

ON POLYNOMIAL SUBMERSIONS OF DEGREE 4 AND THE REAL JACOBIAN CONJECTURE IN \mathbb{R}^2

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ABSTRACT. We prove the following version of the real Jacobian conjecture: “Let $F = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant. If the degree of p is less than or equal to 4, then F is injective”. The approach to prove this result leads to a complete classification, up to affine change of coordinates, of the polynomial submersions of degree 4 in \mathbb{R}^2 whose level sets are not all connected.

1. INTRODUCTION

Let $F = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 map such that its Jacobian determinant, $\det DF(x, y)$, is not zero for all $(x, y) \in \mathbb{R}^2$. From the inverse function theorem, F is a local diffeomorphism. It is not true in general that F is a global diffeomorphism. For instance, the map $F(x, y) = (e^x \cos y, e^x \sin y)$ is neither injective nor surjective, although $\det DF(x, y) = e^{2x}$. It is a natural problem to ask for additional conditions to guarantee that F is a global diffeomorphism. Several branches of Mathematics have presented different partial solutions to this problem, see for instance [9, 10, 15].

In the polynomial case, since every injective map is also surjective (see [2]), it is enough to provide conditions to the injectivity of F . In this case, the *real Jacobian conjecture* claims that the polynomial maps $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\det DF(x, y) \neq 0$ for each $(x, y) \in \mathbb{R}^2$ are globally injective. Nevertheless, in 1994 Pinchuk gave a counterexample to this conjecture in [17]. Now if we assume further that $\det DF(x, y)$ is a non-zero *constant*, to know whether F is injective or not is an open problem. Actually this is a special case of the famous *Jacobian conjecture*, which claims that given K a field of characteristic zero, any polynomial map from K^n to K^n such that its Jacobian determinant is equal to 1 is injective. Jacobian conjecture was stated at the first time in 1939 by Keller ([13]), and up to now it is an open problem if $n \geq 2$. We refer to [8] for further information on the Jacobian conjecture.

In [12] Gwoździewicz proved that the real Jacobian conjecture is true assuming further that the degrees of p and q are less than or equal to 3. In [4] Braun and Santos generalized this result by showing that the injectivity of F follows provided just the degree of p is less than or equal to 3, independently of the degree of q . On the other hand, in the above-mentioned Pinchuk counterexample, the degree of p is 10 and the degree of q is 40. It is natural then to look for the highest degree of p between 4 and 9 for which the injectivity of $F = (p, q)$ is necessary independently

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of the degree of q . In this paper we present a step on the solution of this problem with the following result:

Theorem 1.1. *Let $F = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial mapping such that $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. If the degree of p is less than or equal to 4, then F is injective.*

Theorem 1.1 is the main result of this paper. Since in [4] it was already proved that F is injective if the degree of p is less than or equal to 3, it is enough to prove the theorem assuming that the degree of p is equal to 4. The fundamental tool in our proof is the following known result, which we prove in Section 3.

Lemma 1.2. *Let $F = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial mapping such that $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. The map F is injective if and only if the level sets of p are connected.*

Based on Lemma 1.2, we divide the proof of Theorem 1.1 into two parts. In the first part we classify the polynomial submersions of degree 4 whose level sets are not all connected. Precisely, we prove Theorem 1.3 below. Before stating this result, we observe that by a *submersion* we mean a function without critical points. Moreover, we introduce the following notion of equivalence between functions. We say that $p_1, p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ are *equivalent* if there exist affine change of coordinates $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $p_1(x, y) = f \circ p_2 \circ T^{-1}(x, y)$.

Theorem 1.3. *If $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial submersion of degree 4 whose level sets are not all connected, then p is equivalent to one of the following:*

- (1) $p(x, y) = y + xy^2 + y^4$,
- (2) $p(x, y) = y + a_{02}y^2 + xy^3$, with $a_{02} = 0$ or 1,
- (3) $p(x, y) = y + x^2y^2$,
- (4) $p(x, y) = y + a_{02}y^2 + y^3 + x^2y^2$, with $a_{02}^2 - 3 < 0$.

In the second part we prove that for each polynomial p of the above theorem there is not a polynomial $q : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\det D(p, q)(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$.

The organization of the paper is as follows. The following three sections deal with the first part of the proof of Theorem 1.1. Precisely, in Section 2 we apply the concepts of subresultants of univariate polynomials to develop tools to decide when two special polynomial functions in \mathbb{R}^2 have common zeros. In Section 3 we construct results to analyze the connectedness of level sets of polynomial submersions. Then in Section 4 we use the results of the preceding sections to prove Theorem 1.3.

In Section 5 we deal with the second part of the proof of Theorem 1.1. The arguments are divided into two groups. In the first one we study the polynomials (1) and (2) of Theorem 1.3 and use techniques analogous to [4] to conclude the non-existence of a polynomial q such that $\det D(p, q)(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. These techniques explore the existence of half-Reeb components in the foliation defined by p . In the second group, when we analyse the polynomials (3) and (4), these techniques no longer work. We then transform part of the problem in being able to decide when a special univariate polynomial is positive. Namely Lemma 5.5 asserts that $L(\theta) = \sum_{j=0}^N b_j(2(j+1)\theta + 2j+1)\theta^j$ is not a positive polynomial. We finish Section 5 detailing the proof of Theorem 1.1.

2. COMMON ZEROS OF POLYNOMIALS

We begin recalling the concept of subresultants of polynomials. Let two polynomials $p(x), q(x) \in \mathbb{C}[x]$,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0,$$

and write their Sylvester matrix in the following form

$$Syl(p, q, x) = \left(\begin{array}{cccccc} a_n & a_{n-1} & \cdots & \cdots & a_0 & & & & \\ & a_n & a_{n-1} & \cdots & \cdots & a_0 & & & \\ & & \ddots & \ddots & & & \ddots & & \\ & & & a_n & a_{n-1} & \cdots & \cdots & a_0 & \\ \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \ddots & \ddots & & & \ddots & & \\ & & & b_m & b_{m-1} & \cdots & \cdots & b_0 & \\ b_m & b_{m-1} & \cdots & \cdots & & & & & \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} m \text{ rows} \\ \\ \\ \\ \\ \\ n \text{ rows} \end{array}.$$

For each $k \in \{0, 1, \dots, [(n+m)/2]\}$, we define the k -subresultant of p and q , $R_k(p, q, x)$, by the determinant of the $(n+m-2k) \times (n+m-2k)$ matrix obtained when we delete the first and the latest k columns and rows of $Syl(p, q, x)$. Although the following result is classical, we include its proof for the sake of completeness. It is based on [3] (see also [16]).

Lemma 2.1. *Let $p(x)$ and $q(x)$ as above, with $a_n b_m \neq 0$. Then p and q have exactly k roots in common (counting multiplicity) if and only if*

$$R_0(p, q, x) = \cdots = R_{k-1}(p, q, x) = 0, \quad R_k(p, q, x) \neq 0.$$

Proof. We denote by C_j the j th column of $Syl(p, q, x)$, for $j = 1, \dots, m+n$. By substituting the last column C_{m+n} by $C_{m+n} + \sum_{k=1}^{m+n-1} y^{m+n-k} C_k$, and calculating the determinant using the Laplace expansion on the last column, we get that $R_0(p, q, x) = f(y)p(y) + g(y)q(y)$, where f and g are polynomials of degree less than or equal to $m-1$ and $n-1$, respectively. Since $R_0(p, q, x)$ is a complex number with does not depend on y , it follows that p and q have a common root if and only if $R_0(p, q, x) = 0$.

Now the lemma follows readily from the following assertion: *If $p(x) = (x - \alpha)p_1(x)$ and $q(x) = (x - \alpha)q_1(x)$, $\alpha \in \mathbb{C}$, then $R_i(p_1, q_1, x) = R_{i+1}(p, q, x)$, for $i = 0, 1, \dots$* To prove the assertion, we make the following operations on the columns of $Syl(p_1, q_1, x)$: we change column C_i by $C_i - \alpha C_{i-1}$ for $i = m+n-2, m+n-3, \dots, 2$. Then we observe that this is exactly the matrix $Syl(p, q, x)$ without the first and the last rows and columns. \square

For further reference, we state formulas of $R_0 = R_0(p, q, x)$ and $R_1 = R_1(p, q, x)$ for a pair of polynomials $p(x)$ and $q(x)$. The proof is omitted as it is achieved by straightforward calculations.

