ISOCHRONICITY FOR TRIVIAL QUINTIC AND SEPTIC PLANAR POLYNOMIAL HAMILTONIAN SYSTEMS

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Abstract. In this paper we completely characterize trivial polynomial Hamiltonian isochronous centers of degrees 5 and 7. Precisely, we provide simple formulas, up to linear change of coordinates, for the Hamiltonians of the form $H = \left(f_1^2 + f_2^2\right)/2$, where $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map with $\det Df = 1$, $f(0,0) = (0,0)$ and the degree of f is 3 or 4.

1. Introduction

Let $P(x, y)$ and $Q(x, y)$ be real polynomials in the variables x and y. We say that a polynomial vector field $\mathcal{X} = (P, Q)$ has degree n when $\max\{\deg P, \deg Q\} = n$. Given a polynomial Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ of degree $n + 1$, the associated polynomial Hamiltonian system of degree n is

$$\dot{x} = -H_y(x, y), \quad \dot{y} = H_x(x, y). \tag{1}$$

System (1) has a *center* at $(0, 0)$ if there is a neighbourhood of the origin filled of periodic orbits except the origin. The maximum connected set filled of periodic orbits having in its inner boundary the origin is called the *period annulus* of the center localized at the origin. If the period annulus is $\mathbb{R}^2 \setminus \{(0, 0)\}$, we call the center *global*. We say that a polynomial Hamiltonian system has an *isochronous center* at the origin if $(0, 0)$ is a center of (1) and all the orbits in the period annulus of the center have the same period.

The following characterization of the polynomial Hamiltonian systems possessing an isochronous center at the origin was given in [6]. The polynomial Hamiltonian system (1) has an isochronous center of period 2π at the origin if and only if

$$H(x, y) = \frac{f_1(x, y)^2 + f_2(x, y)^2}{2}, \tag{2}$$

for all (x, y) in a neighborhood N_0 of the origin, where $f = (f_1, f_2) : N_0 \to \mathbb{R}^2$ is an analytic map with Jacobian determinant $\det Df$ constant and equal to 1, and $f(0, 0) = (0, 0)$. We observe that this characterization still holds

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for analytic Hamiltonians. When f can be taken polynomial, we say that the polynomial Hamiltonian isochronous center is *trivial*. In this case, it is clear that f will be defined in all $\mathbb{R}^2$. From [8], when the center is trivial, it is a global center if and only if f is globally injective. Thus the problem of knowing whether a trivial polynomial Hamiltonian isochronous center is global or not is equivalent to the *Jacobian conjecture* in $\mathbb{R}^2$, which stands that a polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with constant Jacobian determinant is globally injective. We mention here that if the degrees of f_1 and f_2 are less than or equal to 101 then f is globally injective (see [7] and, for other results on the Jacobian conjecture, see [3]). Thus *all the trivial polynomial Hamiltonian isochronous centers of degree less than or equal to* 201 *are global ones.*

In [2] it was proved that all isochronous centers of cubic polynomial Hamiltonian systems are trivial (and hence global) and after a linear change of coordinates the Hamiltonian can be written as

$$H(x, y) = (k_1x)^2 + \left(k_2y + k_3x + k_4x^2\right)^2,$$

where $k_1, k_2, k_3, k_4 \in \mathbb{R}$ and $k_1k_2 \neq 0$. We also mention that there are no polynomial Hamiltonian isochronous centers of degree 4, see [4]. Moreover, it was proved in [5] that there are no polynomial Hamiltonian isochronous centers of even degree for which the analytical function f of the Hamiltonian (2) is defined in the whole plane. In particular, *there are no trivial polynomial Hamiltonian isochronous centers of even degree.* On the other hand, there are examples of non-trivial polynomial Hamiltonian isochronous centers of degree $6k + 1$ for all $k \geq 1$, see section 3. We point out that in these examples the map f is defined in the whole plane. The following are thus natural questions.

Open question 1: *Are there non-trivial quintic polynomial Hamiltonian isochronous centers?*

Open question 2: *Are there non-trivial polynomial Hamiltonian isochronous centers with Hamiltonian (2) such that f is not analytical in the whole plane $\mathbb{R}^2$?*

We observe that if the open question 2 has a negative answer, then by [5] there are no polynomial Hamiltonian isochronous centers of even degree.

Our main result is the characterization of the quintic and the septic trivial polynomial Hamiltonian isochronous centers, see theorems 4 and 5 respectively, where we provide formulas for the Hamiltonian of these systems. We also give an alternative formula for the Hamiltonian of the cubic polynomial isochronous centers (with a trivial proof), using that these centers are trivial, see Proposition 3.

2. Trivial polynomial Hamiltonian isochronous centers

We shall use the following technical result.

