (3.9) we get

$$\begin{aligned}\|u\|_s &\leq \|u_1\|_s + \|u_2\|_s + \|Ru\|_s \\ &\leq C_s\|Lu + cu\|_s + C'_s(\rho + \delta)\|u\|_s + K_s\|u\|_{s-1}.\end{aligned}$$

For $C'_s(\rho+\delta) < 1/2$ this gives (3.3) and a density argument easily implies that (3.3) holds as well for $u \in H^{s+1}_c(Q_\delta)$ We have proved.

Lemma 3.1. *Assume $b(0,t) \equiv 0$, $|t| \leq T'$, $0 < T < T'$, $s \in \mathbb{R}$. There exists $\delta > 0$ and $C > 0$ such that*

$$\|u\|_s \leq C\big(\|Lu + cu\|_s + \|u\|_{s-1}\big) \quad u \in H^{s+1}_c(Q_\delta). \quad \square \tag{3.10}$$

We now improve the latter result as follows.

Theorem 3.1. *Assume $b(0,t) \equiv 0$, $|t| \leq T'$, $0 < T < T'$, $s \in \mathbb{R}$. There exists $\delta > 0$ and $C > 0$ such that*

$$u \in H^{s-1}_c(Q_\delta) \text{ and } (L + c)u \in H^s \implies u \in H^s$$

and

$$\|u\|_s \leq C\big(\|(L + c)u\|_s + \|u\|_{s-1}\big). \tag{3.11}$$

Therefore, for any $s \in \mathbb{R}$ and $u \in \mathcal{E}'(Q_\delta)$, $(L + c)u \in H^s \implies u \in H^s$.

Proof: A standard adaptation of the proof of Lemma 3.1 gives the result. Define the norm

$$\|u\|^2_{s,\epsilon} = \frac{1}{(2\pi)^2}\int |\widehat{u}(\xi,\tau)|^2\lambda^{2s}(\xi,\tau)\big(1 + \varepsilon\mu^2(\xi,\tau)\big)^{-2}\,d\xi d\tau$$

which is equivalent to the norm $\|u\|_{s-2}$ but tends to $\|u\|_s$ as $\varepsilon \searrow 0$ and repeat the arguments in the proof of Lemma 3.1 replacing the operator E_s with a pseudo-differential operator with symbol $\alpha(x,t)\lambda^s(\xi,\tau)(1+\epsilon\mu^2(\xi,\tau))^{-1}$ which has order $s-2$ but remains bounded in $S^s_{1,0}$ as $\epsilon \searrow 0$. In this way, one gets the analog of (3.10)

$$\|u\|_{s,\epsilon} \leq C\big(\|(L + c)u\|_{s,\epsilon} + \|u\|_{s-1}\big), \quad u \in H^{s-1}_c(Q_\delta).$$

If $u \in H^{s-1}_c(Q_\delta)$ and $L^t u \in H^s$, one concludes by letting $\varepsilon \searrow 0$ that $\|u\|_s \leq \limsup_{\epsilon\searrow 0}\|u\|_{s,\epsilon} < \infty$ so $u \in H^s_c(Q_\delta)$ and (3.11) holds. $\square$

3.1. **A microlocal version of Theorem 3.1.** In order to prepare for the proof of the semi-global solvability for linear PDO of arbitrary order in two variables we present a microlocal result that follows directly from Theorem 3.1. For this we keep the notation already established at the beginning of this section and let

$$\Sigma \doteq \{(x,t,\xi,0) : T^*Q \setminus 0 : \ b(x,t) = 0, \xi \neq 0\}$$

be the characteristic set of L on $\mathbb{R}^2$. Suppose that $p = (x_0, t_0, \xi_0, \tau_0) \in T^*Q \setminus 0$ and $u \in \mathcal{D}'(Q)$. We recall that $u \in H^s$ *at* p if there exists a properly supported pseudo-differential operator $C(x,t,D_x,D_t)$ of order zero which is microlocally elliptic at p and such that $Cu \in H^s_{\text{loc}}(Q)$. If $u \in H^s$ at p for every point $p \in X \subset T^*\Omega \setminus 0$, we say that $u \in H^s$ at X and we shall then write $u \in H^s_X$.

As before, for $\delta > 0$ small we shall set $Q_\delta = Q \cap \{|x| < \delta\}$.

Theorem 3.2. *Assume* $b(0,\cdot) = 0$ *and let*

$$\gamma_\delta \doteq Q_\delta \times \{(1,0)\} \subset T^*Q \setminus 0.$$

Given $s \in \mathbb{R}$ *there is* $\delta > 0$ *such that*

$$u \in \mathcal{E}'(Q_\delta), \ (L+c)u \in H^s_{\gamma_\delta} \implies u \in H^s_{\gamma_\delta}.$$

Proof: According to Theorem 3.1 given $s \in \mathbb{R}$ there is $\delta > 0$ such that

$$v \in \mathcal{E}'(Q_\delta), \ (L+c)v \in H^s \implies v \in H^s. \tag{3.12}$$

We fix such δ and let u be as in the statement. First we choose $\varepsilon > 0$ appropriately small such that if we set

$$\Gamma = \{(\xi,\tau) : |\tau| < \varepsilon\xi\}$$

then

$$(L+c)u \in H^s_{\text{loc}}(Q_\delta \times \Gamma). \tag{3.13}$$

Notice that, by ellipticity, we have

$$u \in H^{s+1}_{\text{loc}}(Q_\delta \times \dot{\Gamma}), \tag{3.14}$$

where $\dot{\Gamma} = \Gamma \setminus \{(\lambda, 0) : \ \lambda > 0\}$.

Let $\chi \in C^\infty_c(Q_\delta)$ be identically equal to one in a neighborhood of the support of u and let $h(\xi,\tau) \in C^\infty(\mathbb{R}^2)$ be homogeneous of degree 0 for $|\xi|^2 + |\tau|^2 \geq 1$, supported on Γ and equal to one on $\Gamma_1 \cap \{|\xi|^2 + |\tau|^2 \geq 1\}$, where $\Gamma_1 = \{(\xi,\tau) : |\tau| < \varepsilon\xi/2\}$. Now

$$\begin{aligned} (L+c)\left(\chi h(D_x,D_t)u\right) &= (L\chi)h(D_x,D_t)u + \chi(L+c)h(D_x,D_t)u \\ &= (L\chi)h(D_x,D_t)u \\ &+ \ \chi h(D_x,D_t)(L+c)u + \chi[L+c, h(D_x,D_t)]u. \end{aligned}$$

Since $h(D_x,D_t)$ is pseudolocal the distribution $h(D_x,D_t)u$ is smooth in the complement of the support of u and hence $(L\chi)h(D_x,D_t)u \in C^\infty_c(Q_\delta)$. On the other hand the total symbol of $\chi[L+c, h(D_x, D_t)]$ vanishes on $Q_\delta \times \{(\xi,\tau) : |\tau| < \varepsilon\xi/2, |\xi|^2 + |\tau|^2 \geq 1\}$ and hence (3.14) implies that $\chi[L+c, h(D_x, D_t)]u \in H^{s+1}$. By (3.13) we have $\chi h(D_x, D_t)(L+c)u \in H^s$ and hence we conclude that $(L+c)\chi h(D_x, D_t)u \in H^s$. By (3.12), $\chi h(D_x, D_t)u \in H^s$, which concludes the argument. □

Corollary 3.1. *Assume that* $b(0,\cdot) = 0$ *and let* $t_j \in -]T,T[$, $j = 1,2$, $t_1 < t_2$, *and set* $I = \{0\} \times [t_1, t_2]$, $\partial I = \{(0,t_1),(0,t_2)\}$. *If* $w \in \mathcal{D}'(Q)$ *is such that* $w \in H^s_{\partial I \times \{(1,0)\}}$ *and* $f \doteq (L+c)w \in H^s_{I \times \{(1,0)\}}$ *then* $w \in H^s_{I \times \{(1,0)\}}$.