Lemma 2.2. *Let $a, b, c, d, e \in \mathbb{R}$. For the polynomials*

$$p(x) = x^2 + ax + b, \quad q(x) = x^3 + cx^2 + dx + e,$$

we have

$$R_1 = a^2 - ac - b + d, \quad R_0 = -(ab - bc + e)^2 + (a(ab - bc + e) - bR_1)R_1.$$

Lemma 2.3. *Let $f, g, A, B : \mathbb{R} \rightarrow \mathbb{R}$ be C^∞ functions and $\alpha \in \mathbb{R}$ be such that $A(\alpha)^2 - 4B(\alpha) < 0$. Define for each $y \in \mathbb{R}$*

$$R(y) = -f(y)^2 + (A(y)f(y) - B(y)g(y))g(y).$$

Then $R(y) \leq 0$ for each y sufficiently close to α . In particular, if α is an isolated zero of $R(y)$ then $R(y)$ does not change sign at α .

Proof. Observe that for each y , $R(y)$ is a quadratic function in $f(y)$ with leading coefficient negative, and discriminant $\Delta(y) = (A(y)^2 - 4B(y))g(y)^2$.

From hypothesis $\Delta(y) \leq 0$, and hence $R(y) \leq 0$, for all y sufficiently close to α . \square

In the remainder of this section, we apply the preceding results to produce criteria to decide when two special polynomials in two variables have common zeros. In order to apply the above subresultant results, we fix one variable of the polynomials.

Theorem 2.4. *Let $p(x, y)$ and $q(x, y)$ in $\mathbb{R}[x, y]$ as follows*

$$p(x, y) = M(y)x^2 + a(y)x + b(y), \quad q(x, y) = N(y)x^3 + c(y)x^2 + d(y)x + e(y).$$

Write the 0-subresultant in x as $R_0(y) = R_0(p, q, x) = b_0 + b_1y + \dots + b_ly^l$. If

- (1) $R_0(y)$ has no common zeros with $N(y)$,
- (2) there exists $z \in \mathbb{R}$ such that $b_l R_0(z) < 0$,

then there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that $p(\alpha, \beta) = q(\alpha, \beta) = 0$.

Proof. From assumption (2), $R_0(y)$ has at least one real zero. Write β_1, \dots, β_k be the distinct real zeros of $R_0(y)$.

If $M(\beta_i) = 0$ for some $i \in \{1, \dots, k\}$, we have two possibilities: either $a(\beta_i) \neq 0$ or $a(\beta_i) = 0$. In the first possibility, we define $\bar{p}(x) = p(x, \beta_i)$ and $\bar{q}(x) = q(x, \beta_i)$. Observe that $\bar{p}(x)$ and $\bar{q}(x)$ are polynomials of degree 1 and 3 in x , respectively. From the definition of 0-subresultant, we see that in this case, $R_0(\beta_i) = N(\beta_i)R_0(\bar{p}, \bar{q}, x)$. Then as $N(\beta_i) \neq 0$, it follows that $R_0(\bar{p}, \bar{q}, x) = 0$. From Lemma 2.1, \bar{p} and \bar{q} has at least a common zero α . This zero α is real because $\bar{p}(x)$ has degree 1. Thus $p(\alpha, \beta_i) = q(\alpha, \beta_i) = 0$. On the other hand, if $a(\beta_i) = 0$, it is simple to see that $R_0(\beta_i) = N(\beta_i)^2 b(\beta_i)^3$. Thus $b(\beta_i) = 0$, hence $p(x, \beta_i) \equiv 0$. Moreover, there exists $\alpha \in \mathbb{R}$ such that $q(\alpha, \beta_i) = 0$, because $q(x, \beta_i)$ is a polynomial of degree 3 in x . Therefore, if $M(\beta_i) = 0$ for some i , the theorem is proven.

From now on we will suppose $M(\beta_i) \neq 0$ for each $i \in \{1, \dots, k\}$. We denote $R_1(y) = R_1(p, q, x)$. If $R_1(\beta_i) \neq 0$ for some $i \in \{1, \dots, k\}$, Lemma 2.1 shows the existence of $\alpha \in \mathbb{R}$ such that $p(\alpha, \beta_i) = q(\alpha, \beta_i) = 0$, and the theorem follows in this case.

Thus we suppose $R_1(\beta_i) = 0$ for each $i \in \{1, \dots, k\}$. We claim there is $i \in \{1, \dots, k\}$ such that $a(\beta_i)^2 - 4M(\beta_i)b(\beta_i) \geq 0$. In this case, since by Lemma 2.1 the polynomials $p(x, \beta_i)$ and $q(x, \beta_i)$ have exactly two zeros in common (we remark that $R_2(p, q, x)(\beta_i) = M(\beta_i) \neq 0$), these zeros must be real ones and we are done.

Therefore it is enough to prove the claim. Suppose on the contrary that $a(\beta_i)^2 - 4M(\beta_i)b(\beta_i) < 0$ for each $i \in \{1, \dots, k\}$. From the definition of subresultants and from Lemma 2.2, it follows that in a neighborhood of each β_i where $M(y)N(y) \neq 0$ (from assumption (1) $N(\beta_i) \neq 0$),

$$(1) \quad R_0(y) = R_0(p, q, x) = M^3 N^2 R_0\left(\frac{p}{M}, \frac{q}{N}, x\right) = M^3 N^2 (-f^2 + (Af - Bg)g),$$

where $A = a/M$, $B = b/M$, $f = ab/M^2 - bc/(MN) + e/N$ and $g = R_1(p/M, q/N, x)$. Since $a^2 - 4Mb < 0$ if and only if $A^2 - 4B < 0$, it follows from Lemma 2.3 that $R_0(y)$ does not change sign. Hence $b_l R_0(y) \geq 0$ for all $y \in \mathbb{R}$, a contradiction with assumption (2). This proves the claim. \square

The following result uses the 1-subresultant to analyze common zeros of polynomials when hypotheses 2 of the preceding theorem is difficult to be verified.

Corollary 2.5. *Let $p(x, y)$ and $q(x, y)$ as in Theorem 2.4. If*

- (1) $R_0(y)$ has no common zeros with $N(y)$,
- (2) there exists $z \in \mathbb{R}$ such that $R_1(z) = 0$, $b_l M(z) > 0$ and $N(z) \neq 0$,

then there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that $p(\alpha, \beta) = q(\alpha, \beta) = 0$.

Proof. Since $M(z)N(z) \neq 0$, we have, as in equation (1),

$$(2) \quad R_0(y) = R_0(p, q, x) = N^2 M^3 (-f^2 + (Af - Bg)g),$$

in a neighborhood of z . If $f(z) \neq 0$, then $b_l R_0(z) = -N(z)^2 M(z)^2 f(z)^2 b_l M(z) < 0$ (recall that $g = R_1(p/M, q/N, x)$), and thus the result follows from Theorem 2.4.

On the other hand, if $f(z) = 0$, we have $R_0(z) = R_1(z) = 0$, which guarantees two common zeros of $p(x, z)$ and $q(x, z)$. If these zeros are real we are done.

Thus we suppose $a(z)^2 - 4M(z)b(z) < 0$. If $g \equiv 0$, since the zeros of f are isolated (it is a non-identically zero rational function, because if $f \equiv 0$, then $R_0 \equiv 0$ by (2), and hence $b_l = 0$, a contradiction with assumption (2)), we have by (2) for $y \neq z$ near z

$$b_l R_0(y) = -N(y)^2 M(y)^2 f(y)^2 b_l M(y) < 0,$$

and we are under the hypotheses of Theorem 2.4. Now if $g \neq 0$, we have $g(y) \neq 0$ and $a(y)^2 - 4M(y)b(y) < 0$ for $y \neq z$ near z . This gives $(A(y)g(y))^2 - 4B(y)g(y)^2 < 0$ which guarantees $-f(y)^2 + A(y)g(y)f(y) - B(y)g(y)^2 < 0$ for $y \neq z$ near z . This together with (2) and assumption (2) gives

$$b_l R_0(y) = N(y)^2 M(y)^2 b_l M(y) (-f(y)^2 + A(y)g(y)f(y) - B(y)g(y)^2) < 0,$$

for $y \neq z$ near z , and we are again under the hypotheses of Theorem 2.4. \square

Similar results can be obtained for polynomials p and q with degrees 2 in one of the variables.

3. LEVEL SETS: DISCRIMINANTS AND NEWTON POLYGONS

In this section we discuss criteria to decide whether a polynomial submersion has connected level sets. We present two approaches: discriminants and Newton polygons of polynomials.

Since we will deal with level sets, we start proving Lemma 1.2 mentioned in the introduction section.