Lemma 1. *Let* $p, q : \mathbb{R}^2 \to \mathbb{R}$ *be homogeneous polynomials of degree* m *and* n *respectively such that* $\det D(p, q) \equiv 0$*. Let also* $d = \gcd(m, n)$*, we define* $m' = m/d$ *and* $n' = n/d$*. Then there exists a homogeneous polynomial* $r : \mathbb{R}^2 \to \mathbb{R}$ *of degree* d *and constants* $c_p, c_q \in \mathbb{R}$ *such that* $p = c_p r^{m'}$ *and* $q = c_q r^{n'}$*.*

Proof. It is enough to prove that the rational function $f = p^n/q^m$ is constant. In order to do that, it is enough to show that $f_x = f_y = 0$. We have

$$f_x = \frac{p^{n-1}q^{m-1}}{q^{2m}} \left(nqp_x - mpq_x\right) = \frac{p^{n-1}q^{m-1}}{q^{2m}} y \left(p_x q_y - q_x p_y\right) = 0,$$

where in the second equality above we used the Euler's Theorem for homogeneous maps. The proof that $f_y = 0$ is analogous. □

We address the reader to [1] for a much more general version of Lemma 1.

In the proofs of the results of this section, we will have to solve partial differential equations of the form

$$p_x + \beta p_y = h,$$

where $p, h : \mathbb{R}^2 \to \mathbb{R}$ are homogeneous polynomials of degrees k and $k - 1$, respectively, and $\beta \in \mathbb{R}$. By defining $q : \mathbb{R}^2 \to \mathbb{R}$ by

$$p(x, y) = q(x, y - \beta x) = q(x_1, y_1), \tag{3}$$

the original equation turns to

$$q_{x_1} = h(x_1, y_1 + \beta x_1),$$

and hence there is $c \in \mathbb{R}$ such that $q(x_1, y_1) = \int_0^{x_1} h(s, y_1 + \beta s)ds + cy_1^k$, and p is given by (3). For further references, we enunciate this procedure in the following result.

Lemma 2. *Let* $p, h : \mathbb{R}^2 \to \mathbb{R}$ *be homogeneous polynomials of degrees* k *and* $k - 1$*, respectively, satisfying* $p_x + \beta p_y = h$*, with* $\beta \in \mathbb{R}$*. Then there is* $c \in \mathbb{R}$ *such that* $p(x, y) = q(x, y - \beta x)$ *where* $q(x_1, y_1) = \int_0^{x_1} h(s, y_1 + \beta s)ds + cy_1^k$*.*

We begin with Proposition 3, where we give an alternative characterization of the cubic polynomial Hamiltonian isochronous centers using the fact that they are trivial (recall the mentioned result of [2]). Right after the proof, we relate the formula of Proposition 3 and the one presented in [2].

Proposition 3. *Assume that the polynomial Hamiltonian system* (1) *has degree* 3 *and has an isochronous center of period* 2π *at the origin. Then up to a linear change of variables, the Hamiltonian can be written as*

$$H = \frac{P^2 + (y + \lambda P)^2}{2},$$

with $P = x + c_2 y^2$*,* $c_2, \lambda \in \mathbb{R}$ *and* $c_2 \neq 0$*.*

Proof. Since the cubic polynomial Hamiltonian isochronous centers are trivial, there exists $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ a polynomial map of degree 2 with $\det Df = 1$ and $f(0,0) = (0,0)$ such that $H = \left(f_1^2 + f_2^2\right)/2$. After a linear change of variables, it is clear that we can write

$$f_1 = x + p_2, \qquad f_2 = y + q_2,$$

with p_2 and q_2 homogeneous polynomials of degree 2. Without loss of generality we can assume that $p_2 \neq 0$ (otherwise we change the roles of f_1 and f_2 and of x and y). The hypothesis $\det Df = 1$ also gives that

$$p_{2x} + q_{2y} = 0, \qquad p_{2x}q_{2y} - q_{2x}p_{2y} = 0. \tag{4}$$

The second equation of (4) gives from Lemma 1 that there exists $\lambda \neq 0$ such that

$$q_2 = \lambda p_2.$$

Substituting this in the first equation of (4) we obtain $p_{2x} + \lambda p_{2y} = 0$, which solved for a homogeneous polynomial of degree 2 (by Lemma 2) determines $c_2 \in \mathbb{R}$ such that

$$p_2 = c_2(y - \lambda x)^2,$$

with $c_2 \neq 0$. We then apply the change of variables $(x, y) \mapsto (x, y - \lambda x)$, finishing the proof. □

We observe that changing x to y and taking $\sqrt{2}k_1 = 1/\sqrt{1+\lambda^2}$, $\sqrt{2}k_2 = \sqrt{1+\lambda^2}$, $\sqrt{2}k_3 = \lambda/\sqrt{1+\lambda^2}$ and $\sqrt{2}k_4 = c_2\sqrt{1+\lambda^2}$, the formula of Proposition 3 satisfies the mentioned formula for the cubic polynomial Hamiltonian isochronous centers of [2]. On the other hand, the change of coordinates $(x, y) \mapsto \sqrt{2}\,(k_2 y + k_3 x, k_1 x)$ transforms the mentioned formula of [2] in $H(x, y) = \left((x + c_2 y^2)^2 + y^2\right)/2$, with $c_2 = k_4/(\sqrt{2}k_1)$, which is the formula of Proposition 3 with $\lambda = 0$. We observe that in [2] it was not assumed that the isochronous center has period exactly 2π.

