INVOLUTIONS FIXING MANY COMPONENTS: A SMALL CODIMENSION PHENOMENON

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Abstract. Let $T : M \to M$ be a smooth involution on a closed smooth m-dimensional manifold and $F = \bigsqcup_{j=0}^{n} F^j$ the fixed point set of T, where F^j denotes the union of those components of F having dimension j, and thus $n < m$ is the dimension of the component of F of largest dimension. Denote by $\pi_0(F)$ the set of dimensions occurring in F. If $j \in \pi_0(F)$, $0 \leq j \leq n$, we assume that the normal bundle of F^j in M does not bound, because otherwise F^j can be removed via an equivariant surgery. In this paper we prove the following results, which characterize small codimension phenomena: suppose that F has one of the following forms: (i) $0 \in \pi_0(F)$, and all the other components of F (including the top-dimensional, with dimension n) are odd-dimensional; (ii) $1 \in \pi_0(F)$, and all the other components of F (including the top-dimensional) are even-dimensional. If k denotes the codimension of F^n, then $k \leq 1$ in the first case, and $k \leq 2$ in the second case. Further, very simple examples will show that these results are best possible. Other results concerning the small codimension phenomenon are found in the literature, where F has two, three or four components. Together with our results, they inspire a general conjecture that will be presented in Section 1, and if valid, this conjecture is best possible. In fact, i) and ii) are the cases $j = 0$ and $j = 1$ of the conjecture in question. Unlike the above mentioned literature results, note that in our case the number of components of F may not be limited as a function of n. In fact, in the first case, if S is any subset of the set $\{i \mid 0 < i < n,\ i \text{ odd}\}$, then we allow that $S \subset \pi_0(F)$. In the second case, if S is any subset of the set $\{i \mid 0 \leq i < n,\ i \text{ even}\}$, we allow that $S \subset \pi_0(F)$. Because of this, to show that the results are not vacuous and that there many involutions to which the theorem applies, we will show that there are examples where all possible dimensions occur, that is, with S maximal.

1. **Introduction**

Let F be a disjoint (finite) union of smooth and closed manifolds and M be a smooth and closed m-dimensional manifold equipped with a smooth involution

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$T\colon M \to M$ whose fixed point set is F. As commented in the abstract, let us assume that the normal bundle of each F^j in M is not a boundary, where $F^j \subset F$ is the union of those components of F having dimension j. If n is the dimension of a component of F of maximal dimension and k is the codimension of this component, then $k \leq \frac{3}{2}n$, and this estimative is best possible; this follows from the famous Five Halves Theorem of J. Boardman, announced in [2], and its strengthened version of [12]. The generality of this result, which is valid for every $n \geq 1$ and $\pi_0(F)$, allows the possibility that fixed components of all dimensions j, $0 \leq j \leq n$, occur; in this way, it is natural to ask whether there exist better bounds for k when we impose conditions on the set $\pi_0(F)$ (as in the abstract, $\pi_0(F)$ means the set of dimensions occurring in F). This question is inspired by the following results of the literature:

1) (R. E. Stong and C. Kosniowski, [12], 1978): if $\pi_0(F) = \{n\}$, where $n \neq 1$ and $n \neq 3$ (and so the normal bundle over F^n is not a boundary), then $k \leq n$, and this result is best possible.

2) (D. C. Royster, [16], 1980): this is the first case of small codimension phenomenon (see after a formalization about). If $\pi_0(F) = \{0, n\}$ and n is odd, then $k \leq 1$. This small bound is realized by the involution (RP^{n+1}, T), where RP^{n+1} is the $(n+1)$-dimensional real projective space and $T[x_0, x_1, ..., x_{n+1}] = [-x_0, x_1, ..., x_{n+1}]$, with n odd.

