PERIODIC TRAVELING WAVE SOLUTION FOR NON-HOMOGENEOUS BBM AND KdVB EQUATIONS

MARCOS A. DE FARIAS[1], CEZAR I. KONDO[2] AND JOSÉ R. DOS SANTOS FILHO[3]

Abstract. In this paper the Green's function method and results about fixed point are used to get existence results on periodic traveling wave solution for non-homogeneous problems of generalized versions of the BBM and KdVB equations. It is shown through the constructions of explicit Green's functions that the periodic boundary value problems for the traveling wave solutions of the BBM and KdVB equations are equivalent to integral equations which generate compact operators in the space of periodic functions. These integral representations allowed us to prove that if the speed of the wave propagation is suitable chosen, then the BBM and KdVB equations will admit periodic traveling wave solution.

1. Introduction

The Benjamin-Bona-Mahony (BBM) equation

$$u_t + uu_x - \delta u_{xxt} = 0, \tag{1.1}$$

was introduced in [1] as an improvement of the KdV equation, see [4], for modeling long waves of small amplitude in 1+1 dimensions. In a more sophisticated model, we have the Korteweg-de Vries-Burgers (KdVB) equation,

$$u_t + uu_x - \epsilon u_{xx} - \delta u_{xxx} = 0, \tag{1.2}$$

also used as mathematical model for the propagation of waves, see [2]. In this paper, we consider non-homogeneous problems of generalized versions of the BBM equation

$$u_t + (f_0(u))_x - \epsilon u_{xx} - \delta u_{xxt} = h_0(x - \beta t), \qquad x \in \mathbb{R},\ t \geq 0, \tag{1.3}$$

and of the KdVB equation

$$u_t + (f_0(u))_x - \epsilon u_{xx} - \delta u_{xxx} = h_0(x - \beta t), \qquad x \in \mathbb{R},\ t \geq 0, \tag{1.4}$$

where $\epsilon > 0$, $\delta > 0$ and $\beta > 0$ are constants, $f_0 \in C^1(\mathbb{R})$ and h_0 is a continuous function in $\mathbb{R}$, non-identically zero, 2T-periodic for some $T > 0$ and with the property

$$\int_0^{2T} h_0(x)dx = 0. \tag{1.5}$$

In this paper we investigate existence of periodic traveling wave solutions in the form

$$u(x,t) = u(x - \beta t) = \upsilon(\eta), \qquad \eta = x - \beta t, \tag{1.6}$$

here β is called the speed of the wave propagation.

Date: Feb 2,2015. Corresponding author[3]: santos@dm.ufscar.br, Tel:+55-16-33519158.

In the paper [5] using the Green's function method Liu and Pao proved that for $\epsilon = 0$ and $\delta < 0$ the equation (1.4) admits periodic traveling wave solution, if the speed of the wave propagation β is taken large enough. In our paper, we extend this result for the equations (1.3) and (1.4).

In terms of the function υ, in (1.6), the equation (1.3) is reduced to the third order ordinary differential equation

$$\upsilon''' - \alpha\upsilon'' - \lambda^2\upsilon' = -(f_1(\upsilon))' + h_1, \tag{1.7}$$

where $h_1(\eta) = (\delta\beta)^{-1}h_0(\eta)$, $f_1(\upsilon) = (\delta\beta)^{-1}f_0(\upsilon)$, $\lambda^2 = \delta^{-1}$ and $\alpha = \epsilon/(\delta\beta)$.

For the equation (1.4), in terms of the function υ, occurs the reduction

$$\upsilon''' + \theta\upsilon'' + \rho^2\upsilon' = (f_2(\upsilon))' - h_2, \tag{1.8}$$

where $h_2(\eta) = \delta^{-1}h_0(\eta)$, $f_2(\upsilon) = \delta^{-1}f_0(\upsilon)$, $\rho^2 = \beta/\delta$ and $\theta = \epsilon/\delta$.

We seek solutions of (1.7) and (1.8) satisfying the boundary condition

$$\upsilon^{(k)}(0) = \upsilon^{(k)}(2T), \qquad k = 0, 1, 2. \tag{1.9}$$

In addition, for the Green's function method to work, another condition is imposed

$$\int_0^{2T} \upsilon(\eta)d\eta = 0. \tag{1.10}$$

It can be seen that any solution of the boundary value problem consisting of (1.7) and (1.9) can be extended to a 2T-periodic traveling wave solution to (1.3). Likewise, any solution of the boundary value problem consisting of (1.8) and (1.9) can be extended to a 2T-periodic traveling wave solution to (1.4).

Integrating both sides of (1.7) with respect to η, by (1.9) and (1.10), we have

$$\upsilon'' - \alpha\upsilon' - \lambda^2\upsilon = -f_1(\upsilon) + H_1 + \frac{1}{2T}\int_0^{2T} [f_1(\upsilon(\eta)) - H_1(\eta)]\,d\eta, \tag{1.11}$$

$$\upsilon^{(k)}(0) = \upsilon^{(k)}(2T), \qquad k = 0, 1, \tag{1.12}$$

where $H_1 = (\delta\beta)^{-1}H_0$ with H_0 being any primitive of h_0.