Proof: Given $s \in \mathbb{R}$ as in the statement take $\delta > 0$ and $\varepsilon > 0$ small such that

(1) The conclusion of Theorem 3.2 holds for Q_δ;
(2) $f \in H^s_{\gamma_\delta}$ and $w \in H^s_{\tilde{\gamma}_{\delta,\varepsilon}}$, where

$$\tilde{\gamma}_{\delta,\varepsilon} = \{(x,t) : |x| < \delta,\ |t - t_j| < \varepsilon, j = 1,2\} \times \{(1,0)\}.$$

Define $\rho(x) = \int_{-T}^{T} b(x,t)dt$ and take $x_2 \in]0,\delta[$ such that either $\rho(x) \neq 0$ for $|x - x_2| < \eta$, $\eta > 0$ small, or $\rho(x) = 0$ for all x satisfying $|x - x_2| < \eta$. We also take $-\delta < x_1 < 0$ with analogous properties. Next we select functions $\phi(x), \psi(t) \in C_c^\infty(\mathbb{R})$ such that

$$\phi(x) = 1,\ x \in [x_1, x_2],\ \ \operatorname{supp}\phi' \subset [x_1 - \eta, x_1] \cup [x_2, x_2 + \eta];$$

$$\psi(t) = 1,\ t \in [t_1, t_2],\ \ \operatorname{supp}\psi' \subset [t_1 - \varepsilon, t_1] \cup [t_2, t_2 + \varepsilon]$$

and form $u(x,t) = \phi(x)\psi(t)w(x,t) \in \mathcal{E}'(Q_\delta)$. In order to complete the proof of the theorem we must show that $u \in H^s_{\gamma_\delta}$ or, according to Theorem 3.2, that $(L+c)u \in H^s_{\gamma_\delta}$. But we have

$$(L+c)u(x,t) = \phi(x)\psi(t)f(x,t) + \psi'(t)\phi(x)w(x,t) + b(x,t)\psi(t)(D_x\phi)(x)w(x,t).$$

Now, if we select $\eta < \varepsilon$,

- $\phi(x)\psi(t)f(x,t) \in H^s_{\gamma_\delta}$ because $f \in H^s_{\gamma_\delta}$;
- $\psi'(t)\phi(x)w(x,t) \in H^s_{\gamma_\delta}$ because $w \in H^s_{\tilde{\gamma}_{\delta,\varepsilon}}$
- $b(x,t)\psi(t)(D_x\phi)(x)w(x,t) \in H^s_{\gamma_\delta}$, since on $\operatorname{supp}(D_x\phi \otimes \psi)$ the function $b(x,t)$ keeps constant sign, a consequence of condition $(\mathcal{P})$, and hence we can apply Hörmander's propagation result ([Hör1], Theorem 3.5.1).

The proof is complete. □

Remark 3.1: Theorem 3.2 and Corollary 3.1 remain true if we replace $L + c$ by $L + c + q(x,t,D_x,D_t)$, where $q(x,t,D_x,D_t)$ is any (properly) supported pseudo-differential operator of order < 0. This follows from the fact that theorems 2.1 and 3.1 remain valid in this slightly more general situation. This fact will be of foremost importance when dealing with the case of arbitrary order.

4. Proof of Theorem 1.1

Fix $s \in \mathbb{R}$ and let $K \subset\subset \Omega$ satisfy the hypotheses of Theorem 1.1. By Proposition 1.2, it is enough to show that, for some open $U \supset K$,

$$u \in \mathcal{E}'(\overline{U}) \cap H^{-s-1} \text{ and } (L^* + \bar{c})u \in H^{-s} \implies u \in H^{-s}$$

holds (we write just H^s to mean $H^s_{\mathrm{loc}}(\Omega)$). If, to simplify the notation, we write s in the place of $-s$ and put $L + c$ in the place of $L^* + \bar{c}$ (whose principal part is $-\bar{L}$, which also satisfies condition $(\mathcal{P})$) this becomes

$$\text{(a)} \qquad u \in \mathcal{E}'(\overline{U}) \cap H^{s-1} \text{ and } (L + c)u \in H^s \implies u \in H^s,$$

which grants H^{-s} solvability for $L^* + \bar{c}$ and also implies H^{-s} solvability for $L + c_1$ for a convenient c_1 and also that of $L + c$ because c can be changed without changing the theorem's hypotheses. Initially, we will show a weaker form of (a), namely

$$\text{(b)} \qquad u \in \mathcal{E}'(K) \cap H^{s-1} \text{ and } (L + c)u \in H^s \implies u \in H^s.$$

Consider a distribution $u \in \mathcal{E}'(K) \cap H^{s-1}$ such that $(L+c)u \in H^s$. Given an arbitrary point $p \in K$ we will show that[2] $p \notin SS_s u$. This is an immediate consequence of elliptic regularity if $p \in \Omega_0$ so we may assume that $p \in \Omega_1 \cap K$ and look at its class $[p]$. Suppose that $[p]$ is of type (ii). Then $[p]$ is a closed interval of a smooth curve $\gamma(s)$ such that its tangent vector $\gamma'(s)$ is a linear combination of $X = \mathsf{Re}\, L$ and $Y = \mathsf{Im}\, L$ when $\gamma(s) \in [p]$. We may find smooth real functions $\alpha, \beta \in C^\infty(\Omega)$ such that $\alpha X - \beta Y = \gamma'(s)$ on $[p]$ and $\alpha^2 + \beta^2 > 0$ on a neighborhood of $[p]$. Then $\tilde{L} = (\alpha + i\beta)L$ satisfies $(\mathcal{P})$ and $\tilde{X} = \mathsf{Re}\, \tilde{L} = \gamma'(s)$ on $[p]$. Rectifying the integral curves of $\tilde{X}$ around $[p]$ we may find a diffeomorphism on an open neighborhood U_p of $[p]$ made up of integral curves of $\tilde{X}$ that transforms $\tilde{X}$ into ∂_t. It is well known that we may assume also (see, e.g., [BCH, Lemma IV.1.1]) that $\tilde{Y} \doteq \mathsf{Im}\, L = ib(x,t)\partial_x$ with $b(x,t)$ real. To simplify the notation we drop all tildes from now on and write L instead of $\tilde{L}$, X instead of $\tilde{X}$ and so on. Hence, we may assume that in the new variables, U_p contains the closure of the rectangle $Q = (-\delta, \delta) \times (-T, T)$, $\{p\} = (0,0)$, $[p] = \{0\} \times [-\tau, \tau]$ with $0 < \tau < T$, $L = \partial_t - ib(x,t)\partial_x + c(x,t)$ with $b(x,t)$ real and $b(0,-T)b(0,T) \neq 0$ which implies, by shrinking $\delta > 0$ that $b(x,-T)b(x,T) \neq 0$ for $|x| < \delta$. By the invariance of $(\mathcal{P})$ under multiplication by non vanishing scalars and change of variables, we know that $(-T,T) \ni t \mapsto b(x,t)$ does not change sign and conclude that either $b(x,t) \geq 0$ on Q of $b(x,t) \leq 0$ on Q. In other words,

$$\text{(4.1)} \qquad b(x,t) \text{ does not change sign on } Q = (-\delta, \delta) \times (-T, T).$$

