Generalized Kotani's trick for unitary operators

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Abstract

We present generalizations of Kotani's trick for unitary operators, including a Hausdorff continuous version adapted from recent results in the self-adjoint case.

Keywords: rank one perturbations; unitary operators; Kotani's trick; Hausdorff continuity.

1 Introduction and results

Rank one perturbations of a unitary operator U is a subject of physical and mathematical interest. Denote by $|\phi\rangle\langle\phi|$ the projection operator on the subspace spanned by the vector ϕ in a (separable) Hilbert space; for $\lambda \in \mathbb{R}$ such perturbations have the form

$$U_\lambda := U \exp(i\lambda|\phi\rangle\langle\phi|) = U\left[1 + (e^{i\lambda} - 1)|\phi\rangle\langle\phi|\right]. \tag{1.1}$$

For simplicity, we normalize ϕ.

From a physical viewpoint, if H_0 represents an unperturbed self-adjoint operator, U_λ corresponds to the Floquet operator of the periodically kicked Hamiltonian formally given by [6, 7, 8, 3]

$$H(t) = H_0 + \lambda\,|\phi\rangle\langle\phi| \sum_{n\in\mathbb{Z}} \delta(t-n),$$

so that in (1.1) we identify $U = e^{-iH_0}$ and λ is the intensity of the kicks. Note that $\lambda \mapsto U_\lambda$ is 2π-periodic, and so one may restrict the considerations to $\lambda \in [0, 2\pi]$. See [22] for the case of periodically rank-N kicked Hamiltonian and [1] for the question of cyclicity and Alekssandrov-Clark theory related to unitary rank one perturbations.

There is a large literature on rank one perturbations of self-adjoint operators (see the review [26] that is still a key reference), and a useful tool in this setting is the so-called Kotani's trick [18, 19, 20], a result on spectral averaging with many applications (see [5, 29, 10, 14, 12, 11] and references therein). There is a more restricted literature in the case of unitary operators.

The first version of this trick in the unitary situation appeared in [6] (see Lemma 5 there), and it says that a particular choice of an absolutely continuous (with respect to Lebesgue) measure ρ in (1.3) results in Ω_ρ equivalent to Lebesgue measure. It was important in the derivation of a unitary version [6] of the so-called Simon-Wolff criterion [29]. This set of results was applied to different (although related) contexts involving unitary operators: Anderson localization in the random unitary framework [15]; studies of the Chalker-Coddington model [2]; Floquet operators with pure point spectrum and energy instability [9]; singular continuous spectra [3, 4] for some Floquet operators; onset of quantum chaos [23]; adaptation of the fractional moment method to random unitary operators [17]. Such applications, besides the interest in spectral averages in the unitary setting in their own right, motivated us to investigate possible unitary versions of results by Marx [20] on rank one perturbations of self-adjoint operators.

The work [20] has considered the spectral averaging through a more general measure η, by extending the case where η is the Lebesgue measure, and examined how continuity properties of η are inherited by the final result; in particular when Hausdorff dimensional properties are considered. Here we present extensions of such results to the unitary setting (1); although the general ideas are borrowed from [20], the technicalities in the unitary case are more involved and requires precise choice of working functions, as discussed in the proofs ahead.

Our first results (i.e., Theorems 1 and 2) compose a more detailed version of spectral averages when ρ below is Lebesgue measure; denote by $\ell(\cdot)$ the Lebesgue measure restricted to the Borel sets of $[0, 2\pi]$ (with the ends identified), and by ω and ω_λ the spectral measures of U and U_λ, respectively, both with respect to ϕ.

Theorem 1 *Consider the measure Γ defined on Borel sets $B \subset [0, 2\pi]$ by the spectral averaging with respect to Lebesgue measure*

$$\Gamma(B) := \int_0^{2\pi} \omega_\lambda(B)\, \mathrm{d}\lambda. \tag{1.2}$$

Then, $\Gamma = \ell$.

Theorem 2 *Fix a finite measure ρ on the Borel sets B of $[0, 2\pi]$, and define the spectral averaging measure Ω_ρ by*

$$\Omega_\rho(B) := \int_0^{2\pi} \omega_\lambda(B)\, \mathrm{d}\rho(\lambda). \tag{1.3}$$

Then, if $\rho \ll \ell$ one also has $\Omega_\rho \ll \ell$.

For $0 \le \alpha \le 1$, denote by h^α the α-Hausdorff measure restricted to $[0, 2\pi]$ [13, 21, 24]. Recall that a Borel measure μ is *α-Hausdorff continuous* (α-Hc) if $\mu(B) = 0$ for all sets with $h^\alpha(B)=0$, and it is α-Hausdorff singular (α-Hs) if there is a Borel set B with $h^\alpha(B)=0$ and $\mu(B^c)=0$. Also, $h^1=\ell$ on measurable subsets of the real line, so that a measure μ is 1-Hc if, and only if, $\mu \ll \ell$.