Conversely, direct differentiation of (1.11), by (1.12), will give (1.7) and (1.9). Therefore the boundary value problem consisting of (1.7), (1.9) and (1.10) is equivalent to the boundary value problem consisting of (1.11) and (1.12) in the interval $[0, 2T]$. Likewise, integrating both sides of (1.8) with respect to η, by (1.9) and (1.10), we have

$$\upsilon'' + \theta\upsilon' + \rho^2\upsilon = f_2(\upsilon) - H_2 - \frac{1}{2T}\int_0^{2T} [f_2(\upsilon(\eta)) - H_2(\eta)]d\eta, \tag{1.13}$$

where $H_2 = \delta^{-1}H_0$ with H_0 being any primitive of h_0.

By the same argument of the previous case, the boundary value problem consisting of (1.8), (1.9) and (1.10) is equivalent to the boundary value problem consisting of (1.13) and (1.12) in the interval $[0, 2T]$.

In section 2, we will use the Green's function method, see [7], to obtain integral formulations for the boundary value problems (1.11)-(1.12) and (1.13)-(1.12). Finally, in Section 3, we state and prove our results about existence of periodic traveling wave solutions, if the speed of the wave propagation β is suitable chosen.

2. Green's Function Method

In this section we will seek for integral equations which are equivalent to the boundary value problems (1.11)-(1.12) and (1.13)-(1.12). First, we establish a notation for what comes next, we denote the distribution defined by a continuous function $f : X \to \mathbb{R}$ by

$$T_f : C_0^\infty(X) \to \mathbb{R}, \qquad T_f(\varphi) = \int_X f(x)\varphi(x)dx.$$

The space of distributions in X will be denoted by $\mathcal{D}'(X)$ and frequently in this section we will work in $\mathcal{D}'((0,2T))$. The Lemma 1 that will be established below leads us to observe that if a continuous function υ satisfies (1.11) or (1.13) in $\mathcal{D}'((0,2T))$, then $\upsilon \in C^2((0,2T))$ and the equation (1.11) or (1.13) is fulfilled in the classical sense.

Lemma 1. *Let $X \subset \mathbb{R}$ be a open subset. If $u \in \mathcal{D}'(X)$ and*

$$u^{(m)} + a_{m-1}u^{(m-1)} + \cdots + a_0 u = f \in C(X),$$

where the coefficients $a_j \in C^\infty(X)$, then $u \in C^m(X)$ so the equation is fulfilled in the classical sense.

For the proof see [3].

Theorem 2. *There are continuous functions g and q in $[0,2T] \times [0,2T]$ such that, for $\xi \in (0,2T)$, $g_\xi(\eta) = g(\eta,\xi)$ and $q_\xi(\eta) = q(\eta,\xi)$ satisfy*

$$g_\xi'' - \alpha g_\xi' - \lambda^2 g_\xi = -\delta_\xi + \frac{1}{2T}, \qquad \text{in } \mathcal{D}'((0,2T)), \tag{2.14}$$

$$g_\xi^{(k)}(0) = g_\xi^{(k)}(2T), \qquad k = 0,1, \tag{2.15}$$

$$q_\xi'' + \theta q_\xi' + \rho^2 q_\xi = \delta_\xi - \frac{1}{2T}, \qquad \text{in } \mathcal{D}'((0,2T)), \tag{2.16}$$

$$q_\xi^{(k)}(0) = q_\xi^{(k)}(2T), \qquad k = 0,1, \tag{2.17}$$

where δ_ξ denotes the Dirac delta distribution concentrated at ξ.

These functions g and q will be called Green's functions and will be used to obtain the integral formulations of the boundary value problems (1.11)-(1.12) and (1.13)-(1.12) respectively. For their constructions, we will need the next lemma.

Lemma 3. *Let $\xi \in (0,2T)$. The following hold:*

(i) If $g_\xi \in C([0,2T])$ satisfies

$$\lim_{\rho \to 0^+} g_\xi'(\xi+\rho) - g_\xi'(\xi-\rho) = -1, \tag{2.18}$$

and

$$g_\xi''(\eta) - \alpha g_\xi'(\eta) - \lambda^2 g_\xi(\eta) = \frac{1}{2T}, \qquad \forall \eta \in (0,2T)\setminus\{\xi\}, \tag{2.19}$$

then g_ξ will satisfy (2.14). Furthermore, if, in addition, g_ξ satisfies (2.15), then

$$\int_0^{2T} g_\xi(\eta)d\eta = 0. \tag{2.20}$$

(ii) If $q_\xi \in C([0,2T])$ *satisfies*

$$\lim_{\rho\to 0^+} q'_\xi(\xi+\rho) - q'_\xi(\xi-\rho) = 1, \tag{2.21}$$

and

$$q''_\xi(\eta) + \theta q'_\xi(\eta) + \rho^2 q_\xi(\eta) = -\frac{1}{2T}, \qquad \forall \eta \in (0,2T)\setminus\{\xi\}, \tag{2.22}$$

then q_ξ *will satisfy (2.16). Furthermore, if, in addition,* q_ξ *satisfies (2.17), then*

$$\int_0^{2T} q_\xi(\eta)d\eta = 0. \tag{2.23}$$

Proof of Lemma 3. We will prove the results for the function g_ξ, the proof for the function q_ξ is analogous.