This class of problems was introduced by P. Pergher in [13], with its general formulation and where the case $\pi_0(F) = \{0, n\}$ was enlarged in the following (not best possible) way: if $\pi_0(F) = \{0, n\}$ and n is an even number having the form $2p$ with p odd, then $k \leq p + 3$. The case $\pi_0(F) = \{0, n\}$ was completed by R. Stong and P. Pergher in [15], in a best possible way. With the cases $\pi_0(F) = \{n\}$ and $\pi_0(F) = \{0, n\}$ completed, the next natural step is the case $\pi_0(F) = \{j, n\}$, $0 < j < n$. Concerning this case, we find best possible bounds for $\pi_0(F) = \{1, n\}$ in [10] and [11], $\pi_0(F) = \{2, n\}$ in [6], [7] and [8] and $\pi_0(F) = \{n-1, n\}$ in [9]. In [14], one has a (not best possible, but almost best possible) bound for the case $\pi_0(F) = \{j, n\}$, where F^j is indecomposable and $2 \leq j < n$, j not of the form $2^t - 1$.

Among the results found in the above references, there are some that characterize a curious and intriguing phenomenon, which we call a *small codimension phenomenon.* This phenomenon is characterized by the fact that, for some models of $\pi_0(F)$, the codimension k of the top-dimensional component of F (with dimension n) is limited as a function of n, which does not occur with the general Boardman bound. Let us quote such results: i) if $\pi_0(F) = \{0, n\}$ and n is odd, then $k \leq 1$. ii) if $\pi_0(F) = \{1, n\}$ and n is even, then $k \leq 2$. iii) if $\pi_0(F) = \{2, n\}$ and n is odd, then $k \leq 3$. iv) if $\pi_0(F) = \{j, n\}$, where F^j is indecomposable and $2 \leq j < n$, with j not of the form $2^t - 1$, and if $n - j$ is odd, then $k \leq j + 2$.

In addition, concerning situations where the cardinality of $\pi_0(F)$ is greater than two, recently the following result was obtained (see [1]):

Theorem. *Let (M, T) be an involution with $\pi_0(F)$ having one of the possible forms, where $n \geq 4$ is even:*

1) $\pi_0(F) = \{0, 2, 3, n\}$;

2) $\pi_0(F) = \{2, 3, n\}$;

3) $\pi_0(F) = \{0, 3, n\}$;

4) $\pi_0(F) = \{3, n\}$.

Then $k \leq 4$. Further, there are involutions showing that this bound is best possible in the cases 2) and 4), and in the cases 1) and 3) with n of the form $n = 4t$, $t \geq 1$.

The above results inspire the following

Conjecture. *Let (M, T) be an involution with $\pi_0(F)$ having one of the possible forms, where, as before, n is the dimension of the top-dimensional component of F and k is the codimension of this component:*

1) there is an odd number $0 < j < n$ so that $j \in \pi_0(F)$, and all the other components of F (including the top-dimensional) are even-dimensional;

2) there is an even number $0 \leq j < n$ so that $j \in \pi_0(F)$, and all the other components of F (including the top-dimensional) are odd-dimensional.

Then $k \leq j + 1$, and the result is best possible.

Note that all results cited above are special cases of this conjecture; further, as we will see in this paper, the conjecture is valid for $j = 0$ and $j = 1$, and so it makes sense. This shows that small codimension phenomena can occur even when F has many components or, more precisely, in some situations where the number of components of F may not be limited as a function of n. This paper is organized as follows: in Section 2, we prove the bounds "$k \leq 1$" and "$k \leq 2$", corresponding to the cases $j = 0$ and $j = 1$. Further, we present very simple examples showing that, if the general bound "$k \leq j + 1$" of the conjecture is valid, then it is best possible.

As mentioned in the abstract, if S is any subset of the set $\{i \mid 0 < i < n,\ i \text{ odd}\}$, in the case $j = 0$, and any subset of the set $\{i \mid 0 \leq i < n,\ i \text{ even}\}$, in the case $j = 1$, then our results allow that $S \subset \pi_0(F)$. To show that the results are not vacuous, in Section 3 we show that there are examples with S maximal, and thus with the number of components of F being not limited as a function of n.