Proof of Lemma 1.2. If the polynomial map F is injective, it is surjective by [2]. Thus F is a global diffeomorphism. Hence for each $c \in \mathbb{R}$, it follows that $p^{-1}\{c\} = F^{-1}(\{c\} \times \mathbb{R})$ is a connected set.

On the other hand, assume that the level sets of p are all connected. Let $A \neq B$ in \mathbb{R}^2 such that $p(A) = p(B)$. Then A and B are in the same connected component of a level set of p . This component is a curve that can be parametrized as $\gamma(t)$ satisfying $\gamma'(t) = (-p_y, p_x)$. Since $(q \circ \gamma)'(t) = \det DF(\gamma(t)) \neq 0$, it follows that q

is monotone along the curve $\gamma(t)$. In particular $q(A) \neq q(B)$. This proves that F is injective. \square

Lemma 3.1. *Let $M \subset \mathbb{R}^2$ be an open set. If $p : M \rightarrow \mathbb{R}$ is a C^∞ submersion then the connected components of the level sets of p induce a C^∞ foliation of dimension 1 of M .*

Proof. See, for instance, [5]. \square

In particular, when $M = \mathbb{R}^2$, any connected component of a level set of p is an unbounded curve in both directions.

Lemma 3.2. *Let $p(x, y) = a_n(y)x^n + \dots + a_0(y)$ be a C^∞ function. Then for each interval $[c, d] \subset \{y \in \mathbb{R} \mid a_n(y) \neq 0\}$ there exists an interval $[a, b]$ such that*

$$\{(x, y) \in \mathbb{R}^2 \mid y \in [c, d], p(x, y) = 0\} \subset [a, b] \times [c, d].$$

Proof. Define

$$A_i = \sup_{y \in [c, d]} \left| \frac{a_i(y)}{a_n(y)} \right|, \quad i = 0, 1, \dots, n-1, \quad B = 1 + \sum_{k=1}^n (nA_{n-k})^{\frac{1}{k}}.$$

If $y \in [c, d]$ and $|x| > B$, then

$$p(x, y) = a_n(y)x^n \left(1 + \frac{a_{n-1}(y)}{a_n(y)} \frac{1}{x} + \dots + \frac{a_0(y)}{a_n(y)} \frac{1}{x^n} \right) \neq 0.$$

The lemma follows by taking $[a, b] = [-B, B]$. \square

Proposition 3.3. *Let $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the C^∞ submersion*

$$p(x, y) = A(y)x^2 + B(y)x + C(y).$$

If

- (1) $A(y) \neq 0, \forall y \in \mathbb{R}$,
- (2) $\Delta(y) = B(y)^2 - 4A(y)C(y)$ is a polynomial with odd degree,

then $p^{-1}\{0\}$ is connected.

Proof. We suppose that the leader coefficient of $\Delta(y)$ is positive. The proof in the other case is analogous. Therefore there exists $c \in \mathbb{R}$ such that $\Delta(c) = 0$ and $\Delta(y) < 0, \forall y < c$. In particular, $p^{-1}\{0\} \subset \mathbb{R} \times [c, \infty)$ and there exists exactly one $x_c \in \mathbb{R}$ such that $p(x_c, c) = 0$. Let Γ be the connected component of $p^{-1}\{0\}$ which contains (x_c, c) and consider $Q = (q_1, q_2) \in \mathbb{R} \times (c, \infty)$ such that $p(Q) = 0$. It is enough to prove that $Q \in \Gamma$.

By Lemma 3.2, there exists an interval $[a, b]$ such that

$$(3) \quad p^{-1}\{0\} \cap (\mathbb{R} \times [c, q_2]) \subset (a, b) \times [c, q_2].$$

Now from Lemma 3.1, both ends of Γ must escape the compact $[a, b] \times [c, q_2]$. Therefore, by (3), Γ will cut the line $\mathbb{R} \times \{q_2\}$ in two points, say (x_1, q_2) and (x_2, q_2) . Since Γ does not have self intersections, $x_1 \neq x_2$. In particular $\Delta(q_2) > 0$ and $q_1 \in \{x_1, x_2\}$, hence $Q \in \Gamma$. \square

We now discuss the approach using Newton polygons. Let $p(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ be a polynomial in $\mathbb{R}[x, y]$. The *Newton polygon* of p , N_p , is the convex hull of the set $\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$. For each face S of N_p we consider the *exterior normal vector* (a, b) of S with co-prime coordinates $a, b \in \mathbb{Z}$. If N_p is unidimensional, then

both normal vectors are considered exterior. We say that S is an outer face of N_p if $a > 0$ or $b > 0$.

For each face S of N_p we define $p_S(x, y) = \sum_{(i,j) \in S} a_{i,j} x^i y^j$. We factorize $p_S(x, y)$ in the following way, see [14],

$$p_S(x, y) = \delta x^r y^s \prod_{j=1}^{\ell} (y^a - \xi_j x^b)^{\nu_j},$$

where $\xi_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \dots, \ell$, are mutually distinct. If each ν_j above is equal to 1 we say that p is *non-degenerate on S* . Observe that if the only points of $S \cap \{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$ are the end-points of S , then p is non-degenerate on S .

We write $\tilde{p}(x, y) = p(x, y) - p(0, 0)$. We say that p is *convenient* if $\tilde{p}(x, 0)\tilde{p}(0, y) \neq 0$. Observe that this is equivalent to N_p intercepting both the axis away from the origin.

Lemma 3.4. *Let $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a convenient polynomial submersion. If p is non-degenerate on each outer face of N_p , then all the level sets of p are connected.*

Proof. From [6, Corollary 1.3], see also [14, Corollary 3.4.1], it follows that p does not have critical values from the infinity, and hence p has trivial fibration. In particular, since \mathbb{R} is a contractible set, we have $p(\mathbb{R}^2) = \mathbb{R}$ and, for each $c \in \mathbb{R}$, $p^{-1}\{c\} \times \mathbb{R}$ is homeomorphic to \mathbb{R}^2 , which guarantees that all the level sets of p are connected, proving the lemma. \square

4. THE PROOF OF THEOREM 1.3

In this section we prove that every polynomial $p(x, y) \in \mathbb{R}[x, y]$ satisfying: (i) $p(x, y)$ has degree 4, (ii) $\nabla p(x, y) \neq (0, 0)$ in \mathbb{R}^2 and (iii) $p(x, y)$ has not all its level sets connected, is equivalent to one of the polynomials in Theorem 1.3.

Clearly a given polynomial of degree 4 can be written as $p_3(x, y) + h(x, y)$, where $p_3(x, y)$ is a polynomial of degree less than or equal to 3 and $h(x, y)$ is a non-zero homogeneous polynomial of degree 4.

The following result is Theorem 2.6 of [7]:

Lemma 4.1 (See [7]). *Any homogeneous polynomial of degree 4 is up to a linear change of variables one of the following:*

- (I) $h(x, y) = x^4 + 6\mu x^2 y^2 + y^4$, $\mu < -1/3$,
- (II) $h(x, y) = \alpha (x^4 + 6\mu x^2 y^2 + y^4)$, $\alpha = \pm 1$, $\mu > -1/3$, $\mu \neq 1/3$,
- (III) $h(x, y) = x^4 + 6\mu x^2 y^2 - y^4$, $\mu \in \mathbb{R}$,
- (IV) $h(x, y) = \alpha y^2 (6x^2 + y^2)$, $\alpha = \pm 1$,
- (V) $h(x, y) = \alpha y^2 (6x^2 - y^2)$, $\alpha = \pm 1$,
- (VI) $h(x, y) = \alpha (x^2 + y^2)^2$, $\alpha = \pm 1$,
- (VII) $h(x, y) = 6\alpha x^2 y^2$, $\alpha = \pm 1$,
- (VIII) $h(x, y) = 4x^3 y$,
- (IX) $h(x, y) = \alpha x^4$, $\alpha = \pm 1$.

As a consequence of Lemma 4.1, it follows that it is enough to analyze the polynomials of the form $p(x, y) = p_3(x, y) + h(x, y)$, with $h(x, y)$ being one of the

polynomials of Lemma 4.1. Our strategy to prove Theorem 1.3 will be to take each polynomial of this form and impose the conditions (ii) $\nabla p(x, y) \neq 0$ in \mathbb{R}^2 and (iii) the level sets of $p(x, y)$ are not all connected. We will always work up to the equivalence notion defined in the introduction section. We emphasize that when we say below that we *apply an affine change of coordinates* $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to a given polynomial $p(x, y)$, we mean the operation $p \circ T^{-1}(x, y)$.