For *trivial* polynomial Hamiltonian isochronous centers of degrees 5 and 7 we have similar formulas to the Hamiltonians, see the following theorems 4 and 5.

Theorem 4. *Assume that the polynomial Hamiltonian system* (1) *has degree* 5 *and has a trivial isochronous center of period* 2π *at the origin. Then up to a linear change of variables, the Hamiltonian can be written as*

$$H = \frac{P^2 + (y + \lambda P)^2}{2},$$

with $P = x + c_2 y^2 + c_3 y^3$, $c_2, c_3, \lambda \in \mathbb{R}$ *and* $c_3 \neq 0$.

Proof. We consider $H = \left(f_1^2 + f_2^2\right)/2$, with $f = (f_1, f_2)$ a polynomial map of degree 3 satisfying $f(0,0) = (0,0)$ and $\det Df(x, y) = 1$ for all $(x, y) \in \mathbb{R}^2$. It is clear that after a linear change of variables we can write

$$f_1 = x + p_2 + p_3, \qquad f_2 = y + q_2 + q_3,$$

where p_i and q_i are homogeneous polynomials of degree i, $i = 2, 3$. Without loss of generality we can suppose that $p_3 \neq 0$. The assumption $\det Df = 1$ gives

$$\begin{aligned} p_{2x} + q_{2y} &= 0, \\ p_{3x} + q_{3y} + p_{2x}q_{2y} - q_{2x}p_{2y} &= 0, \\ p_{3x}q_{2y} - q_{2x}p_{3y} + p_{2x}q_{3y} - q_{3x}p_{2y} &= 0, \\ p_{3x}q_{3y} - q_{3x}p_{3y} &= 0. \end{aligned} \tag{5}$$

The last equation gives by Lemma 1 that there exists $\lambda \in \mathbb{R}$ such that

$$q_3 = \lambda p_3. \tag{6}$$

Substituting this in the third equation in (5), we obtain

$$p_{3x} \left(q_2 - \lambda p_2\right)_y - \left(q_2 - \lambda p_2\right)_x p_{3y} = 0. \tag{7}$$

Here we have two possibilities: either $q_2 \neq \lambda p_2$ or $q_2 = \lambda p_2$.

In the first one, equation (7) gives by Lemma 1 that there exist $a, b, c_2, c_3 \in \mathbb{R}$, with $c_2c_3(a^2 + b^2) \neq 0$ such that

$$q_2 = \lambda p_2 + c_2 \left(ax + by\right)^2, \qquad p_3 = c_3 \left(ax + by\right)^3. \tag{8}$$

Using then the first equation of (8) and the first one of (5), we obtain

$$p_{2x} + \lambda p_{2y} = -2bc_2 \left(ax + by\right).$$

From Lemma 2, we obtain $d_2 \in \mathbb{R}$ such that

$$p_2 = -bc_2 \left((a + b\lambda)x + 2b(y - \lambda x)\right) x + d_2(y - \lambda x)^2.$$

Then we substitute p_2, (8) and (6) in the second equation of (5) and, after dividing by $ax + by$ and equating the coefficients of x and y to 0, we obtain the system

$$\begin{pmatrix} a & \lambda \\ b & -1 \end{pmatrix} \begin{pmatrix} 3\left(a + b\lambda\right) c_3 \\ 4\big(b^3c_2 + (a + b\lambda)d_2\big)c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $a + b\lambda = 0$, it follows that $b = 0$ (since $c_2 \neq 0$), and hence $a = 0$, a contradiction. On the other hand if $a + b\lambda \neq 0$, it follows that $c_3 = 0$, which is again a contradiction.

If now $q_2 = \lambda p_2$, it follows from (6) and from the first and the second equations in (5) that

$$p_{2x} + \lambda p_{2y} = 0, \qquad p_{3x} + \lambda q_{3y} = 0.$$

Solving these equations for homogeneous polynomials of degrees 2 and 3 (see Lemma 2), respectively, we obtain $c_2, c_3 \in \mathbb{R}$ such that

$$p_2 = c_2 \left(y - \lambda x\right)^2, \qquad p_3 = c_3 \left(y - \lambda x\right)^3,$$

with $c_3 \neq 0$, because $p_3 \neq 0$. So

$$\begin{aligned} f_1 &= x + c_2 \left(y - \lambda x\right)^2 + c_3 \left(y - \lambda x\right)^3, \\ f_2 &= y + \lambda \Big(c_2 \left(y - \lambda x\right)^2 + c_3 \left(y - \lambda x\right)^3\Big). \end{aligned}$$

Applying the linear change of variables $(x, y) \mapsto (x, y - \lambda x)$, we end the proof of the theorem. $\square$

The following is a characterization of trivial polynomial Hamiltonian isochronous centers of degree 7.