The main tools to be used are in the context started with the famous work [17] of René Thom about cobordism theory, which gave him the Fields medal in 1958, and continued ten years later with the monumental work [5] of P. Conner and E. Floyd, which extended the previous Thom results to the setting of singular cobordism of spaces and equivariant cobordism.

2. **Small codimension phenomena**

Keeping in mind the objects discussed in Section 1, from [5], a pair (M, T) represents an unoriented cobordism class in the m-dimensional unoriented Z_2-equivariant cobordism group $\mathcal{N}_m^{Z_2}(*)$, and the normal bundle of F^j in M, which is an $(m-j)$-dimensional real vector bundle $\eta^{m-j} \to F^j$, is a *singular manifold* in $BO(m-j)$, and represents an unoriented cobordism class in the j-dimensional unoriented cobordism group of $BO(m-j)$, $\mathcal{N}_j(BO(m-j))$. One has the Conner and Floyd short exact sequence

$$0 \to \mathcal{N}_m^{Z_2}(*) \xrightarrow{j_*} \bigoplus_{j=0}^m \mathcal{N}_j(BO(m-j)) \xrightarrow{\partial} \mathcal{N}_{m-1}(BO(1)) \to 0,$$

where j_* maps $[(M,T)]$ into the cobordism class $\sum_{j=0}^{m}[\eta^{m-j} \to F^j]$, and ∂ maps $[\eta^{m-j} \to F^j]$ into the cobordism class $[\lambda \to RP(\eta^{m-j})]$; here, $RP(\eta^{m-j})$ is the total space of the real projective bundle associated to η^{m-j}, and λ is the canonical Hopf line bundle over it (in particular, ∂ is zero on the $j = m$ summand). Then $\mathcal{N}_m^{Z_2}(*)$ is isomorphic to the kernel of the map ∂, and ∂ is split by lifting to the $j = m-1$ summand. To deal with cobordism classes of bundles, one needs some basic facts from [5]. For a vector bundle $\eta \to N$ and a natural number $p \geq 1$, write $p\eta \to N$ for the Whitney sum of p copies of η; $R \to N$ will denote the one dimensional trivial vector bundle over N. If η is k-dimensional, write $W(\eta) = 1 + w_1(\eta) + w_2(\eta) + ... + w_k(\eta) \in H^*(N, Z_2)$ for the total Stiefel-Whitney class of η; if N is a closed smooth n-dimensional manifold, $W(N) = 1 + w_1(N) + w_2(N) + ... + w_n(N)$ will denote the total Stiefel-Whitney class of the tangent bundle of N. In this case, one has an algebraic scheme to determine the cobordism class of $\eta \to N$, given by the set of *Whitney numbers* (or *characteristic numbers*) of η; such modulo 2 numbers are obtained by evaluating n-dimensional Z_2-cohomology classes of the form

$$w_{i_1}(N)w_{i_2}(N)...w_{i_r}(N)w_{j_1}(\eta)w_{j_2}(\eta)...w_{j_s}(\eta) \in H^n(N, Z_2)$$

(that is, with $i_1 + i_2 + ... + i_r + j_1 + j_2 + ... + j_s = n$) on the fundamental homology class $[N] \in H_n(N, Z_2)$. The structure of the Z_2-cohomology ring and the Stiefel-Whitney class of $RP(\eta)$ are also necessary: if $\lambda \to RP(\eta)$ is the Hopf line bundle, set $w_1(\lambda) = c$. From [4; page 517], one has

$$W(RP(\eta)) = (1 + w_1(N) + ... + w_n(N))[(1+c)^k + (1+c)^{k-1}w_1(\eta) + ... + w_k(\eta)],$$

where here we are suppressing bundle maps. Further, this gives the relation

$$c^k + c^{k-1}w_1(\eta) + c^{k-2}w_2(\eta) + \cdots + w_k(\eta) = 0.$$

The structure of the Z_2-cohomology ring of $RP(\eta)$ is then determined by the above relation and from the Leray-Hirsch Theorem (see [3], p. 129), which gives that $H^*(RP(\eta), Z_2)$ is a free graded $H^*(N, Z_2)$-module with basis $1, c, c^2, ..., c^{k-1}$.