Then, by Hörmander's theorem on propagation of singularities ([Hör1], Theorem 3.5.1) , the intersection of $SS_s u$ with Q must propagate vertically either upwards or downwards. In other words, given a point $(x_0, t_0) \in Q$ either $(x_0, t) \in SS_s u$ for

[2] Here and in the sequel we shall denote by $SS_s u$ the H^s-*singular support of* u, that is, the complement of the set of all points p in Ω for which u is in H^s_{loc} in a neighborhood of p.

all $t \geq t_0$ or $(x_0, t) \in SS_s u$ for all $t \leq t_0$. Since L is elliptic on the set $(-\delta, \delta) \times \{T\} \cup (-\delta, \delta) \times \{-T\}$, we conclude that $Q \cap SS_s u = \emptyset$ and in particular $p \notin SS_s u$. A similar argument shows that if $[p]$ is of type (i) then $p \notin SS_s u$.

Assume next that $[p]$ is of type (iii), i.e., $[p]$ is a closed one-dimensional orbit. By hypothesis $[p] \not\subset K$, and we may embed a closed interval $I \subset [p]$ containing p in its interior with both endpoints outside K. Arguing as before, i.e., taking a convenient multiple $\tilde{L} = \tilde{X} + i\tilde{Y}$ of L, rectifying the flow of $\tilde{X}$ around I, etc., we are led to consider, after appropriate change of coordinates, the case of a distribution $u \in H^{s-1}_{\mathrm{loc}}(U)$ defined on a neighborhood U of the closure of a cube $Q' = (-a, a) \times (-T', T')$ such that u vanishes identically on $(-a, a) \times (T, T') \cup (-a, a) \times (-T', -T)$ for some $0 < T < T'$ and satisfies $(L + c)u \in H^s_{\mathrm{loc}}(U)$, where

$$L + c = \frac{\partial}{\partial t} - ib(x, t)\frac{\partial}{\partial x} + c(x, t),$$

it is initially defined on a neighborhood of the closure of Q' and then extended to $\mathbb{R}^2$ and satisfies

($\star$) $b(x, t)$ and $c(x, t)$ are smooth and bounded with bounded derivatives;

($\mathcal{P}$) $b(x, t)$ is real and $t \mapsto b(x, t)$ does not change sign for any $x \in \mathbb{R}$;

and $b(0, t) \equiv 0$ for $-T' \leq t \leq T'$. Modifying the distribution off $\overline{Q'}$ we may assume without loss of generality that $u \in H^{s-1}(\mathbb{R}^2)$ and $(L + c)u \in H^s(\mathbb{R}^2)$. Set

$$\mathcal{X} = \{x \in (-a, a) : \beta(x) = 0\}, \quad \beta(x) = \sup_{|t| \leq T'} |b(x, t)|,$$
$$\mathcal{Y} = \{x \in (-a, a) : \beta(x) > 0\}.$$

Notice that the sets $\mathcal{X}$ and $\mathcal{Y}$ form a partition of the interval $(-a, a)$ and consider a point $q = (x_0, t_0) \in Q'$ with $x_0 \in \mathcal{Y}$. Hence, there exists $t_1 \in (-T', T')$ such that $b(x_0, t_1) \neq 0$, say $b(x_0, t_1) > 0$. For some $\varepsilon > 0$, $b(x, t_1) > 0$ if $|x - x_0| < \varepsilon$, so invoking condition ($\mathcal{P}$) we see that $b(x, t)$ does not change sign on $(x_0 - \varepsilon, x_0 + \varepsilon) \times (-T', T')$. By the previous arguments, we see that $q \notin SS_s u$ if $x_0 \in \mathcal{Y}$ so

$$SS_s u \cap \big(\mathcal{Y} \times (-T, T)\big) = \emptyset. \tag{4.2}$$

On the other hand, if $x_0 \in \operatorname{Int} \mathcal{X}$, then $b(x, t) \equiv 0$ for x close enough to x_0 and $|t| \leq T$, so again $b(x, t)$ does not change sign on $(x_0 - \varepsilon, x_0 + \varepsilon) \times (-T', T')$ for some $\varepsilon > 0$ and we derive that

$$SS_s u \cap \big(\operatorname{Int} \mathcal{X} \times (-T, T)\big) = \emptyset. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$SS_s u \subset \big((-a, a) \setminus (\mathcal{Y} \cup \operatorname{Int} \mathcal{X})\big) \times (-T, T). \tag{4.4}$$

Thus, if $q = (x_0, t_0) \in SS_s u$, then $x_0 \in \mathcal{X}$ and there exists a pair of sequences $x_j \nearrow x_0$, $x'_j \searrow x_0$ with $x_j, x'_j \in \operatorname{Int} \mathcal{X} \cup \mathcal{Y}$, $j = 1, 2, \ldots$. Hence, given $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \subset (-a, a)$, we may find $\phi(x) \in C^\infty_c(x_0 - \delta, x_0 + \delta)$ such that

$\phi(x) \equiv 1$ in a neighborhood of x_0 and $\operatorname{supp} \phi'(x) \subset \mathcal{Y} \cup \operatorname{Int} \mathcal{X}$. Set $v = \phi u$. Then $Lv = \phi Lu + uL\phi \in H^s$ because H^s is invariant under multiplication by test functions, $\operatorname{supp} L\phi \subset \operatorname{supp} \phi' \times (-T, T) \subset (\mathcal{Y} \cup \operatorname{Int} \mathcal{X}) \times (-T, T)$ and the restriction of u to $(\mathcal{Y} \cup \operatorname{Int} \mathcal{X}) \times (-T, T)$ is locally in H^s by (4.4). Furthermore, $q \in SS_s u$ if and only if $q \in SS_s v$. It follows that to prove that $q = (x_0, t_0) \notin SS_s u$, which will show that $SS_s u \cap Q' = \emptyset$, it is enough to prove that there exists $\delta > 0$ such that

$$\text{for any } v \in \mathcal{E}'\big((x_0 - \delta, x_0 + \delta) \times (-T', T')\big) \cap H^{s-1} \quad (L + c)v \in H^s \Longrightarrow v \in H^s. \tag{4.5}$$

We have reduced the proof of $q \notin SS_s u$ to proving (4.5) where we need only deal with distributions compactly supported in Q'. The validity of (4.5) is now granted by Theorem 3.1. Since q is arbitrary we conclude that $SS_s u \cap Q' = \emptyset$.

