Norm resolvent approximation of thin homogeneous tubes by heterogeneous ones

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Abstract

We study the operator $-\mathrm{div}(A(x)\nabla\psi(x))$ restricted to a waveguide $\Omega_\varepsilon \subset \mathbb{R}^3$, with heterogeneous function $A(x)$ constant in the longitudinal direction. The purpose is to obtain an effective operator, in the norm resolvent sense, when the diameter of Ω_ε tends to zero with heterogeneity approaching a homogeneous situation (i.e., a constant function A). The effective operator presents a potential that, besides the traditional dependence on waveguide geometric properties, there is also a contribution from $A(x)$ which results, when combined with the curvature, for example, in the possibility of a repulsive interaction.

1 Introduction

Let $I = (a, b)$ be an open interval of $\mathbb{R}$; the cases $a = -\infty$ and/or $b = +\infty$ are included. Let $r : I \to \mathbb{R}^3$ be a curve of class C^3 parametrized by its arc-length parameter s. Denote by $k(s)$ and $\tau(s)$ its curvature and torsion at the point $r(s)$, respectively. Pick S an open, bounded and connected subset of $\mathbb{R}^2$ with "transverse" coordinates $y = (y_1, y_2)$; we require that $0 \in S$. We build a tube (waveguide) in $\mathbb{R}^3$ by properly moving the region S along $r(s)$; at each point $r(s)$ the cross-section region S may present a (continuously differentiable) rotation angle $\alpha(s)$ (see details in Section 2). We demand that the functions $k(s), (\tau + \alpha')(s), (\tau + \alpha')'(s) \in L^\infty(I)$.

Given $\varepsilon > 0$, denote by Ω_ε the waveguide generated by the cross-section εS. Let $A : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The purpose of this paper is study the (Dirichlet) operator

$$\psi(x) \mapsto -\mathrm{div}\,(A(x)\nabla\psi(x)), \qquad \psi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon), \tag{1}$$

and to find an effective operator in the limit $\varepsilon \to 0$ for particular choices of the function A; the latter characterizes the heterogeneity of the waveguide. Here, $\nabla\psi$ denotes the gradient vector of ψ in Ω_ε. Since the sequence of tubes Ω_ε "approaches" the curve $r(s)$, as $\varepsilon \to 0$, it is expected that the effective operator could be identified with a one-dimensional operator in $L^2(I)$.

The most studied situation is of a homogeneous tube, which may be characterized by a constant function $A(x) = 1$; ahead we review some relevant results in this situation. The case of strongly heterogeneous tubes has also been considered, that is, $A(x) = \tilde{a}(y/\varepsilon)$, where $\tilde{a} : \mathbb{R}^2 \to \mathbb{R}$ is a periodic function and $y = (y_1, y_2)$ are the variables in the transverse section S. More specifically, in [8] the authors analyze the interaction between thickness ε and axial δ-periodic heterogeneities as both ε and δ tend to zero, for bounded A depending only on one of the transverse variables and in the three possible regimes: $\varepsilon \ll \delta$, $\varepsilon \gg \delta$ and $\varepsilon \approx \delta$ (such work is a counterpart of [7] that has considered a 3D-2D reduction). See also [12] for spectral analysis in related thin heterogeneous tubes, for bounded and uniformly coercive A depending now on the two transverse coordinates; by applying the Γ-convergence technique, it was found that the results depend critically on the choice of the (above) regime considered. In [2, 14] the homogenization, also under coerciveness of A and other conditions, is studied in any dimension and different techniques. Other related works can be traced through the cited references.

However, here we are interested in a simpler question, that is, whether there is any nontrivial contribution when one approaches thin homogeneous tubes from heterogeneous ones; more specifically, we roughly consider a nonconstant function $A(\varepsilon y)$ of the transverse coordinates and with differentiable coefficients (see precise details ahead), so that as $\varepsilon \to 0$ the homogeneous case is recovered due to the formal approximation $A(\varepsilon y) \approx A(0) = 1$. In fact, as discussed ahead, this process presents a contribution to effective operators and, when combined with curvature, repulsive and attractive potentials may arise.

Now we comment on the homogeneous case $A(x) = 1$, i.e.,

$$\psi(x) \mapsto -\Delta\psi(x), \qquad \psi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon), \tag{2}$$

where Δ denote the Laplacian operator in Ω_ε. This situation was studied in several works [3, 4, 5, 6, 9, 10]. Note that, as $\varepsilon \to 0$, the region Ω_ε becomes narrower and the whole spectrum of $-\Delta$ diverges. To control this kind of divergence the following strategy is usual. Let λ_0 be the first (i.e., the lowest) eigenvalue of the Dirichlet Laplacian $-\Delta_y$ restricted to S and u_0 the (positive) associated normalized eigenfunction, that is,

$$-\Delta_y u_0 = \lambda_0 u_0, \quad u_0 \in H_0^1(S), \quad \lambda_0 > 0, \quad \int_S |u_0|^2 \,\mathrm{d}y = 1.$$

In order to obtain a meaningful spectral convergence, one needs to subtract $(\lambda_0/\varepsilon^2)\mathbf{1}$ from (2); $\mathbf{1}$ denotes the identity operator. Thus, the operator turns out to be

$$\psi(x) \mapsto -(\Delta\psi)(x) - \frac{\lambda_0}{\varepsilon^2}\psi(x), \qquad \psi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon). \tag{3}$$

Put

$$V(s) := C(S)(\tau + \alpha')^2(s) - \frac{k^2(s)}{4}, \quad s \in I, \tag{4}$$

where $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C(S) := \int_S |\langle \nabla_y u_0, Ry \rangle|^2 \,\mathrm{d}y$. Consider the one-dimensional self-adjoint operator

$$(Tw)(s) = -w''(s) + V(s)w(s), \quad w \in H^2(I) \cap H_0^1(I).$$

Let $\mathcal{L} := \{w(s)u_0(y) : w \in L^2(I)\}$ be a closed subspace of $L^2(I \times S)$ and denote by $\mathbf{0}$ the null operator on its orthogonal complement $\mathcal{L}^\perp$. One has

$$-\Delta - \frac{\lambda_0}{\varepsilon^2}\mathbf{1} \approx T \oplus \mathbf{0}, \qquad \varepsilon \to 0, \tag{5}$$

i.e., T is the effective operator in this situation. The approximation (5) was proved in different aspects. In particular, for bounded tubes, in [3] the authors have used the Γ-convergence technique to obtain T as effective operator and to find an asymptotic behavior of the eigenvalues of $-\Delta$. In [6], where unbounded tubes were considered, the approximation in (5) was proven in the norm resolvent sense; the technique used there comes from [10].

