Norm resolvent approximation of thin homogeneous tubes by heterogeneous ones

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Abstract

We study the operator $-\text{div}(A(x)\nabla\psi(x))$ restricted to a waveguide $\Omega_\varepsilon \subset \mathbb{R}^3$, with heterogeneous function $A(x)$ constant in the longitudinal direction. The purpose is to obtain an effective operator, in the norm resolvent sense, when the diameter of $\Omega_\varepsilon$ tends to zero with heterogeneity approaching a homogeneous situation (i.e., a constant function $A$). The effective operator presents a potential that, besides the traditional dependence on waveguide geometric properties, there is also a contribution from $A(x)$ which results, when combined with the curvature, for example, in the possibility of a repulsive interaction.

1 Introduction

Let $I = (a, b)$ be an open interval of $\mathbb{R}$; the cases $a = -\infty$ and/or $b = +\infty$ are included. Let $r : I \to \mathbb{R}^3$ be a curve of class $C^3$ parametrized by its arc-length parameter $s$. Denote by $k(s)$ and $\tau(s)$ its curvature and torsion at the point $r(s)$, respectively. Pick $S$ an open, bounded and connected subset of $\mathbb{R}^2$ with “transverse” coordinates $y = (y_1, y_2)$; we require that $0 \in S$. We build a tube (waveguide) in $\mathbb{R}^3$ by properly moving the region $S$ along $r(s)$; at each point $r(s)$ the cross-section region $S$ may present a (continuously differentiable) rotation angle $\alpha(s)$ (see details in Section 2). We demand that the functions $k(s), (\tau + \alpha'(s)), (\tau + \alpha')' \in L^\infty(I)$.

Given $\varepsilon > 0$, denote by $\Omega_\varepsilon$ the waveguide generated by the cross-section $\varepsilon S$. Let $A : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The purpose of this paper is study the (Dirichlet) operator

$$\psi(x) \mapsto -\text{div} (A(x)\nabla\psi(x)), \quad \psi \in H^2(\Omega_\varepsilon) \cap H^1_U(\Omega_\varepsilon),$$

and to find an effective operator in the limit $\varepsilon \to 0$ for particular choices of the function $A$; the latter characterizes the heterogeneity of the waveguide. Here, $\nabla\psi$ denotes the gradient vector of $\psi$ in $\Omega_\varepsilon$. Since the sequence of tubes $\Omega_\varepsilon$ “approaches” the curve $r(s)$, as $\varepsilon \to 0$, it is expected that the effective operator could be identified with a one-dimensional operator in $L^2(I)$.
The most studied situation is of a homogeneous tube, which may be characterized by a constant function \( A(x) = 1 \); ahead we review some relevant results in this situation. The case of strongly heterogeneous tubes has also been considered, that is, \( A(x) = \tilde{a}(y/\varepsilon) \), where \( \tilde{a} : \mathbb{R}^2 \to \mathbb{R} \) is a periodic function and \( y = (y_1, y_2) \) are the variables in the transverse section \( S \). More specifically, in \([8]\) the authors analyze the interaction between thickness \( \varepsilon \) and axial \( \delta \)-periodic heterogeneities as both \( \varepsilon \) and \( \delta \) tend to zero, for bounded \( A \) depending only on one of the transverse variables and in the three possible regimes: \( \varepsilon \ll \delta, \varepsilon \gg \delta \) and \( \varepsilon \approx \delta \) (such work is a counterpart of \([7]\) that has considered a 3D-2D reduction). See also \([12]\) for spectral analysis in related thin heterogeneous tubes, for bounded and uniformly coercive \( A \) depending now on the two transverse coordinates; by applying the \( \Gamma \)-convergence technique, it was found that the results depend critically on the choice of the (above) regime considered. In \([2, 14]\) the homogenization, also under coerciveness of \( A \) and other conditions, is studied in any dimension and different techniques. Other related works can be traced through the cited references.

However, here we are interested in a simpler question, that is, whether there is any nontrivial contribution when one approaches thin homogeneous tubes from heterogeneous ones; more specifically, we roughly consider a nonconstant function \( A(\varepsilon y) \) of the transverse coordinates and with differentiable coefficients (see precise details ahead), so that as \( \varepsilon \to 0 \) the homogeneous case is recovered due to the formal approximation \( A(\varepsilon y) \approx A(0) = 1 \). In fact, as discussed ahead, this process presents a contribution to effective operators and, when combined with curvature, repulsive and attractive potentials may arise.

Now we comment on the homogeneous case \( A(x) = 1 \), i.e.,

\[ \psi(x) \mapsto -\Delta \psi(x), \quad \psi \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon), \tag{2} \]

where \( \Delta \) denote the Laplacian operator in \( \Omega_\varepsilon \). This situation was studied in several works \([3, 4, 5, 6, 9, 10]\). Note that, as \( \varepsilon \to 0 \), the region \( \Omega_\varepsilon \) becomes narrower and the whole spectrum of \( -\Delta \) diverges. To control this kind of divergence the following strategy is usual. Let \( \lambda_0 \) be the first (i.e., the lowest) eigenvalue of the Dirichlet Laplacian \( -\Delta_y \) restricted to \( S \) and \( u_0 \) the (positive) associated normalized eigenfunction, that is,

\[ -\Delta_y u_0 = \lambda_0 u_0, \quad u_0 \in H^1_0(S), \quad \lambda_0 > 0, \quad \int_S |u_0|^2 \, dy = 1. \]

In order to obtain a meaningful spectral convergence, one needs to subtract \((\lambda_0/\varepsilon^2)1\) from \( (2) \); \( 1 \) denotes the identity operator. Thus, the operator turns out to be

\[ \psi(x) \mapsto - (\Delta \psi)(x) - \frac{\lambda_0}{\varepsilon^2} \psi(x), \quad \psi \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon). \tag{3} \]

Put

\[ V(s) := C(S)(\tau + \alpha')^2(s) - \frac{k^2(s)}{4}, \quad s \in I, \tag{4} \]

where \( R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( C(S) := \int_S |(\nabla_y u_0, Ry)|^2 \, dy \). Consider the one-dimensional self-adjoint operator

\[ (Tw)(s) = -w''(s) + V(s)w(s), \quad w \in H^2(I) \cap H^1_0(I). \]
Let $\mathcal{L} := \{w(s)u_0(y) : w \in L^2(I)\}$ be a closed subspace of $L^2(I \times S)$ and denote by 0 the null operator on its orthogonal complement $\mathcal{L}^\perp$. One has

$$-\Delta - \frac{\lambda_0}{\varepsilon^2} 1 \approx T \oplus 0, \quad \varepsilon \to 0,$$

i.e., $T$ is the effective operator in this situation. The approximation (5) was proved in different aspects. In particular, for bounded tubes, in [3] the authors have used the $\Gamma$-convergence technique to obtain $T$ as effective operator and to find an asymptotic behavior of the eigenvalues of $-\Delta$. In [6], where unbounded tubes were considered, the approximation in (5) was proven in the norm resolvent sense; the technique used there comes from [10].