Given a test function $\varphi \in C_0^\infty((0,2T))$, let $a, b \in \mathbb{R}$ such that $0 < a < b < 2T$ and $\operatorname{supp}(\varphi) \subset [a,b]$. If $\xi \notin [a,b]$, then $\varphi(\xi) = 0$ and by (2.19) we have

$$(g''_\xi - \alpha g'_\xi - \lambda^2 g_\xi)(\varphi) = T_{\frac{1}{2T}}(\varphi) = -\delta_\xi(\varphi) + T_{\frac{1}{2T}}(\varphi).$$

Now, supposing $\xi \in [a,b]$ we obtain

$$\begin{aligned}(g''_\xi - \alpha g'_\xi - \lambda^2 g_\xi)(\varphi) &= T_{g_\xi}(\varphi'' + \alpha\varphi' - \lambda^2\varphi)\\ &= \int_a^b g_\xi(\eta)\left[\varphi''(\eta) + \alpha\varphi'(\eta) - \lambda^2\varphi(\eta)\right] d\eta.\end{aligned} \tag{2.24}$$

By using the continuity of g_ξ in $[0,2T]$ and integrating by parts we get

$$\begin{aligned}\int_a^b g_\xi(\eta)\varphi''(\eta)d\eta &= \lim_{\rho\to 0^+}\left[g_\xi(\xi-\rho)\varphi'(\xi-\rho) - g_\xi(\xi+\rho)\varphi'(\xi+\rho)\right]\\ &\quad + \lim_{\rho\to 0^+}\left[g'_\xi(\xi+\rho)\varphi(\xi+\rho) - g'_\xi(\xi-\rho)\varphi(\xi-\rho)\right]\\ &\quad + \lim_{\rho\to 0^+}\left[\int_a^{\xi-\rho} g''_\xi(\eta)\varphi(\eta)d\eta + \int_{\xi+\rho}^b g''_\xi(\eta)\varphi(\eta)d\eta\right].\end{aligned}$$

By continuity,

$$\lim_{\rho\to 0^+}\left[g_\xi(\xi-\rho)\varphi'(\xi-\rho) - g_\xi(\xi+\rho)\varphi'(\xi+\rho)\right] = 0,$$

also by boundedness of g'_ξ and by (2.18),

$$\lim_{\rho\to 0^+}\left[g'_\xi(\xi+\rho)\varphi(\xi+\rho) - g'_\xi(\xi-\rho)\varphi(\xi-\rho)\right] = -\varphi(\xi).$$

Thus,

$$\begin{aligned}\int_a^b g_\xi(\eta)\varphi''(\eta)d\eta &= -\varphi(\xi)\\ &\quad + \lim_{\rho\to 0^+}\left[\int_a^{\xi-\rho} g''_\xi(\eta)\varphi(\eta)d\eta + \int_{\xi+\rho}^b g''_\xi(\eta)\varphi(\eta)d\eta\right].\end{aligned} \tag{2.25}$$

Similarly we have

$$\int_a^b g_\xi(\eta)\varphi'(\eta)d\eta = -\lim_{\rho\to 0^+}\left[\int_a^{\xi-\rho} g'_\xi(\eta)\varphi(\eta)d\eta + \int_{\xi+\rho}^b g'_\xi(\eta)\varphi(\eta)d\eta\right]. \tag{2.26}$$

Therefore (2.19), (2.24), (2.25) and (2.26) show that

$$(g''_\xi - \alpha g'_\xi - \lambda^2 g_\xi)(\varphi) = (-\delta_\xi + T_{\frac{1}{2T}})(\varphi),$$

so we have (2.14).

Now, if, in addition, g_ξ satisfies (2.15), then we can extend g_ξ periodically in $\mathbb{R}$ to a function $\widetilde{g}_\xi \in C(\mathbb{R})$ such that $\widetilde{g}_\xi \in C^2(\mathbb{R}\setminus\{\xi + 2nT : n \in \mathbb{Z}\})$. Thus, (2.18) and (2.19) yield

$$\lim_{\rho\to 0^+} \widetilde{g}'_\xi(\xi + 2nT + \rho) - \widetilde{g}'_\xi(\xi + 2nT - \rho) = -1, \qquad \forall n \in \mathbb{Z}, \tag{2.27}$$

and

$$\widetilde{g}''_\xi(\eta) - \alpha\widetilde{g}'_\xi(\eta) - \lambda^2\widetilde{g}_\xi(\eta) = \frac{1}{2T}, \qquad \forall \eta \in \mathbb{R}\setminus\{\xi + 2nT : n \in \mathbb{Z}\}. \tag{2.28}$$

Similarly to the proof of (2.14), by (2.27) and (2.28), we obtain

$$\widetilde{g}''_\xi - \alpha\widetilde{g}'_\xi - \lambda^2\widetilde{g}_\xi = -\sum_{n\in\mathbb{Z}} \delta_{\xi+2nT} + \frac{1}{2T}, \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.29}$$