The case in which $[p]$ is of type (iv) is similar to the case of type (iii) and need not be described in detail. This is so because, in view of Lemma 1.1, $\mathcal{O}_p$ is an embedding $j(s)$, $s \in \mathbb{R}$, $j(0) = p$, with $j(s_-), j(s_+) \notin K$ for some $s_- < 0 < s_+$.

Suppose now that $[p]$ is of type (v). If we proceed as in the case of type (iii) we end up with a cube $Q' = (-a, a) \times (-T', T')$ and an operator

$$L + c = \frac{\partial}{\partial t} - ib(x, t)\frac{\partial}{\partial x} + c(x, t)$$

with the following difference: we know that u vanishes identically on, say, $(-a, a) \times (T, T')$ for some $0 < T < T'$ but do not have the same information for t close to $-T$. However, this is compensated for by the fact that we may assume, after shrinking $a > 0$ and taking $T < T'$ closer to T', that L is elliptic on $(-a, a) \times [-T', -T]$. If $\eta(t) \in C^\infty(\mathbb{R})$ is such that $\eta(t) \equiv 0$ for $t \leq -T'$ and $\eta(t) \equiv 1$ for $t \geq -T$, it follows that $u_1 \doteq \eta(t)u(x, t)$ still satisfies $(L + c)u_1 \in H^s$ and vanishes identically on $(-a, a) \times (T, T') \cup (-a, a) \times (-T', -T)$. Hence, we may reason with u_1 as we did with u for the case in which $[p]$ was of type (iii) to conclude that u_1 is in H^s and so is u where $\eta \neq 0$.

Summing up, we have shown that if $u \in \mathcal{E}'(K) \cap H^{s-1}$ and $(L + c)u \in H^s$ then $u \in H^s$. For $\varepsilon > 0$, let $K(\varepsilon)$ denote the set of points whose distance to K is $\leq \varepsilon$ and set $U = \operatorname{Int} K(\varepsilon) \supset K$. It is not difficult to show that if for every $\varepsilon > 0$, $K(\varepsilon)$ contains a closed one-dimensional orbit of L then the intersection $\bigcap_{\varepsilon>0} K(\varepsilon) = K$ also does[3], contradicting the hypotheses of the theorem. Hence, for some small $\varepsilon > 0$, (b) holds with $K(\varepsilon)$ in the place of K and we conclude that (a) holds if we take $U = \operatorname{Int} K(\varepsilon)$, as we wished to prove. □

[3] For each ν we take a point q_ν belonging to a closed one dimensional orbit contained in $K(1/\nu)$ and in such a way that $q_\nu \to q \in K$. Then the orbit through q must be contained in K and it is one-dimensional, for two dimensional orbits are open. Furthermore, taking a curve through q transversal to the vector field $X = \operatorname{Re} L$ and analyzing the first-return map of the orbits of X one concludes that the orbit through q must be closed because the orbits through q_ν are. This implies our claim.

5. Applications, remarks and examples

Concerning the hypotheses of Theorem 1.1, it is well known that condition $(\mathcal{P})$ is necessary for local solvability. Suppose now that $K \subset\subset \Omega$ contains a closed orbit γ. We may choose coordinates on a tubular neighborhood of γ periodic in t so that a multiple of L has the form

$$L = \frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}, \quad |x| < a, \quad |t| \le 1, \quad (x,-1) \sim (x,1),$$

where $b(x,-1) \equiv b(x,1)$, $b(0,t) \equiv 0$, and $\gamma(\sigma) = (0,\sigma)$. If $s > 2$ and $u \in H^s_{\text{loc}}(\Omega)$ (so it is of class C^1 by Sobolev's embedding) and satisfies $Lu = f$ we see that

$$\int_{-1}^{1} f(0,\sigma)\, d\sigma = \int_{-1}^{1} \partial_s u(0,\sigma)\, d\sigma = 0, \tag{5.1}$$

showing that $Lu = f$ cannot be solved in H^s on a neighborhood of γ if (5.1) does not hold.

When L is elliptic and $(L+c)u = f \in H^s_{\text{loc}}$, the standard elliptic regularity properties imply that $u \in H^{s+1}_{\text{loc}}$. This optimal situation is termed "solvability with no loss of derivatives" and justifies the terminology "solvability with loss of one derivative" when we can only guarantee that u is in H^s_{loc}. Of course, for general equations, we cannot expect to lose less that one derivative, as in the case of $L = \partial_t$ on $\mathbb{R}^2$.

The semi-global solvability results with loss of one derivative imply global solvability results with loss of one derivative.

5.1. **Global solvability.** In this subsection we deal with a first-order operator $P = L + c$ defined on a connected, orientable, non compact, two-dimensional smooth manifold Ω, here L is a complex vector field without zeros and c is a complex smooth function. We assume

(i) L satisfies $(\mathcal{P})$ and
(ii) L does not have any closed one-dimensional orbit.

In general, L may have orbits of dimension one or two. The orbits of dimension two are open sets whose point set boundary is a union of orbits of dimension one. A consequence of (ii) is that L has no relatively compact orbit of any dimension (we refer to [Hou3] and [Hou2] for additional information). We want to solve the equation

$$P = Lu + cu = f, \quad f \in H^s_{\text{loc}}(\Omega), \tag{5.2}$$

on Ω with $u \in H^s_{\text{loc}}(\Omega)$. In the sequel, we will write $S(u)$ to denote the support of a distribution $u \in \mathcal{D}'(\Omega)$.

Theorem 5.1. *Let L be as above and $s \in \mathbb{R}$. The following properties are equivalent:*

(1) $(L+c)H^s_{\text{loc}}(\Omega) = H^s_{\text{loc}}(\Omega)$;

(2) *For every compact set $K \subset \Omega$ there is a compact set $K' \subset \Omega$ such that: $u \in \mathcal{E}'(\Omega)$ and $S((L^t + c)u) \subset K$ implies $S(u) \subset K'$;*

(2') *For every compact set $K \subset \Omega$ there is a compact set $K' \subset \Omega$ such that: $u \in H_c^{-s}(\Omega)$ and $S((L^t + c)u) \subset K$ implies $S(u) \subset K'$;*

(3) *For every compact set $K \subset \Omega$ there is a compact set $K' \subset \Omega$ such that if γ is a one-dimensional orbit in $\Omega \setminus K$ with endpoints in K then $\gamma \subset K'$.*

We begin by describing the strategy of the proof. The equivalence (2) $\Longleftrightarrow$ (3) that furnishes a geometric characterization (3) of the $(L+c)$-convexity property (2) can be found in ([Hou3], Lemma 2.5) and (2) implies (2') trivially, so it is enough to prove (1) $\Longrightarrow$ (2) and (2') $\Longrightarrow$ (1). Note that (1) means that $P = L + c$ maps $H^s_{\rm loc}(\Omega)$ onto $H^s_{\rm loc}(\Omega)$ while property (2) is the famous P-convexity property —also called P-convexity for supports— (see Definition 10.6.1 in [Hör4], where the transpose P^t is expressed as $P(-D)$ because the operator has constant coefficients).

It is well known that (1) implies (2) —i.e., Ω is $(L+c)$-convex— as follows from a known application of the uniform boundedness theorem. This is proved, e.g., for operators with constant coefficients in ([Hör4], Theorem 10.6.6), but the basic argument extends without changes to general differential operators.