Our goal in this work is to obtain an approximation in the norm resolvent sense for the operator (1) under suitable conditions on nonconstant $A(x)$. Namely, we suppose that there exists a coercive and $C^2$ function $a : S \to \mathbb{R}$, that is, $a(y) \geq r > 0$, for all $y$, so that the resulting operator is elliptic and

$$A_\varepsilon(s, y) := (A \circ f_\varepsilon)(s, y) = a(\varepsilon y), \quad \forall (s, y) \in I \times S,$$

where $f_\varepsilon$ is the natural diffeomorphism given by (10) in Section 2. This condition implies that the heterogeneity $A \circ f_\varepsilon$ is constant in the longitudinal direction and, in particular, without loss it is assumed that $A(r(s)) = a(0) = 1$, for all $s \in I$. We will see that the effective operator in this situation presents an additional potential which comes from a derivative of $A$, i.e., the variation of the heterogeneity at the reference curve.

Given $\varepsilon > 0$, consider the following eigenvalue equation in the cross-section $S$:

$$-\text{div} (a(\varepsilon y) \nabla_y u) = \lambda u, \quad u \in H^2(S) \cap H^1_0(S).$$

Denote by $\lambda_0^\varepsilon$ the first eigenvalue of the operator and by $u_0^\varepsilon$ the corresponding eigenfunction. In particular, $\lambda_0^\varepsilon \to \lambda_0$ and $u_0^\varepsilon \to u_0$ in $L^2(S)$, as $\varepsilon \to 0$. As a consequence, for all $\varepsilon > 0$ small enough, $\lambda_0^\varepsilon > 0$ and it is a simple eigenvalue; see Section 3 for more details.

We then pass to study the sequence of operators

$$\psi(x) \mapsto -\text{div}(A(x) \nabla \psi(x)) - \frac{\lambda^\varepsilon_0}{\varepsilon^2} \psi(x), \quad \psi \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon).$$

Let

$$z_\alpha(s) := (\cos \alpha(s), -\sin \alpha(s)), \quad V_\alpha(s) := -\frac{k(s)}{2} \langle (\nabla_y a)(0), z_\alpha(s) \rangle,$$

where $\nabla_y$ denotes the gradient vector in $S$.

Define the one-dimensional operator

$$(Tw)(s) := -w''(s) + V(s)w(s) + V_\alpha(s)w(s), \quad \text{dom } T = H^2(I) \cap H^1_0(I).$$

Let $U_\varepsilon$ and $V_\varepsilon$ be the unitary operators given, respectively, by (11) and (12) in Section 2. The main result of this work is
Theorem 1. As $\varepsilon \to 0$, one has
\[
\left\|V_\varepsilon U_\varepsilon \left(\begin{array}{c}
\frac{\partial}{\partial x} - \frac{\lambda_0}{\varepsilon^2} I + \frac{1}{\varepsilon} \frac{\partial}{\partial y_1} \nabla \end{array}\right)^{-1}
\right\| - \left[ T^{-1} \oplus 0 \right] \to 0,
\] (7)
where $0$ is the null operator on the subspace $L^\perp$.

Remark 1. If $(\alpha + \alpha') = 0$, for all $s \in I$, (7) follows directly by Theorem 4 in Section 5. Otherwise, we need an additional step; see Section 3 and Lemma 1 in Section 6.

In the homogeneous case $A(x) = 1$ for all $x$, it is known (and we see from (4)) that the bending property (i.e., nonzero curvature) generates an attractive interaction $-k^2/4$ whereas the twisting property (i.e., $\tau + \alpha' \neq 0$) a repulsive one [13]. However, we see that the approximation by the heterogeneous case (1) may have important consequences. For example, suppose $\Omega_\varepsilon$ is a planar bent and nontwisted tube with $k \neq 0$ and $\tau = \alpha = 0$. In this case the effective operator is
\[
(Tw)(s) = -w''(s) - \left( \frac{k^2(s)}{4} + \frac{\partial a}{\partial y_1}(0) \frac{k(s)}{2} \right) w(s).
\]
Suppose the local variation $(\partial a/\partial y_1)(0) < 0$; if either the curvature $k(s)$ is small enough or the absolute value of such variation is large, then $A(x)$ combined with curvature gives a net repulsive contribution as $\varepsilon \to 0$.

This text is organized as follows. In Section 2 we discuss details of the standard tube $\Omega_\varepsilon$ construction and perform the change of variables necessary to study the operator (1); for example, straightening $\Omega_\varepsilon$. In Section 3 we study the operator restricted to the cross-section $S$ in order to control the divergent energies from the transverse spectrum of $-\Delta$, as $\varepsilon \to 0$. In Sections 4 and 5 we present some preliminary results to obtain Theorem 1. Finally, in Section 6, we show how to find the effective operator $T$ and conclude Theorem 1. Along the text the symbol $K$ is used to denote different constants.