For each $k \in \mathbb{N}$, let φ_k be a function in $C_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\varphi_k) \subset \left[-\frac{1}{k}, 2T + \frac{1}{k}\right], \quad 0 \le \varphi_k \le 1 \quad \text{and} \quad \varphi_k(\eta) = 1, \quad \forall \eta \in [0, 2T].$$

From (2.29), for k large enough, if $\xi \in (0, 2T)$ it follows that

$$\left(\widetilde{g}''_\xi - \alpha\widetilde{g}'_\xi - \lambda^2\widetilde{g}_\xi\right)(\varphi_k) = -\varphi_k(\xi) + \frac{1}{2T}\int_{-\frac{1}{k}}^{2T+\frac{1}{k}} \varphi_k(\eta)d\eta,$$

hence

$$\lim_{k\to+\infty} \left(\widetilde{g}''_\xi - \alpha\widetilde{g}'_\xi - \lambda^2\widetilde{g}_\xi\right)(\varphi_k) = 0. \tag{2.30}$$

On the other hand,

$$\begin{aligned}\left(\widetilde{g}''_\xi - \alpha\widetilde{g}'_\xi - \lambda^2\widetilde{g}_\xi\right)(\varphi_k) &= T_{\widetilde{g}_\xi}(\varphi''_k + \alpha\varphi'_k - \lambda^2\varphi_k) \\ &= \int_{-\frac{1}{k}}^{2T+\frac{1}{k}} \widetilde{g}_\xi(\eta)[\varphi''_k(\eta) + \alpha\varphi'_k(\eta) - \lambda^2\varphi_k(\eta)]d\eta,\end{aligned}$$

integrating by parts, we obtain

$$\lim_{k\to+\infty} \left(\widetilde{g}''_\xi - \alpha\widetilde{g}'_\xi - \lambda^2\widetilde{g}_\xi\right)(\varphi_k) = -\lambda^2\int_0^{2T} g_\xi(\eta)d\eta. \tag{2.31}$$

Thus, by (2.30) and (2.31), we have (2.20). □

Proof of Theorem 2. By Lemma 3, it suffices to seek for continuous functions g and q in $[0, 2T] \times [0, 2T]$ such that $g_\xi(\eta) = g(\eta, \xi)$ satisfies (2.15), (2.18) and (2.19); and $q_\xi(\eta) = q(\eta, \xi)$ satisfies (2.17), (2.21) and (2.22).

For $\eta \neq \xi$ a solution for (2.19) is

$$g_\xi(\eta) = \begin{cases} c_1e^{z_1\eta} + c_2e^{z_2\eta} - \dfrac{1}{2T\lambda^2}, & 0 \le \eta < \xi, \\[2ex] c_3e^{z_1\eta} + c_4e^{z_2\eta} - \dfrac{1}{2T\lambda^2}, & \xi < \eta \le 2T, \end{cases} \tag{2.32}$$

where c_1, c_2, c_3 and c_4 are constants that depend on ξ, $z_1 = \frac{\alpha+(\alpha^2+4\lambda^2)^{\frac{1}{2}}}{2}$ and $z_2 = \frac{\alpha-(\alpha^2+4\lambda^2)^{\frac{1}{2}}}{2}$.

It follows from (2.15) and (2.32) that

$$c_1 = c_3 e^{2Tz_1} \qquad \text{and} \qquad c_2 = c_4 e^{2Tz_2}. \tag{2.33}$$

A continuous function g_ξ in $[0, 2T]$ implies that

$$\lim_{\eta\to\xi^-} g_\xi(\eta) = \lim_{\eta\to\xi^+} g_\xi(\eta),$$

hence, by (2.32) and (2.33), we have

$$c_3 = \frac{c_4\left[e^{z_2(\xi+2T)} - e^{z_2\xi}\right]}{e^{z_1\xi} - e^{z_1(\xi+2T)}}. \tag{2.34}$$

Using (2.18), (2.32), (2.33) and (2.34), we obtain

$$g_\xi(\eta) = \begin{cases} \dfrac{1}{\gamma}\left[\dfrac{e^{z_1\eta - z_1\xi + 2Tz_1}}{(e^{2Tz_1}-1)} - \dfrac{e^{z_2\eta - z_2\xi + 2Tz_2}}{(e^{2Tz_2}-1)}\right] - \dfrac{1}{2T\lambda^2}, & 0 \le \eta \le \xi, \\[2ex] \dfrac{1}{\gamma}\left[\dfrac{e^{z_1\eta - z_1\xi}}{(e^{2Tz_1}-1)} - \dfrac{e^{z_2\eta - z_2\xi}}{(e^{2Tz_2}-1)}\right] - \dfrac{1}{2T\lambda^2}, & \xi \le \eta \le 2T, \end{cases}$$

where $\gamma = (\alpha^2 + 4\lambda^2)^{\frac{1}{2}}$. Hence, taking

$$G(\eta, z) = \frac{e^{z\eta}}{\gamma\,(e^{2Tz}-1)},$$

we can write

$$g(\eta,\xi) = \begin{cases} G(\eta - \xi + 2T, z_1) - G(\eta - \xi + 2T, z_2) - \dfrac{1}{2T\lambda^2}, & 0 \le \eta \le \xi, \\[2ex] G(\eta - \xi, z_1) - G(\eta - \xi, z_2) - \dfrac{1}{2T\lambda^2}, & \xi \le \eta \le 2T. \end{cases} \tag{2.35}$$

For the function q, we proceed as in the construction above. However, in this situation there are three cases to consider.