The approach we follow in order to show that (2') implies (1) —which is usually referred to as the Mittag-Leffler method by its resemblance with the technique he used to construct meromorphic functions with prescribed poles— is widely known for operators with constant coefficients (see, e.g., ([Hör4], Section 10.6, p.41) and is flexible enough to deal with operators with variable coefficients once the key hypotheses are proved. Roughly speaking, the main technical step in proving that (2') implies (1) is to show that given an open set Ω_1 relatively compact in Ω one may use (2') to find an open set Ω_2, also relatively compact in Ω, $\Omega_1 \subset \Omega_2 \subset \Omega$, with the following property: if $u \in H^s_{\rm loc}(\Omega_2)$ satisfies the homogeneous equation $(L+c)u = 0$ on Ω_1, then u can be approximated on a substantial open subset of Ω_1 in the $H^s_{\rm loc}$ topology by restrictions to Ω_1 of distributions $v \in H^s_{\rm loc}(\Omega_2)$ that satisfy the homogeneous equation $(L+c)v = 0$ on Ω_2. The key lemma is an adaptation to our context of a result first proved by Bernard Malgrange in his thesis ([M], Théorème 1, p.328).

Lemma 5.1. *Let $\Omega_j \subset \Omega$, $j = 1, 2, 3$, be open sets such that*

$$\Omega_0 \subset \overline{\Omega}_0 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \Omega$$

and assume that every $u \in H_c^{-s}(\overline{\Omega}_2)$ such that $(L^t + c)u \in H_c^{-s}(\overline{\Omega}_0)$ belongs to $H_c^{-s}(\Omega_1)$. If $u \in H^s_{\rm loc}(\Omega_1)$ satisfies the equation

$$(L+c)u = 0 \quad \text{on } \Omega_1$$

then u can be approximated in $H^s_{\rm loc}(\Omega_0)$ by distributions $v \in H^s_{\rm loc}(\Omega_2)$ satisfying the equation $(L+c)v = 0$ on Ω_2.

Proof: Let us write $P = L + c$,

$$\mathcal{N}_j = \{u \in H^s_{\mathrm{loc}}(\Omega_j) : \ Pu = 0\}, \quad j = 0, 1, 2,$$

and let $\mathcal{N}'_k \subset \mathcal{N}_0$, $k = 1, 2$, be the set of all restrictions of elements in $\mathcal{N}_k$ to Ω_0. By the Hahn-Banach theorem, the density of $\mathcal{N}'_2$ in $\mathcal{N}'_1$ will follow if every continuous linear functional on $H^s_{\mathrm{loc}}(\Omega_0)$ that vanishes on $\mathcal{N}'_2$ also vanishes on $\mathcal{N}'_1$. Such a linear functional may be represented by $\nu \in H^{-s}_c(\Omega_0)$ where the pairing between $H^s_{\mathrm{loc}}(\Omega_0)$ and $H^{-s}_c(\Omega_0)$ is the extension to $H^s_{\mathrm{loc}}(\Omega_0) \times H^{-s}_c(\Omega_0)$ of the bilinear map

$$C^\infty_c(\Omega_0) \times C^\infty_c(\Omega_0) \ni (\phi, \psi) \mapsto \int \phi(x)\psi(x)\, dx.$$

We shall prove that there exists $\mu \in H^{-s}_c(\Omega_1)$ such that

$$P^t\mu = \nu. \tag{5.3}$$

Consider the subspace $\mathcal{X} \subset H^{-s}_c(\Omega_2)$ defined as

$$\mathcal{X} = \{P^t\mu : \ \mu \in H^{-s}_c(\Omega_2) \text{ such that } \ P^t\mu \in H^{-s}_c(\Omega_2)\}$$

whose annihilator $\mathcal{X}^\perp \doteq \{v \in H^s_{\mathrm{loc}}(\Omega_2) : \ \langle v, \nu\rangle = 0, \ \forall v \in \mathcal{X}\}$ satisfies $\mathcal{X}^\perp \subset \mathcal{N}_2$ because $P^tC^\infty_c(\Omega_2) \subset \mathcal{X}$. Since $\nu \perp \mathcal{N}_2$ and $\nu \in \mathcal{X}^{\perp\perp} = \overline{\mathcal{X}}$ —note that $H^{-s}_c(\Omega_2)$ is a reflexive space— it follows that $\nu \in \overline{\mathcal{X}}$. We will show below that

$$\overline{\mathcal{X}} \subset \{P^t\mu : \ \mu \in H^{-s}_c(\overline{\Omega}_2) \text{ such that } \ P^t\mu \in H^{-s}_c(\overline{\Omega}_2)\}. \tag{5.4}$$

Therefore, ν belongs to the set on the right hand side of (5.4) so (5.3) holds for some $\mu \in H^{-s}_c(\overline{\Omega}_2)$ and then, using the hypothesis in the lemma, we may solve (5.3) with $\mu \in H^{-s}_c(\Omega_1)$. Now, for every $u \in \mathcal{N}_1$ we have

$$\langle \nu, u\rangle = \langle P^t\mu, u\rangle = \langle \mu, Pu\rangle = 0$$

showing that ν vanishes on $\mathcal{N}_1$ (and thus on $\mathcal{N}'_1$) as we wished to prove. It remains to prove (5.4). By the proof of Theorem 1.1 we know that

$$u \in H^{-s-1}_c(\overline{\Omega}_2) \cap \text{ and } P^tu \in H^{-s}_c(\overline{\Omega}_2) \implies u \in H^{-s}_c(\overline{\Omega}_2)$$

and an application of the closed graph theorem to this inclusion map gives an estimate

$$\|u\|_{-s} \le C(\|P^tu\|_{-s} + \|u\|_{-s-1}), \quad u \in H^{-s-1}_c(\overline{\Omega}_2). \tag{5.5}$$

Arguing as in the proof of Proposition 1.2, (5.5) can be refined to the stronger estimate

$$\|u\|_{-s} \le C\|P^tu\|_{-s}, \quad \text{if } u \in H^{-s-1}_c(\overline{\Omega}_2) \text{ and } P^tu \in H^{-s}_c(\overline{\Omega}_2). \tag{5.6}$$

If $\nu \in \overline{\mathcal{X}}$, there exists a sequence $P^t\mu_j \in \mathcal{X}$ such that $P^t\mu_j \to \nu$ in the H^{-s} norm. Applying (5.6) to $\mu_j - \mu_k$ we conclude that μ_j is a Cauchy sequence in $H^{-s}_c(\overline{\Omega}_2)$ that converges to an element $\mu \in H^{-s}_c(\overline{\Omega}_2)$ such that $P^t\mu = \nu$ so (5.4) is proved. $\square$

If (2') holds and a nonempty relatively compact open set $\Omega_0 \subset \Omega$ is given, we may find open relatively compact sets Ω_1, Ω_2 such that the sets Ω_j, $j = 0, 1, 2$ fulfill the hypothesis of Lemma 5.1. To do so, set $K_0 = \overline{\Omega}_0$ and let K_0' be the set granted by (2'). If we choose as Ω_1 a relatively compact open neighborhood of K_0' and Ω_2 is any relatively compact open neighborhood of $\overline{\Omega}_1$ the triplet $\{\Omega_0, \Omega_1, \Omega_2\}$ does the job. Note that we may take Ω_2 as large as we wish in the sense that it may contain any preselected compact set, in particular, if $K_1 \doteq \overline{\Omega}_1$ and K_1' is the set granted by (2'), we may assume that $K_1' \subset \Omega_2$. Proceeding along these lines it is easy to construct relatively compact open sets $\Omega_j \subset \overline{\Omega}_j \subset \Omega_{j+1}$, $j = 0, 1, 2, \ldots$, such that for every $j \geq 1$ the triplet $\{\Omega_{j-1}, \Omega_j, \Omega_{j+1}\}$ satisfies the hypothesis Lemma 5.1 and $\bigcup_j \Omega_j = \Omega$.