2 Geometry of the domain and change of coordinates

As already mentioned in the Introduction, let $I = (a, b)$, $-\infty \leq a < b \leq +\infty$, be an interval of $\mathbb{R}$. Let $r : I \to \mathbb{R}^3$ be a simple $C^3$ curve in $\mathbb{R}^3$ parametrized by its arc-length parameter $s$. The curvature of $r$ at the position $s$ is $k(s) := \|r''(s)\|$. We choose the usual orthonormal triad of vector fields $\{T(s), N(s), B(s)\}$, the so-called Frenet frame, given the tangent, normal and binormal vectors, respectively, moving along the curve and defined by
\[
T = r'; \quad N = k^{-1} T'; \quad B = T \times N.
\] (8)
To justify the construction (8), it is assumed that $k \neq 0$, but if $r$ has a piece of a straight line (i.e., $k = 0$ identically in this piece), usually one can choose a constant
Frenet frame instead. It is possible to combine constant frames with the Frenet frame (8) and so obtaining a global $C^2$ Frenet frame; see [11], Theorem 1.3.6. In each situation we assume that a global Frenet frame exists and that the Frenet equations are satisfied, that is,

$$
\begin{pmatrix}
T' \\
N' \\
B'
\end{pmatrix} =
\begin{pmatrix}
0 & k & 0 \\
-k & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B
\end{pmatrix},
$$

(9)

where $\tau(s)$ is the torsion of $r(s)$, actually defined by (9). Choose $s_0 \in I$ and consider $\alpha : I \to \mathbb{R}$ a $C^1$ function so that $\alpha(s_0) = 0$. We suppose that $\tau + \alpha'$ is derivable and $(\tau + \alpha')' \in L^\infty(I)$. Let $S$ be an open, bounded, connected and smooth (nonempty) subset of $\mathbb{R}^2$. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$
\tilde{x}(s, y) = r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s)
$$

and consider the domain

$$
\Omega_\varepsilon = \{ \tilde{x}(s, y) \in \mathbb{R}^3 : s \in I, y = (y_1, y_2) \in S \},
$$

where

$$
N_\alpha(s) := \cos \alpha(s) N(s) + \sin \alpha(s) B(s),
$$

$$
B_\alpha(s) := -\sin \alpha(s) N(s) + \cos \alpha(s) B(s).
$$

Hence, this tube $\Omega_\varepsilon$ is obtained by putting the region $\varepsilon S$ along the curve $r(s)$, which is simultaneously rotated by an angle $\alpha(s)$ with respect to the cross section at the position $r(s)$.

Now, consider the sequence of operators defined by (1) in Introduction. Our technique corresponds to the study of the sequence of the corresponding quadratic forms. Namely,

$$
\tilde{b}_\varepsilon(\tilde{\psi}) := \int_{\Omega_\varepsilon} A(x) |\nabla \tilde{\psi}|^2 dx, \quad \text{dom } \tilde{b}_\varepsilon = H_0^1(\Omega_\varepsilon).
$$

We perform a change of variables so that the region $\Omega_\varepsilon$ turns to be a straight cylinder $I \times S$. For this, consider the mapping

$$
f_\varepsilon : I \times S \to \Omega_\varepsilon
$$

$$
(s, y) \mapsto r(s) + \varepsilon y_1 N_\alpha(s) + \varepsilon y_2 B_\alpha(s).
$$

(10)

The price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon$ which is induced by $f_\varepsilon$, i.e.,

$$
G = (G_{ij}), \quad G_{ij} = \langle e_i, e_j \rangle = G_{ji}, \quad 1 \leq i, j \leq 3,
$$

where

$$
e_1 = \frac{\partial f_\varepsilon}{\partial s}, \quad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \quad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.
$$

Some calculations show that in the Frenet frame

$$
J = \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = \begin{pmatrix}
\beta_\varepsilon & -\varepsilon(\tau + \alpha') \langle z_\alpha^1, y \rangle & \varepsilon(\tau + \alpha') \langle z_\alpha, y \rangle \\
0 & \varepsilon \cos \alpha & \varepsilon \sin \alpha \\
0 & -\varepsilon \sin \alpha & \varepsilon \cos \alpha
\end{pmatrix},
$$

5
where
\[ \beta_\varepsilon(s, y) = 1 - \varepsilon k(s) (z_\alpha, y), \quad z_\alpha = (\cos \alpha, -\sin \alpha), \quad \text{and} \quad z_\perp^\alpha = (\sin \alpha, \cos \alpha). \]

The inverse matrix of \( J \) is given by
\[
J^{-1} = \begin{pmatrix}
\frac{1}{\beta_\varepsilon} (\tau + \alpha') y_2 / \beta_\varepsilon & - (\tau + \alpha') y_1 / \beta_\varepsilon \\
0 & (1/\varepsilon) \cos \alpha \\
0 & (1/\varepsilon) \sin \alpha 
\end{pmatrix}.
\]

Note that \( JJ' = G \) and \( \det J = |\det G|^{1/2} = \varepsilon^2 \beta_\varepsilon. \) Since \( k \) is a bounded function, for \( \varepsilon \) small enough \( f_\varepsilon \) does not vanish in \( I \times S \). Thus, \( \beta_\varepsilon > 0 \) and \( f_\varepsilon \) is a local diffeomorphism. In case \( f_\varepsilon \) is injective (again by requiring that \( \varepsilon > 0 \) is small), a global diffeomorphism is obtained.

Introducing the notation
\[
\| \hat{\psi} \|^2_G := \int_{I \times S} |\hat{\psi}|^2 \beta_\varepsilon(s, y) ds dy,
\]
and the unitary transformation
\[
U_\varepsilon : L^2(\Omega_\varepsilon) \rightarrow L^2(I \times S, \beta_\varepsilon(s, y) ds dy) \quad \mapsto \quad \varepsilon \hat{\psi} \circ f_\varepsilon,
\]
we obtain the quadratic form
\[
\hat{b}_\varepsilon(U_\varepsilon \hat{\psi}) = \| \sqrt{a_\varepsilon(\varepsilon y)} J^{-1} \nabla (U_\varepsilon \hat{\psi}) \|^2_G, \quad \text{dom } \hat{b}_\varepsilon = H^1_0(I \times S).
\]

Recall the condition (6) in Introduction; there exists a positive and \( C^2 \) function \( a : S \rightarrow \mathbb{R} \) so that \((A \circ f_\varepsilon)(s, y) = a(\varepsilon y)\), for all \((s, y) \in I \times S\).