Our goal in this work is to obtain an approximation in the norm resolvent sense for the operator (1) under suitable conditions on nonconstant $A(x)$. Namely, we suppose that there exists a coercive and C^2 function $a : \overline{S} \to \mathbb{R}$, that is, $a(y) \geq r > 0$, for all y, so that the resulting operator is elliptic and

$$A_\varepsilon(s,y) := (A \circ f_\varepsilon)(s,y) = a(\varepsilon y), \qquad \forall (s,y) \in I \times S, \tag{6}$$

where f_ε is the natural diffeomorphism given by (10) in Section 2. This condition implies that the heterogeneity $A \circ f_\varepsilon$ is constant in the longitudinal direction and, in particular, without loss it is assumed that $A(r(s)) = a(0) = 1$, for all $s \in I$. We will see that the effective operator in this situation presents an additional potential which comes from a derivative of A, i.e., the variation of the heterogeneity at the reference curve.

Given $\varepsilon > 0$, consider the following eigenvalue equation in the cross-section S:

$$-\mathrm{div}\,(a(\varepsilon y)\nabla_y u) = \lambda u, \qquad u \in H^2(S) \cap H_0^1(S).$$

Denote by λ_ε^0 the first eigenvalue of the operator and by u_ε^0 the corresponding eigenfunction. In particular, $\lambda_\varepsilon^0 \to \lambda_0$ and $u_\varepsilon^0 \to u_0$ in $L^2(S)$, as $\varepsilon \to 0$. As a consequence, for all $\varepsilon > 0$ small enough, $\lambda_\varepsilon^0 > 0$ and it is a simple eigenvalue; see Section 3 for more details.

We then pass to study the sequence of operators

$$\psi(x) \mapsto -\mathrm{div}(A(x)\nabla\psi(x)) - \frac{\lambda_\varepsilon^0}{\varepsilon^2}\psi(x), \qquad \psi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon).$$

Let

$$z_{\alpha(s)} := (\cos\alpha(s), -\sin\alpha(s)), \quad V_a(s) := -\frac{k(s)}{2}\left\langle (\nabla_y a)(0), z_{\alpha(s)} \right\rangle,$$

where ∇_y denotes the gradient vector in S.

Define the one-dimensional operator

$$(Tw)(s) := -w''(s) + V(s)w(s) + V_a(s)w(s), \quad \mathrm{dom}\, T = H^2(I) \cap H_0^1(I).$$

Let U_ε and V_ε be the unitary operators given, respectively, by (11) and (12) in Section 2. The main result of this work is

Theorem 1. *As $\varepsilon \to 0$, one has*

$$\left\| \left[V_\varepsilon U_\varepsilon \left(-\mathrm{div}(A(x)\nabla) - \frac{\lambda_\varepsilon^0}{\varepsilon^2}\mathbf{1} \right)^{-1} (V_\varepsilon U_\varepsilon)^{-1} \right] - \left[T^{-1} \oplus \mathbf{0} \right] \right\| \longrightarrow 0, \qquad (7)$$

where $\mathbf{0}$ is the null operator on the subspace $\mathcal{L}^\perp$.

Remark 1. *If $(\tau + \alpha') = 0$, for all $s \in I$, (7) follows directly by Theorem 4 in Section 5. Otherwise, we need an additional step; see Section 3 and Lemma 1 in Section 6.*

In the homogeneous case $A(x) = 1$ for all x, it is known (and we see from (4)) that the bending property (i.e., nonzero curvature) generates an attractive interaction $-k^2/4$ whereas the twisting property (i.e., $\tau + \alpha' \neq 0$) a repulsive one [13]. However, we see that the approximation by the heterogeneous case (1) may have important consequences. For example, suppose Ω_ε is a planar bent and nontwisted tube with $k \neq 0$ and $\tau = \alpha = 0$. In this case the effective operator is

$$(Tw)(s) = -w''(s) - \left(\frac{k^2(s)}{4} + \frac{\partial a}{\partial y_1}(0)\frac{k(s)}{2} \right) w(s).$$

Suppose the local variation $(\partial a/\partial y_1)(0) < 0$; if either the curvature $k(s)$ is small enough or the absolute value of such variation is large, then $A(x)$ combined with curvature gives a net repulsive contribution as $\varepsilon \to 0$.

This text is organized as follows. In Section 2 we discuss details of the standard tube Ω_ε construction and perform the change of variables necessary to study the operator (1); for example, straightening Ω_ε. In Section 3 we study the operator restricted to the cross-section S in order to control the divergent energies from the transverse spectrum of $-\Delta$, as $\varepsilon \to 0$. In Sections 4 and 5 we present some preliminary results to obtain Theorem 1. Finally, in Section 6, we show how to find the effective operator T and conclude Theorem 1. Along the text the symbol K is used to denote different constants.

2 Geometry of the domain and change of coordinates

As already mentioned in the Introduction, let $I = (a, b)$, $-\infty \leq a < b \leq +\infty$, be an interval of $\mathbb{R}$. Let $r : I \to \mathbb{R}^3$ be a simple C^3 curve in $\mathbb{R}^3$ parametrized by its arc-length parameter s. The curvature of r at the position s is $k(s) := \|r''(s)\|$. We choose the usual orthonormal triad of vector fields $\{T(s), N(s), B(s)\}$, the so-called Frenet frame, given the tangent, normal and binormal vectors, respectively, moving along the curve and defined by

$$T = r'; \quad N = k^{-1}T'; \quad B = T \times N. \qquad (8)$$

To justify the construction (8), it is assumed that $k \neq 0$, but if r has a piece of a straight line (i.e., $k = 0$ identically in this piece), usually one can choose a constant