If $P = P(x_1, x_2, ..., x_{m-1}, t)$ is a polynomial of dimension $m-1$ in the variables $x_1, x_2, ..., x_{m-1}$ and t, where each x_i has degree i and t has degree

1, then P determines a homomorphism $P : \mathcal{N}_{m-1}(BO(1)) \to Z_2$, given by $P([\lambda \to V^{m-1}] = P(w_1(V^{m-1}), w_2(V^{m-1}), ..., w_{m-1}(V^{m-1}), w_1(\lambda))[V^{m-1}]$. Then the composition

$\bigoplus_{j=0}^{m} \mathcal{N}_j(BO(m-j)) \xrightarrow{\partial} \mathcal{N}_{m-1}(BO(1)) \xrightarrow{P} Z_2$

is obviously zero if restricted to the kernel of ∂. This is the key point to obtain the bounds "$k \leq 1$" and "$k \leq 2$", and also the announced examples with many components. Let us denote by $\lambda_j \to RP^j$ the canonical line bundle over the j-dimensional real projective space RP^j, and write $\alpha_j \in H^1(RP^j, Z_2)$ for the generator. To avoid excessive notation, we always write $1 + w_1 + w_2 + ...$, $1 + v_1 + v_2 + ...$ and $1 + c$, respectively, for the Stiefel-Whitney classes of components F^j of F, of normal bundles of the F^j in M and of Hopf line bundles over projective space bundles, $\lambda \to RP(\eta)$.

First we consider the $j = 0$ case, showing that $k \leq 1$. Take then an involution (M, T) with $\pi_0(F)$ corresponding to the $j = 0$ case. From [5] one has $\chi(M) \equiv \chi(F) \bmod 2$, where χ is the Euler characteristic. So $\chi(M) \equiv 1 \bmod 2$ and m is even. Consider the homomorphism $P : \mathcal{N}_{m-1}(BO(1)) \to Z_2$ corresponding to the polynomial $P(x_1, x_2, ..., x_{m-1}, t) = (x_1 + t)^{m-1}$. One has $\partial([\eta^m \to F^0] = [\lambda_{m-1} \to RP^{m-1}]$, and, because m is even, $w_1(\lambda_{m-1}) = \alpha_{m-1}$ and $w_1(RP^{m-1}) = 0$. Thus $P([\lambda_{m-1} \to RP^{m-1}]) = \alpha_{m-1}^{m-1}[RP^{m-1}] = 1$. Now consider $0 < j < m-1$ odd, and let $\lambda \to RP(\eta^{m-j})$ be the Hopf line bundle. Then $w_1(RP(\eta^{m-j})) = w_1 + \binom{m-j}{1}c + v_1 = w_1 + c + v_1$ because $m - j$ is odd, and thus $w_1(RP(\eta^{m-j})) + w_1(\lambda) = w_1 + v_1$. Since $w_1 + v_1$ comes from the cohomology of F^j and $j < m-1$, $(w_1 + v_1)^{m-1} = 0$. It follows that $P([\lambda \to RP(\eta^{m-j})] = 0$, and thus necessarily $P([\lambda \to RP(\eta^{m-1})] = 1$, which means that F has an $(m-1)$-component. This proves that $k \leq 1$ (in fact, $k = 1$).