Theorem 3 *Let ρ and Ω_ρ be as in Theorem 2 and $0<\alpha<1$. If ρ is α-Hausdorff continuous, then Ω_ρ is δ-Hausdorff continuous for all $0<\delta<\alpha<1$.*

The rest of this work is dedicated to the proofs of the above theorems. In Section 2 we discuss some analogous to Poisson and conjugate Poisson transforms [16] of measures suitable to the unitary setting that we are interested in. The proofs are concluded in Section 3.

2 Transforms

For the proofs of the theorems stated in the Section 1, it is convenient to introduce suitable transforms of measures. Recall that ρ denotes a Borel finite measure on $[0,2\pi]$; we represent by $z=re^{i\theta}$ a complex number. We will make use of the following *Poisson* Φ_ρ and *conjugate Poisson* Π_ρ transforms of ρ,

$$\Phi_\rho(z) := \int_0^{2\pi} \frac{e^{i\theta'}+z}{e^{i\theta'}-z}\, \mathrm{d}\rho(\theta'), \quad |z|<1, \tag{2.4}$$

and its real part

$$\Pi_\rho(re^{i\theta}) := \mathrm{Re}\,\Phi_\rho(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\, \mathrm{d}\rho(\theta'), \quad r<1. \tag{2.5}$$

Sometimes Φ_ρ is known as the *Cauchy* transform [9] of ρ.

Recall also that the upper α-derivative of a measure η is given by

$$\overline{D}^\alpha_\eta(\theta) := \limsup_{\epsilon\to 0} \; \frac{\eta(\theta-\epsilon,\theta+\epsilon)}{\epsilon^\alpha}\,.$$

The following theorem is crucial and its proof can be found in [24, 21].

Theorem 4 *Given a σ-finite Borel measure η on $\mathbb{R}$, consider the Borel sets*

$$T^\alpha_{\eta\infty} := \{x\in\mathbb{R} : \overline{D}^\alpha_\eta(x)=\infty\}, \quad T^\alpha_{\eta+} := \mathbb{R}\setminus T^\alpha_{\eta\infty},$$

denote by χ_α characteristic function of $T^\alpha_{\eta\infty}$, and define

$$\mathrm{d}\eta_{\alpha\mathrm{Hs}} := \chi_\alpha \mathrm{d}\eta, \quad \mathrm{d}\eta_{\alpha\mathrm{Hc}} := (1-\chi_\alpha)\,\mathrm{d}\eta.$$

Then, $\eta_{\alpha\mathrm{Hc}} \perp \eta_{\alpha\mathrm{Hs}}$, $\eta=\eta_{\alpha\mathrm{Hc}}+\eta_{\alpha\mathrm{Hs}}$, with $\eta_{\alpha\mathrm{Hc}}$ an α-Hc measure whereas $\eta_{\alpha\mathrm{Hs}}$ is α-Hs.

Next a result corresponding to a known one in the self-adjoint setting, that is, Theorem 3.1 in [10].

Proposition 1 *Fix* $\alpha \in [0,1)$ *and* $\theta \in (0,2\pi)$*. Then* $\limsup_{r\to 1^-}(1-r)^{1-\alpha}\Pi_\rho(re^{i\theta})$ *and* $\overline{D}^\alpha_\rho(\theta)$ *are simultaneously finite or infinite.*

Proof. The proof will follow from two facts. First fact: denote $1-r$ by ϵ, then if $\epsilon > 0$ is small enough, there is $K > 0$ so that

$$(1-r)^{1-\alpha}\Pi_\rho(re^{i\theta}) \geq \frac{K}{2}\frac{\rho(\theta-\epsilon,\theta+\epsilon)}{\epsilon^\alpha}. \tag{2.6}$$

In fact, by the positivity of the integrand below, we have

$$\begin{aligned}
\Pi_\rho(re^{i\theta}) &= \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\,\mathrm{d}\rho(\theta) \\
&\geq \int_{|\theta-\theta'|<\epsilon} \frac{\epsilon(2-\epsilon)}{1+(1-\epsilon)^2-2(1-\epsilon)\cos(\epsilon)}\,\mathrm{d}\rho(\theta') \\
&= \rho(\theta-\epsilon,\theta+\epsilon)\left[\frac{\epsilon(2-\epsilon)}{1+(1-\epsilon)^2-2(1-\epsilon)\cos(\epsilon)}\right] \\
&= \rho(\theta-\epsilon,\theta+\epsilon)\left[\frac{\epsilon(2-\epsilon)}{2\epsilon^2+O(\epsilon^3)}\right] \\
&= \frac{\rho(\theta-\epsilon,\theta+\epsilon)}{2\epsilon}\left[\frac{(2-\epsilon)}{2}\right](1-O(\epsilon)) \\
&\geq \frac{K}{2}\frac{\rho(\theta-\epsilon,\theta+\epsilon)}{\epsilon}\,.
\end{aligned}$$

It was used that $\cos\epsilon = 1-\epsilon^2/2+O(\epsilon^4)$ and that there is $K > 0$ so that, for positive and ϵ small enough, one has $[\frac{(2-\epsilon)}{2}][1-O(\epsilon)] \geq K$. Now (2.6) is immediate.