We denote \( \hat{\psi} := U_\varepsilon \tilde{\psi} \) and, to simplify the notation, sometimes we write \( a_\varepsilon(y) := a(\varepsilon y) \). Thus,
\[
\hat{b}_\varepsilon(\hat{\psi}) = \| \sqrt{a_\varepsilon(\varepsilon y)} J^{-1} \nabla \hat{\psi} \|^2_G
= \int_{I \times S} \frac{a_\varepsilon}{\beta_\varepsilon} \hat{\psi}' + \langle \nabla_y \hat{\psi}, R_y \rangle (\tau + \alpha)' \|^2 ds dy
+ \int_{I \times S} \frac{a_\varepsilon \beta_\varepsilon}{\varepsilon^2} |\nabla_y \hat{\psi}|^2 ds dy,
\]
\[ \text{dom } \hat{b}_\varepsilon = H^1_0(I \times S). \]

Recall that \( R \) is the rotation matrix that appears after (4). Note that \( \text{dom } \hat{b}_\varepsilon \) is a subspace of the Hilbert space \( L^2(I \times S, \beta_\varepsilon ds dy) \). The measure \( \beta_\varepsilon ds dy \) comes from of the Riemannian metric obtained from the change of variables \( f_\varepsilon \).

Now, we consider a change of variables so that we can work in the Hilbert space \( L^2(I \times S) \) with the usual measure. Define the isometry
\[
V_\varepsilon : L^2(I \times S, \beta_\varepsilon(s, y) ds dy) \rightarrow L^2(I \times S) \quad \mapsto \quad \beta_\varepsilon^{1/2} \hat{\psi},
\]
\[ \text{dom } V_\varepsilon = \beta_\varepsilon^{1/2} H^1_0(I \times S), \]
\[ \text{dom } V_\varepsilon = H^1_0(I \times S). \]
and denote $\psi := V_2 \hat{\psi}$. With this change, the quadratic form $\tilde{b}_e(\hat{\psi})$ becomes

$$\tilde{b}_e(\psi) = \int_{I \times S} \frac{a_e}{\beta_e} \left| \psi' - \frac{\psi \partial \beta_e}{2 \partial S} \frac{1}{\beta_e} + \left( \langle \nabla_y \psi, R_y \rangle + \psi \beta_e^{1/2} \langle \nabla_y \left( 1/\beta_e^{1/2} \right), R_y \rangle \right) (\tau + \alpha')(s) \right|^2 \, ds \, dy + \int_{I \times S} \frac{a_e \beta_e}{\varepsilon^2} \left| \nabla_y \left( \psi / \beta_e^{1/2} \right) \right|^2 \, ds \, dy,$$

dom $\tilde{b}_e = H^1_0(I \times S)$.

We draw attention to the last integral in the expression of $\tilde{b}_e(\psi)$. For this term we present more details of the change of variables (12).

Some calculations show that

$$\int_{I \times S} \frac{a_e \beta_e}{\beta_y} \left( \frac{\partial}{\partial y_1} \left( \psi \beta_e^{-1/2} \right) \right)^2 \, ds \, dy = \int_{I \times S} \frac{a_e \beta_e}{\beta_y} \left( \frac{\partial \psi}{\partial y_1} \beta_e^{-1/2} - \frac{\partial \beta_e}{\partial y_1} \frac{\psi}{\beta_e^{3/2}} \right)^2 \, ds \, dy$$

$$= \int_{I \times S} \frac{a_e \beta_e}{\beta_y} \left[ \left( \frac{\partial \psi}{\partial y_1} \right)^2 \beta_e^{-1} - \beta_e^{-2} \frac{\partial \beta_e}{\partial y_1} \frac{\partial \psi}{\partial y_1} + \beta_e^{-3} \left( \frac{\partial \beta_e}{\partial y_1} \right)^2 \psi^2 \right] \, ds \, dy$$

$$= \int_{I \times S} \left[ a_e \left( \frac{\partial \psi}{\partial y_1} \right)^2 - \frac{1}{4} a_e \beta_e^{-2} \varepsilon^2 k^2(s) \cos^2 \alpha(s) \psi^2 - \frac{\varepsilon^2}{2} \beta_e^{-1} \frac{\partial a}{\partial y_1} (\varepsilon y) k(s) \cos \alpha(s) \psi^2 \right] \, ds \, dy.$$

In these passages, we have performed an integration by parts to obtain

$$- \int_{I \times S} \frac{a_e \beta_e}{\beta_y} \frac{\partial \psi}{\partial y_1} \psi \, ds \, dy = - \frac{1}{2} \int_{I \times S} \frac{a_e \beta_e}{\beta_y} \frac{\partial \psi}{\partial y_1} \psi \, ds \, dy$$

$$= - \int_{I \times S} \left( \frac{\varepsilon^2}{2} \beta_e^{-1} \frac{\partial a}{\partial y_1} (\varepsilon y) k(s) \cos \alpha(s) \psi^2 + a_e \beta_e^{-2} \varepsilon^2 k^2(s) \cos^2 \alpha(s) \psi^2 \right) \, ds \, dy.$$

Similarly, one can show

$$\int_{I \times S} \frac{a_e \beta_e}{\beta_y} \left( \frac{\partial}{\partial y_2} \left( \psi \beta_e^{-1/2} \right) \right)^2 \, ds \, dy$$

$$= \int_{I \times S} \left[ a_e \left( \frac{\partial \psi}{\partial y_2} \right)^2 - \frac{1}{4} a_e \beta_e^{-2} \varepsilon^2 k^2(s) \sin^2 \alpha(s) \psi^2 + \frac{\varepsilon^2}{2} \beta_e^{-1} \frac{\partial a}{\partial y_2} (\varepsilon y) k(s) \sin \alpha(s) \psi^2 \right] \, ds \, dy.$$

Thus,

$$\int_{I \times S} \frac{a_e \beta_e}{\varepsilon^2} \left| \nabla_y \left( \psi / \beta_e^{1/2} \right) \right|^2 \, ds \, dy$$

$$= \int_{I \times S} \left[ \frac{a_e}{\varepsilon^2} \nabla_y \psi^2 - \frac{\beta_e^{-2} k^2(s) |\psi|^2}{4} - \frac{\beta_e^{-1}}{2} \left( \langle \nabla_y a(\varepsilon y), z_{\alpha(s)}(k(s)|\psi|^2 \rangle \right) \, ds \, dy,$$

and the quadratic form $\tilde{b}(\psi)$ becomes

$$\tilde{b}_e(\psi) = \int_{I \times S} \frac{a_e}{\beta_e} \left| \psi' - \frac{\psi \partial \beta_e}{2 \partial S} \frac{1}{\beta_e} + \left( \langle \nabla_y \psi, R_y \rangle + \psi \beta_e^{1/2} \langle \nabla_y \left( 1/\beta_e^{1/2} \right), R_y \rangle \right) (\tau + \alpha')(s) \right|^2 \, ds \, dy + \int_{I \times S} \frac{a_e}{\beta_e} \left| \nabla_y \psi^2 - \frac{\beta_e^{-2} k^2(s) |\psi|^2}{4} - \frac{\beta_e^{-1}}{2} \left( \langle \nabla_y a(\varepsilon y), z_{\alpha(s)}(k(s)|\psi|^2 \rangle \right) \, ds \, dy,$$

dom $\tilde{b}_e = H^1_0(I \times S)$. 