Frenet frame instead. It is possible to combine constant frames with the Frenet frame (8) and so obtaining a global C^2 Frenet frame; see [11], Theorem 1.3.6. In each situation we assume that a global Frenet frame exists and that the Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{9}$$

where $\tau(s)$ is the torsion of $r(s)$, actually defined by (9). Choose $s_0 \in I$ and consider $\alpha : I \to \mathbb{R}$ a C^1 function so that $\alpha(s_0) = 0$. We suppose that $\tau + \alpha'$ is derivable and $(\tau + \alpha')' \in L^\infty(I)$. Let S be an open, bounded, connected and smooth (nonempty) subset of $\mathbb{R}^2$. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$\vec{x}(s, y) = r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s)$$

and consider the domain

$$\Omega_\varepsilon = \{\vec{x}(s, y) \in \mathbb{R}^3 : s \in I, y = (y_1, y_2) \in S\},$$

where

$$\begin{aligned} N_\alpha(s) &:= \cos\alpha(s)N(s) + \sin\alpha(s)B(s), \\ B_\alpha(s) &:= -\sin\alpha(s)N(s) + \cos\alpha(s)B(s). \end{aligned}$$

Hence, this tube Ω_ε is obtained by putting the region εS along the curve $r(s)$, which is simultaneously rotated by an angle $\alpha(s)$ with respect to the cross section at the position $r(s)$.

Now, consider the sequence of operators defined by (1) in Introduction. Our technique corresponds to the study of the sequence of the corresponding quadratic forms. Namely,

$$\tilde{b}_\varepsilon(\tilde{\psi}) := \int_{\Omega_\varepsilon} A(x)|\nabla\tilde{\psi}|^2 \mathrm{d}x, \qquad \mathrm{dom}\ \tilde{b}_\varepsilon = H_0^1(\Omega_\varepsilon).$$

We perform a change of variables so that the region Ω_ε turns to be a straight cylinder $I \times S$. For this, consider the mapping

$$\begin{aligned} f_\varepsilon : \ I \times S &\to \Omega_\varepsilon \\ (s, y) &\mapsto r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s). \end{aligned} \tag{10}$$

The price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon$ which is induced by f_ε, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j\rangle = G_{ji}, \qquad 1 \le i, j \le 3,$$

where

$$e_1 = \frac{\partial f_\varepsilon}{\partial s}, \quad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \quad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$J = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \beta_\varepsilon & -\varepsilon(\tau + \alpha')\langle z_\alpha^\perp, y\rangle & \varepsilon(\tau + \alpha')\langle z_\alpha, y\rangle \\ 0 & \varepsilon\cos\alpha & \varepsilon\sin\alpha \\ 0 & -\varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix},$$

where

$$\beta_\varepsilon(s,y) = 1 - \varepsilon k(s)\langle z_\alpha, y\rangle, \quad z_\alpha = (\cos\alpha, -\sin\alpha), \quad \text{and} \quad z_\alpha^\perp = (\sin\alpha, \cos\alpha).$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} 1/\beta_\varepsilon & (\tau+\alpha')y_2/\beta_\varepsilon & -(\tau+\alpha')y_1/\beta_\varepsilon \\ 0 & (1/\varepsilon)\cos\alpha & -(1/\varepsilon)\sin\alpha \\ 0 & (1/\varepsilon)\sin\alpha & (1/\varepsilon)\cos\alpha \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2\beta_\varepsilon$. Since k is a bounded function, for ε small enough f_ε does not vanish in $I \times S$. Thus, $\beta_\varepsilon > 0$ and f_ε is a local diffeomorphism. In case f_ε is injective (again by requiring that $\varepsilon > 0$ is small), a global diffeomorphism is obtained.

Introducing the notation

$$\|\hat\psi\|_G^2 := \int_{I\times S} |\hat\psi|^2 \beta_\varepsilon(s,y)\mathrm{d}s\mathrm{d}y,$$

and the unitary transformation

$$\begin{array}{rccc} U_\varepsilon: & L^2(\Omega_\varepsilon) & \to & L^2(I\times S, \beta_\varepsilon(s,y)\mathrm{d}s\mathrm{d}y) \\ & \tilde\psi & \mapsto & \varepsilon\,\tilde\psi \circ f_\varepsilon \end{array}, \qquad (11)$$

we obtain the quadratic form

$$\hat b_\varepsilon(U_\varepsilon\tilde\psi) = \|\sqrt{a_\varepsilon(\varepsilon y)}J^{-1}\nabla(U_\varepsilon\tilde\psi)\|_G^2, \quad \mathrm{dom}\ \hat b_\varepsilon = H_0^1(I\times S).$$

Recall the condition (6) in Introduction; there exists a positive and C^2 function $a : \overline{S} \to \mathbb{R}$ so that $(A\circ f_\varepsilon)(s,y) = a(\varepsilon y)$, for all $(s,y) \in I \times S$.

We denote $\hat\psi := U_\varepsilon\tilde\psi$ and, to simplify the notation, sometimes we write $a_\varepsilon(y) := a(\varepsilon y)$. Thus,

$$\begin{aligned} \hat b_\varepsilon(\hat\psi) &= \|\sqrt{a(\varepsilon y)}\,J^{-1}\nabla\hat\psi\|_G^2 \\ &= \int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon}\left|\hat\psi' + \langle\nabla_y\hat\psi, Ry\rangle(\tau+\alpha)'\right|^2 \mathrm{d}s\mathrm{d}y \\ &+ \int_{I\times S} \frac{a_\varepsilon\beta_\varepsilon}{\varepsilon^2}|\nabla_y\hat\psi|^2\,\mathrm{d}s\mathrm{d}y, \end{aligned}$$

$\mathrm{dom}\ \hat b_\varepsilon = H_0^1(I\times S)$. Recall that R is the rotation matrix that appears after (4). Note that $\mathrm{dom}\ \hat b_\varepsilon$ is a subspace of the Hilbert space $L^2(I\times S, \beta_\varepsilon \mathrm{d}s\mathrm{d}y)$. The measure $\beta_\varepsilon\mathrm{d}s\mathrm{d}y$ comes from of the Riemannian metric obtained from the change of variables f_ε.