Case 1: $\theta^2 - 4\rho^2 > 0$, i.e. $\beta < \frac{\epsilon^2}{4\delta}$.

$$q(\eta,\xi) = \begin{cases} \dfrac{1}{\mu}\left[\dfrac{e^{w_1\eta - w_1\xi + 2Tw_1}}{(1-e^{2Tw_1})} - \dfrac{e^{w_2\eta - w_2\xi + 2Tw_2}}{(1-e^{2Tw_2})}\right] - \dfrac{1}{2T\rho^2}, & 0 \le \eta \le \xi, \\[2ex] \dfrac{1}{\mu}\left[\dfrac{e^{w_1\eta - w_1\xi}}{(1-e^{2Tw_1})} - \dfrac{e^{w_2\eta - w_2\xi}}{(1-e^{2Tw_2})}\right] - \dfrac{1}{2T\rho^2}, & \xi \le \eta \le 2T, \end{cases} \tag{2.36}$$

where $\mu = (\theta^2 - 4\rho^2)^{\frac{1}{2}}$, $w_1 = \frac{-\theta+\mu}{2}$ and $w_2 = \frac{-\theta-\mu}{2}$. Hence, taking

$$H(\eta, w) = \frac{e^{w\eta}}{\mu\,(1-e^{2Tw})},$$

we can write

$$q(\eta,\xi)=\begin{cases} H(\eta-\xi+2T,w_1)-H(\eta-\xi+2T,w_2)-\dfrac{1}{2T\rho^2}, & 0\le\eta\le\xi,\\[2ex] H(\eta-\xi,w_1)-H(\eta-\xi,w_2)-\dfrac{1}{2T\rho^2}, & \xi\le\eta\le 2T. \end{cases} \tag{2.37}$$

Case 2: $\theta^2-4\rho^2=0$, i.e. $\beta=\frac{\epsilon^2}{4\delta}$.

$$q(\eta,\xi)=\begin{cases} \frac{\theta^2[(\xi+2T)e^{-\theta T}-\xi]e^{\frac{-\theta(\eta-\xi+2T)}{2}}}{4\rho^2(1-e^{-\theta T})^2}+\frac{\theta^2(\eta+2T)e^{\frac{-\theta(\eta-\xi+2T)}{2}}}{4\rho^2(1-e^{-\theta T})}-\frac{1}{2T\rho^2}, & 0\le\eta\le\xi,\\[2ex] \frac{\theta^2[(\xi+2T)e^{-\theta T}-\xi]e^{\frac{-\theta(\eta-\xi)}{2}}}{4\rho^2(1-e^{-\theta T})^2}+\frac{\theta^2\eta e^{\frac{-\theta(\eta-\xi)}{2}}}{4\rho^2(1-e^{-\theta T})}-\frac{1}{2T\rho^2}, & \xi\le\eta\le 2T. \end{cases} \tag{2.38}$$

Case 3: $\theta^2-4\rho^2<0$, i.e. $\beta>\frac{\epsilon^2}{4\delta}$.

$$q(\eta,\xi)=\begin{cases} \frac{2e^{\frac{-\theta(\eta-\xi+2T)}{2}}[\sin\frac{\omega(\eta-\xi+2T)}{2}-e^{-\theta T}\sin\frac{\omega(\eta-\xi)}{2}]}{\omega(e^{-2\theta T}-2e^{-\theta T}\cos\omega T+1)}-\frac{1}{2T\rho^2}, & 0\le\eta\le\xi,\\[2ex] \frac{2e^{\frac{-\theta(\eta-\xi)}{2}}[\sin\frac{\omega(\eta-\xi)}{2}-e^{-\theta T}\sin\frac{\omega(\eta-\xi-2T)}{2}]}{\omega(e^{-2\theta T}-2e^{-\theta T}\cos\omega T+1)}-\frac{1}{2T\rho^2}, & \xi\le\eta\le 2T, \end{cases} \tag{2.39}$$

where $\omega=(4\rho^2-\theta^2)^{\frac{1}{2}}$. $\square$

Now, we describe some useful properties of our Green's functions.