7
The cross section problem

The purpose of this section is to find a way to control the second integral in the definition of \( \bar{b}_\varepsilon(\psi) \) above.

As a strategy, we study the (elliptic) eigenvalue problem

\[
-\text{div}(a(\varepsilon y)\nabla y u) = \lambda u, \quad u \in H^1_0(S) \cap H^2(S).
\]

For each \( \varepsilon > 0 \), consider the self-adjoint operator \( D_\varepsilon u := -\text{div}(a(\varepsilon y)\nabla y u) \), \( \text{dom} D_\varepsilon = H^1_0(S) \cap H^2(S) \). Each \( D_\varepsilon \) has compact resolvent and is bounded from below; so the spectrum \( \sigma(D_\varepsilon) \) is purely discrete. Denote by \( \lambda^n_\varepsilon \) the \( n \)-th eigenvalue of \( D_\varepsilon \) counted with multiplicity and \( u^n_\varepsilon(y) \) the corresponding normalized eigenfunction. Namely,

\[ -\text{div}(a(\varepsilon y)\nabla y u^n_\varepsilon) = \lambda^n_\varepsilon u^n_\varepsilon, \quad 0 \leq \lambda^0_\varepsilon \leq \lambda^1_\varepsilon \leq \lambda^n_\varepsilon, \quad \lambda^n_\varepsilon \to \infty \ (n \to \infty). \tag{13} \]

Now, consider the auxiliary eigenvalue problem

\[ -\Delta_y u = \lambda u, \quad u \in H^1_0(S) \cap H^2(S). \]

Recall \( -\Delta_y \) is the Dirichlet Laplacian operator restricted to \( S \). Denote by \( \lambda_n \) the \( n \)-th eigenvalue of \( -\Delta_y \) counted with multiplicity and \( u_n(y) \) the corresponding normalized eigenfunction. We have

\[ -\Delta_y u_n = \lambda_n u_n, \quad 0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_n \to \infty \ (n \to \infty). \]

The geometry of \( S \) ensures that \( \lambda_0 \) is simple.

Since \( a \) is a \( C^2 \) function, there exists \( K > 0 \), so that,

\[ |a(\varepsilon y) - 1| \leq K \varepsilon, \quad \forall y \in S, \tag{14} \]

for all \( \varepsilon > 0 \) small enough.

**Theorem 2.** There exists \( K > 0 \), so that, for all \( \varepsilon > 0 \) small enough,

\[ \| (D_\varepsilon + \xi 1)^{-1} - (D + \xi 1)^{-1} \| \leq K \varepsilon, \]

for all \( \xi > 0 \).

**Proof.** Consider the quadratic forms

\[ d_\varepsilon(u) = \int_S a(\varepsilon y)|\nabla_y u|^2 \, dy, \quad d(u) = \int_S |\nabla_y u|^2 \, dy, \]

\( \text{dom} d_\varepsilon = \text{dom} d = H^1_0(S) \), associated to \( D_\varepsilon \) and \( -\Delta_y \), respectively.

Since \( D_\varepsilon \) and \( -\Delta_y \) are positive operators, given \( \xi > 0, -\xi \in \rho(D_\varepsilon), \rho(-\Delta_y) \), i.e., \( \xi \) belongs to the resolvent set of these operators. Thus, for all \( \xi > 0 \) and for all \( u \in H^1_0(S) \),

\[ |(d_\varepsilon + \xi)(u) - (d + \xi)(u)| = |d_\varepsilon(u) - d(u)| \leq \int_S |a(\varepsilon y) - 1||\nabla_y u|^2 \, dy \leq K \varepsilon (d + \xi)(u). \]

Now, the result follows by Theorem 3 in [1]. \( \square \)
As a consequence of this theorem, for each \( n \in \mathbb{N}_0 := \{0, 1, 2, 3, \cdots \} \),

(i) \( \lambda^n_\varepsilon \to \lambda_n \), as \( \varepsilon \to 0 \);

(ii) \( u^n_\varepsilon \to u_n \), in \( L^2(S) \), as \( \varepsilon \to 0 \).

Since \( \lambda_0 > 0 \), the condition (i) imply that \( \lambda^0_\varepsilon > 0 \) and is a simple eigenvalue, for all \( \varepsilon > 0 \) small enough.

Further, (i) and (ii) imply that, for each \( n \in \mathbb{N}_0 \),

(iii) \( (\nabla u^n_\varepsilon)_\varepsilon \) is a bounded sequence in \( L^2(S) \).

In fact, just note that

\[
\int_S a(\varepsilon y)|\nabla u^n_\varepsilon|^2 \, dy = \lambda^n_\varepsilon. \tag{15}
\]