Theorem 5. *Assume that the polynomial Hamiltonian system* (1) *has degree* 7 *and has a trivial isochronous center of period* 2π *at the origin. Then up to a linear change of variables the Hamiltonian* H *has one of the following two forms:*

$$H = \frac{P_1^2 + (y + \lambda P_1)^2}{2}, \qquad P_1 = x + c_2 y^2 + c_3 y^3 + c_4 y^4,$$
$$H = \frac{P_2^2 + (Q + \lambda P_2)^2}{2}, \qquad P_2 = x + \beta_1 Q + \beta_2 Q^2,$$

where $Q = y + \Gamma x^2$, $c_2, c_3, c_4, \beta_1, \beta_2, \lambda, \Gamma \in \mathbb{R}$ *and* $c_4 \beta_2 \Gamma \neq 0$.

Proof. Let f_1 and f_2 be polynomials of degree 4 such that $\det D(f_1, f_2) = 1$ and $f_1(0,0) = f_2(0,0) = 0$. After a linear change of variables, it is clear we can write

$$f_1 = x + p_2 + p_3 + p_4, \qquad f_2 = y + q_2 + q_3 + q_4, \tag{9}$$

where p_i and q_i are homogeneous polynomials of degree i for $i = 2, 3, 4$. Moreover, since the homogeneous terms of positive degrees of the Jacobian determinant of (f_1, f_2) are zero, we obtain the following equations:

$$p_{2x} + q_{2y} = 0, \tag{10}$$
$$p_{3x} + q_{3y} + p_{2x} q_{2y} - q_{2x} p_{2y} = 0, \tag{11}$$
$$p_{4x} + q_{4y} + p_{2x} q_{3y} - q_{3x} p_{2y} + p_{3x} q_{2y} - q_{2x} p_{3y} = 0, \tag{12}$$
$$p_{2x} q_{4y} - q_{4x} p_{2y} + p_{4x} q_{2y} - q_{2x} p_{4y} + p_{3x} q_{3y} - q_{3x} p_{3y} = 0, \tag{13}$$
$$p_{3x} q_{4y} - q_{4x} p_{3y} + p_{4x} q_{3y} - q_{3x} p_{4y} = 0, \tag{14}$$
$$p_{4x} q_{4y} - q_{4x} p_{4y} = 0. \tag{15}$$

Equation (15) and Lemma 1 give that

$$q_4 = \lambda p_4. \tag{16}$$

Substituting this in equation (14) yields

$$p_{4x} (q_3 - \lambda p_3)_y - (q_3 - \lambda p_3)_x \, p_{4y} = 0.$$

We have two possibilities, either

$$q_3 = \lambda p_3 \tag{17}$$

or, from Lemma 1, there exist $a, b, c_3, c_4 \in \mathbb{R}$ such that

$$p_4 = c_4 (ax + by)^4, \quad q_3 = \lambda p_3 + c_3 (ax + by)^3, \tag{18}$$

with $(a^2 + b^2) c_3 c_4 \neq 0$.

Assuming (17), we obtain from (13) that

$$p_{4x}\left(q_2-\lambda p_2\right)_y - p_{4y}\left(q_2-\lambda p_2\right)_x = 0.$$

We have then another two possibilities, either

$$q_2 = \lambda p_2 \tag{19}$$

or, from Lemma 1 there exist $a, b, c, c_2, c_4 \in \mathbb{R}$ such that

$$p_4 = c_4\left(ax^2+2bxy+cy^2\right)^2, \qquad q_2 = \lambda p_2 + c_2\left(ax^2+2bxy+cy^2\right), \tag{20}$$

with $(a^2+b^2+c^2)c_2c_4 \neq 0$.

Assuming (19), it follows from (16) and (17) that equations (10), (11) and (12) turn to

$$p_{2x}+\lambda p_{2y}=0, \qquad p_{3x}+\lambda p_{3y}=0, \qquad p_{4x}+\lambda p_{4y}=0,$$

respectively. Solving these equations for homogeneous polynomials of degrees 2, 3 and 4, we obtain that

$$p_2 = c_2(y-\lambda x)^2, \qquad p_3 = c_3(y-\lambda x)^3, \qquad p_4 = c_4(y-\lambda x)^4.$$

By applying the linear change of coordinates $(x, y) \mapsto (x, y-\lambda x)$ in (9), we obtain the first Hamiltonian of the theorem.