Second fact: if $\overline{D}^\alpha_\rho(\theta) < \infty$, then $\limsup_{r\to 1^-}(1-r)^{1-\alpha}\Pi_\rho(re^{i\theta}) < \infty$. Furthermore, if $\overline{D}^\alpha_\rho(\theta) = 0$, then $\limsup_{r\to 1^-}(1-r)^{1-\alpha}\Pi_\rho(re^{i\theta}) = 0$.

To verify this, note initially that

$$\lim_{r\to 1^-}\left[\frac{\delta^{\alpha+1}}{((1-r)^2+r\delta^2)^2}-\frac{\sin(\delta)\delta^{\alpha}}{(1-2r\cos\delta+r^2)^2}\right] = \delta^{\alpha}\left[\frac{\delta}{\delta^4}-\frac{\sin\delta}{2^4\sin(\frac{\delta}{2})^4}\right]$$

$$= \delta^{\alpha}\left[\frac{\delta}{\delta^4}-\frac{\delta+O(\delta^3)}{\delta^4+O(\delta^6)}\right]$$

$$= \delta^{\alpha}\left[\frac{\delta}{\delta^4}-\frac{\delta}{\delta^4}\frac{1+O(\delta^2)}{1+O(\delta^2)}\right]$$

$$= O(\delta^{\alpha-1}).$$

The case $\alpha=0$ follows directly from item i) of Theorem 5 below.

Suppose now that $\alpha\in(0,1)$; in this case denote $f_\theta(\delta)=\rho[\theta,\theta+\delta)$ if $\delta\geq 0$, and $f_\theta(\delta)=-\rho[\theta+\delta,\theta)$ if $\delta\leq 0$, and by considering a Stieltjes integral with respect to $f_\theta(\delta)$, we have

$$\limsup_{r\to 1^-}(1-r)^{1-\alpha}\Pi_\rho(re^{i\theta})=\limsup_{r\to 1^-}(1-r)^{1-\alpha}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\,\mathrm{d}\rho(\theta')$$

$$= \limsup_{r\to 1^-}\int_{-\theta}^{2\pi-\theta}(1-r)^{1-\alpha}\frac{1-r^2}{1-2r\cos(\delta)+r^2}\,\mathrm{d}f_\theta(\delta)$$

$$=\limsup_{r\to 1^-}\int_{-\delta_0}^{\delta_0}(1-r)^{1-\alpha}\frac{2r(1-r^2)\sin\delta f_\theta(\delta)}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta$$

(by dominated convergence, since outside the interval $(-\delta_0,\delta_0)$ the integrand is bounded and vanishes; then an integration by parts was done, in which the boundary terms vanish as $r\to 0$)

$$= \limsup_{r\to 1^-}\Big[\int_0^{\delta_0}\frac{(1-r)^{1-\alpha}2r(1-r^2)(\sin\delta)\rho[\theta,\theta+\delta)}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta$$

$$+ \int_{-\delta_0}^{0}\frac{(1-r)^{1-\alpha}2r(1-r^2)(\sin\delta)(-\rho[\theta+\delta,\theta))}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta\Big]$$

$$= \limsup_{r\to 1^-}\Big[\int_0^{\delta_0}\frac{(1-r)^{1-\alpha}2r(1-r^2)(\sin\delta)\rho[\theta,\theta+\delta)}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta$$

$$+ \int_0^{\delta_0}\frac{(1-r)^{1-\alpha}2r(1-r^2)(\sin\delta)\rho[\theta-\delta,\theta)}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta\Big]$$

$$\leq \limsup_{r\to 1^-}2C\left[(1-r)^{1-\alpha}2r(1-r^2)\right]\int_0^{\delta_0}\frac{(\delta^{\alpha}\sin\delta)}{(1-2r\cos(\delta)+r^2)^2}\,\mathrm{d}\delta$$

(it was used that $\rho(\theta-\delta,\theta+\delta)\le C\delta^\alpha$ if $\delta\le\delta_0$ and certain $C>0$ since $\overline{D}^\alpha_\rho(\theta)<\infty$)

$$= \limsup_{r\to 1^-} 2C[(1-r)^{1-\alpha}2r(1-r^2)]\int_0^{\delta_0}\frac{\delta^{\alpha+1}}{((1-r)^2+r\delta^2)^2}\,\mathrm{d}\delta,$$

by the remark at the beginning and the fact that $O(\delta^{\alpha-1})$ is integrable in a neighborhood of the origin for $\alpha>0$, and that the multiplicative r-dependent factor vanishes as $r\to 1$. Now, after a suitable change of variable, the above integral takes the form

$$\le \limsup_{r\to 1^-} 2C\left[(1-r)^{1-\alpha}2r(1-r^2)\right]\left[\frac{(1-r)^{\alpha-2}}{r^{\frac{\alpha+2}{2}}}\right]\int_0^{\frac{r^{\frac{1}{2}}}{1-r}\delta_0}\frac{u^{\alpha+1}}{(u^2+1)^2}\,\mathrm{d}u$$

$$\le\; 8C\int_0^\infty \frac{u^{\alpha+1}}{(u^2+1)^2}\,\mathrm{d}u<\infty,$$

and one concludes that the limit is finite and if $\overline{D}^\alpha_\rho(\theta)=0$; since C may be taken arbitrary small, the above term vanishes. This concludes the proof. □

Now an important result that parallels a famous one [26, 28] for measures on the real line. Few points of the proof of Theorem 5 are missing in the references, but they are simple counterparts of the corresponding self-adjoint cases and will be skipped.