Now, we consider a change of variables so that we can work in the Hilbert space $L^2(I\times S)$ with the usual measure. Define the isometry

$$\begin{array}{rccc} V_\varepsilon: & L^2(I\times S, \beta_\varepsilon(s,y)\mathrm{d}s\mathrm{d}y) & \to & L^2(I\times S) \\ & \hat\psi & \mapsto & \beta_\varepsilon^{1/2}\hat\psi \end{array}, \qquad (12)$$

and denote $\psi := V_\varepsilon \hat{\psi}$. With this change, the quadratic form $\hat{b}_\varepsilon(\hat{\psi})$ becomes

$$\begin{aligned}\overline{b}_\varepsilon(\psi) &= \int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon^2}\left|\psi' - \frac{\psi}{2}\frac{\partial \beta_\varepsilon}{\partial s}\frac{1}{\beta_\varepsilon} + \left(\langle \nabla_y \psi, Ry\rangle + \psi \beta_\varepsilon^{1/2}\langle \nabla_y\left(1/\beta_\varepsilon^{1/2}\right), Ry\rangle\right)(\tau+\alpha)'(s)\right|^2 \mathrm{d}s\mathrm{d}y \\ &+ \int_{I\times S} \frac{a_\varepsilon \beta_\varepsilon}{\varepsilon^2}\left|\nabla_y\left(\psi/\beta_\varepsilon^{1/2}\right)\right|^2 \mathrm{d}s\mathrm{d}y,\end{aligned}$$

$\mathrm{dom}\, \overline{b}_\varepsilon = H_0^1(I\times S)$.

We draw attention to the last integral in the expression of $\overline{b}_\varepsilon(\psi)$. For this term we present more details of the change of variables (12).

Some calculations show that

$$\begin{aligned}&\int_{I\times S} a_\varepsilon \beta_\varepsilon \left(\frac{\partial}{\partial y_1}\left(\psi \beta_\varepsilon^{-1/2}\right)\right)^2 \mathrm{d}s\mathrm{d}y = \int_{I\times S} a_\varepsilon \beta_\varepsilon \left(\frac{\partial \psi}{\partial y_1}\beta_\varepsilon^{-1/2} - \frac{\partial \beta_\varepsilon}{\partial y_1}\frac{\psi}{2}\beta_\varepsilon^{-3/2}\right)^2 \mathrm{d}s\mathrm{d}y \\ =& \int_{I\times S} a_\varepsilon \beta_\varepsilon \left[\left(\frac{\partial \psi}{\partial y_1}\right)^2 \beta_\varepsilon^{-1} - \beta_\varepsilon^{-2}\frac{\partial \beta_\varepsilon}{\partial y_1}\frac{\partial \psi}{\partial y_1}\psi + \frac{\beta_\varepsilon^{-3}}{4}\left(\frac{\partial \beta_\varepsilon}{\partial y_1}\right)^2 \psi^2\right] \mathrm{d}s\mathrm{d}y \\ =& \int_{I\times S} \left[a_\varepsilon \left(\frac{\partial \psi}{\partial y_1}\right)^2 - \frac{1}{4}a_\varepsilon \beta_\varepsilon^{-2}\varepsilon^2 k^2(s)\cos^2\alpha(s)\,\psi^2 - \frac{\varepsilon^2}{2}\beta_\varepsilon^{-1}\frac{\partial a}{\partial y_1}(\varepsilon y)k(s)\cos\alpha(s)\psi^2\right] \mathrm{d}s\mathrm{d}y.\end{aligned}$$

In these passages, we have performed an integration by parts to obtain

$$\begin{aligned}&-\int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon}\frac{\partial \beta_\varepsilon}{\partial y_1}\frac{\partial \psi}{\partial y_1}\psi\, \mathrm{d}s\mathrm{d}y = -\frac{1}{2}\int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon}\frac{\partial \beta_\varepsilon}{\partial y_1}\frac{\partial \psi^2}{\partial y_1} \mathrm{d}s\mathrm{d}y \\ =& -\int_{I\times S} \left(\frac{\varepsilon^2}{2}\beta_\varepsilon^{-1}\frac{\partial a}{\partial y_1}(\varepsilon y)k(s)\cos\alpha(s)\,\psi^2 + a_\varepsilon \beta_\varepsilon^{-2}\varepsilon^2 k^2(s)\cos^2\alpha(s)\frac{\psi^2}{2}\right) \mathrm{d}s\mathrm{d}y.\end{aligned}$$

Similarly, one can show

$$\begin{aligned}&\int_{I\times S} a_\varepsilon \beta_\varepsilon \left(\frac{\partial}{\partial y_2}\left(\psi \beta_\varepsilon^{-1/2}\right)\right)^2 \mathrm{d}s\mathrm{d}y \\ =& \int_{I\times S} \left[a_\varepsilon \left(\frac{\partial \psi}{\partial y_2}\right)^2 - \frac{1}{4}a_\varepsilon \beta_\varepsilon^{-2}\varepsilon^2 k^2(s)\sin^2\alpha(s)\,\psi^2 + \frac{\varepsilon^2}{2}\beta_\varepsilon^{-1}\frac{\partial a}{\partial y_2}(\varepsilon y)k(s)\sin\alpha(s)\,\psi^2\right] \mathrm{d}s\mathrm{d}y.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{I\times S} \frac{a_\varepsilon \beta_\varepsilon}{\varepsilon^2}\left|\nabla_y\left(\psi/\beta_\varepsilon^{1/2}\right)\right|^2 \mathrm{d}s\mathrm{d}y \\ =& \int_{I\times S} \left[\frac{a_\varepsilon}{\varepsilon^2}|\nabla_y \psi|^2 - \frac{\beta_\varepsilon^{-2}}{4}a_\varepsilon k^2(s)|\psi|^2 - \frac{\beta_\varepsilon^{-1}}{2}\langle (\nabla_y a)(\varepsilon y), z_{\alpha(s)}\rangle k(s)|\psi|^2\right] \mathrm{d}s\mathrm{d}y,\end{aligned}$$

and the quadratic form $\overline{b}(\psi)$ becomes

$$\begin{aligned}\overline{b}_\varepsilon(\psi) &= \int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon^2}\left|\psi' - \frac{\psi}{2}\frac{\partial \beta_\varepsilon}{\partial s}\frac{1}{\beta_\varepsilon} + \langle \nabla_y \psi, Ry\rangle(\tau+\alpha')(s)\right. \\ &+ \left.\psi \beta_\varepsilon^{1/2}\langle \nabla_y\left(1/\beta_\varepsilon^{1/2}\right), Ry\rangle(\tau+\alpha')(s)\right|^2 \mathrm{d}s\mathrm{d}y \\ &+ \int_{I\times S} \frac{a_\varepsilon}{\varepsilon^2}|\nabla_y \psi|^2 \mathrm{d}s\mathrm{d}y - \int_{I\times S} \frac{1}{4}\frac{a_\varepsilon}{\beta_\varepsilon^2}k^2(s)|\psi|^2\, \mathrm{d}s\mathrm{d}y \\ &- \int_{I\times S} \frac{1}{2\beta_\varepsilon}\langle (\nabla_y a)(\varepsilon y), z_{\alpha(s)}\rangle k(s)|\psi|^2 \mathrm{d}s\mathrm{d}y,\end{aligned}$$

$\mathrm{dom}\, \overline{b}_\varepsilon = H_0^1(I\times S)$.