Lemma 4. *The functions $G(\eta,z)$ and $H(\eta,w)$ have the following properties:*

(i) $G(\eta,z_1)-G(\eta,z_2)\ge 0,\ \forall\eta\ge 0.$

(ii) $\displaystyle\int_0^\eta G(\eta-\xi,z)d\xi+\int_\eta^{2T}G(\eta-\xi+2T,z)d\xi=\frac{1}{\gamma z},\ \forall\eta\in[0,2T],\forall z\in\mathbb{R}.$

(iii) $H(\eta,w_1)-H(\eta,w_2)\ge 0,\ \forall\eta\ge 0.$

(iv) $\displaystyle\int_0^\eta H(\eta-\xi,w)d\xi+\int_\eta^{2T}H(\eta-\xi+2T,w)d\xi=-\frac{1}{\mu w},\ \forall\eta\in[0,2T],\forall w\in\mathbb{R}.$

Proof. Observing that $z_1>0$, $z_2<0$ and $w_2<w_1<0$ we obtain (i) and (iii). The proofs of (ii) and (iv) follow by direct calculation. $\square$

Lemma 5. *Let $\upsilon\in C([0,2T])$ such that $\upsilon\in C^2((0,2T))$ and $\upsilon(0)=\upsilon(2T)$. If υ satisfies (1.10) and (1.11), or* (1.10) *and* (1.13), *then υ' and υ'' are bounded in $(0,2T)$ and hence υ have first and second derivative at* 0 *and* $2T$, *from the right and from the left respectively. Furthermore, $\upsilon'(0)=\upsilon'(2T)$.*

Proof. Suppose that υ satisfies (1.10) and (1.11), the other case follows analogously. For $0<a<b<2T$, integrating (1.11) from a to b, we get

$$\upsilon'(b)-\upsilon'(a)= \tag{2.40}$$
$$\alpha\left(\upsilon(b)-\upsilon(a)\right)+\int_a^b\left\{\lambda^2\upsilon(\eta)-f_1(\upsilon(\eta))+H_1(\eta)+\tfrac{1}{2T}\int_0^{2T}[f_1(\upsilon(\xi))-H_1(\xi)]d\xi\right\}d\eta.$$

Therefore, υ' is bounded in $(0, 2T)$ and by the equation (1.11) we have the boundedness of υ''. Hence, υ have first and second derivative in the boundary of $[0, 2T]$. Furthermore, by making $a \to 0^+$ and $b \to 2T^-$ in (2.40), by (1.10), it follows that $\upsilon'(0) = \upsilon'(2T)$. □

Finally, the integral formulations of the boundary value problems (1.11)-(1.12) and (1.13)-(1.12) are given by the next theorem.

Theorem 6. *Let υ be a continuous function on $[0, 2T]$.*

(i) υ is a solution of the boundary value problem (1.11)-(1.12) in $[0, 2T]$ if, and only if, it is a solution of the integral equation

$$\upsilon(\eta) = \int_0^{2T} g(\eta, \xi)[f_1(\upsilon(\xi)) - H_1(\xi)]d\xi, \qquad 0 \le \eta \le 2T, \tag{2.41}$$

where $g(\eta, \xi)$ is the function constructed in the Theorem 2.

(ii) υ is a solution of the boundary value problem (1.13)-(1.12) in $[0, 2T]$ if, and only if, it is a solution of the integral equation

$$\upsilon(\eta) = \int_0^{2T} q(\eta, \xi)[f_2(\upsilon(\xi)) - H_2(\xi)]d\xi, \qquad 0 \le \eta \le 2T, \tag{2.42}$$

where $q(\eta, \xi)$ is the function constructed in the Theorem 2.

Proof. Let υ be a solution of the boundary value problem (1.11)-(1.12). By taking $\kappa(\eta) = \int_0^{2T} g(\eta, \xi)[f_1(\upsilon(\xi)) - H_1(\xi)]d\xi$, we can work in $\mathcal{D}'((0, 2T))$ and to use (2.14) to obtain

$$\kappa'' - \alpha\kappa' - \lambda^2\kappa = -f_1(\upsilon) + H_1 + \frac{1}{2T}\int_0^{2T} [f_1(\upsilon(\xi)) - H_1(\xi)]d\xi, \qquad \mathcal{D}'((0, 2T)).$$

Thus, taking $\vartheta(\eta) = \kappa(\eta) - \upsilon(\eta)$ we conclude that

$$\vartheta''(\eta) - \alpha\vartheta'(\eta) - \lambda^2\vartheta(\eta) = 0, \qquad \eta \in (0, 2T), \tag{2.43}$$

$$\vartheta(0) = \vartheta(2T), \tag{2.44}$$

$$\int_0^{2T} \vartheta(\eta)d\eta = 0. \tag{2.45}$$

Since the null function is the only solution of (2.43), (2.44) and (2.45), we have

$$\upsilon(\eta) = \int_0^{2T} g(\eta, \xi)[f_1(\upsilon(\xi)) - H_1(\xi)]d\xi, \qquad \forall \eta \in [0, 2T].$$

Conversely, if υ satisfies (2.41), by (2.14) we obtain

$$\upsilon'' - \alpha\upsilon' - \lambda^2\upsilon = -f_1(\upsilon) + H_1 + \frac{1}{2T}\int_0^{2T} [f_1(\upsilon(\xi)) - H_1(\xi)]d\xi, \qquad \mathcal{D}'((0, 2T)).$$

Thus, by Lemma 1, it follows that $\upsilon \in C^2((0, 2T))$ and (1.11) is satisfied in the classical sense. From (2.20), (2.41) and Fubini's theorem we obtain (1.10). Thence (2.15), (2.41) and Lemma 5 imply (1.12).