Theorem 5 ([27, 28]) *Let ρ be a finite measure on the Borel sets of $[0,2\pi]$. The following statements hold true:*

i) $\rho(\{\theta_0\})=\lim_{r\to 1^-}\frac{(1-r)}{2}\Pi_\rho(re^{i\theta_0})$, *for all* $\theta_0\in[0,2\pi]$.

ii) *The* $\Pi^-_\rho(e^{i\theta}):=\lim_{r\to 1^-}\Pi_\rho(re^{i\theta})$ *exists* ℓ*-a.e.*

iii) $\Pi_\rho(re^{i\theta})\frac{\mathrm{d}\theta}{2\pi}\to \mathrm{d}\rho$ *weakly as* $r\to 1^-$. *I.e., for all continuous and* 2π*-periodic function* f *on* $\mathbb{R}$*, one has*

$$\int_0^{2\pi} f(\theta)\Pi_\rho(re^{i\theta})\frac{\mathrm{d}\theta}{2\pi}\longrightarrow\int_0^{2\pi}f(\theta)\,\mathrm{d}\rho(\theta).$$

iv) *The singular part of* ρ *is supported in* $\left\{\theta:\lim_{r\to 1^-}|\Phi_\rho(re^{i\theta})|=\infty\right\}$.

v) *The absolutely continuous (with respect to Lebesgue) part of* ρ *is given by* $\mathrm{d}\rho_{\mathrm{ac}}(\theta)=\Pi^-_\rho(e^{i\theta})\frac{\mathrm{d}\theta}{2\pi}$.

Now a version of the Aronszajn-Krein formula in the context of unitary operators. See Lemma 5 in [9].

Proposition 2 *Let ω and ω_λ be the spectral measures as in the Introduction, and pick $z \in \mathbb{C}$ with $|z| \neq 1$. Then, one has*

$$\Phi_{\omega_\lambda}(z) \;=\; \frac{(e^{i\lambda}-1)+(e^{i\lambda}+1)\Phi_\omega(z)}{(e^{i\lambda}+1)+(e^{i\lambda}-1)\Phi_\omega(z)} \tag{2.7}$$

$$= \frac{e^{i\lambda}+\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}}{e^{i\lambda}-\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}}\,. \tag{2.8}$$

Proposition 3 *Let ρ be as before and Ω_ρ given by* (1.3)*. Then,*

$$\Phi_{\Omega_\rho}(z) = \Phi_\rho\left(\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}\right). \tag{2.9}$$

Proof. Put $w = \frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}$; it follows directly from the definition of Ω_ρ, that for any characteristic function of a borelian χ_E,

$$\int_0^{2\pi} \chi_E(\theta)\,\mathrm{d}\Omega_\rho(\theta) = \int_0^{2\pi}\left(\int_0^{2\pi}\chi_E(\theta)\,\mathrm{d}\omega_\lambda(\theta)\right)\mathrm{d}\rho(\lambda),$$

then it also holds true for any measurable simple function, and by taking limits for any integrable function in place of χ_E. In particular, we have,

$$\Phi_{\Omega_\rho}(z) = \int_0^{2\pi} \Phi_{\omega_\lambda}(z)\,\mathrm{d}\rho(\lambda)\,;$$

hence, by (2.8), it follows that

$$\Phi_{\Omega_\rho}(z) = \int_0^{2\pi}\frac{e^{i\lambda}+w}{e^{i\lambda}-w}\,\mathrm{d}\rho(\lambda) = \Phi_\rho(w),$$

which is the result to be proven. □

Corollary 1 *Denote by $\omega_{\lambda\mathrm{s}}$ the singular part of the spectral measure ω_λ. Then, for $\lambda \in [0, 2\pi)$, $\omega_{\lambda\mathrm{s}}$ is supported in the set*

$$\left\{\theta : \lim_{r\to 1^-}\Phi_\omega(re^{i\theta}) = -\frac{e^{i\lambda}+1}{e^{i\lambda}-1} = i\cot\frac{\lambda}{2}\right\}$$

(the case $\lambda = 0$ can be taken in the sense of $\lambda \to 0^+$); in particular, the family $\{\omega_{\lambda\mathrm{s}}\}_{\lambda\in[0,2\pi)}$ is mutually singular.

Proof. It is an immediate consequence of Theorem 5 iv) and Proposition 2. □

This has the following spin-off:

Corollary 2 *If ρ is a continuous measure, then Ω_ρ is continuous as well.*

We need to recall another concept; a Borel measure η on $[0,2\pi]$ is *uniformly α-Hölder continuous* (UαHC), for some $0 \le \alpha \le 1$, if there exists $C > 0$ so that $\eta(I) \le C|I|^\alpha$, for all subinterval I with $|I| < 1$.