3 The cross section problem

The purpose of this section is to find a way to control the second integral in the definition of $\overline{b}_\varepsilon(\psi)$ above.

As a strategy, we study the (elliptic) eigenvalue problem

$$-\mathrm{div}(a(\varepsilon y)\nabla_y u) = \lambda u, \qquad u \in H_0^1(S) \cap H^2(S).$$

For each $\varepsilon > 0$, consider the self-adjoint operator $D_\varepsilon u := -\mathrm{div}(a(\varepsilon y)\nabla_y u)$, $\mathrm{dom}\, D_\varepsilon = H_0^1(S) \cap H^2(S)$. Each D_ε has compact resolvent and is bounded from below; so the spectrum $\sigma(D_\varepsilon)$ is purely discrete. Denote by λ_ε^n the n-th eigenvalue of D_ε counted with multiplicity and $u_\varepsilon^n(y)$ the corresponding normalized eigenfunction. Namely,

$$-\mathrm{div}(a(\varepsilon y)\nabla_y u_\varepsilon^n) = \lambda_\varepsilon^n u_\varepsilon^n, \quad 0 \le \lambda_\varepsilon^0 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^n \cdots, \quad \lambda_\varepsilon^n \to \infty \ (n \to \infty). \tag{13}$$

Now, consider the auxiliary eigenvalue problem

$$-\Delta_y u = \lambda u, \qquad u \in H_0^1(S) \cap H^2(S).$$

Recall $-\Delta_y$ is the Dirichlet Laplacian operator restricted to S. Denote by λ_n the n-th eigenvalue of $-\Delta_y$ counted with multiplicity and $u_n(y)$ the corresponding normalized eigenfunction. We have

$$-\Delta_y u_n = \lambda_n u_n, \quad 0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_n \to \infty \ (n \to \infty).$$

The geometry of S ensures that λ_0 is simple.

Since a is a C^2 function, there exists $K > 0$, so that,

$$|a(\varepsilon y) - 1| \le K\,\varepsilon, \quad \forall y \in S, \tag{14}$$

for all $\varepsilon > 0$ small enough.

Theorem 2. *There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\|(D_\varepsilon + \xi\mathbf{1})^{-1} - (D + \xi\mathbf{1})^{-1}\right\| \le K\,\varepsilon,$$

for all $\xi > 0$.

Proof. Consider the quadratic forms

$$d_\varepsilon(u) = \int_S a(\varepsilon y)|\nabla_y u|^2\,\mathrm{d}y, \quad d(u) = \int_S |\nabla_y u|^2\,\mathrm{d}y,$$

$\mathrm{dom}\, d_\varepsilon = \mathrm{dom}\, d = H_0^1(S)$, associated to D_ε and $-\Delta_y$, respectively.

Since D_ε and $-\Delta_y$ are positive operators, given $\xi > 0$, $-\xi \in \rho(D_\varepsilon), \rho(-\Delta_y)$, i.e., ξ belongs to the resolvent set of these operators. Thus, for all $\xi > 0$ and for all $u \in H_0^1(S)$,

$$|(d_\varepsilon + \xi)(u) - (d + \xi)(u)| = |d_\varepsilon(u) - d(u)| \le \int_S |a(\varepsilon y) - 1||\nabla_y u|^2\,\mathrm{d}y \le K\,\varepsilon(d + \xi)(u).$$

Now, the result follows by Theorem 3 in [1]. $\square$

As a consequence of this theorem, for each $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\}$,

(i) $\lambda_\varepsilon^n \to \lambda_n$, as $\varepsilon \to 0$;

(ii) $u_\varepsilon^n \to u_n$, in $L^2(S)$, as $\varepsilon \to 0$.

Since $\lambda_0 > 0$, the condition (i) imply that $\lambda_\varepsilon^0 > 0$ and is a simple eigenvalue, for all $\varepsilon > 0$ small enough.

Further, (i) and (ii) imply that, for each $n \in \mathbb{N}_0$,

(iii) $(\nabla u_\varepsilon^n)_\varepsilon$ is a bounded sequence in $L^2(S)$.

In fact, just note that

$$\int_S a(\varepsilon y)|\nabla u_\varepsilon^n|^2 \, \mathrm{d}y = \lambda_\varepsilon^n. \tag{15}$$

4 Reformulation of the problem

In case of collapsing homogeneous waveguides, it is known that the first eigenvalue diverges as ε^{-2} for small ε [3, 6] (see, e.g., equation (3)); since, by Section 3, the sequence $(\lambda_\varepsilon^0)_\varepsilon$ is bounded, a natural "renormalization" guess is to subtract $(\lambda_\varepsilon^0/\varepsilon^2) \int_{I\times S} |\psi|^2 \, \mathrm{d}s\mathrm{d}y$ from $\overline{b}_\varepsilon(\psi)$. This is a standard procedure to extract a meaningful limit of this very singular problem with the presence of regions that scale in different manners. We also add a constant $c > \| -k^2(s)/4 + V_a(s)\|_\infty$; this fact will be convenient later on to guarantee that zero is in the resolvent set of the related operators. Thus, we pass to study the quadratic form