We divide the analysis of the nine cases of Lemma 4.1 in five groups, each of them in one of the subsections below. Theorem 1.3 will be a direct consequence of propositions 4.3, 4.5, 4.7, 4.9 and 4.11 below.

To establish notation, we write $p_3(x, y)$ in the following form:

$$p_3 = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3.$$

Observe that applying our equivalence notion we do not need to write the constant term.

4.1. Cases (I), (II), (III) and (VI).

Lemma 4.2. *The polynomials of cases (I), (II), (III) and (VI) are equivalent to*

$$h(x, y) = x^4 + \gamma x^2 y^2 + \theta y^4, \quad \theta = \pm 1, \quad \gamma \in \mathbb{R}.$$

Moreover, if $\theta = 1$, then $\gamma \neq -2$.

Proof. Polynomial (I) is counted above taking $\gamma = 6\mu$ and $\theta = 1$. Dividing polynomial (II) by α and taking $\gamma = 6\mu$, we conclude that it is also counted in the lemma, with $\theta = 1$. Case (III) is counted above with $\gamma = 6\mu$ and $\theta = -1$. Finally, dividing polynomial (VI) by α , it is in the form of the lemma with $\gamma = 2$ and $\theta = 1$. \square

Observe that if $\theta = 1$ and $\gamma = -2$, the application of $T(x, y) = (x - y, x + y)$ transforms $h(x, y)$ in the polynomial of case (VII).

Proposition 4.3. *The polynomials $p = p_3 + h$, where h is from cases (I), (II), (III) and (VI), are not submersions.*

Proof. We analyze $p(x, y) = p_3(x, y) + h(x, y)$, with $h(x, y)$ from Lemma 4.2.

Observe that $p_y(x, y) = M(y)x^2 + a(y)x + b(y)$ and $p_x(x, y) = N(y)x^3 + c(y)x^2 + d(y)x + e(y)$, with $N(y) \equiv 4$. With the notations of Theorem 2.4, observe that

$$R_0(y) = R_0(p_y, p_x, x) = b_0 + b_1y + \cdots + b_8y^8 + b_9y^9,$$

with $b_9 = -64\theta(4\theta - \gamma^2)^2$. If $b_9 \neq 0$, it follows from Theorem 2.4 that there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that $p_y(\alpha, \beta) = p_x(\alpha, \beta) = 0$, and hence p is not a submersion. We suppose thus $b_9 = 0$, i.e. $\theta = 1$ and $\gamma = 2$, see Lemma 4.2. In this case we have $b_8 = 0$ and $b_7 = -576((a_{12} - a_{30})^2 + (a_{21} - a_{03})^2)$. From the same Theorem 2.4, if $a_{12} \neq a_{30}$ or $a_{21} \neq a_{03}$, it follows that p is not a submersion.

On the other hand, if $a_{30} = a_{12}$ and $a_{03} = a_{21}$, we have $b_6 = 0$ and $b_5 = -16((a_{12}^2 + 4a_{02} - 4a_{20} - a_{21}^2)^2 + 4(a_{12}a_{21} - 2a_{11})^2)$. If $b_5 \neq 0$, we conclude again from Theorem 2.4 that p is not a submersion.

Now if $b_5 = 0$, i.e. $a_{02} = -a_{12}^2/4 + a_{20} + a_{21}^2/4$ and $a_{11} = a_{12}a_{21}/2$, we have $b_4 = 0$ and $b_3 = -((8a_{10} + a_{12}^3 - 4a_{20}a_{12})^2 + (a_{12}^2a_{21} - 4a_{21}a_{20} + 8a_{01})^2)$. As above, if $b_3 \neq 0$, then p is not a submersion.

Finally if $a_{10} = -a_{12}^3/8 + a_{20}a_{12}/2$ and $a_{01} = -a_{12}^2a_{21}/8 + a_{20}a_{21}/2$, it follows that $x = -a_{12}/4$ annihilates $p_x(x, y)$ and $y = -a_{21}/4$ annihilates $p_y(x, y)$. Then the point $(-a_{12}/4, -a_{21}/4)$ is a zero of $\nabla p(x, y)$. Hence p is not a submersion. \square

4.2. Cases (IV) and (V).

Lemma 4.4. *The polynomials $p = p_3 + h$, with h from cases (IV) or (V), are equivalent to*

$$p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{03}y^3 + x^4 + \theta x^2y^2,$$

with $\theta = \pm 1$ and $a_{03} = 1$ or 0 .

Proof. Dividing $p = p_3 + h$, with h coming from cases (IV) and (V), by α and $-\alpha$, respectively, and applying $T(x, y) = (y, \sqrt{6}x)$, we conclude that p is equivalent to

$$p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^4 + \theta x^2y^2,$$

with $\theta = \pm 1$. Then we apply $T(x, y) = (x + a_{12}/(2\theta), y + a_{21}/(2\theta))$ to eliminate a_{12} and a_{21} . In case $a_{03} = 0$ we are done. On the other hand, if $a_{03} \neq 0$, we apply further the transformation $T(x, y) = (x/a_{03}, y/a_{03})$. Then we divide the obtained polynomial by a_{03}^4 and obtain the form of the lemma in this case. \square

Proposition 4.5. *If the polynomials $p = p_3 + h$, where h is from cases (IV) or (V), are submersions, then all their level sets are connected.*

Proof. We consider the polynomial of Lemma 4.4. If $a_{01} = a_{02} = a_{03} = 0$, it is simple to conclude that p is not a submersion: if $a_{11} \neq 0$, then $p_y(0, y) \equiv 0$ and $p_x(0, -a_{10}/a_{11}) = 0$. If $a_{11} = 0$, then $p_y(x, 0) \equiv 0$ and $p_x(x, 0)$ is a polynomial of degree 3.

On the other hand, if $a_{01}^2 + a_{02}^2 + a_{03}^2 > 0$, the outer faces of the possible Newton polygons, N_p , for this polynomial are in Figure 1. Since the interior part of each

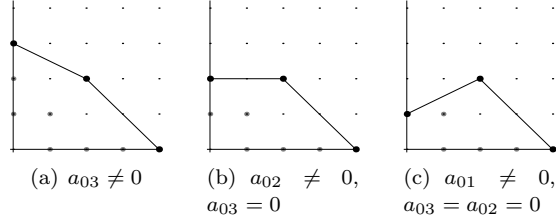


FIGURE 1. Outer faces of N_p for Lemma 4.4.

outer face of N_p does not contain lattice points where $a_{i,j} \neq 0$, it follows that p is non-degenerate on the outer faces. Hence from Lemma 3.4, if p is a submersion, all its level sets are connected. \square

4.3. Case (VII).

Lemma 4.6. *The polynomial $p = p_3 + h$, where h is of case (VII), is equivalent to one of the following:*

- (4) $p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + y^3 + x^2y^2,$
 $a_{30} = 1$ or $0,$
- (5) $p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + x^2y^2, \quad a_{02} = \pm 1,$
- (6) $p(x, y) = a_{10}x + a_{01}y + a_{11}xy + x^2y^2.$

Proof. We first divide $p(x, y)$ by 6α . Then we apply the transformation $T(x, y) = (x + a_{12}/2, y + a_{21}/2)$ to conclude that p is equivalent to

$$p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{03}y^3 + x^2y^2.$$

If $a_{03}a_{30} \neq 0$, we first apply $T(x, y) = (x/\sqrt[3]{a_{30}a_{03}^2}, y/\sqrt[3]{a_{30}^2a_{03}})$ to $p(x, y)$. Then we multiply the obtained polynomial by $a_{30}^{-2}a_{03}^{-2}$ and conclude that p is equivalent to the first case of (4). If $a_{30} = 0$ and $a_{03} \neq 0$, we apply $T(x, y) = (x/\sqrt[3]{a_{03}}, \sqrt[3]{a_{03}}y)$ to obtain the second case of (4). If $a_{30} \neq 0$ and $a_{03} = 0$, we change x by y to get the case just studied.

If now $a_{30} = a_{03} = 0$, and $a_{02} \neq 0$, we apply $T(x, y) = (1/\sqrt{|a_{02}|}x, \sqrt{|a_{02}|}y)$ to obtain case (5). If $a_{20} \neq 0$, change x by y to get the case just studied.