Proposition 4 *Suppose that the Borel measure ρ on $[0,2\pi]$ is uniformly α-Hölder continuous. Then, there exists $C_\alpha \ge 0$ so that*

$$\Pi_{\Omega_\rho}(z) \le C_\alpha \left(\frac{|W(z)|^2}{\Pi_\omega(z)}\right)^{1-\alpha}, \tag{2.10}$$

with $W(z) = \frac{1}{2}(1 + \Phi_\omega(z))$.

Proof. By following the arguments presented in the "Second fact" in the proof of Proposition 1, one checks that under the hypothesis $\rho[\theta, \theta+\delta) \le C\delta^\alpha$, we have

$$\limsup_{r\to 1^-} \left((1-r)^{1-\alpha}\Pi_\rho(re^{i\theta})\right) \;\le\; 8C \int_0^\infty \frac{u^{\alpha+1}}{(u^2+1)^2}\,\mathrm{d}u,$$

for $\theta \in (0,2\pi)$. Note that the hypothesis in the proposition implies that $\rho[\theta,\theta+\delta) \le C\delta^\alpha$, with C independent of θ, hence one may conclude that there exists a positive constant C_1' so that

$$\Pi_\rho(re^{i\theta}) \le C_1'(1-r)^{\alpha-1},$$

for $r \in [r_0, 1)$ and $\theta \in (0,2\pi)$, and certain $r_0 \ge 0$. Since the function $F(r,\theta) \equiv (1-r)^{1-\alpha}\Pi_\rho(re^{i\theta})$ is continuous on the compact set $[0,r_0] \times [0,2\pi]$, it is bounded, say by C_2'. Thus, by taking $C' = \max\{C_1', C_2'\}$, one has

$$\Pi_\rho(re^{i\theta}) \le C'(1-r)^{\alpha-1},$$

for $r \in [0,1)$ and $\theta \in (0,2\pi)$. Since the function on the left hand side of this inequality is continuous with respect to θ, the inequality extends also to $\theta = 0$ and $\theta = 2\pi$.

Thus, by Proposition 3,

$$\begin{aligned}
\Pi_{\Omega_\rho}(z) &= \Pi_\rho\left(\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}\right) \\
&\le C'\left(\frac{1}{1-|\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}|}\right)^{1-\alpha} = C'\left(\frac{1}{1-|1-\frac{1}{W(z)}|}\right)^{1-\alpha} \\
&\le 2^{1-\alpha}C'\left(\frac{1}{1-|1-\frac{1}{W(z)}|^2}\right)^{1-\alpha} = 2^{1-\alpha}C'\left(\frac{1}{\frac{2\mathrm{Re}\,W(z)-1}{|W(z)|^2}}\right)^{1-\alpha} \\
&= 2^{1-\alpha}C'\left(\frac{|W(z)|^2}{\Pi_\omega(z)}\right)^{1-\alpha},
\end{aligned}$$

and in the second equality we have multiplied both numerator and denominator by

$$1+\left|1-\frac{1}{W(z)}\right| = 1+\left|\frac{\Phi_\omega(z)-1}{\Phi_\omega(z)+1}\right| \leq 2.$$

Therefore, it is enough to take $C_\alpha = 2^{1-\alpha}C'$ to complete the proof. □

Theorem 6 *If ρ is uniformly 1-Hölder continuous, then Ω_ρ is uniformly 1-Hölder continuous as well.*

Proof. By Corollary 2, we can assume that I is an open interval. Let $f \geq 0$ be a continuous 2π-periodic function; then we have,

$$\begin{aligned}
\int_0^{2\pi} f(\theta)\mathrm{d}\theta &\geq \limsup_{r\to 1^-} {C_1}^{-1} \int_0^{2\pi} f(\theta)\Pi_{\Omega_\rho}(re^{i\theta})\mathrm{d}\theta \\
&= \limsup_{r\to 1^-} {C_1}^{-1} \int_0^{2\pi} \left(\int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\mathrm{d}\theta \right) \mathrm{d}\Omega_\rho(\theta') \\
&\geq {C_1}^{-1} \int_0^{2\pi} \left(\lim_{r\to 1^-} (\int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\mathrm{d}\theta \right) \mathrm{d}\Omega_\rho(\theta') \\
&= \frac{2\pi}{C_1} \int_0^{2\pi} f(\theta')\,\mathrm{d}\Omega_\rho(\theta').
\end{aligned}$$

The above inequality follows by Fatou's Lemma; in the last equality we have used the following well-known property (see [25] for a proof): for all continuous and 2π-periodic function f, one has

$$\lim_{r\to 1^-} \int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\,\frac{\mathrm{d}\theta}{2\pi} = f(\theta'), \tag{2.11}$$

with uniform limit. So, for any open interval $I \subset [0, 2\pi]$, we can take a monotonic sequence of 2π-periodic continuous functions $\{f_n\}$ that converges to the characteristic function χ_I (when restricted to $[0, 2\pi]$), and by passing to the limit in the inequality above we obtain

$$\Omega_\rho(I) \leq \frac{C_1}{2\pi}|I|\,.$$

This finishes the proof. □

3 Proofs of the main results

The time is ripe for proving the main theorems.