4 Reformulation of the problem

In case of collapsing homogeneous waveguides, it is known that the first eigenvalue diverges as \( \varepsilon^{-2} \) for small \( \varepsilon \in [3, 6] \) (see, e.g., equation (3)); since, by Section 3, the sequence \( (\lambda^0_\varepsilon)_\varepsilon \) is bounded, a natural “renormalization” guess is to subtract \( \left( \lambda^0_\varepsilon/\varepsilon^2 \right) \int_{I \times S} |\psi|^2 \, ds \, dy \) from \( b_\varepsilon(\psi) \). This is a standard procedure to extract a meaningful limit of this very singular problem with the presence of regions that scale in different manners. We also add a constant \( c > \| - k^2(s)/4 + V_\varepsilon(s) \|_\infty \); this fact will be convenient later on to guarantee that zero is in the resolvent set of the related operators. Thus, we pass to study the quadratic form

\[
b_\varepsilon(\psi) := \overline{b}_\varepsilon(\psi) - \frac{\lambda^0_\varepsilon}{\varepsilon^2} \int_{I \times S} |\psi|^2 \, ds \, dy + c \int_{I \times S} |\psi|^2 \, ds \, dy \\
= \int_{I \times S} \frac{a_\varepsilon}{\varepsilon^2} |\psi|^2 - \psi \frac{\partial \beta_\varepsilon}{\partial x} \frac{1}{2} \partial_s \beta_\varepsilon + \langle \nabla_y \psi, R_y \rangle (\tau + \alpha')(s) \\
+ \psi \beta_\varepsilon^{1/2} \langle \nabla_y (1/\beta_\varepsilon^{1/2}), R_y \rangle (\tau + \alpha')(s) \bigg|^2 \, ds \, dy \\
+ \int_{I \times S} \frac{a_\varepsilon}{\varepsilon^2} \left( |\nabla_y \psi|^2 - \lambda^0_\varepsilon |\psi|^2 \right) \, ds \, dy - \int_{I \times S} \frac{1}{4 \beta_\varepsilon^2} k^2(s) |\psi|^2 \, ds \, dy \\
- \int_{I \times S} \frac{1}{2 \beta_\varepsilon} \langle \nabla_y a(\varepsilon y), z_{\alpha(s)} \rangle k(s)|\psi|^2 \, ds \, dy + c \int_{I \times S} |\psi|^2 \, ds \, dy,
\]

\( \text{dom } b_\varepsilon = H_0^1(I \times S) \). Denote by \( B_\varepsilon \) the self-adjoint operator associated with it.

To simplify the calculations we perform the following approximation. Consider the quadratic form

\[
h_\varepsilon(\psi) = \int_{I \times S} |\psi' + \langle \nabla_y \psi, R_y \rangle (\tau + \alpha')(s)|^2 \, ds \, dy \\
+ \int_{I \times S} \frac{a_\varepsilon}{\varepsilon^2} \left( |\nabla_y \psi|^2 - \lambda^0_\varepsilon |\psi|^2 \right) \, ds \, dy - \int_{I \times S} \frac{1}{4 \beta_\varepsilon^2} k^2(s) |\psi|^2 \, ds \, dy \\
- \int_{I \times S} \frac{1}{2 \beta_\varepsilon} \langle \nabla_y a(0), z_{\alpha(s)} \rangle k(s)|\psi|^2 \, ds \, dy + c \int_{I \times S} |\psi|^2 \, ds \, dy.
\]
dom $h_\varepsilon = H^1_0(I \times S)$, and denote by $H_\varepsilon$ the self-adjoint operator associated with it.

**Theorem 3.** There exist $K > 0$ so that

$$\|B_\varepsilon^{-1} - H_\varepsilon^{-1}\| \leq K \varepsilon,$$

for all $\varepsilon > 0$ small enough.

The choice of $c$ ensures that $0 \in \rho(B_\varepsilon) \cap \rho(H_\varepsilon)$, for all $\varepsilon > 0$ small enough. We have two considerations here. First, condition (14) implies that since $a$ is a $C^2$ function,

$$\sup_y \|(\nabla y a)(\varepsilon y) - (\nabla y a)(0)\|_{R^2} \leq K \varepsilon,$$

for some $K > 0$. Second, $\beta_\varepsilon(s, y) \to 1$ uniformly as $\varepsilon \to 0$, and there exists $K > 0$ so that

$$\left\| \frac{\partial \beta_\varepsilon}{\partial s} \right\|_{\infty} \leq K \varepsilon, \quad \sup_{s, y} \|\nabla y (1/\beta_\varepsilon^{1/2})\|_{R^2} \leq K \varepsilon,$$

for $\varepsilon > 0$ small enough. Now, the proof of Theorem 3 is quite similar to proof of Theorem 3.1 in [4] and will not be presented here.

## 5 Reduction of dimension

Recall that $u_0^\varepsilon$ is the eigenfunction associated with the first eigenvalue $\lambda_0^\varepsilon$ of $D_\varepsilon$. Define the closed subspace $L_\varepsilon := \{w(s)u_0^\varepsilon(y) : w \in L^2(I)\}$ and consider the orthogonal decomposition

$$L^2(I \times S) = L_\varepsilon \oplus L^\perp_\varepsilon.$$

Thus, each $\psi \in L^2(I \times S)$ can be written as

$$\psi(s, y) = w(s)u_0^\varepsilon(y) + \eta_\varepsilon(s, y), \quad w \in L^2(I), \eta_\varepsilon \in L^\perp_\varepsilon. \quad (16)$$

Consider the positive one-dimensional quadratic form

$$t_\varepsilon(w) := h_\varepsilon(wu_0^\varepsilon), \quad \text{dom } t_\varepsilon = H^1_0(I).$$

Denote by $T_\varepsilon$ the self-adjoint operator associated with it. In this case, $\text{dom } T_\varepsilon = H^2(I) \cap H^1_0(I)$.

Taking in count that

$$\int_S u_0^\varepsilon(\nabla y u_0^\varepsilon, Ry) \, dy = 0,$$

we have

$$t_\varepsilon(w) = \int_I \left\{ |w'|^2 + \left[ C_\varepsilon(S)(\tau + \alpha')^2 - \frac{k^2(s)}{4} - \frac{k(s)}{2}(\langle \nabla y a(0), z_{\alpha(s)} \rangle + c) \right] |w|^2 \right\} \, ds,$$

where

$$C_\varepsilon(S) := \int_S |(\nabla y u_0^\varepsilon, Ry)|^2 \, dy.$$
**Theorem 4.** There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,

$$\|H_\varepsilon^{-1} - (T_\varepsilon^{-1} \oplus 0)\| \leq K \varepsilon,$$

where $0$ is the null operator on the subspace $L^\perp_\varepsilon$.