If $f \in H^s_{\mathrm{loc}}(\Omega)$, we may solve the equations $(L + c)u_j = f$ on Ω_j with $u_j \in H^s_{\mathrm{loc}}(\Omega)$ by a crucial application of Theorem 1.1 with $K = \overline{\Omega_j}$. Formally, the desired global solution could be given by the telescoping series $u = u_0 + \sum_{j=0}^{\infty}(u_{j+1} - u_j)$ but to obtain a true solution one would need that $u_{j+1} - u_j$ is small enough to make the series convergent. Thus, one sets $u_0 \doteq u_0'$, $u_1 \doteq u_1'$ and for $j \geq 1$ replaces inductively each u_{j+1} by a new solution u_{j+1}' of $(L + c)u_{j+1}' = f$ on Ω_{j+1}. If u_j' has already been defined, we may take $u_{j+1}' \doteq u_{j+1} - v_{j+1}$ where v_{j+1} satisfies $(L + c)v_{j+1} = 0$ on Ω_{j+1} and is very close to the restriction to Ω_{j-1} of $u_{j+1} - u_j'$, which is possible by an application of Lemma 5.1 because $(L + c)(u_{j+1} - u_j') = 0$ on Ω_j. More precisely, choosing test functions $\psi_j \in C_c^\infty(\Omega_{j-1})$ that are identically equal to one on $\overline{\Omega}_{j-2}$, $j \geq 2$, one may choose the distributions v_{j+1} so that, $\|\psi_j(u_{j+1}' - u_j')\|_s \leq 2^{-j}$. Thus, the series $u_0' + \sum_j(u_{j+1}' - u_j')$ converges in $H^s_{\mathrm{loc}}(\Omega)$ to a global solution $u \in H^s_{\mathrm{loc}}(\Omega)$ of $(L + c)u = f$. This proves that (2') implies (1). ☐

5.2. **Examples.** Consider the real vector field with polynomial coefficients defined on $\mathbb{R}^2$

$$L = 2x\frac{\partial}{\partial t} + (1 - x^2)\frac{\partial}{\partial x}$$

with a global first integral $Z(x, t) = (1 - x^2)e^t$. All orbits are one-dimensional and homeomorphic to $\mathbb{R}$; they are the leaves of the foliation induced on $\mathbb{R}^2$ by the level sets $Z(x, t) = \lambda$ =constant. If $\lambda \leq 0$ each level set has two connected components which are one-dimensional orbits while when $\lambda > 0$ the level set is the graph $t = \log(\lambda/(1 - x^2))$ that is a convex curve contained in the strip $|x| < 1$ (the set of all leaves of the latter kind is usually called a Reeb component of the foliation). Set $K = [-1, 1] \times \{0\}$ and pick $\lambda > 0$ small. Then the part of the graph $t = \log(\lambda/(1 - x^2))$ below the x-axis is an arc γ with vertex at the point $v(\lambda) = (0, \log \lambda)$ and endpoints $e_+ = (+\sqrt{1 - \lambda}, 0)$, $e_- = (-\sqrt{1 - \lambda}, 0) \in K$. It is easy to construct a distribution u that satisfies $Su = \gamma$ and $S(L^t u) = e_+ \cup e_-$ (take a uniform distribution of mass concentrated in γ). Since $v(\lambda) \to -\infty$ as $\lambda \searrow 0$, we see that $\mathbb{R}^2$ is not L-convex and the equation $Lu = f$ cannot be globally solved in $H^s_{\mathrm{loc}}(\mathbb{R}^2)$. On the other hand, it is easy to check that the restriction of L to any of the three sets $\Omega^+ = \{(x,t) : \ x > 1\}$, $\Omega^- = \{(x,t) : \ x < -1\}$, $\Omega_0 = \{(x,t) : \ -1 < x < 1\}$ is L-convex and L is then H^s_{loc}-solvable in anyone of them.

We now show examples of the different types of equivalence classes described in Proposition 1.1. Let $b(x,t) \geq 0$ a smooth real function and consider the vector field

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x}, \quad (x,t) \in \mathbb{R}^2.$$

If $b(x,t) > 0$ off the origin but $b(0,0) = 0$ it follows that $\mathcal{O}_0 = \mathbb{R}^2$ and $[0] = \{(0,0)\}$. If $b(x,t) = 0$ precisely when $x = 0$ and $t_0 \leq t \leq t_1$, $t_0 \leq 0 \leq t_1$, $t_0 < t_1$, then $\mathcal{O}_0 = \mathbb{R}^2$ and $[0] = \{0\} \times [t_0, t_1]$. If $b(x,t) = 0$ precisely when $x = 0$ and $t \geq 0$ then $\mathcal{O}_0 = \mathbb{R}^2$ and $[0] = \{0\} \times [0, \infty)$. These are the three types of equivalence classes that are not orbits. If $b(x,t) = 0$ precisely when $x = 0$ then L has a pair of two-dimensional orbits, $\{x > 0\}$ and $\{x < 0\}$ and a one-dimensional orbit $\mathcal{O}_0 = [0] = \{0\} \times \mathbb{R}$. To give an example of closed 1-orbit of a vector field defined on $\mathbb{R}^2$ we depart from the Cauchy-Riemann vector field which in polar coordinates is given by

$$\mathcal{C} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right).$$

If $b(r) \geq 0$ is a smooth function of r, such that $b(r) = 1$ for $|r-1| \geq 1/2$, and $b(r) = 0$ if and only if $r = 1$. Then, the perturbation of $\mathcal{C}$ given by

$$L = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + \frac{ib(r)}{r}\frac{\partial}{\partial \theta}\right)$$

has two 2-orbits, namely $r > 1$ and $r < 1$, and one closed 1-orbit which is also a class of type (iii). Note that L is well defined at the origin because it coincides with $\mathcal{C}$ in a neighborhood of the origin.

For vector fields with real analytic coefficients, equivalence classes of the types (ii) and (v) described in Proposition 1.1 do not occur.

6. Semi-global solvability for LPDO in two variables

Let Ω be an open subset of $\mathbb{R}^2$ and let $P = P(y, D_y)$ be a linear partial differential operator on Ω of order $m \geq 1$. We assume that P has simple real characteristics, which means that $d_\eta p_m(y,\eta) \neq 0$ in $T^*\Omega \setminus 0$, and also that $P(y, D_y)$ satisfies the Nirenberg-Treves condition $(\mathcal{P})$ in Ω (cf. [T] for its definition).