Proof. (Theorem 1) Recall that Γ is defined through (1.2) and note that by (1.3) one has $\Gamma = \Omega_\ell$. By Theorem 6, Γ is U1HC and so absolutely continuous; hence, by Theorem 5 v),

$$d\Gamma(\theta) = \frac{\Pi_\Gamma^-(e^{i\theta})}{2\pi}\, \mathrm{d}\theta\,.$$

However, from (2.11) one has that $\Pi_\ell^-(\cdot)$ is constant and equals 2π, so by Proposition 3 it follows that $\mathrm{d}\Gamma(\theta) = \mathrm{d}\theta$, i.e., $\Gamma = \ell$. □

We need a known result of approximations of α-Hc measures by UαHC ones, as described in Theorem 7; for the proof see [24, 21].

Theorem 7 *Let η be a σ-finite Borel measure on $[0, 2\pi]$. If η is α-Hc, $\alpha \in [0, 1]$, then, given $\epsilon > 0$, there exist mutually singular measures η_1 and η_2 so that η_1 is uniformly α-Hölder continuous, $\eta_2([0, 2\pi]) < \epsilon$ and $\eta = \eta_1 + \eta_2$.*

Proof. (Theorem 2) In this case ρ is 1-Hc. So, for each $\epsilon > 0$ one may write $\rho = \rho_1 + \rho_2$ as in Theorem 7, and ρ_1 is uniformly 1-Hölder continuous $\rho_2([0, 2\pi]) < \epsilon$. Thus, $\Omega(B) = \Omega_{\rho_1}(B) + \Omega_{\rho_2}(B)$ and, by Theorem 6, Ω_{ρ_1} is also U1HC. If $\ell(B) = 0$, it then follows that $\Omega_{\rho_1}(B) = 0$; hence, since $\omega_\lambda([0, 2\pi]) = \|\phi\|^2 = 1$, for all λ, one has

$$\Omega_\rho(B) = \Omega_{\rho_2}(B) \le \Omega_{\rho_2}([0, 2\pi]) < \epsilon\,,$$

and since $\epsilon >$ is arbitrary, $\Omega_\rho(B) = 0$ and so $\Omega_\rho \ll \ell$. □

Proof. (Theorem 3) Suppose initially that ρ is uniformly α-Hölder continuous. We first look at points outside the support of the measure ω (such support will be denoted by $\operatorname{supp}\omega$).

Proposition 5 *Pick $\alpha \in (0, 1)$. If ρ is uniformly α-Hölder continuous, then Ω_ρ is α-Hc outside* $\operatorname{supp}\omega \cup \{0, 2\pi\}$.

Proof. Fix θ outside $\operatorname{supp}\omega \cup \{0, 2\pi\}$. Then, there are $\Delta_1, \Delta_2 > 0$ so that, for $r < 1$ close enough to 1,

$$|W(re^{i\theta})|^2 \le \Delta_1, \qquad \Pi_\omega(re^{i\theta}) \ge \Delta_2(1 - r).$$

In fact, select $\epsilon_0 > 0$ with $\omega(\theta - \epsilon_0, \theta + \epsilon_0) = 0$; thus,

$$\begin{aligned} |W(re^{i\theta})| &\le \int_{[0,2\pi]\cap|\theta-\theta'|\ge\epsilon_0} \frac{\mathrm{d}\omega(\theta')}{[(1 - 2r\cos(\theta - \theta') + r^2]^{\frac{1}{2}}} \\ &\le \frac{1}{[2(1 - \epsilon_0)(1 - \cos\epsilon_0)]^{\frac{1}{2}}} \end{aligned}$$

for $r \geq 1-\epsilon_0$. Moreover,

$$\lim_{r\to 1^-} \frac{\Pi_\omega(re^{i\theta})}{1-r} = \int_{[0,2\pi]\cap|\theta-\theta'|\geq\epsilon_0} \frac{\mathrm{d}\omega(\theta')}{1-\cos(\theta-\theta')} > 0$$

is finite since $0 < 1-\cos(\theta-\theta') \leq 2$ if $|\theta-\theta'| \geq \epsilon_0 > 0$, and $\theta \neq 0, 2\pi$.

By combining these facts with Proposition 4 one has, for θ as in the hypotheses,

$$\limsup_{r\to 1^-} (1-r)^{1-\alpha} \Pi_{\Omega_\rho}(re^{i\theta}) \leq C_\alpha \left(\frac{\Delta_1}{\Delta_2}\right)^{1-\alpha} < \infty \,,$$

and the proof is complete after combining this with Theorem 4 and Proposition 1. □

Lemma 1 *Pick $\alpha \in (0,1)$ and assume that ρ is uniformly α-Hölder continuous. Fix $\beta \in (0,1)$. Then, Ω_ρ is γ-Hc in $T^\beta_{\omega 0^+} \setminus \{0, 2\pi\}$, with*

$$\gamma = \gamma(\alpha,\beta) = \alpha - 2(1-\beta)(1-\alpha),$$

for $\beta > \max\left\{0, \frac{2-3\alpha}{2(1-\alpha)}\right\}$.