Thus we suppose $a_{30} = a_{03} = a_{20} = a_{02} = 0$, and obtain case (6). \square

Proposition 4.7. *If $p = p_3 + h$, with h of case (VII), is a submersion and has at least one disconnected level set, then p is equivalent to one of the following:*

$$p(x, y) = y + a_{02}y^2 + y^3 + x^2y^2, \quad a_{02}^2 - 3 < 0,$$

$$p(x, y) = y + x^2y^2.$$

Proof. We consider each case of Lemma 4.6. We observe that the interior of the outer faces of N_p for the polynomial (4) when either $a_{30} \neq 0$, $a_{30} = 0$ and $a_{20} \neq 0$, or $a_{30} = a_{20} = 0$ and $a_{10} \neq 0$ do not contain lattice points with $a_{i,j} \neq 0$, see the possible outer faces of N_p in (a) of Figure 2. Hence it follows from Lemma 3.4 that

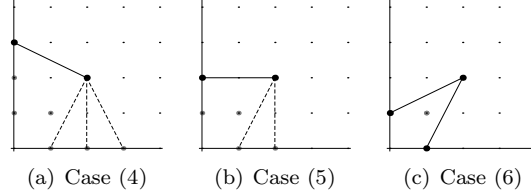


FIGURE 2. Outer faces of N_p for Lemma 4.6.

if p is a submersion, all its level sets are connected.

On the other hand if $a_{30} = a_{20} = a_{10} = 0$, it follows that $p_x(x, 0) \equiv 0$ and $p_y(x, 0) = a_{01} + a_{11}x$. Assume that p is a submersion. Then $a_{11} = 0$ and $a_{01} \neq 0$. Since $p_x(0, y) \equiv 0$ and $p_y(0, y) = a_{01} + 2a_{02}y + 3y^2$, it follows that $a_{02}^2 - 3a_{01} < 0$. In particular $a_{01} > 0$. We multiply p by $1/\sqrt{a_{01}^3}$ and apply the change $(x, y) \mapsto (x/\sqrt[3]{a_{01}}, y/\sqrt[3]{a_{01}})$. This gives the first case of this proposition.

For the polynomial (5), when either $a_{20} \neq 0$ or $a_{20} = 0$ and $a_{10} \neq 0$, see the possible outer faces of N_p in (b) of Figure 2, respectively. From Lemma 3.4 it follows that in both cases, if p is a submersion, its level sets are connected. On the other hand, if $a_{20} = a_{10} = 0$, it is simple to conclude that p is not a submersion.

Finally we consider the polynomial (6). Assume that p is a submersion. If $a_{10}a_{01} \neq 0$, it follows from Lemma 3.4 that the level sets of p are connected, see diagram (c) in Figure 2. On the other hand, we assume that $a_{10} = 0$ and $a_{01} \neq 0$. If $a_{11} \neq 0$, then $\nabla p(-a_{01}/a_{11}, 0) = (0, 0)$. Finally, if $a_{11} = 0$, we apply $T(x, y) = (x/a_{01}, a_{01}y)$ to obtain the second case of this proposition. \square

4.4. Case (VIII).

Lemma 4.8. *The polynomial $p = p_3 + h$, where h is of case (VIII), is equivalent to one of the following:*

$$(7) \quad \begin{aligned} p &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3 + x^3y, \\ a_{03}^2 + a_{12}^2 + a_{02}^2 &> 0, \end{aligned}$$

$$(8) \quad p = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + x^3y, \quad 4a_{11}^3 + 27a_{01}^2 \neq 0,$$

$$(9) \quad p = a_{10}x + 2y + a_{20}x^2 - 3xy + x^3y,$$

$$(10) \quad p = a_{10}x + a_{20}x^2 + x^3y, \quad a_{20} = 0 \text{ or } 1.$$

Proof. By multiplying $p(x, y)$ by $1/4$ and applying the change $T(x, y) = (x + a_{21}/3, y + a_{30})$, we obtain that $p(x, y)$ is equivalent to

$$p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3 + x^3y.$$

If $a_{03}^2 + a_{12}^2 + a_{02}^2 > 0$, we obtain case (7), whereas if these coefficients are zero, we could have $a_{01}^2 + a_{11}^2$ zero or not. In the first case we get (10) (if $a_{20} \neq 0$, simply divide $p(x, y)$ by a_{20} and apply the change $T(x, y) = (x, y/a_{20})$) and in the second case, if $4a_{11}^3 + 27a_{01}^2 \neq 0$, we get case (8). On the other hand, if $4a_{11}^3 + 27a_{01}^2 = 0$, then $a_{01} \neq 0$ and the change $T(x, y) = (\sqrt[3]{2/a_{01}}x, a_{01}y/2)$ gives case (9). \square

Proposition 4.9. *If $p = p_3 + h$, with h of case (VIII), is a submersion and has not all its level sets connected, then p is equivalent to*

$$p(x, y) = y + a_{02}y^2 + xy^3, \quad a_{02} = 0 \text{ or } 1.$$

Proof. We now analyze the polynomials of Lemma 4.8. We first observe that if $a_{10} = a_{20} = 0$, it follows that $p_x(x, 0) \equiv 0$ and $p_y(x, 0)$ is a polynomial of degree 3, and thus p is not a submersion. We assume thus that $a_{10}^2 + a_{20}^2 > 0$.

In case (7) if $a_{03} \neq 0$, then the possible outer faces of N_p are shown in (a) of Figure 3. If $a_{03} = 0$ and $a_{02} \neq 0$, the possibilities are in (b) of Figure 3. If $a_{03} = a_{02} = 0$ and $a_{01} \neq 0$, the possible outer faces are shown in (c) of Figure 3. We observe that no outer face contains lattice points in its interior. Thus from Lemma 3.4, it follows that if p is a submersion, all its level sets are connected. Finally we assume $a_{03} = a_{02} = a_{01} = 0$. Observe that $p_y(0, y) \equiv 0$ and $p_x(0, y) =$

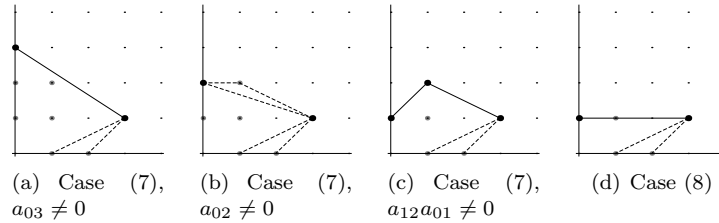


FIGURE 3. Outer faces of N_p in cases (7) and (8) of Lemma 4.8

$a_{10} + a_{11}y + a_{12}y^2$. Thus if $d = a_{11}^2 - 4a_{12}a_{10} \geq 0$, it follows that p is not a submersion. On the other hand, assume $d < 0$ and observe that if $y = -(a_{11} + x^2)/(2a_{12})$, then $p_y(x, y) \equiv 0$ and $p_x(x, y)$ is a polynomial of degree 4 of the form

$-d/(4a_{12}) + \dots - 5x^4/(4a_{12})$. This polynomial has a zero as $d < 0$. Hence p is not a submersion.

Now in case (8), if $a_{01} \neq 0$, observe that if S is the horizontal outer face of N_p (see (d) of Figure 3), then $p_S(x, y)$ can be factorized as $y(x - \xi_1)(x - \xi_2)(x - \xi_3)$, where $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$ are the roots of $x^3 + a_{11}x + a_{01}$. Since the discriminant of this cubic polynomial is a multiple of $4a_{11}^3 + 27a_{01}^2 \neq 0$, it follows that ξ_1, ξ_2, ξ_3 are mutually distinct. Hence p is non-degenerate on S . Clearly, p is non-degenerate on the other outer face, thus from Lemma 3.4, it follows that if p is a submersion, all its level sets are connected. On the other hand, if $a_{01} = 0$, we observe that $\nabla p(0, -a_{10}/a_{11}) = (0, 0)$.

In case (9), observe that $p_y(-2, y) \equiv 0$, and $p_x(-2, y)$ is a polynomial of degree 1, hence p is not a submersion.