$$\begin{aligned} b_\varepsilon(\psi) \;&:=\; \overline{b}_\varepsilon(\psi) - \frac{\lambda_\varepsilon^0}{\varepsilon^2}\int_{I\times S} |\psi|^2 \, \mathrm{d}s\mathrm{d}y + c\int_{I\times S} |\psi|^2 \, \mathrm{d}s\mathrm{d}y \\ &= \int_{I\times S} \frac{a_\varepsilon}{\beta_\varepsilon^2}\Big|\psi' - \frac{\psi}{2}\frac{\partial \beta_\varepsilon}{\partial s}\frac{1}{\beta_\varepsilon} + \langle \nabla_y \psi, Ry\rangle(\tau + \alpha')(s) \\ &+ \psi \beta_\varepsilon^{1/2} \langle \nabla_y \left(1/\beta_\varepsilon^{1/2}\right), Ry\rangle(\tau + \alpha')(s)\Big|^2 \, \mathrm{d}s\mathrm{d}y \\ &+ \int_{I\times S} \frac{a_\varepsilon}{\varepsilon^2}\left(|\nabla_y \psi|^2 - \lambda_\varepsilon^0 |\psi|^2\right) \mathrm{d}s\mathrm{d}y - \int_{I\times S} \frac{1}{4}\frac{a_\varepsilon}{\beta_\varepsilon^2} k^2(s)|\psi|^2 \, \mathrm{d}s\mathrm{d}y \\ &- \int_{I\times S} \frac{1}{2\beta_\varepsilon}\langle (\nabla_y a)(\varepsilon y), z_{\alpha(s)}\rangle k(s)|\psi|^2 \mathrm{d}s\mathrm{d}y + c\int_{I\times S} |\psi|^2 \, \mathrm{d}s\mathrm{d}y\,, \end{aligned}$$

dom $b_\varepsilon = H_0^1(I \times S)$. Denote by B_ε the self-adjoint operator associated with it.

To simplify the calculations we perform the following approximation. Consider the quadratic form

$$\begin{aligned} h_\varepsilon(\psi) \;&=\; \int_{I\times S} |\psi' + \langle \nabla_y \psi, Ry\rangle(\tau + \alpha')(s)|^2 \, \mathrm{d}s\mathrm{d}y \\ &+ \int_{I\times S} \frac{a_\varepsilon}{\varepsilon^2}\left(|\nabla_y \psi|^2 - \lambda_\varepsilon^0|\psi|^2\right) \mathrm{d}s\mathrm{d}y - \int_{I\times S} \frac{1}{4} k^2(s)|\psi|^2 \, \mathrm{d}s\mathrm{d}y \\ &- \int_{I\times S} \frac{1}{2}\langle (\nabla_y a)(0), z_{\alpha(s)}\rangle k(s)|\psi|^2 \, \mathrm{d}s\mathrm{d}y + c\int_{I\times S} |\psi|^2 \, \mathrm{d}s\mathrm{d}y \end{aligned}$$

dom $h_\varepsilon = H_0^1(I \times S)$, and denote by H_ε the self-adjoint operator associated with it.

Theorem 3. *There exist $K > 0$ so that*

$$\left\| B_\varepsilon^{-1} - H_\varepsilon^{-1} \right\| \leq K\,\varepsilon,$$

for all $\varepsilon > 0$ small enough.

The choice of c ensures that $0 \in \rho(B_\varepsilon) \cap \rho(H_\varepsilon)$, for all $\varepsilon > 0$ small enough. We have two considerations here. First, condition (14) implies that since a is a C^2 function,

$$\sup_y \|(\nabla_y a)(\varepsilon y) - (\nabla_y a)(0)\|_{\mathbb{R}^2} \leq K\,\varepsilon,$$

for some $K > 0$. Second, $\beta_\varepsilon(s, y) \to 1$ uniformly as $\varepsilon \to 0$, and there exists $K > 0$ so that

$$\left\| \frac{\partial \beta_\varepsilon}{\partial s} \right\|_\infty \leq K\,\varepsilon, \quad \sup_{s,y} \|\nabla_y (1/\beta_\varepsilon^{1/2})\|_{\mathbb{R}^2} \leq K\,\varepsilon,$$

for $\varepsilon > 0$ small enough. Now, the proof of Theorem 3 is quite similar to proof of Theorem 3.1 in [4] and will not be presented here.

5 Reduction of dimension

Recall that u_ε^0 is the eigenfunction associated with the first eigenvalue λ_ε^0 of D_ε. Define the closed subspace $\mathcal{L}_\varepsilon := \{w(s) u_\varepsilon^0(y) : w \in L^2(I)\}$ and consider the orthogonal decomposition

$$L^2(I \times S) = \mathcal{L}_\varepsilon \oplus \mathcal{L}_\varepsilon^\perp.$$

Thus, each $\psi \in L^2(I \times S)$ can be written as

$$\psi(s, y) = w(s) u_\varepsilon^0(y) + \eta_\varepsilon(s, y), \quad w \in L^2(I), \eta_\varepsilon \in \mathcal{L}_\varepsilon^\perp. \tag{16}$$

Consider the positive one-dimensional quadratic form

$$t_\varepsilon(w) := h_\varepsilon(w u_\varepsilon^0), \qquad \mathrm{dom}\; t_\varepsilon = H_0^1(I).$$

Denote by T_ε the self-adjoint operator associated with it. In this case, dom $T_\varepsilon = H^2(I) \cap H_0^1(I)$.

Taking in count that

$$\int_S u_\varepsilon^0 \langle \nabla_y u_\varepsilon^0, Ry \rangle \,\mathrm{d}y = 0,$$

we have

$$t_\varepsilon(w) = \int_I \left\{ |w'|^2 + \left[C_\varepsilon(S)(\tau + \alpha')^2 - \frac{k^2(s)}{4} - \frac{k(s)}{2} \langle (\nabla_y a)(0), z_{\alpha(s)} \rangle + c \right] |w|^2 \right\} \mathrm{d}s,$$

where

$$C_\varepsilon(S) := \int_S |\langle \nabla_y u_\varepsilon^0, Ry \rangle|^2 \,\mathrm{d}y.$$

Theorem 4. *There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\| H_\varepsilon^{-1} - \left(T_\varepsilon^{-1} \oplus \mathbf{0} \right) \right\| \leq K\, \varepsilon,$$

where $\mathbf{0}$ is the null operator on the subspace $\mathcal{L}_\varepsilon^\perp$.