Proof. For points in $T^\beta_{\omega 0^+} \setminus \{0, 2\pi\}$ that do not belong to the support of ρ, the result follows by Proposition 5. Take then $\theta \in (T^\beta_{\omega 0^+} \setminus \{0, 2\pi\}) \cap \operatorname{supp}\rho$; by Proposition 4, one has

$$(1-r)^\gamma \Pi_{\Omega_\rho}(re^{i\theta}) \leq C_\alpha \left[\frac{(1-r)^{\frac{1-\gamma}{1-\alpha}} |W(re^{i\theta})|^2}{\Pi_\omega(re^{i\theta})}\right]^{1-\alpha} .$$

Since $\theta \neq 0, 2\pi$ and $\overline{D}^\beta_\omega(\theta) < \infty$, arguments similar to those in the “Second fact” in the proof of Proposition 1, it leads to

$$\limsup_{r\to 1^-} (1-r)^{1-\beta} |W(re^{i\theta})| < \infty \Rightarrow |W(re^{i\theta})| \leq C_\theta (1-r)^{\beta-1},$$

for $r \in [r_0, 1)$, and certain $r_0 > 0$. Hence, there is $C_{\theta\alpha} > 0$ so that

$$(1-r)^\gamma \Pi_{\Omega_\rho}(re^{i\theta}) \leq C_{\theta\alpha} \left[\frac{(1-r)^{\frac{1-\gamma}{1-\alpha}+2(\beta-1)}}{\Pi_\omega(re^{i\theta})}\right]^{1-\alpha} = C_{\theta\alpha} \left[\frac{(1-r)}{\Pi_\omega(re^{i\theta})}\right]^{1-\alpha} ,$$

since for γ as in the statement of the lemma, one has $\frac{1-\gamma}{1-\alpha} + 2(\beta-1) = 1$. Since $\theta \in \operatorname{supp}\omega \setminus \{0, 2\pi\}$,

$$\lim_{r\to 1^-} \frac{\Pi_\omega(re^{i\theta})}{1-r} = \int_0^{2\pi} \frac{\mathrm{d}\omega(\theta')}{1-\cos(\theta-\theta')} > 0 \,,$$

thus,

$$\limsup_{r\to 1^-} (1-r)^\gamma \Pi_{\Omega_\rho}(re^{i\theta}) \leq \lim_{r\to 1^-} C_{\theta\alpha} \left[\frac{(1-r)}{\Pi_\omega(re^{i\theta})}\right]^{1-\alpha} < \infty \,.$$

Finally, $\gamma > 0$ since $\beta > \max\{0, \frac{2-3\alpha}{2(1-\alpha)}\}$, and the proof ends by combining this with Theorem 4 and Proposition 1. □

We have then concluded the proof in case the measure ρ is UαHC. Denote $\delta = \alpha(1-\epsilon)$, $\epsilon \in (0,1)$; for the general case, it is enough to prove the result for all ϵ small enough, since if a measure is γ-Hc, then it is γ'-Hc for all $\gamma' \le \gamma$. Pick β so that $\gamma(\alpha,\beta) = \delta$, that is, $\beta = 1 - \frac{\alpha\epsilon}{2(1-\alpha)}$, then for $\epsilon > 0$ sufficiently small, $\beta > \max\{0, \frac{2-3\alpha}{2(1-\alpha)}\}$ and so we may apply the above lemma.

Let B be a Borel set with $h^\delta(B) = 0$; we shall show that $\Omega_\rho(B) = 0$. In fact,

$$\begin{aligned}\Omega_\rho(B) &= \Omega_\rho(B \cap (T^\beta_{\omega 0^+} - \{0, 2\pi\})) + \Omega_\rho(B \cap T^\beta_{\omega\infty} \cup \{0, 2\pi\}) = \Omega_\rho(B \cap T^\beta_{\omega\infty}) \\ &= \int_0^{2\pi} \omega_{\lambda s}(B \cap T^\beta_{\omega\infty})\, \mathrm{d}\rho(\lambda) \\ &\le \int_0^{2\pi} \omega_{\lambda s}(T^1_{\omega\infty})\, \mathrm{d}\rho(\lambda) = 0;\end{aligned}$$

the second equality follows by Lemma 1 since ρ is continuous and it implies that Ω_ρ is continuous, and so $\Omega_\rho(\{0, 2\pi\}) = 0$; the inequality follows from $T^\beta_{\omega\infty} \subset T^1_{\omega\infty}$, and the last equality from the fact that $T^1_{\omega\infty} = \operatorname{supp} \omega_s$, ρ is continuous and since the measures $\omega_{\lambda s}$ are mutually singular. This completes the proof in this case.