Now consider the $j = 1$ case, and let (M, T) be an involution with $\pi_0(F)$ corresponding to this case. First consider m even, and take the homomorphism $P : \mathcal{N}_{m-1}(BO(1)) \to Z_2$ corresponding to the polynomial $P(x_1, x_2, ..., x_{m-1}, t) = x_1^{m-2}t$. As before, $\partial([\eta^m \to F^0] = [\lambda_{m-1} \to RP^{m-1}]$, $w_1(\lambda_{m-1}) = \alpha_{m-1}$ and $w_1(RP^{m-1}) = 0$, so $P([\lambda_{m-1} \to RP^{m-1}]) = 0$. Without loss, we can suppose $\eta^{m-1} \to F^1 = \lambda_1 \oplus (m-2)R \to RP^1$. One has $W(RP(\eta^{m-1})) =$

$(1+c)^{m-1}+(1+c)^{m-2}\alpha_1$, so $w_1(RP(\eta^{m-1})) = \binom{m-1}{1}c+\alpha_1 = c+\alpha_1$. Hence $(w_1(RP(\eta^{m-1})))^{m-2}c = (c+\alpha_1)^{m-2}c = c^{m-1}+\binom{m-2}{1}c^{m-2}\alpha_1 = c^{m-1}$. But $H^*(RP(\eta^{m-1}),Z_2)$ is the free $H^*(RP^1,Z_2)$-module on 1, c, c^2, ..., c^{m-2}, so $c^{m-2}\alpha_1$ is the generator of $H^{m-1}(RP(\eta^{m-1}),Z_2)$. Further, one has the relation $c^{m-1} = c^{m-2}\alpha_1$, which means that c^{m-1} is nonzero. It follows that $P([\lambda \to RP(\eta^{m-1}]) = c^{m-1}[RP(\eta^{m-1})] = 1$. Now consider $1 < j < m-2$ even. Then $w_1(RP(\eta^{m-j}) = w_1 + \binom{m-j}{1}c + v_1 = w_1+v_1$. Again, since w_1+v_1 comes from the cohomology of F^j and $j < m-2$, $(w_1+v_1)^{m-2} = 0$, and thus $P([\lambda \to RP(\eta^{m-j})] = 0$. It follows that $P([\lambda \to RP(\eta^{m-2})] = 1$, which means that F has an $(m-2)$-component. This means that $k=2$ in this case. Now suppose m odd, and take the homomorphism $P : \mathcal{N}_{m-1}(BO(1)) \to Z_2$ corresponding to the polynomial $P(x_1,x_2,...,x_{m-1},t) = (x_1+t)^{m-2}x_1$. In this case, $w_1(\lambda_{m-1}) = \alpha_{m-1} = w_1(RP^{m-1})$, which gives $P\partial([\eta^m \to F^0]) = 0$. Also, over F^1, $w_1(RP(\eta^{m-1})) = \binom{m-1}{1}c+\alpha_1 = \alpha_1$ and $(\alpha_1+c)^{m-2}\alpha_1 = c^{m-2}\alpha_1$, which is the generator of $H^{m-1}(RP(\eta^{m-1}),Z_2)$. Thus $P([\lambda \to RP(\eta^{m-1}]) = 1$. If $1 < j < m-1$ is even, $w_1(RP(\eta^{m-j})) = w_1 + \binom{m-j}{1}c + v_1 = w_1+c+v_1$ and $(w_1+c+v_1+c)^{m-2}(w_1+c+v_1) = (w_1+v_1)^{m-2}(w_1+c+v_1)$. But $j < m-1$ and $m-1$ and j even give $j \le m-3$. Hence $(w_1+v_1)^{m-2} = 0$ and $P([\lambda \to RP(\eta^{m-j})] = 0$. Thus $P([\lambda \to RP(\eta^{m-1})] = 1$, which means that F has an $(m-1)$-component and $k=1$. This completes the proof.