Now if we assume (17) and (20), equation (10) turns to

$$p_{2x}+\lambda p_{2y}+2c_2(bx+cy)=0.$$

Using Lemma 2, we obtain $d_2 \in \mathbb{R}$ such that

$$p_2 = -c_2\big(bx+cy+c(y-\lambda x)\big)x + d_2(y-\lambda x)^2. \tag{21}$$

Substituting this in (11), using (17) and (20), we obtain a partial differential equation of the form $p_{3x}+\lambda p_{3y}=h$, with

$$h(x, y+\lambda x) = 4(b^2-ac)c_2^2x^2 + 4c_2(bcc_2+ad_2+bd_2\lambda+L\lambda)xy + 4c_2Ly^2,$$

where

$$L = c^2c_2 + d_2(b+c\lambda). \tag{22}$$

Then from Lemma 2, we obtain $d_3 \in \mathbb{R}$ such that

$$\begin{aligned} p_3 = {} & \frac{4}{3}c_2^2(b^2-ac)x^3 + 2c_2(bcc_2+ad_2+bd_2\lambda+L\lambda)x^2(y-\lambda x) \\ & + 4c_2Lx(y-\lambda x)^2 + d_3(y-\lambda x)^3. \end{aligned} \tag{23}$$

We finally substitute (21) and (23), and also (16), (17) and (20) in equation (12), and, after dividing it by 2, we obtain the following homogeneous identity:

$$
\begin{aligned}
0 = \Big(&(a+b\lambda)\big(2ac_4 - 3c_2d_3\lambda^2 + 2c_2^2\,(3(L\lambda - bcc_2) - d_2(a+3b\lambda))\,\big)\\
&+ 4b^3c_2^3\Big)x^3 + \Big(2c_4(3ab + \lambda(2b^2+ac)) + 3c_2d_3\lambda(2a+\lambda(b-c\lambda))\\
&+ 2c_2^2\big(d_2(a+b\lambda)(b-3c\lambda) + 3(bcc_2 - L\lambda)(b-c\lambda)\\
&- 2a(c^2c_2 + 2L)\big)\Big)x^2y + \Big(2c_4\left(3b(b+c\lambda) + ac - b^2\right)\\
&- 3c_2d_3(a - b\lambda - 2c\lambda^2) - 4c_2^2\left((b^2-ac)d_2 + 3c\lambda L\right)\Big)xy^2\\
&+ \Big((2cc_4 - 3c_2d_3)(b+c\lambda) + 4cc_2^2L\Big)y^3
\end{aligned} \tag{24}
$$

Recalling that $(a^2+b^2+c^2)c_2c_4 \neq 0$, we will divide the analysis in two cases: either $b = -c\lambda$ or $b + c\lambda \neq 0$.

In the first possibility, the coefficient of y^3 in (24) is $4cc_2^2L$. Since $L = c^2c_2$, it follows that $c = 0$, and hence $L = 0$ and $b = 0$. Then the coefficient of xy^2 turns to $-3ac_2d_3$, which gives that $d_3 = 0$, and hence the coefficient of x^3 gives that $c_4 = c_2^2d_2$. In particular, $d_2 \neq 0$. With these information, it follows from (21), (23) and (20) that

$$p_2 = d_2(y-\lambda x)^2, \quad p_3 = 2ac_2d_2x^2(y-\lambda x), \quad p_4 = a^2c_2^2d_2x^4,$$

and q_2, q_3 and q_4 are given by (20), (17) and (16), respectively. Then by applying the change of coordinates $(x, y) \mapsto (x, y-\lambda x)$ in (9), it follows that in the new variables

$$
\begin{aligned}
f_1 &= x + d_2(y^2 + 2ac_2x^2y + a^2c_2^2x^4),\\
f_2 &= y + \lambda\left(x + d_2(y^2 + 2ac_2x^2y + a^2c_2^2x^4)\right) + ac_2x^2.
\end{aligned}
$$

By defining $\beta_2 = d_2$, $\Gamma = ac_2$ and $Q = y + \Gamma x^2$, we clearly obtain the second Hamiltonian of the theorem with $\beta_1 = 0$.

Now we analyze the second possibility $b + c\lambda \neq 0$. The coefficients of y^3 and xy^2 in (24) give the following linear system (recall that L is given by (22))

$$A\begin{pmatrix} c_4 \\ d_3 \end{pmatrix} = \begin{pmatrix} -4cc_2^2L \\ 4c_2^2\left((b^2-ac)d_2 + 3c\lambda L\right) \end{pmatrix}, \tag{25}$$

where

$$A = \begin{pmatrix} 2c\,(b+c\lambda) & -3c_2\,(b+c\lambda) \\ 2\left(3b(b+c\lambda) + ac - b^2\right) & -3c_2\left(a - b\lambda - 2c\lambda^2\right) \end{pmatrix}.$$