**Proof.** Due to the decomposition (16), for $\psi \in \text{dom } h_\varepsilon$,

$$\psi(s, y) = w(s) u_\varepsilon^0(y) + \eta_\varepsilon(s, y), \quad w \in H^0_0(I), \quad \eta_\varepsilon \in \text{dom } h_\varepsilon \cap L^\perp_\varepsilon.$$

Thus, $h_\varepsilon(\psi)$ can be rewritten as

$$h_\varepsilon(\psi) = t_\varepsilon(w) + h_\varepsilon(wu_\varepsilon^0, \eta_\varepsilon) + h_\varepsilon(\eta_\varepsilon, wu_\varepsilon^0) + h_\varepsilon(\eta_\varepsilon).$$

We need to check that there are $c_0 > 0$ and functions $0 \leq q(\varepsilon), 0 \leq p(\varepsilon)$ and $c(\varepsilon)$ so that $t_\varepsilon(w), h_\varepsilon(\eta_\varepsilon)$ and $h_\varepsilon(wu_\varepsilon^0, \eta_\varepsilon)$ satisfy the following conditions:

$$t_\varepsilon(w) \geq c(\varepsilon)\|wu_\varepsilon^0\|_{L^2(I \times S)}^2, \quad \forall w \in \text{dom } t_\varepsilon, \quad c(\varepsilon) \geq c_0 > 0; \quad (17)$$

$$h_\varepsilon(\eta_\varepsilon) \geq p(\varepsilon)\|\eta_\varepsilon\|_{L^2(I \times S)}^2, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap L^\perp_\varepsilon; \quad (18)$$

$$|h_\varepsilon(wu_\varepsilon^0, \eta_\varepsilon)|^2 \leq q(\varepsilon)^2 t_\varepsilon(w) h_\varepsilon(\eta_\varepsilon), \quad \forall \psi \in \text{dom } h_\varepsilon; \quad (19)$$

and with

$$p(\varepsilon) \to \infty, \quad c(\varepsilon) = O(p(\varepsilon)), \quad q(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \quad (20)$$

Under these conditions, Proposition 3.1 in [10] guarantees that, for $\varepsilon > 0$ small enough, we have for the operator norm

$$\|H_\varepsilon^{-1} - (T_\varepsilon^{-1} \oplus 0)\| \leq p(\varepsilon)^{-1} + K q(\varepsilon) c(\varepsilon)^{-1},$$

for some $K > 0$.

Clearly, $t_\varepsilon(w) \geq c\|w\|_{L^2(I)}^2 = c\|wu_\varepsilon^0\|_{L^2(I \times S)}$, for all $w \in \text{dom } t_\varepsilon$. Take $c(\varepsilon) := c$. By the Min-Max Principle,

$$\int_S (a_\varepsilon |\nabla_y \eta_\varepsilon|^2 - \lambda_\varepsilon^0 |\eta_\varepsilon|^2) \, dy \geq (\lambda_\varepsilon^1 - \lambda_\varepsilon^0) \int_S |\eta_\varepsilon|^2 \, dy, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap L^\perp_\varepsilon,$$

for a.e. $s$. Since $\lambda_\varepsilon^1 - \lambda_\varepsilon^0 \to \lambda_1 - \lambda_0 > 0$, as $\varepsilon \to 0$, there exists $K > 0$, so that,

$$\int_S (a_\varepsilon |\nabla_y \eta_\varepsilon|^2 - \lambda_\varepsilon^0 |\eta_\varepsilon|^2) \, dy \geq K \int_S |\eta_\varepsilon|^2 \, dy, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap L^\perp_\varepsilon,$$

for all $\varepsilon > 0$ small enough and a.e. $s$. Thus,

$$h_\varepsilon(\eta_\varepsilon) \geq \frac{K}{\varepsilon^2} \int_{I \times S} |\eta_\varepsilon|^2 \, ds \, dy, \quad \forall \eta_\varepsilon \in \text{dom } h_\varepsilon \cap L^\perp_\varepsilon,$$

for all $\varepsilon > 0$ small enough. Just take $p(\varepsilon) := K/\varepsilon^2$.

Since $\eta_\varepsilon \in H^0_0(I \times S) \cap L^\perp_\varepsilon$,

$$\int_S \eta_\varepsilon(s, y) u_\varepsilon^0(y) \, dy = 0, \quad \text{a.e. } s,$$

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and, consequently,
\[ \int_S \eta'_\varepsilon(s,y)w_\varepsilon^0(y) \, dy = 0, \quad \text{a.e. } s. \]

These conditions and a integration by parts show that
\[ h_\varepsilon(wu_\varepsilon^0, \eta_\varepsilon) \]
\[ = \int_{I \times S} \left( w' u_\varepsilon^0 + w \langle \nabla y u_\varepsilon^0, Ry \rangle (\tau + \alpha')(s) \right) \left( \eta'_\varepsilon + \langle \nabla y \eta_\varepsilon, Ry \rangle (\tau + \alpha')(s) \right) \, ds \, dy \]
\[ = \int_{I \times S} w' u_\varepsilon^0 \langle \nabla y \eta_\varepsilon, Ry \rangle (\tau + \alpha')(s) \, ds \, dy \]
\[ - \int_{I \times S} (w' (\tau + \alpha')(s) + w (\tau + \alpha')(s)) \eta_\varepsilon \langle \nabla y u_\varepsilon^0, Ry \rangle \, ds \, dy \]
\[ + \int_{I \times S} w \langle \nabla y u_\varepsilon^0, Ry \rangle \langle \nabla \eta_\varepsilon, Ry \rangle (\tau + \alpha')^2(s) \, ds \, dy. \]

Now, note that there exists \( K > 0 \) so that,
\[ \int_{I \times S} |\nabla \eta_\varepsilon|^2 \, ds \, dy \leq \sup_{y \in S} \left\{ \frac{1}{a(y)} \right\} \int_{I \times S} a(\varepsilon y)|\nabla \eta_\varepsilon|^2 \, ds \, dy \]
\[ = \sup_{y \in S} \left\{ \frac{1}{a(y)} \right\} \left[ \int_{I \times S} (a(\varepsilon y)|\nabla \eta_\varepsilon|^2 - \lambda_\varepsilon^0 |\eta_\varepsilon|^2) \, ds \, dy \right] \]
\[ + \lambda_\varepsilon^0 \int_{I \times S} |\eta_\varepsilon|^2 \, ds \, dy \]
\[ \leq K \varepsilon^2 h_\varepsilon(\eta_\varepsilon), \]
for all \( \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^+, \) for all \( \varepsilon > 0 \) small enough. We also have \( \int_I |w'|^2 \, ds \leq t_\varepsilon(w), \)
\[ \int_I |w|^2 \, ds \leq K t_\varepsilon(w), \]
for all \( w \in \text{dom } t_\varepsilon, \) and all \( \varepsilon > 0 \) small enough.