A curve Γ in $T^*\Omega \setminus 0$ is called a *semi-bicharacteristic* (for P) if there is a non vanishing positively homogeneous smooth function q on $T^*\Omega \setminus 0$ such that Γ is an integral curve of $H_{\mathsf{Re}\,(qp_m)}$ for which $\mathsf{Re}\,(qp_m)|_\Gamma = 0$.

Our goal in this section is to prove the following theorem:

Theorem 6.1. *Let K be a compact subset of Ω satisfying the following property:*

- *Every characteristic point for P over K lies on a compact semi-bicharacteristic interval with no characteristic endpoints over K.*

Given $s \in \mathbb{R}$ and $u \in \mathcal{E}'(K)$ such that $Pu \in H^s(K)$ it follows that $u \in H^{s+m-1}(K)$.

In particular this result implies that $\ker(P^*) \cap \mathcal{E}'(K)$ is a finite dimensional subspace of $C_c^\infty(K)$ and that, according to the argument that proves Proposition 1.2 (cf. also [Hör3], p.64), whenever f belongs to $H^s_{\rm loc}(\Omega)$ and is orthogonal to $\ker(P^*)\cap\mathcal{E}'(K)$ the equation $Pu = f$ has a solution $u \in H^{s+m-1}_{\rm loc}(\Omega)$ near K.

In order to prove Theorem 6.1 we follow the arguments in [Hör3], proof of Theorem 26.11.2. Fix s and u as in the statement. We must show that if $(y, \eta) \in T^*\Omega \setminus 0$ lies over K then $u \in H^{s+m-1}_{(y,\eta)}$. Denote by $\mathcal{N}$ the characteristic set of P. If $(y, \eta) \notin \mathcal{N}$ then $u \in H^{s+m}_{(y,\eta)}$ by ellipticity and we are done. Next assume $(y, \eta) \in \mathcal{N}$ and choose a semi-bicharacteristic interval γ containing (y, η) and with no characteristic endpoints over K. Then $u \in H^{s+m}_{\partial\gamma}$. If γ is not contained in $\mathcal{N}$ it follows that $u \in H^{s+m-1}_{(y,\eta)}$ thanks to Theorem 26.6.2 in [Hör3]. Hence we can assume that $\gamma \subset \mathcal{N}$ and thus we have reduced our proof to Theorem 6.2 below.

Theorem 6.2. *If γ is a compact semi-bicharacteristic interval for P contained in $\mathcal{N}$ and with no characteristic endpoints over K and if $u \in \mathcal{D}'(\Omega)$ is such that $Pu \in H^s_\gamma$ and $u \in H^{s+m-1}_{\partial\gamma}$ then $u \in H^{s+m-1}_\gamma$.*

Remark 6.1: If γ is a compact semi-bicharacteristic interval for P contained in $\mathcal{N}$ and with no characteristic endpoints over K then necessarily the endpoints of $\wp \circ \gamma$ lie outside K. Here $\wp : T^*\Omega \to \Omega$ denotes the base projection.

6.1. **The local representation of a semi-bicharacteristic curve contained in the characteristic set.** Let Γ be a semi-bicharacteristic contained in $\mathcal{N}$ and let $(y_0, \eta_0) \in \Gamma$. By the representation of P given in [T], we can select local coordinates (y_1, y_2) centered at y_0, defined for $\Omega_0 = \{|y_j| < \varepsilon\}$, such that in the dual coordinates (η_1, η_2) we have $\eta_0 = (1, 0)$ and such that the principal symbol p_m of p factors out in the form

$$p_m(y, \eta) = q_{m-1}(y, \eta)(\eta_2 + ib(y)\eta_1),\ \ (y, \eta) \in \Omega_0 \times \mathbb{R}^2,$$

where the homogeneous polynomial q_{m-1} does not vanish for $y \in \Omega_0$ and $\eta \in \mathcal{U}$, a conic neighborhood of $(1, 0)$ in $\mathbb{R}^2$. By condition $(\mathcal{P})$ each function $y_2 \mapsto b(y_1, y_2)$ does not change its sign. In $\Omega_0 \times \mathcal{U}$ the curve Γ is also a semi-bicharacteristic for $\eta_2 + ib(y)\eta_1$ and hence there is $A + iB$ smooth, positively homogeneous and nowhere 0 such that Γ is the integral curve through $(0, 0, 1, 0)$ of the vector field $H_{\eta_2 A - b\eta_1 B}$.

Notice now that

$$\mathcal{N} \cap (\Omega_0 \times \mathcal{U}) = \{(y, \eta) : \eta_2 = 0, b(y) = 0\};$$

moreover condition $(\mathcal{P})$ also gives $(\partial b/\partial y_2)(y) = 0$ when $b(y) = 0$. Consequently since Γ is contained in $\mathcal{N}$

$$H_{\eta_2 A - b\eta_1 B} = A\frac{\partial}{\partial y_2} - b_{y_1} B\eta_1 \frac{\partial}{\partial \eta_1} \text{ on } \Gamma.$$

In particular we obtain $\Gamma(\sigma) = (0, y_2(\sigma), \eta_1(\sigma), 0)$ satisfying $(y_2(0), \eta_1(0)) = (0, 1)$, $b(0, y_2(\sigma)) = 0$ and

$$\begin{cases} \dot{y}_2(\sigma) = A(0, y_2(\sigma), \eta_1(\sigma), 0) = A(0, y_2(\sigma), 1, 0)\eta_1(\sigma)^\kappa \\ \\ \dot{\eta}_1(\sigma) = -B(0, y_2(\sigma), \eta_1(\sigma), 0)b_{y_1}(0, y_2(\sigma))\eta_1(\sigma) \\ \qquad = -B(0, y_2(\sigma), 1, 0)b_{y_1}(0, y_2(\sigma))\eta_1(\sigma)^{\kappa+1}. \end{cases}$$

Here κ is the homogeneity degree of $A + iB$. We have two cases to consider:

(1) Suppose $A(0, 0, 1, 0) = 0$. Then $\Gamma(\sigma) = (0, 0, \eta_1(\sigma), 0)$, where $\eta_1(\sigma)$ solves

$$\dot{\eta}_1(\sigma) = -B(0, 0, 1, 0)b_{y_1}(0, 0)\eta_1(\sigma)^{\kappa+1}.$$

In this case the whole semi-bicharacteristic Γ projects into $\{(0, 0)\}$.

(2) Suppose $A(0, 0, 1, 0) \neq 0$. In this case we can reparametrize Γ, in a neighborhood of (y_0, η_0), in the form $]-\varepsilon_1, \varepsilon_1[\ni y_2 \mapsto (0, y_2, \tilde{\eta}_1(y_2), 0)$.

Summing up we have proved:

Proposition 6.1. *Let Γ be a semi-bicharacteristic curve for $P(y, D_y)$ contained in $\mathcal{N}$ and assume that its base projection does not reduce to a single point. Given $y_0 \in \wp \circ \Gamma$ we can find local coordinates (y_1, y_2) centered at y_0 and defined on $\Omega_1 = \{|y_j| < \varepsilon_1\}$ such that $(\wp \circ \Gamma) \cap \Omega_1 = \{(0, y_2) : |y_2| < \varepsilon_1\}$.*

In particular it follows that $\wp \circ \Gamma$ is an embedded one-dimensional submanifold in Ω.