The proof of (ii) is similar to (i). □

Corollary 7. *We have:*

(i) Any continuous solution υ of (2.41) can be extended to a $2T$-periodic function $u \in C^3(\mathbb{R})$ such that $u(x-\beta t)$ is a periodic traveling wave solution for (1.3).

(ii) Any continuous solution υ of (2.42) can be extended to a $2T$-periodic function $u \in C^3(\mathbb{R})$ such that $u(x-\beta t)$ is a periodic traveling wave solution for (1.4).

3. **The Existence Results**

Let C_{2T} be the space of the real-valued continuous functions, $\upsilon(\eta)$, on $[0,2T]$ such that $\upsilon(0)=\upsilon(2T)$ equipped with the supremum norm $\|\cdot\|$ and let $B_r(0)$ be the closed ball in C_{2T} with center zero and radius $r>0$.

For $\upsilon \in C_{2T}$, define the operators

$$A\upsilon(\eta) = \int_0^{2T} g(\eta,\xi)[f_1(\upsilon(\xi)) - H_1(\xi)]d\xi, \qquad \forall \eta \in [0,2T], \tag{3.46}$$

and

$$U\upsilon(\eta) = \int_0^{2T} q(\eta,\xi)[f_2(\upsilon(\xi)) - H_2(\xi)]d\xi, \qquad \forall \eta \in [0,2T]. \tag{3.47}$$

Lemma 8. *The operator A has the following properties:*

(i) $A : C_{2T} \to C_{2T}$.

(ii) $\overline{A(B_r(0))}$ is compact in C_{2T}, $\forall r > 0$.

(iii) $A : B_r(0) \to C_{2T}$ is continuous, $\forall r > 0$.

Proof. The proof of (i) follows by (2.15) and by the uniform continuity of g. To obtain (ii), we use the inequality

$$|A\upsilon(\eta)| \le 2T \sup_{x,\xi\in[0,2T]} |g(x,\xi)| \sup_{(z,\xi)\in[-r,r]\times[0,2T]} |f_1(z) - H_1(\xi)|,$$

to conclude that $A(B_r(0))$ is uniformly bounded. The uniform continuity of g in $[0,2T]\times[0,2T]$ and the inequality

$$|A\upsilon(\eta) - A\upsilon(z)| \le \int_0^{2T} |g(\eta,\xi) - g(z,\xi)|\, d\xi \sup_{(z,\xi)\in[-r,r]\times[0,2T]} |f_1(z) - H_1(\xi)|,$$

imply that $A(B_r(0))$ is equicontinuous, thus by the Arzelà-Ascoli theorem we conclude that $\overline{A(B_r(0))}$ is compact in C_{2T}. For (iii), we use the inequality

$$\|A\upsilon - Au\| \le 2T \sup_{\eta,\xi\in[0,2T]} |g(\eta,\xi)| \sup_{z\in[-r,r]} |f_1'(z)|\, \|\upsilon - u\|, \qquad \forall \upsilon, u \in B_r(0). \quad \square$$

Remark: These properties also hold to the operator U.

Theorem 9. *Let $\mathcal{M}$ be a non-empty convex subset of a normed space $\mathcal{N}$. Let T be a continuous mapping of $\mathcal{M}$ into a compact set $\mathcal{K} \subset \mathcal{M}$. Then T has a fixed point.*

For the proof see [6].

Now we state the two main theorems of this paper. From these theorems we can conclude that for β suitable chosen, the equations (1.3) and (1.4) will admit periodic traveling wave solutions with speed of the wave propagation equal to β. For $r > 0$, setting

$$M_r = \sup_{(w,\xi)\in[-r,r]\times[0,2T]} |f_0(w) - H_0(\xi)|,$$

we have the following theorems.

Theorem 10. *For any $\beta \geq \frac{2M_r}{r}$ the equation (1.3) admits periodic traveling wave solution in the form $u(x-\beta t)$ such that $|u(x-\beta t)| \leq r$.*

Proof. By (i) of Lemma 4, we obtain

$$|g(\eta,\xi)| \leq \begin{cases} G(\eta-\xi+2T, z_1) - G(\eta-\xi+2T, z_2) + \frac{1}{2T\lambda^2}, & 0 \leq \eta \leq \xi, \\ G(\eta-\xi, z_1) - G(\eta-\xi, z_2) + \frac{1}{2T\lambda^2}, & \xi \leq \eta \leq 2T, \end{cases}$$

thus, by (ii) of Lemma 4,

$$\sup_{\eta\in[0,2T]} \int_0^{2T} |g(\eta,\xi)|\, d\xi \leq 2\delta.$$

Therefore, for $\upsilon \in B_r(0)$, since $f_1 = \frac{1}{\delta\beta} f_0$ and $H_1 = \frac{1}{\delta\beta} H_0$ we have

$$\|A\upsilon\| \leq \sup_{\eta\in[0,2T]} \int_0^{2T} |g(\eta,\xi)|\, d\xi \, \|f_1(\upsilon) - H_1\| \leq \frac{2M_r}{\beta} \leq r.$$

Hence $A(B_r(0)) \subset B_r(0)$ and by the Lemma 8 and Theorem 9, A has a fixed point $\upsilon \in B_r(0)$. Concluding, by the Corollary 7 we obtain the proof. □