The determinant of A is $12c_2\,(b+c\lambda)^3 \neq 0$. Thus c_4 and d_3 are given by inverting A in (25). We substitute them in the coefficients of x^3 and x^2y of (24) and obtain, respectively, using (22), that

$$\frac{2c_2^3(b^2-ac)^2(2b^2+ac+3bc\lambda)}{(b+c\lambda)^3} = 0, \qquad \frac{6cc_2^3(b^2-ac)^2}{(b+c\lambda)^2} = 0,$$

which gives that $c \neq 0$ and $a = b^2/c$. We finally obtain

$$a = \frac{b^2}{c}, \qquad c_4 = \frac{c_2^2 L}{b + c\lambda}, \qquad d_3 = \frac{2cc_2 L}{b + c\lambda},$$

with $Lc(b + c\lambda) \neq 0$. Now from (23) and from (20) (after some calculations)

$$\begin{aligned} p_3 &= \frac{2c_2 L}{c(b + c\lambda)}(bx + cy)^2(y - \lambda x), \\ p_4 &= \frac{c_2^2 L}{c^2(b + c\lambda)}(bx + cy)^4, \\ q_2 &= \lambda p_2 + \frac{c_2}{c}(bx + cy)^2. \end{aligned} \tag{26}$$

We consider the change of variables $(x, y) \mapsto \big((bx + cy)/(b + c\lambda), y - \lambda x\big)$, whose inverse is the transformation $(x, y) \mapsto \big(x - cy/(b + c\lambda), \lambda x + by/(b + c\lambda)\big)$. Observe that the determinant of the change is 1. By applying the transformation in (21) and in (26), we obtain that

$$\begin{aligned} p_2 &= -c_2(b + c\lambda)x^2 + \frac{L}{b + c\lambda}y^2, \\ p_3 &= \frac{2c_2(b + c\lambda)L}{c}x^2 y, \\ p_4 &= \frac{c_2^2(b + c\lambda)^3 L}{c^2}x^4, \\ q_2 &= \lambda p_2 + \frac{c_2(b + c\lambda)^2}{c}x^2. \end{aligned} \tag{27}$$

Therefore

$$\begin{aligned} p_2 + p_3 + p_4 &= -c_2(b + c\lambda)x^2 + \frac{L}{b + c\lambda}\left(y + \frac{c_2(b + c\lambda)^2}{c}x^2\right)^2 \\ &= -c_2(b + c\lambda)x^2 + \beta_2 Q^2, \end{aligned} \tag{28}$$

with $\beta_2 = L/(b + c\lambda)$ and $Q = y + \Gamma x^2$, where $\Gamma = c_2(b + c\lambda)^2/c$. Then from (9), in the new variables,

$$\begin{aligned} f_1 &= x - \frac{c}{b + c\lambda}y - c_2(b + c\lambda)x^2 + \beta_2 Q^2 \\ &= x - \frac{c}{b + c\lambda}Q + \beta_2 Q^2. \end{aligned}$$

Similarly, substituting the last equation of (27) and equations (17), (16) and (28) in (9), we get that in the new variables

$$\begin{aligned} f_2 &= \frac{b}{b + c\lambda}y + \frac{c_2(b + c\lambda)^2}{c}x^2 + \lambda\left(x - c_2(b + c\lambda)x^2 + \beta_2 Q^2\right) \\ &= Q + \lambda\left(x - \frac{c}{b + c\lambda}Q + \beta_2 Q^2\right). \end{aligned}$$

By defining $\beta_1 = -c/(b + c\lambda)$, we obtain that the above f_1 and f_2 satisfy the second Hamiltonian of the theorem (now with $\beta_1 \neq 0$).

We have yet to analyze possibility (18). The remaining part of the proof will be to show that if we assume this possibility we get a contradiction. We will treat this analyzing two cases: $a = -b\lambda$ and $a + b\lambda \neq 0$.

In the first case, we consider new c_3 and c_4 not zero in order that (18) turn to

$$p_4 = c_4(y - \lambda x)^4, \qquad q_3 = \lambda p_3 + c_3 \left(y - \lambda x\right)^3 . \tag{29}$$

We take the change of coordinates $(x, y) \mapsto (x, y - \lambda x)$. Then using (29) we write equations (10), (11), (12) and (13) in these new variables, where we denote $p_i(x, y) = \overline{p_i}(x, y - \lambda x)$ and $q_i(x, y) = \overline{q_i}(x, y - \lambda x)$, $i = 2, 3, 4$:

$$\begin{aligned} \overline{p}_{2x} + (\overline{q}_2 - \lambda\overline{p}_2)_y &= 0, \\ \overline{p}_{3x} + \overline{p}_{2x}\overline{q}_{2y} - \overline{q}_{2x}\overline{p}_{2y} + 3c_3y^2 &= 0, \\ \overline{p}_{3x}(\overline{q}_2 - \lambda\overline{p}_2)_y - (\overline{q}_2 - \lambda\overline{p}_2)_x\overline{p}_{3y} + 3c_3\overline{p}_{2x}y^2 &= 0, \\ \left(-4c_4(\overline{q}_2 - \lambda\overline{p}_2)_x y + 3c_3\overline{p}_{3x}\right) y^2 &= 0. \end{aligned} \tag{30}$$