6.2. **Semi-global reduction near a semi-bicharacteristic interval.** Let Γ be as in the statement of Proposition 6.1. Let $\Gamma_1 \subset \Gamma$ be a semi-bicharacteristic compact interval containing γ in its interior. By Proposition 6.1 it follows that $\wp \circ \Gamma_1$ is diffeomorphic to a compact interval in the real line and this allows us to choose coordinates (x, t), defined on the chart $V_0 = \{(x, t) : |t| < a, |t| < T_0\}$, such that we have

$$\wp \circ \Gamma_1 = \{(0, t) : |t| < T_0\}, \quad \wp \circ \gamma = \{(0, t) : |t| \leq T\}, \quad 0 < T < T_0.$$

We shall write the dual coordinates to (x, t) as (ξ, τ). Notice moreover that, according to the argument that led to the proof of Proposition 6.1, we can then parametrize Γ_1 as

$$\Gamma_1(t) = (0, t, \xi_\bullet(t), \tau_\bullet(t)), \ |t| < T_0. \tag{6.1}$$

Denote by $p_m(x,t,\xi,\tau)$ the principal symbol of P. The fact that Γ_1 is a semi-bicharacteristic for P means that Γ_1' is everywhere tangent to the flow of the vector field $\mathsf{Re}\, H_{(A+iB)p_m}$, where $|A| \geq c > 0$ near the closure of Γ_1. Since Γ_1 is contained in $\mathcal{N}$ and $\Gamma_1'(t) = (0,1,\xi_\bullet'(t),\tau_\bullet'(t))$ it follows easily that $\partial p_m/\partial\tau \neq 0$ on Γ_1. By Euler's identity we obtain

$$\frac{\partial p_m}{\partial \xi}(\Gamma_1(t))\xi_\bullet(t) + \frac{\partial p_m}{\partial \tau}(\Gamma_1(t))\tau_\bullet(t) = m\, p_m(\Gamma_1(t)) = 0$$

and hence necessarily we must have $\xi_\bullet(t) \neq 0$ for all t. Set $\lambda(t) = \tau_\bullet(t)/\xi_\bullet(t)$. Then $p_m(0,t,1,\lambda(t)) = 0$ for every t and hence Taylor's formula gives

$$p_m(0,t,1,\tau) = g(t,\tau)(\tau - \lambda(t)). \tag{6.2}$$

Notice that $g(t,\tau)$ is a polynomial in τ of degree $m-1$ and also that $g(t,\lambda(t)) \neq 0$ for every t. For this last claim just notice that if $g(t_0,\lambda(t_0)) = 0$ for some t_0 then $\partial p_m/\partial\tau = 0$ at $(0,t_0,1,\lambda(t_0))$ by (6.2) and hence we also would have $\partial p_m/\partial\xi = 0$ at $(0,t_0,1,\lambda(t_0))$ by Euler's identity, which contradicts the fact that P has simple real characteristics, that is, $d_{(\xi,\tau)}p_m \neq 0$ everywhere.

We have, for $\xi \neq 0$,

$$p_m(0,t,\xi,\tau) = G(t,\xi,\tau)(\tau - \lambda(t)\xi), \tag{6.3}$$

where

$$G(t,\xi,\tau) = \xi^{m-1} g(t,\tau/\xi).$$

Notice that (6.3) persists when $\xi = 0$, since G is indeed a homogeneous polynomial of degree $m-1$, smooth in t. Notice also that $G(t,1,\lambda(t)) = g(t,\lambda(t)) \neq 0$.

Next we choose $\eta > 0$ small such that $-\lambda(t)$ extends a smooth function $\Lambda(x,t)$, for $|x| < \eta$, $|t| < T + \eta$, which is a single root of the polynomial $p_m(x,t,1,\cdot)$. We can also extend $G(t,\tau,\xi)$ as a smooth function $Q_{m-1}(x,t,\xi,\tau)$ such that

$$p_m(x,t,\xi,\tau) = Q_{m-1}(x,t,\xi,\tau)(\tau + \Lambda(x,t)\xi),$$

holds (x,t), for $|x| < \eta, |t| < T + \eta$. Notice that

$$Q_{m-1}(0,t,\xi_\bullet(t),\tau_\bullet(t)) = \xi_\bullet^{m-1}(t) G(t,1,\lambda(t)) \neq 0$$

for every t. Hence we can assume, reducing $\eta > 0$ if necessary, that $Q_{m-1}(x,t,\xi,\tau) \neq 0$ for $|x| < \eta, |t| < T + \eta$ and (ξ,τ) is a conic neighborhood of γ.

Finally we perform a change of variables of the form $x' = x'(x,t), t' = t$ in order to reduce $\tau + \mathsf{Re}\,\Lambda(x,t)\xi$ to a new covariable τ. After recalling the invariance of the Nirenberg-Treves condition (P) we can state:

Proposition 6.2. *Let γ be a semi-bicharacteristic interval contained in a semi-bicharacteristic curve $\Gamma \subset \mathcal{N}$. Denote by γ^* its projection on the sphere bundle $S^*\Omega$. Then there is a coordinate system (x,t) around $\wp \circ \gamma$, defined on $V_0 = \{(x,t) : |x| < \sigma, |t| < T + \sigma\}$, such that, if (ξ,τ) denote the corresponding dual coordinates:*

(1) $\gamma^* = \{(0,t,1,0) : |t| \leq T\}$;

(2) *There is a conic neighborhood* $\mathcal{C}_0$ *of* $(1,0)$ *in* $\mathbb{R}^2$ *such that on* $V_0 \times \mathcal{C}_0$*, we can write*

$$p_m(x,t) = q_{m-1}(x,t,\xi,\tau)(\tau + ib(x,t)\xi)$$

where q_{m-1} *is a homogeneous polynomial in* (ξ,τ)*, elliptic in* $V_0 \times \mathcal{C}_0$*,* b *is real-valued and* $b(x,\cdot)$ *keeps constant sign.*

Next we proceed as in [G, Lemme 1, p.225]; noticing that the proof of this result can easily be carried out around γ^* allows us to state:

Proposition 6.3. *There are an open neighborhood* $V_0 \times \mathcal{C}_0$ *of* γ^**, a smooth function* $g(x,t)$ *on* V_0 *and pseudo-differential operators* $R_{m-1}(x,t,D_x,D_t) \in \Psi^{m-1}(V_0)$*,* $G_\sharp(x,t,D_x,D_t) \in \Psi^{-1}(V_0)$*, both with symbols supported in* $V_0 \times \mathcal{C}_0$*, such that* R_{m-1} *is elliptic near* γ^* *and*

$$P \equiv (D_t + b(x,t)D_x + g(x,t) + G_\sharp)R_{m-1} \text{ in } V_0 \times \mathcal{C}_0,$$

where $\equiv$ *means equality modulo a smooth regularizing operator.*

6.3. **Proof of Theorem 6.2.** The proof of Theorem 6.2 is now an immediate consequence of propositions 6.2, 6.3 in conjunction with Corollary 3.1 and the remark that follows it. □

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