Finally, in the case (10), it is clear that if $a_{10} = 0$ then p is not a submersion, whereas if $a_{10} \neq 0$, we multiply p by $1/a_{10}^2$ and apply the change $(x, y) \mapsto (a_{10}y, x/a_{10})$ to obtain the polynomial of the proposition, which is a submersion. It is clear that $p^{-1}\{0\}$ is not connected. \square

4.5. Case (IX).

Lemma 4.10. *The polynomials of the form $p = p_3 + h$, where h is in case (IX), are equivalent to one of the following:*

$$(11) \quad a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + y^3 + x^4,$$

$$(12) \quad a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + x^4, \\ a_{12} \neq 0, \text{ or } a_{12} = a_{21} = 0,$$

$$(13) \quad a_{10}x + a_{20}x^2 + a_{02}y^2 + a_{30}x^3 + x^2y + x^4, \quad a_{02} \neq 0,$$

$$(14) \quad a_{10}x + a_{01}y + a_{20}x^2 + a_{30}x^3 + x^2y + x^4, \quad a_{01} \neq 0,$$

$$(15) \quad a_{10}x + x^2y + x^4,$$

Proof. First we divide $p(x, y)$ by α . Then we get the case (11) above if $a_{03} \neq 0$ by applying $T(x, y) = (x, \sqrt[3]{a_{03}y})$. If $a_{03} = 0$ and $a_{12} \neq 0$, we get the first part of case (12) above, whereas if $a_{03} = a_{12} = a_{21} = 0$, we obtain the second one. Now if $a_{03} = a_{12} = 0$ and $a_{21} \neq 0$, we apply the transformation $T(x, y) = (x + a_{11}/(2a_{21}), a_{21}y)$ to obtain

$$p(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{30}x^3 + x^2y + x^4.$$

If $a_{02} \neq 0$ we apply $T(x, y) = (x, y + a_{01}/(2a_{02}))$ to obtain case (13). On the other hand, if $a_{02} = 0$ we obtain case (14) if $a_{01} \neq 0$ and, if $a_{01} = 0$, the transformation $T(x, y) = (x, y + a_{30}x + a_{20})$ gives case (15). \square

Proposition 4.11. *If $p = p_3 + h$, where h is in case (IX), is a submersion and has at least one disconnected level set, then p is equivalent to*

$$p(x, y) = y + xy^2 + y^4.$$

Proof. We analyze each case of Lemma 4.10.

The only outer face of N_p of case (11) is shown in (a) of Figure 4. Since it does not contain lattice points, if p is a submersion, it follows from Lemma 3.4 that all the level sets of p are connected.

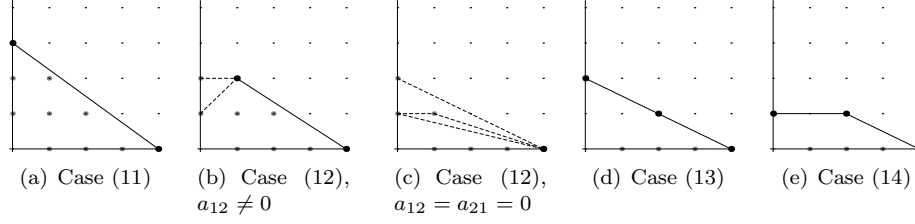


FIGURE 4. Outer faces of N_p of Lemma 4.10.

In case (12), first assume that $a_{01}^2 + a_{02}^2 > 0$. If either $a_{12} \neq 0$ or $a_{12} = a_{21} = 0$, we see the possibilities for the outer faces of N_p in (b) and (c) of Figure 4, respectively. Since the inner part of any outer face contains lattice points with $a_{i,j} \neq 0$, it follows from Lemma 3.4 that if p is a submersion then all its levels sets are connected. Now assume that $a_{01} = a_{02} = 0$. If $a_{12} \neq 0$ then taking $y = -(a_{11} + a_{21}x)/(2a_{12})$, it follows that $p_y(x, y) \equiv 0$ and $p_x(x, y)$ is a polynomial of degree 3 in x . On the other hand, if $a_{12} = a_{21} = 0$, it is simple to conclude also that p is not a submersion.

In case (13), observe that if $y = -x^2/(2a_{02})$ then $p_y(x, y) \equiv 0$ and

$$(16) \quad p_x(x, y) = a_{10} + 2a_{20}x + 3a_{30}x^2 + \frac{4a_{02} - 1}{a_{02}}x^3.$$

Thus if $a_{02} \neq 1/4$, it follows that p is not a submersion. We assume now that $a_{02} = 1/4$. Observe that for each $c \in \mathbb{R}$, $p(x, y) - c$ is a polynomial of degree 2 in y , with discriminant $\Delta(x) = -a_{30}x^3 - a_{20}x^2 - a_{10}x + c$. From Proposition 3.3 (with x changed by y), if either $a_{30} \neq 0$ or $a_{30} = a_{20} = 0$ and $a_{10} \neq 0$, it follows that the levels sets of p are connected, if p is a submersion. On the other hand, if $a_{30} = 0$ and $a_{20} \neq 0$, it follows from (16) that p is not a submersion.

In this case (13), if from the beginning we assume that p is a submersion, we can conclude that all its level sets are connected if $a_{02} \neq 1/4$, by analyzing the only outer face S of N_p , see (d) of Figure 4, and by using Lemma 3.4. But if $a_{02} = 1/4$, then $p_S(x, y) = (y + 2x^2)^2/4$, and we can not apply the lemma.

The outer faces of N_p of case (14) is shown in (e) of Figure 4. As above, it follows that if p is a submersion then its level sets are connected.

Finally, case (15) with $a_{10} = 0$ is not a submersion, whereas if $a_{10} \neq 0$, we divide p by $\sqrt[3]{a_{10}^4}$ and then the change $(x, y) \mapsto (y/\sqrt[3]{a_{10}^2}, x/\sqrt[3]{a_{10}})$ gives the polynomial of the proposition. \square

5. THE POLYNOMIALS OF THEOREM 1.3 AND THE PROOF OF THEOREM 1.1

Now we arrive to the second part of the paper. In this section we prove that if p is one of the polynomials of Theorem 1.3, then there is not a polynomial q such that $\det D(p, q)(x, y) > 0$, for each $(x, y) \in \mathbb{R}^2$, see Proposition 5.6. Then we detail the proof of Theorem 1.1 at the end of the section.

Our approach is inspired in [4] and, as there, we deal with half-Reeb components of foliations. We begin recalling this concept.

Let $M \subset \mathbb{R}^2$ be an open set and $f : M \rightarrow \mathbb{R}$ be a C^∞ submersion. We denote by $\mathcal{F}(f)$ the foliation of M given by the connected components of the level sets of f , see Lemma 3.1.

Definition 5.1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ submersion, $h_0 : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $h_0(x, y) = xy$, and

$$B = \{(x, y) \in [0, 2] \times [0, 2] \mid 0 < x + y \leq 2\}.$$

We say that $\mathcal{A} \subset \mathbb{R}^2$ is a *half-Reeb component*, or simply a *hRc*, of $\mathcal{F}(f)$ if there exists a homeomorphism $T : B \rightarrow \mathcal{A}$ which is a topological equivalence between $\mathcal{F}(h_0)|_B$ and $\mathcal{F}(f)|_{\mathcal{A}}$ with the following properties:

- (1) The segment $\{(x, y) \in B \mid x + y = 2\}$ is sent by T onto a transversal section to the leaves of $\mathcal{F}(f)$ in the complement of $T(1, 1)$. This section is called the *compact edge* of \mathcal{A} .
- (2) Both the segments $\{(x, y) \in B \mid x = 0\}$ and $\{(x, y) \in B \mid y = 0\}$ are sent by T onto full half leaves of $\mathcal{F}(f)$, called the *non-compact edges* of \mathcal{A} .

Figure 5 below illustrates this definition.

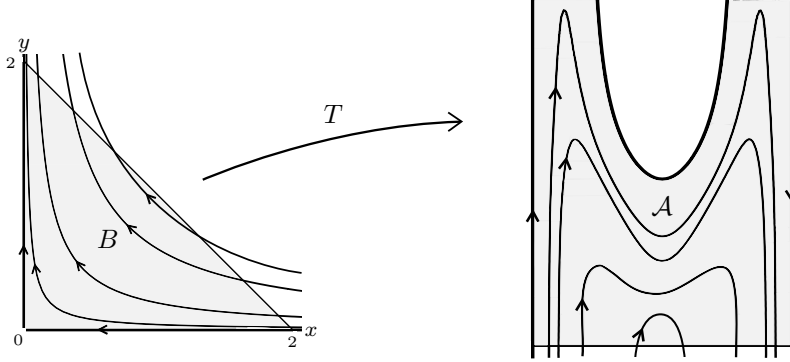


FIGURE 5. Definition of hRc.

The existence of hRc is equivalent to the existence of inseparable leaves on the foliation $\mathcal{F}(f)$, see [11] for details.

The following proposition of [4] relates the existence of hRc with connectedness of level sets.