Integrating the fourth equation of (30) in y, we obtain $d_3 \in \mathbb{R}$ such that

$$\overline{p}_3 = \frac{4c_4}{3c_3}(\overline{q}_2 - \lambda\overline{p}_2)y + d_3y^3.$$

On the other hand, defining $\overline{p}_2 = a_1x^2 + 2a_2xy + a_3y^2$ and integrating in y the first equation of (30), we obtain $a_4 \in \mathbb{R}$ such that

$$\overline{q}_2 = a_4x^2 + 2\left(a_2\lambda - a_1\right)xy + \left(a_3\lambda - a_2\right)y^2.$$

Then we substitute the above $\overline{q}_2$ and $\overline{p}_3$ in the second and in the third equations of (30), obtaining two identically zero homogeneous polynomials of degrees 2 and 3, respectively. We denote this by $h_2 = 0$ and $h_3 = 0$, respectively. The coefficient of y^3 of h_3 is $-8c_4(a_4 - a_1\lambda)^2/(3c_3)$. Since $c_4 \neq 0$, it follows that $a_4 = a_1\lambda$. Then the coefficient of x^2 of h_2 turns to $-4a_1^2$, and hence $a_1 = 0$. Finally, the coefficients of y^2 and y^3 of h_2 and h_3, respectively, are $-4a_2^2 + 3c_3$ and $6a_2c_3$, implying that $a_2 = c_3 = 0$, which is a contradiction.

In the second case we consider the change of variables $(x, y) \to \big(ax + by, y - \lambda x\big)$ and write $\overline{p_i}$ and $\overline{q}_i$, $i = 2, 3, 4$ the maps p_i and q_i in these new variables, i.e. $p_i(x, y) = \overline{p_i}(ax + by, y - \lambda x)$, $q_i(x, y) = \overline{q_i}(ax + by, y - \lambda x)$. Then denoting the new variables by (x, y) again, it follows that $\overline{p}_4 = c_4x^4$, $\overline{q}_4 = \lambda\overline{p}_4$ and $\overline{q}_3 = \lambda\overline{p}_3 + c_3x^3$, and equations (10), (11), (12) and (13) turn, respectively, to

$$
\begin{aligned}
a\overline{p}_{2x} - \lambda\overline{p}_{2y} + b\overline{q}_{2x} + \overline{q}_{2y} &= 0,\\
(a+b\lambda)\left(\overline{p}_{3x} + \overline{p}_{2x}\overline{q}_{2y} - \overline{q}_{2x}\overline{p}_{2y}\right) + 3bc_3x^2 &= 0,\\
(a+b\lambda)\Big(\overline{p}_{3x}(\overline{q}_2 - \lambda\overline{p}_2)_y - (\overline{q}_2 - \lambda\overline{p}_2)_x\overline{p}_{3y} + 4c_4x^3 &\\
-3c_3\overline{p}_{2y}x^2\Big) &= 0,\\
(a+b\lambda)\left(4c_4(\overline{q}_2 - \lambda\overline{p}_2)_yx - 3c_3\overline{p}_{3y}\right)x^2 &= 0.
\end{aligned} \tag{31}
$$

The second and the fourth equations of (31) give, respectively, that

$$
\overline{p}_{3x} = -(\overline{p}_{2x}\overline{q}_{2y} - \overline{q}_{2x}\overline{p}_{2y}) - \frac{3bc_3}{a+b\lambda}x^2, \qquad \overline{p}_{3y} = \frac{4c_4}{3c_3}(\overline{q}_2 - \lambda\overline{p}_2)_yx. \tag{32}
$$

Substituting (32) in the third equation of (31), we obtain that

$$
\begin{aligned}
\left(\overline{p}_{2x}\overline{q}_{2y} - \overline{p}_{2y}\overline{q}_{2x} + \frac{4c_4}{3c_3}(\overline{q}_2 - \lambda\overline{p}_2)_xx + \frac{3bc_3}{a+b\lambda}x^2\right)(\overline{q}_2 - \lambda\overline{p}_2)_y &\\
-4c_4x^3 + 3c_3\overline{p}_{2y}x^2 &= 0.
\end{aligned} \tag{33}
$$

Then defining $\overline{p}_2 = a_1x^2 + 2a_2xy + a_3y^2$ the first equation of (31) will be a differential equation in $\overline{q}_2$. Solving it for a homogeneous polynomial of degree 2 (now it is similar to Lemma 2, but we apply the change of variables $x \mapsto x - by$ instead and integrate in y), it follows that there exists $a_4 \in \mathbb{R}$ such that

$$
\overline{q}_2 = a_4(x-by)^2 - 2(aa_1 - a_2\lambda)(x-by)y - \left(a(a_2 + a_1b) - (a_3 + a_2b)\lambda\right)y^2.
$$