Proof. Due to the decomposition (16), for $\psi \in \text{dom } h_\varepsilon$,

$$\psi(s, y) = w(s)\, u_\varepsilon^0(y) + \eta_\varepsilon(s, y), \quad w \in H_0^1(I), \quad \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^\perp.$$

Thus, $h_\varepsilon(\psi)$ can be rewritten as

$$h_\varepsilon(\psi) = t_\varepsilon(w) + h_\varepsilon(w u_\varepsilon^0, \eta_\varepsilon) + h_\varepsilon(\eta_\varepsilon, w u_\varepsilon^0) + h_\varepsilon(\eta_\varepsilon).$$

We need to check that there are $c_0 > 0$ and functions $0 \leq q(\varepsilon), 0 \leq p(\varepsilon)$ and $c(\varepsilon)$ so that $t_\varepsilon(w)$, $h_\varepsilon(\eta_\varepsilon)$ and $h_\varepsilon(w u_\varepsilon^0, \eta_\varepsilon)$ satisfy the following conditions:

$$t_\varepsilon(w) \geq c(\varepsilon) \| w u_\varepsilon^0 \|^2_{\mathrm{L}^2(I \times S)}, \quad \forall w \in \text{dom } t_\varepsilon, \quad c(\varepsilon) \geq c_0 > 0; \tag{17}$$

$$h_\varepsilon(\eta_\varepsilon) \geq p(\varepsilon) \| \eta_\varepsilon \|^2_{\mathrm{L}^2(I \times S)}, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^\perp; \tag{18}$$

$$|h_\varepsilon(w u_\varepsilon^0, \eta_\varepsilon)|^2 \leq q(\varepsilon)^2\, t_\varepsilon(w)\, h_\varepsilon(\eta_\varepsilon), \quad \forall \psi \in \text{dom } h_\varepsilon; \tag{19}$$

and with

$$p(\varepsilon) \to \infty, \quad c(\varepsilon) = O(p(\varepsilon)), \quad q(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{20}$$

Under these conditions, Proposition 3.1 in [10] guarantees that, for $\varepsilon > 0$ small enough, we have for the operator norm

$$\left\| H_\varepsilon^{-1} - \left(T_\varepsilon^{-1} \oplus \mathbf{0} \right) \right\| \leq p(\varepsilon)^{-1} + K\, q(\varepsilon)\, c(\varepsilon)^{-1},$$

for some $K > 0$.

Clearly, $t_\varepsilon(w) \geq c \|w\|^2_{L^2(I)} = c \| w u_\varepsilon^0 \|_{L^2(I \times S)}$, for all $w \in \text{dom } t_\varepsilon$. Take $c(\varepsilon) := c$. By the Min-Max Principle,

$$\int_S \left(a_\varepsilon |\nabla_y \eta_\varepsilon|^2 - \lambda_\varepsilon^0 |\eta_\varepsilon|^2 \right) \mathrm{d}y \geq (\lambda_\varepsilon^1 - \lambda_\varepsilon^0) \int_S |\eta_\varepsilon|^2 \, \mathrm{d}y, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^\perp,$$

for a.e. s. Since $\lambda_\varepsilon^1 - \lambda_\varepsilon^0 \to \lambda_1 - \lambda_0 > 0$, as $\varepsilon \to 0$, there exists $K > 0$, so that,

$$\int_S \left(a_\varepsilon |\nabla_y \eta_\varepsilon|^2 - \lambda_\varepsilon^0 |\eta_\varepsilon|^2 \right) \mathrm{d}y \geq K \int_S |\eta_\varepsilon|^2 \, \mathrm{d}y, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^\perp,$$

for all $\varepsilon > 0$ small enough and a.e. s. Thus,

$$h_\varepsilon(\eta_\varepsilon) \geq \frac{K}{\varepsilon^2} \int_{I \times S} |\eta_\varepsilon|^2 \mathrm{d}s \mathrm{d}y, \qquad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^\perp, \tag{21}$$

for all $\varepsilon > 0$ small enough. Just take $p(\varepsilon) := K/\varepsilon^2$.

Since $\eta_\varepsilon \in H_0^1(I \times S) \cap \mathcal{L}_\varepsilon^\perp$,

$$\int_S \eta_\varepsilon(s, y) u_\varepsilon^n(y) \, \mathrm{d}y = 0, \qquad \text{a.e. s},$$

and, consequently,

$$\int_S \eta'_\varepsilon(s,y) u^n_\varepsilon(y)\,\mathrm{d}y = 0, \qquad \text{a.e. s.}$$

These conditions and a integration by parts show that

$$\begin{aligned} & h_\varepsilon(wu^0_\varepsilon, \eta_\varepsilon) \\ = & \int_{I\times S} \left(w' u^0_\varepsilon + w\langle \nabla_y u^0_\varepsilon, Ry\rangle(\tau+\alpha')(s)\right)\left(\eta'_\varepsilon + \langle \nabla_y \eta_\varepsilon, Ry\rangle(\tau+\alpha')(s)\right)\,\mathrm{d}s\mathrm{d}y \\ = & \int_{I\times S} w' u^0_\varepsilon \langle \nabla_y \eta_\varepsilon, Ry\rangle(\tau+\alpha')(s)\,\mathrm{d}s\mathrm{d}y \\ - & \int_{I\times S} (w'\,(\tau+\alpha')(s) + w\,(\tau+\alpha')'(s))\eta_\varepsilon \langle \nabla_y u^0_\varepsilon, Ry\rangle\,\mathrm{d}s\mathrm{d}y \\ + & \int_{I\times S} w\langle \nabla u^0_\varepsilon, Ry\rangle\langle \nabla \eta_\varepsilon, Ry\rangle(\tau+\alpha')^2(s)\,\mathrm{d}s\mathrm{d}y. \end{aligned}$$

Now, note that there exists $K>0$ so that,

$$\begin{aligned} \int_{I\times S} |\nabla \eta_\varepsilon|^2\,\mathrm{d}s\mathrm{d}y & \leq \sup_{y\in S}\left\{\frac{1}{a(\varepsilon y)}\right\}\int_{I\times S} a(\varepsilon y)|\nabla\eta_\varepsilon|^2\,\mathrm{d}s\mathrm{d}y \\ & = \sup_{y\in S}\left\{\frac{1}{a(\varepsilon y)}\right\}\Big[\int_{I\times S}\left(a(\varepsilon y)|\nabla\eta_\varepsilon|^2 - \lambda^0_\varepsilon|\eta_\varepsilon|^2\right)\,\mathrm{d}s\mathrm{d}y \\ & \quad + \lambda^0_\varepsilon \int_{I\times S} |\eta_\varepsilon|^2\,\mathrm{d}s\mathrm{d}y\Big] \\ & \leq K\,\varepsilon^2\,h_\varepsilon(\eta_\varepsilon), \end{aligned}$$

for all $\eta_\varepsilon \in \mathrm{dom}\ h_\varepsilon \cap \mathcal{L}^\perp_\varepsilon$, for all $\varepsilon > 0$ small enough. We also have $\int_I |w'|^2 \mathrm{d}s \leq t_\varepsilon(w)$, $\int_I |w|^2 \leq K t_\varepsilon(w)$, for all $w \in \mathrm{dom}\ t_\varepsilon$ and all $\varepsilon>0$ small enough.