Suppose finally that ρ is α-Hc. Take $\delta < \alpha$ and $\epsilon > 0$, and ρ_1, ρ_2 so that $\rho = \rho_1 + \rho_2$, as in Theorem 7, that is, ρ_1 is UαHC and $\rho_2([0, 2\pi]) < \epsilon$. Pick a Borel set B so that $h^\delta(B) = 0$. The first part of this proof implies

$$\int_0^{2\pi} \omega_\lambda(B)\, \mathrm{d}\rho_1(\lambda) = 0,$$

hence,

$$\Omega_\rho(B) = \int_0^{2\pi} \omega_\lambda(B)\, \mathrm{d}\rho_2(\lambda) < \epsilon.$$

From this one concludes that $\Omega_\rho(B) = 0$. The proof of the theorem is complete. □

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References

[1] E. Abakumov, C. Liaw and A. Poltoratski: Cyclicity in rank-1 perturbation problems, *J. London Math. Soc.* **88**, 523–537 (2013).

[2] J. Asch, O. Bourget and A. Joye: Localization properties of the Chalker-Coddington model, *Ann. Henri Poincaré* **11**, 1341–1373 (2010).

[3] O. Bourget: Singular continuous Floquet operator for systems with increasing gaps, *J. Math. Anal. Appl.* **276**, 28–39 (2002).

[4] O. Bourget: Singular continuous Floquet operator for periodic quantum systems, *J. Math. Anal. Appl.* **301**, 65–83 (2005).

[5] R. Carmona: One-dimensional Schrödinger operators with random or deterministic potentials: New spectral types, *J. Func. Anal.* **51**, 229–258 (1983).

[6] M. Combescure: Spectral properties of a periodically kicked quantum Hamiltonian, *J. Stat. Phys.* **59**, 679–690 (1990).

[7] M. Combescure: Recurrent versus diffusive dynamics for a kicked quantum oscillator, *Ann. Inst. H. Poincaré A: Phys. Théor.* **57**, 67–87 (1992).

[8] C. R. de Oliveira: On kicked systems modulated along the Thue-Morse sequence, *J. Phys. A: Math. Gen.* **27**, L847–L851 (1994).

[9] C. R. de Oliveira and M. S. Simsen: A Floquet operator with purely point spectrum and energy instability, *Ann. H. Poincaré* **8**, 1255–1277 (2007).

[10] R. del Rio, S. Jitomirskaya, Y. Last and B. Simon: Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization, *J. Anal. Math.* **69**, 153–200 (1996).

[11] R. del Rio, C. Martinez and H. Schulz-Baldes: Spectral averaging techniques for Jacobi matrices, *J. Math. Phys.* **49**, 023507, 13 pp. (2008).

[12] R. del Rio and O. Tchebotareva: Sturm-Liouville operators in the half axis with local perturbations, *J. Math. Anal. Appl.* **329**, 557–566 (2007).

[13] K. J. Falconer: *Fractal Geometry*, Wiley, Chichester 1990.

[14] F. Gesztesy and K. A. Makarov: $\mathrm{SL}(2,\mathrm{C})$, exponential Herglotz representations, and spectral averaging, *S. Petersburg Math. J.* **15**, 393–418 (2004).

[15] E. Hamza, A. Joye and G. Stolz: Localization for random unitary operators, *Lett. Math. Phys.* **75**, 255–272 (2006).

[16] P. W. Jones and A. G. Poltoratski: Asymptotic growth of Cauchy transforms, *Ann. Acad. Sci. Fenn. Math.* **29**, 99–120 (2004).

[17] A. Joye: Fractional Moment Estimates for Random Unitary Operators, *Lett. Math. Phys.* **72**, 51–64 (2005).

[18] S. Kotani: Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *Stochastic Analysis* (Katata/Kyoto, 1982), 225–247, North-Holland Math. Library **32**, North-Holland, Amsterdam 1984.

[19] S. Kotani: Lyapunov exponents and spectra for one dimensional random Schrödinger operators, *Contemporary Math.* (AMS) **50**, 277–286 (1984).

[20] C. A. Marx: Continuity of spectral averaging, *Proc. Amer. Math. Soc.* **139**, 283–291 (2011).

[21] P. Mattila: *Geometry of Sets and Measures in Euclidean Spaces, Fractals and rectifiability*, Cambridge Univ. Press, Cambridge 1995.

[22] J. McCaw and B. H. J. McKellar: Pure point spectrum for the time evolution of a periodically rank-N kicked Hamiltonian, *J. Math. Phys.* **46**, 032108, 24 pp. (2005).

[23] B. Milek and P. Šeba: Singular continuous quasi-energy spectrum in the kicked rotator with separable perturbation: possibility of the onset of quantum chaos, *Phys. Rev. A* **42**, 3213–3220 (1990).

[24] C. A. Rogers, *Hausdorff Measures*, Cambridge Univ. Press., Cambridge 1970.

[25] W. Rudin: *Real and Complex Analysis*, 3rd ed., p. 239, McGraw-Hill, New York 1987.

[26] B. Simon: Spectral analysis of rank one perturbations and applications, *CRM Proceedings Lectures Notes* **8**, pp. 109–149, Amer. Math. Soc., Providence 1995.

[27] B. Simon, Analogs of the m-function in the theory of orthogonal polynomials on the unit circle, *J. Comput. Appl. Math.* **171**, 411–424 (2004).

[28] B. Simon: *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs **120**, Amer. Math. Soc., Providence 2005.

[29] B. Simon and T. Wolff: Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, *Comm. Pure Appl. Math.* **39**, 75–90 (1986).