Proposition 5.2. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ submersion. Then $\mathcal{F}(f)$ has a hRc if and only if there exists $c \in \mathbb{R}$ such that $f^{-1}\{c\}$ is not connected.*

A particular version of the following result is already contained in [4].

Proposition 5.3. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^∞ submersion, $\mathcal{A} \subset \mathbb{R}^2$ be a hRc of $\mathcal{F}(f)$ and U be a neighborhood of \mathcal{A} . If $h : U \rightarrow [0, \infty)$ is a measurable function such that*

$$\int_{\mathcal{A}} h = \infty,$$

then there is not a differentiable $g : U \rightarrow \mathbb{R}$ such that $\det D(f, g) = h$ at U .

Proof. Let $\gamma : [0, 1] \rightarrow \mathcal{A}$ be an injective curve that parametrizes the compact edge of \mathcal{A} . For n big enough, the leaf of $\mathcal{F}(f)$ passing through $\gamma(1/n)$ cuts γ again in $\gamma(t_n)$, with a $t_n > 1/n$. We denote by β_n the interval of this leaf between $\gamma(1/n)$ and $\gamma(t_n)$, and by γ_n the interval of γ between this two points. We denote yet by B_n the compact region bounded by γ_n and β_n . From the monotone convergence theorem and from definition of hRc, it follows that

$$(17) \quad \int_{\mathcal{A}} h = \lim_{n \rightarrow \infty} \int_{B_n} h.$$

If there is g such that $h = \det D(f, g)$ in U , Green's theorem gives

$$\int_{B_n} h = \int_{\gamma_n} -g(f_x dx + f_y dy) \leq \int_{\gamma} -g(f_x dx + f_y dy),$$

because β_n is orthogonal to ∇f . Thus $\int_{B_n} h$ is uniformly bounded, a contradiction with (17) and with $\int_{\mathcal{A}} h = \infty$. \square

We will also need the following simple result (a reduced version of it was already used in [4]).

Lemma 5.4. *Let $b_1, b_2 > 0$, and $\phi_1 : (0, b_1) \rightarrow \mathbb{R}$ and $\phi_2 : (b_2, \infty) \rightarrow \mathbb{R}$ be defined by $\phi_1(t) = \phi_2(t) = \sum_{j=k_1}^{k_2} c_j t^j$, where $k_1, k_2 \in \mathbb{Z}$, $k_1 \leq k_2$, and $c_{k_1}, c_{k_2} \neq 0$. If $\phi_i(t) > 0$ for all t in the domain of ϕ_i , then*

- (1) $c_{k_i} > 0$ for $i = 1, 2$.
- (2) If $k_1 \leq -1$, then $\int_0^{b_1} \phi_1(t) dt = \infty$.
- (3) If $k_2 \geq -1$, then $\int_{b_2}^{\infty} \phi_2(t) dt = \infty$.

Proof. By multiplying $\phi_i(t)$ by t^{-k_i} , we get $\sum_{j=k_1}^{k_2} c_j t^{j-k_i} > 0$ for all t in the domain of ϕ_i . Taking $t \rightarrow 0$ if $i = 1$, or $t \rightarrow \infty$ if $i = 2$, it follows that $c_{k_i} > 0$, proving 1. The proof of (2) and (3) follows by Hölder's inequality: defining $I_1 = (0, b_1)$ and $I_2 = (b_2, \infty)$, we have $b_i^{-1-k_i} \int_{I_i} \phi_i(t) dt \geq \int_{I_i} t^{-1-k_i} \phi_i(t) dt = \infty$, since for both $i = 1, 2$, the last integral is $\int_{I_i} c_{k_i} t^{-1} dt$ plus a finite integral. \square

We now apply these results to analyze each polynomial of Theorem 1.3 in each of the subsections below.

5.1. Case 1. We consider the polynomial $p(x, y) = y + xy^2 + y^4 = y(1 + xy + y^3)$. It is simple to see that the following set is a hRc of $\mathcal{F}(p)$:

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq y < 0 \text{ and } 0 \leq x \leq -1/y - y^2\} \cup \{(x, 0) \mid x \geq 0\}.$$

We claim that given a polynomial $h(x, y) = \sum_{i+j \leq k} b_{ij} x^i y^j$ such that $h(x, y) > 0$, $\forall (x, y) \in \mathbb{R}^2$, then $\int_{\mathcal{A}} h = \infty$. Thus from Proposition 5.3, there does not exist a polynomial q such that $\det D(p, q)(x, y) > 0$ in \mathbb{R}^2 .

To prove the claim, we define

$$(18) \quad \tau = \min \{j - i - 1 \mid 0 \leq i + j \leq k \text{ and } b_{ij} \neq 0\}.$$

Applying the change of variables $(x, y) \mapsto (-xy/(1+y^3), -y)$ in the interior of \mathcal{A} , we obtain

$$(19) \quad \begin{aligned} \int_{\mathcal{A}} h &= \int_0^1 \int_0^1 \sum_{\substack{i+j \leq k \\ j-i-1=\tau}} b_{ij} (-1)^j x^i y^{j-i-1} (1-y^3)^{i+1} dx dy \\ &= \int_0^1 \int_0^1 (s(x)y^\tau + s_1(x)y^{\tau+1} + \dots) dy dx, \end{aligned}$$

where

$$s(x) = \sum_{\substack{i+j \leq k \\ j-i-1=\tau}} b_{ij} (-1)^j x^i,$$

and $s_1(x), s_2(x), \dots$ are suitable polynomials in x .

From (18), the polynomial $s(x)$ is not identically zero, hence there exists a $c \in \mathbb{R}$, $0 < c \leq 1$, such that $s(x) \neq 0 \forall x \in (0, c)$. Since $h(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$, we can apply Lemma 5.4 for each $x \in (0, c)$. Hence from statement (1) of Lemma 5.4, we get $s(x) > 0$ for all $x \in (0, c)$. Moreover, since $b_{00} \neq 0$, we have $\tau \leq -1$. Thus by applying statement (2) of Lemma 5.4, it follows that, for each $x \in (0, c)$

$$\int_0^1 (s(x)y^\tau + s_1(x)y^{\tau+1} + \dots) dy = \infty.$$

Then

$$\int_0^c \int_0^1 (s(x)y^\tau + s_1(x)y^{\tau+1} + \dots) dy dx = \infty,$$

which from (19) gives that $\int_{\mathcal{A}} h = \infty$, and the claim is proven.

5.2. Case 2. We consider the quite analogous case of the polynomial $p(x, y) = y + a_{02}y^2 + xy^3$, with $a_{02} = 0$ or 1 . To simplify the analysis, we apply the change $x \mapsto -x$ to this polynomial, obtaining $p(x, y) = y + a_{02}y^2 - xy^3$, $a_{02} = 0$ or 1 . It is simple to conclude that the closure of the following set is a hRc of $\mathcal{F}(p)$.

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq y < 0 \text{ and } 1 - a_{02} \leq x \leq 1/y^2 + a_{02}/y\}.$$

We observe that $\mathcal{A} \cup R \subset \mathcal{B} \cup R$, where \mathcal{A} is the hRc of case 1 and $R = [0, 2] \times [-1, 0]$. Hence for each positive polynomial $h(x, y)$ we have

$$\int_{\mathcal{B} \cup R} h \geq \int_{\mathcal{A} \cup R} h = \infty.$$

Since R is a bounded set it follows that $\int_{\mathcal{B}} h = \infty$. Therefore, from Proposition 5.3, there is not a polynomial q such that $\det D(p, q) > 0$ in \mathbb{R}^2 .

For the next two cases, we need the following result.

Lemma 5.5. *Given $N \in \mathbb{N}$ and $b_0, \dots, b_N \in \mathbb{R}$, let $L(\theta)$ be the polynomial*

$$L(\theta) = \sum_{j=0}^N b_j (2(j+1)\theta + 2j+1)\theta^j.$$

If $L(\theta)$ is not identically zero, there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that $L(\theta_1) < 0 < L(\theta_2)$.

Proof. Let the non-identically zero polynomial $f(\theta) = \sum_{j=0}^N b_j \theta^j$. The following identity is straightforward

$$L(\theta) = \frac{((\theta + \theta^2)f(\theta^2))'}{f(\theta)}.$$

Letting α and β two consecutive zeros of $g(\theta) = (\theta + \theta^2)f(\theta)^2$, it follows that the derivative of $g(\theta)$ changes sign in the interval (α, β) (see also [1, Lemma 1.1.11]). Hence $L(\theta)$ changes sign in (α, β) . \square

5.3. Case 3. We take now the polynomial $p(x, y) = y + x^2y^2$. The analysis here will be quite different from the former ones. We first observe that the following set is a hRc of $\mathcal{F}(p)$:

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1 \text{ and } -1/x^2 \leq y \leq 0\}.$$

We notice that $\int_{\mathcal{A}} 1 = \int_1^\infty 1/x^2 dx < \infty$, and thus Proposition 5.3 can not be used to prove that there is not a polynomial q such that $\det D(p, q) = 1$, for example. But we will use this proposition to eliminate candidates to be the polynomial q . (Then by using a different argument we will show there is not such a q).