These inequalities, the conditions (i), (ii) and (iii) in Section 3 and (21), imply that
\[ |h_\varepsilon(wu_\varepsilon^0, \eta_\varepsilon)| \leq K \varepsilon (t_\varepsilon(w))^{1/2} (h_\varepsilon(\eta_\varepsilon))^{1/2}, \]
for some \( K > 0, \) for all \( w \in \text{dom } t_\varepsilon, \) all \( \eta_\varepsilon \in \text{dom } h_\varepsilon \cap \mathcal{L}_\varepsilon^+ \) and all \( \varepsilon > 0 \) small enough.

Take \( q(\varepsilon) := K \varepsilon. \)

Since (17), (18), (19) and (20) are satisfied, the result follows. \( \square \)

6 Effective operator

In this section we show the convergence of the sequence \( T_\varepsilon \) in the norm resolvent sense as \( \varepsilon \to 0; \) the effective operator is found.

Define the quadratic form
\[ t(w) = \int_I \left\{ |w'|^2 + \left[ C(S)(\tau + \alpha')^2(s) - \frac{k^2(s)}{4} - \frac{k(s)}{2} \langle (\nabla y a)(0), z_{\alpha(s)} \rangle + c \right] |w|^2 \right\} \, ds, \]
dom \( t = H^1_0(I), \) and denote by \( T \) its associated self-adjoint operator. Namely,
\[ (T w)(s) = -w''(s) + \left[ C(S)(\tau + \alpha')^2(s) - \frac{k^2(s)}{4} - \frac{k(s)}{2} \langle (\nabla y a)(0), z_{\alpha(s)} \rangle + c \right] w(s), \]
dom $T = H^2(I) \cap H^1_0(I)$.

Note that if $(\tau + \alpha') = 0$, for all $s \in I$, then we have $T_\varepsilon = T$, for all $\varepsilon > 0$, and the conclusion of Theorem 1 follows. For the general case we shall make use of the following results:

**Lemma 1.** The sequence $\langle C_\varepsilon(S) \rangle_\varepsilon$ converges to $C(S)$, as $\varepsilon \to 0$.

**Proof.** Since $a(\varepsilon y) \to 1$ uniformly, by (i) and (15) in Section 3, we have $\int_S |\nabla_y u_\varepsilon^0|^2 \, dy \to \lambda_0$. Also in Section 3, the condition (iii) implies that any subsequence of $(\nabla_y u_\varepsilon^0)_\varepsilon$ has a weakly converging subsequence $(\nabla_y u_\varepsilon^0)_{\varepsilon'}$ with $\nabla_y u_\varepsilon^0 \rightharpoonup \nabla_y u_0$ in $L^2(S)$. Now we introduce a seminorm $\| \cdot \|$ on $L^2(S)$ as

$$\| \cdot \| := \left( \int_S |\langle \cdot, Ry \rangle|^2 \, dy \right)^{\frac{1}{2}},$$

and note that $C_\varepsilon^2(S) = [\nabla_y u_\varepsilon^0]$ and $C(S) = [\nabla_y u_0]$. Thus, by combining the just mentioned facts, we have

$$\left| C_\varepsilon^2(S) - C(S) \right|^2 = \left| [\nabla_y u_\varepsilon^0] - [\nabla_y u_0] \right|^2 \leq \left| [\nabla_y u_\varepsilon^0] - \nabla_y u_0 \right|^2$$

$$= \int_S \left( |\langle \nabla_y u_\varepsilon^0 - \nabla_y u_0, Ry \rangle|^2 \right) \, dy \leq D_S \int_S |\nabla_y u_\varepsilon^0 - \nabla_y u_0|^2 \, dy$$

$$= D_S \int_S \left( |\nabla_y u_\varepsilon^0|^2 + |\nabla_y u_0|^2 - 2\langle \nabla_y u_\varepsilon^0, \nabla_y u_0 \rangle \right) \, dy \xrightarrow{\varepsilon \to 0} D_S (\lambda_0 + \lambda_0 - 2\lambda_0) = 0,$$

where $D_S$ is obtained from $|Ry| = |y|$ and the boundedness of $S$. Hence, $C_\varepsilon^2(S) \to C(S)$, as $\varepsilon \to 0$.

By repeating the above arguments, one sees that any subsequence of $(C_\varepsilon(S))_\varepsilon$ has a converging subsequence with the uniquely determined limit $C(S)$, and so Urysohn’s subsequence principle implies that $C_\varepsilon(S) \to C(S)$ as $\varepsilon \to 0$. \hfill $\square$

**Proposition 1.** There exist $K > 0$ and a function $s(\varepsilon) > 0$, so that, $s(\varepsilon) \to 0$, as $\varepsilon \to 0$, and

$$\| T_\varepsilon^{-1} - T^{-1} \| \leq K s(\varepsilon),$$

for all $\varepsilon > 0$ small enough.

**Proof.** Lemma 1 ensures that there exists $\tilde{s}(\varepsilon) > 0$, so that, $\tilde{s}(\varepsilon) \to 0$, as $\varepsilon \to 0$, and

$$|C_\varepsilon(S) - C(S)| \leq \tilde{s}(\varepsilon),$$

for all $\varepsilon > 0$ small enough. Thus, since $(\tau + \alpha') \in L^\infty(I)$, there exists $K > 0$, so that

$$|t_\varepsilon(w) - t(w)| \leq K \int_I |C_\varepsilon(S) - C(S)| \, |w|^2 \, ds \leq K \tilde{s}(\varepsilon) \int_I |w|^2 \, ds \leq K \tilde{s}(\varepsilon) \, t(w).$$

Take $s(\varepsilon) := (K/c)\tilde{s}(\varepsilon)$. The results follows by Theorem 3 of [1]. \hfill $\square$

The proof of Theorem 1 follows by combining Theorem 4 and Proposition 1.
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References