Substituting the above $\overline{p_2}$ and $\overline{q_2}$ in (33) we obtain an identically zero homogeneous polynomial of degree 3. The coefficient of x^3 of this polynomial gives that

$$
-4c_4 - \frac{2}{3}(aa_1 + a_4b - a_2\lambda)\left(\frac{8a_4c_4}{c_3} + \frac{9bc_3}{a+b\lambda}\right) = 0.
$$

Now we substitute $\overline{p_2}$ and $\overline{q_2}$ in the equation $\overline{p}_{3xy} - \overline{p}_{3yx} = 0$ given by (32) and obtain an identically zero homogeneous polynomial of degree 1. The coefficient of x of this polynomial gives that

$$
\frac{16c_4\,(aa_1 + a_4b - a_2\lambda)}{3c_3} = 0.
$$

Combining the last two equations, we obtain that $c_4 = 0$, a contradiction. □

We observe that the two hamiltonians that appear in Theorem 5 can not be transformed in each other using a linear change of coordinates. This is because applying the linear change $(x, y) \mapsto (ax + by, cx + dy)$ in the second Hamiltonian, the only way to make the monomial x^8 disappear is to take

$a = 0$. Then to make the monomial x^4 disappear, we have to take $c = 0$. But then we no longer have a change of coordinates.

We also observe that our results give formulas (up to linear change of coordinates) for all the polynomial maps $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\det Df = 1$, $f(0,0) = (0,0)$ and degree of f is 2, 3 or 4. Using these formulas it is very simple to see that such maps are injective.

3. Examples of polynomial Hamiltonian isochronous centers

The following example shows that there exist non trivial polynomial Hamiltonian isochronous centers of degree $6k+1$, for all $k \in \{1, 2 \ldots\}$.

Example 6. *Let $\lambda \in \mathbb{R}$ and $k \in \{1, 2 \ldots\}$. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$f_1 = \frac{x + \lambda y^k}{\sqrt{1 + (x + \lambda y^k)^2}}, \qquad f_2 = \frac{\left(x + \lambda y^k\right)^2 + \left(1 + (x + \lambda y^k)^2\right)^2 y}{\sqrt{1 + (x + \lambda y^k)^2}}.$$

It follows that the Jacobian determinant of f is constant and equal to 1. Moreover, taking $H = \left(f_1^2 + f_2^2\right)/2$, it is simple to see that $2H$ is the polynomial

$$(x + \lambda y^k)^2 + 2y(x + \lambda y^k)^2 \left(1 + (x + \lambda y^k)^2\right) + y^2 \left(1 + (x + \lambda y^k)^2\right)^3,$$

which clearly has degree $6k+2$ if $\lambda \neq 0$ (and degree 8 if $\lambda = 0$). Thus system (1) *with Hamiltonian H has an isochronous center of degree $6k+1$ at the origin if $\lambda \neq 0$ (degree 7 if $\lambda = 0$).*

Lemma 7. *The isochronous center presented in Example 6 is non-trivial.*

Proof. Suppose on the contrary that the center is trivial. Then there exists a *polynomial* map $g = (g_2, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $\det Dg = 1$ and $g(0,0) = (0,0)$ such that $g_1^2 + g_2^2 = 2H$. The map $h = g \circ T$, with $T(x, y) = (x - \lambda y^k, y)$ is also polynomial, $\det Dh = 1$, $h(0,0) = (0,0)$ and

$$\widetilde{H} = \frac{h_1^2 + h_2^2}{2} = \frac{x^2 + 2yx^2(1 + x^2) + y^2(1 + x^2)^3}{2}$$

is a polynomial of degree 8. Thus system (1) with Hamiltonian $\widetilde{H}$ has a trivial isochronous center at the origin. Since h is globally injective (by the mentioned result of [7] or by Theorem 5), it follows from the mentioned result of [8] that this center is global. This is not possible, because the level curve $H = 1/2$ is not bounded (it is formed by the curves $y = -1/(1 + x^2)$ and $y = (1 - x^2)/(1 + x^2)^2$). □

Example 6 with $\lambda = 0$ has already appeared in [2].

The following example provide trivial isochronous centers for all even degrees.

Example 8. *Let* $k \in \{2, 3, \ldots, \}$*,* $\lambda, c_2, \ldots, c_k \in \mathbb{R}$*, with* $c_k \neq 0$ *and* $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$f_1 = x + c_2 y^2 + \cdots + c_k y^k, \qquad f_2 = y + \lambda f_1.$$

It is clear that $f(0, 0) = (0, 0)$ *and* $\det Df = 1$*. Thus system* (1) *with the Hamiltonian given by* $H = (f_1^2 + f_2^2)/2$ *has a trivial polynomial Hamiltonian isochronous center of degree* $2k-1$ *at the origin. Since* f *is clearly injective, this center is global.*

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