We suppose there exists $q(x, y) = \sum_{i+j \leq k} b_{ij} x^i y^j$ such that

$$h(x, y) = \det D(p, q)(x, y) = \sum_{i+j \leq k} b_{ij} (2(j-i)x^{i+1}y^{j+1} - ix^{i-1}y^j) > 0,$$

for each $(x, y) \in \mathbb{R}^2$. We define

$$\tau = \max \{i - 2j - 3 \mid 0 \leq i + j \leq k \text{ and } b_{ij} \neq 0\},$$

and we claim that $\tau < -1$. Indeed, calculating $\int_{\mathcal{A}} h$ by applying the change of variables $(x, y) \mapsto (x, -x^2y)$, we obtain

$$\begin{aligned} \int_{\mathcal{A}} h &= \int_0^1 \int_1^\infty \sum_{i+j \leq k} b_{ij} (-1)^{j+1} (2(j-i)y + i) y^j x^{i-2j-3} dx dy \\ &= \int_0^1 \int_1^\infty (s(y)x^\tau + s_1(y)x^{\tau-1} + \dots) dx dy, \end{aligned}$$

where

$$s(y) = \sum_{\substack{i+j \leq k \\ i-2j-3=\tau}} b_{ij} (-1)^{j+1} (2(j-i)y + i) y^j$$

and $s_i(y)$ are suitable polynomials in y . If $\tau \geq -1$, we conclude that $s(y)$ is not identically zero and thus there exists $c \leq 1$ such that $s(y) \neq 0$ in $(0, c)$. Moreover, since $h(x, y)$ is a positive polynomial, it follows from Lemma 5.4 that $s(y) > 0$ in $(0, c)$. By the same lemma, it follows that the above integral is infinite, a contradiction with the Proposition 5.3 (since we suppose there exists q such that $\det D(p, q) = h$). Hence the claim is proven.

Now still supposing $h(x, y) > 0$, for all $(x, y) \in \mathbb{R}^2$, and considering $y = \theta x^{-2}$ we have

$$h(x, \theta x^{-2}) = \sum_{i+j \leq k} b_{ij} (2(j-i)\theta - i) \theta^j x^{i-2j-1} > 0$$

for all $x, \theta \in \mathbb{R}$, with $x \neq 0$. Since $b_{ij} = 0$ for $i - 2j - 1 > 0$, because from the claim $\tau < -1$, we have

$$(20) \quad L(\theta) = \sum_{\substack{i+j \leq k \\ i-2j-1=0}} b_{ij} (2(j-i)\theta - i) \theta^j \geq 0,$$

for all $\theta \in \mathbb{R}$. Then from Lemma 5.5, it follows that $L(\theta)$ is identically zero, which guarantees in particular that $b_{10} = 0$. But this is a contradiction with

$-b_{10} = h(0, 0) > 0$. Hence we conclude there is not a polynomial $q(x, y)$ such that $\det D(p, q)(x, y) > 0$, for all $(x, y) \in \mathbb{R}^2$.

5.4. Case 4. Finally, we consider $p(x, y) = y + a_{02}y^2 + y^3 + x^2y^2$, with $a_{02}^2 < 3$. We shall use the notations of Subsection 5.3. We first observe that the closure of

$$\mathcal{A} = \left\{ (x, y) \in \mathbb{R}^2 \mid -1 \leq y < 0 \text{ and } \sqrt{2 - a_{02}} \leq x \leq \sqrt{-1/y - a_{02} - y} \right\}$$

is a half-Reeb component of $\mathcal{F}(p)$. Given $m < 1$, since the inequality $m^2 < 1 + a_{02}y + y^2$ is valid for all $y \in (b, 0)$ and for some $b \geq -1$, it follows that

$$(21) \quad m\sqrt{-1/y} < \sqrt{-1/y - a_{02} - y}, \quad \forall y \in (b, 0).$$

We suppose also that $b \geq -m^2$ and define

$$\tilde{\mathcal{A}} = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq m/\sqrt{-b} \text{ and } -m^2/x^2 \leq y < 0 \right\}.$$

Now if $(x, y) \in \tilde{\mathcal{A}}$, then $b \leq y < 0$ and $m/\sqrt{-b} \leq x \leq m\sqrt{-1/y}$. Hence by (21), it follows that $(x, y) \in \mathcal{A} \cup C$, where C is the bounded set $C = \{-1 \leq y < 0 \text{ and } 1 \leq x \leq \sqrt{2 - a_{02}}\}$ if $1 \leq \sqrt{2 - a_{02}}$, or $C = \emptyset$. Thus $\int_{\mathcal{A} \cup C} h \geq \int_{\tilde{\mathcal{A}}} h$, for all positive function h defined in \mathbb{R}^2 . In particular, if $\int_{\tilde{\mathcal{A}}} h = \infty$ then $\int_{\mathcal{A}} h = \infty$.

As in the preceding subsection, we suppose there is a polynomial $q(x, y) = \sum_{i+j \leq k} b_{ij}x^i y^j$ such that $h(x, y) = \det D(p, q)(x, y) > 0$ in \mathbb{R}^2 . Then we define $\tau = \max\{i - 2j - 3 \mid b_{ij} \neq 0\}$ and we claim that $\tau < -1$. Indeed calculating $\int_{\tilde{\mathcal{A}}} h$ by applying the change of variables $(x, y) \mapsto (x, -x^2y)$, we get

$$\int_{\tilde{\mathcal{A}}} h = \int_0^{m^2} \int_{\frac{m}{\sqrt{-b}}}^{\infty} \sum_{i+j \leq k} b_{ij}(-1)^{j+1} ((2(j-i)y + i)x^{i-2j-3}y^j + r_{ij}(x, y)) dx dy,$$

where $r_{ij}(x, y) = 3ix^{i-2j-7}y^{j+2} - 2a_{02}ix^{i-2j-5}y^{j+1}$. As in Subsection 5.3, it follows that if $\tau \geq -1$ then this integral is infinite, which from the reasoning above gives that $\int_{\mathcal{A}} h = \infty$. But this contradicts Proposition 5.3. Thus the claim is proven.

By considering again $y = \theta x^{-2}$, we have

$$\sum_{i+j \leq k} b_{ij}((2(j-i)\theta - i)\theta^j x^{i-2j-1} - 2a_{02}\theta^{j+1}x^{i-2j-3} - 3i\theta^{j+2}x^{i-2j-5}) > 0,$$

for all $\theta, x \in \mathbb{R}$, with $x \neq 0$. Since $\tau < -1$, it follows in particular that the polynomial $L(\theta)$ defined in (20) is such that $L(\theta) \geq 0$ for all $\theta \in \mathbb{R}$. Therefore, by Lemma 5.5, $L(\theta)$ is identically zero, hence $b_{10} = 0$, which is a contradiction with $-b_{10} = h(0, 0) > 0$. Thus we conclude that there is not a polynomial q such that $\det D(p, q) > 0$.

From the previous four subsections, we directly obtain the following result.

Proposition 5.6. *If $p(x, y)$ is one of the polynomials of Theorem 1.3, then there is not a polynomial $q(x, y)$ such that $\det D(p, q)(x, y) > 0$, $\forall (x, y) \in \mathbb{R}^2$.*

We are finally ready to prove our main theorem. Indeed, we just need to reorganize the results we obtained throughout the paper.

Proof of Theorem 1.1. Let $F = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial map such that the degree of p is less than or equal to 4 and such that $\det DF(x, y) \neq 0$ in \mathbb{R}^2 . It is immediate from the hypothesis that p is a submersion. If the degree of p is less than 4, then F is injective from the main result of [4].

If F is not injective, it follows thus that the degree of p is 4 and from Lemma 1.2, p have some disconnected level set. From Theorem 1.3 it follows that p is equivalent to one of the four polynomials in that theorem. But then from Proposition 5.6, there is no q such that $\det D(p, q)(x, y) \neq 0$ in \mathbb{R}^2 . This contradiction with the hypothesis finishes the proof. \square

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