These inequalities, the conditions (i), (ii) and (iii) in Section 3 and (21), imply that

$$|h_\varepsilon(wu^0_\varepsilon, \eta_\varepsilon)| \leq K\varepsilon (t_\varepsilon(w))^{1/2}(h_\varepsilon(\eta_\varepsilon))^{1/2},$$

for some $K>0$, for all $w\in \mathrm{dom}\ t_\varepsilon$, all $\eta_\varepsilon \in \mathrm{dom}\ h_\varepsilon \cap \mathcal{L}^\perp_\varepsilon$ and all $\varepsilon > 0$ small enough. Take $q(\varepsilon) := K\,\varepsilon$.

Since (17), (18), (19) and (20) are satisfied, the result follows. □

6 Effective operator

In this section we show the convergence of the sequence T_ε in the norm resolvent sense as $\varepsilon \to 0$; the effective operator is found.

Define the quadratic form

$$t(w) = \int_I \left\{ |w'|^2 + \left[C(S)(\tau+\alpha')^2(s) - \frac{k^2(s)}{4} - \frac{k(s)}{2}\langle(\nabla_y a)(0), z_{\alpha(s)}\rangle + c\right]|w|^2\right\}\mathrm{d}s,$$

dom $t = H^1_0(I)$, and denote by T its associated self-adjoint operator. Namely,

$$(Tw)(s) = -w''(s) + \left[C(S)(\tau+\alpha')^2(s) - \frac{k^2(s)}{4} - \frac{k(s)}{2}\langle(\nabla_y a)(0), z_{\alpha(s)}\rangle + c\right]w(s),$$

dom $T = H^2(I) \cap H_0^1(I)$.

Note that if $(\tau + \alpha') = 0$, for all $s \in I$, then we have $T_\varepsilon = T$, for all $\varepsilon > 0$, and the conclusion of Theorem 1 follows. For the general case we shall make use of the following results:

Lemma 1. *The sequence $(C_\varepsilon(S))_\varepsilon$ converges to $C(S)$, as $\varepsilon \to 0$.*

Proof. Since $a(\varepsilon y) \to 1$ uniformly, by (i) and (15) in Section 3, we have $\int_S |\nabla_y u_\varepsilon^0|^2 \, dy \to \lambda_0$. Also in Section 3, the condition (iii) implies that any subsequence of $(\nabla_y u_\varepsilon^0)_\varepsilon$ has a weakly converging subsequence $(\nabla_y u_{\varepsilon'}^0)_{\varepsilon'}$ with $\nabla_y u_{\varepsilon'}^0 \rightharpoonup \nabla_y u_0$ in $L^2(S)$. Now we introduce a seminorm $[\cdot]$ on $L^2(S)$ as

$$[\cdot] := \left(\int_S |\langle \cdot, Ry \rangle|^2 \, dy \right)^{\frac{1}{2}},$$

and note that $C_\varepsilon(S)^{\frac{1}{2}} = [\nabla_y u_\varepsilon^0]$ and $C(S)^{\frac{1}{2}} = [\nabla_y u_0]$. Thus, by combining the just mentioned facts, we have

$$\begin{aligned}
\left| C_{\varepsilon'}(S)^{\frac{1}{2}} - C(S)^{\frac{1}{2}} \right|^2 &= \big| [\nabla_y u_{\varepsilon'}^0] - [\nabla_y u_0] \big|^2 \le [\nabla_y u_{\varepsilon'}^0 - \nabla_y u_0]^2 \\
&= \int_S |\langle \nabla_y u_{\varepsilon'}^0 - \nabla_y u_0, Ry \rangle|^2 \, dy \le D_S \int_S |\nabla_y u_{\varepsilon'}^0 - \nabla_y u_0|^2 \, dy \\
&= D_S \int_S \left(|\nabla_y u_{\varepsilon'}^0|^2 + |\nabla_y u_0|^2 - 2\langle \nabla_y u_{\varepsilon'}^0, \nabla_y u_0 \rangle \right) dy \\
&\xrightarrow{\varepsilon \downarrow 0} D_S \left(\lambda_0 + \lambda_0 - 2\lambda_0 \right) = 0,
\end{aligned}$$

where D_S is obtained from $|Ry| = |y|$ and the boundedness of S. Hence, $C_{\varepsilon'}(S) \to C(S)$, as $\varepsilon' \to 0$.

By repeating the above arguments, one sees that any subsequence of $(C_\epsilon(S))_\epsilon$ has a converging subsequence with the uniquely determined limit $C(S)$, and so Urysohn's subsequence principle implies that $C_\epsilon(S) \to C(S)$ as $\epsilon \to 0$. □

Proposition 1. *There exist $K > 0$ and a function $s(\varepsilon) > 0$, so that, $s(\varepsilon) \to 0$, as $\varepsilon \to 0$, and*

$$\left\| T_\varepsilon^{-1} - T^{-1} \right\| \le K \, s(\varepsilon),$$

for all $\varepsilon > 0$ small enough.

Proof. Lemma 1 ensures that there exists $\tilde{s}(\varepsilon) > 0$, so that, $\tilde{s}(\varepsilon) \to 0$, as $\varepsilon \to 0$, and

$$|C_\varepsilon(S) - C(S)| \le \tilde{s}(\varepsilon),$$

for all $\varepsilon > 0$ small enough. Thus, since $(\tau + \alpha') \in L^\infty(I)$, there exists $K > 0$, so that,

$$|t_\varepsilon(w) - t(w)| \le K \int_I |C_\varepsilon(S) - C(S)| \, |w|^2 ds \le K \, \tilde{s}(\varepsilon) \int_I |w|^2 ds \le K \, \tilde{s}(\varepsilon) \, t(w).$$

Take $s(\varepsilon) := (K/c)\tilde{s}(\varepsilon)$. The results follows by Theorem 3 of [1]. □

The proof of Theorem 1 follows by combining Theorem 4 and Proposition 1.

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