Theorem 11. *For either of the three cases:*

(i) $\dfrac{2M_r}{r} \leq \beta < \dfrac{\epsilon^2}{4\delta}$,

(ii) $\left[\dfrac{3T^2\epsilon^2}{\delta^2(1-e^{\frac{-\epsilon T}{\delta}})^2} + \dfrac{2T^2\epsilon^2}{\delta^2(1-e^{\frac{-\epsilon T}{\delta}})} + 1\right] \dfrac{M_r}{r} \leq \beta = \dfrac{\epsilon^2}{4\delta}$,

(iii) $\left[\frac{8T}{(4\beta\delta-\epsilon^2)^{\frac{1}{2}}(1-e^{\frac{-\epsilon T}{\delta}})^2} + \frac{1}{\beta}\right] \frac{M_r}{r} \leq 1$ *and* $\beta > \frac{\epsilon^2}{4\delta}$.

The equation (1.4) *admits periodic traveling wave solution in the form $u(x-\beta t)$ such that $|u(x-\beta t)| \leq r$.*

Proof. Assume (i), then the Green's function in this case is the function (2.37). By (iii) of Lemma 4, we obtain

$$|q(\eta,\xi)| \leq \begin{cases} H(\eta-\xi+2T, w_1) - H(\eta-\xi+2T, w_2) + \frac{1}{2T\rho^2}, & 0 \leq \eta \leq \xi, \\ H(\eta-\xi, w_1) - H(\eta-\xi, w_2) + \frac{1}{2T\rho^2}, & \xi \leq \eta \leq 2T, \end{cases}$$

thus, by (iv) of Lemma 4,

$$\sup_{\eta\in[0,2T]}\int_0^{2T}|q(\eta,\xi)|\,d\xi \le \frac{2\delta}{\beta}.$$

Therefore, for $v \in B_r(0)$,

$$\|Uv\| \le \sup_{\eta\in[0,2T]}\int_0^{2T}|q(\eta,\xi)|\,d\xi\,\|f_2(v)-H_2\|,$$

since $f_2 = \frac{1}{\delta}f_0$ and $H_2 = \frac{1}{\delta}H_0$, we have

$$\|Uv\| \le \frac{2M_r}{\beta} \le r.$$

Hence $U(B_r(0)) \subset B_r(0)$ and by the Lemma 8 and Theorem 9, U has a fixed point $v \in B_r(0)$. Thus, by the Corollary 7, the equation (1.4) admits periodic traveling wave solution in the form $u(x-\beta t)$ such that $|u(x-\beta t)| \le r$.

Now, assuming (ii), the Green's function is the function (2.38). Thus, observing that $\theta = \frac{\epsilon}{\delta}$ and $\rho^2 = \frac{\beta}{\delta}$, we obtain

$$|q(\eta,\xi)| \le \frac{\delta}{\beta}\left[\frac{3T\epsilon^2}{2\delta^2(1-e^{\frac{-\epsilon T}{\delta}})^2} + \frac{T\epsilon^2}{\delta^2(1-e^{\frac{-\epsilon T}{\delta}})} + \frac{1}{2T}\right],\ \forall \eta,\xi \in [0,2T].$$

Therefore, for $v \in B_r(0)$,

$$\|Uv\| \le \sup_{\eta\in[0,2T]}\int_0^{2T}|q(\eta,\xi)|\,d\xi\,\|f_2(v)-H_2\| \le r.$$

Hence $U(B_r(0)) \subset B_r(0)$ and by the Lemma 8 and Theorem 9, U has a fixed point $v \in B_r(0)$. Thus, by the Corollary 7, we obtain the desired result.

For (iii), the Green's function is the function (2.39). Therefore,

$$|q(\eta,\xi)| \le \frac{4}{\omega(1-e^{\frac{-\epsilon T}{\delta}})^2} + \frac{1}{2T\rho^2},$$

then, for $v \in B_r(0)$,

$$\|Uv\| \le \left[\frac{8T}{(4\beta\delta-\epsilon^2)^{\frac{1}{2}}(1-e^{\frac{-\epsilon T}{\delta}})^2} + \frac{1}{\beta}\right]M_r \le r.$$

Concluding as in the two other cases, the result follows. □

Since Theorem 9 does not guarantee uniqueness of the fixed point we can not add uniqueness in the statements of our results. The additional hypotheses

$$2T\sup_{\eta,\xi\in[0,2T]}|g(\eta,\xi)|\sup_{z\in[-r,r]}|f_1'(z)| < 1$$

and

$$2T\sup_{\eta,\xi\in[0,2T]}|q(\eta,\xi)|\sup_{z\in[-r,r]}|f_2'(z)| < 1$$

when imposed implies that the operators A and U, see (3.46) and (3.47), are contractions, therefore getting uniqueness.

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Universidade Federal de São Carlos P. O. Box 676, 13565-905, São Carlos-SP, Brazil
E-mail address: marcosfarias@dm.ufscar.br[1]
E-mail address: dcik@dm.ufscar.br[2]
E-mail address: santos@dm.ufscar.br[3]