Now we prove that, if the bound $k \le j+1$ of the conjecture is valid, then it is best possible. Let $(RP^{j+n+1}, T_{j,n})$ be the involution defined in homogeneous coordinates by

$$T_{j,n}[x_0,x_1,\ldots,x_{j+n+1}] = [-x_0,-x_1,\ldots,-x_j,x_{j+1},\ldots,x_{j+n+1}].$$

The fixed point set of $T_{j,n}$ with normal bundles is $((n+1)\lambda_j \to RP^j) \sqcup ((j+1)\lambda_n \to RP^n)$. If $j \ge 1$ is odd and $n > 1$ is even, then $w_1((n+1)\lambda_j) = \alpha_j$ and so $(w_1((n+1)\lambda_j))^j[RP^j] = 1$, which means that $(n+1)\lambda_j$ does not bound. Also, $(j+1)\lambda_n$ does not bound because RP^n does not bound. Thus, the bound $k \le j+1$ corresponding to part 1) of the conjecture is best possible. Now, if $j \ge 0$ is even and $n > j$ is odd, by similar reasons the bundles $(n+1)\lambda_j$ and

$(j+1)\lambda_n$ do not bound, and thus the bound corresponding to part 2) of the conjecture is also best possible.

3. **Examples with many components**

In this section we show that there are examples corresponding to the $j=0$ and $j=1$ cases with $\pi_0(F)$ maximal. First we consider the $j=0$ case; take $m \geq 2$ even. If $m=2$, the involution $(RP^2, T_{0,1})$ works, so we suppose $m \geq 4$. For each $1 \leq i \leq m-3$ odd, the bundle $\lambda_i \oplus ((m-i-1)R) \to RP^i$ does not bound. Consider the cobordism class $X = [mR \to \{point\}] + \sum_{i=1}^{m-3}[\lambda_i \oplus ((m-i-1)R) \to RP^i]$, where in the sum each $1 \leq i \leq m-3$ is odd. Call $\partial(X) = Y \in \mathcal{N}_{m-1}(BO(1))$. Then $\partial(X+Y) = Y+Y = 0$, which means that the disjoint union $(mR \to \{point\}) \cup (\cup_{i=1}^{m-3} \lambda_i \oplus ((m-i-1)R) \to RP^i) \cup (\lambda \to V^{m-1})$, where in the sum each $1 \leq i \leq m-3$ is odd and $\lambda \to V^{m-1}$ is a representative of Y, is the fixed point set with normal bundles of some involution (M,T), where M is m-dimensional. From Section 2, one has $P(Y) = 1$, where P is the homomorphism corresponding to the polynomial $P = (x_1 + t)^{m-1}$. Then $Y \neq 0$ and (M,T) is the required example.

Now we consider the $j=1$ case; first take $m \geq 3$ odd. In this case, the argument is quite similar to the above; in fact, consider the cobordism class $X = [mR \to \{point\}] + [\lambda_1 \oplus ((m-2)R) \to RP^1] + \sum_{i=2}^{m-3}[\lambda_i \oplus ((m-i-1)R) \to RP^i]$, where in the sum each $2 \leq i \leq m-3$ is even. Call $\partial(X) = Y \in \mathcal{N}_{m-1}(BO(1))$. Then $(mR \to \{point\}) \cup (\lambda_1 \oplus ((m-2)R) \to RP^1) \cup (\cup_{i=2}^{m-3} \lambda_i \oplus ((m-i-1)R) \to RP^i) \cup (\lambda \to V^{m-1})$, where in the sum each $2 \leq i \leq m-3$ is even and $\lambda \to V^{m-1}$ is a representative of Y, is the fixed point set with normal bundles of some involution (M,T), which again is the required example.

Finally, take $m \geq 4$ even. In this case, the argument is a bit more elaborate. For $0 \leq i \leq m-1$, let $A_i = [(m-i)\lambda_i \to RP^i] \in \mathcal{N}_i(BO(m-i))$. Note that $RP((m-i)\lambda_i)$ can be identified with $RP^i \times RP^{m-i-1}$, and the Hopf line bundle $\lambda \to RP((m-i)\lambda_i)$ with the tensor product $\lambda_i \otimes \lambda_{m-i-1}$ (again we are suppressing bundle maps). From the symmetry, we see that $\partial(A_i) = \partial(A_{m-i-1})$. In particular, $\partial(A_1) = \partial(A_{m-2})$, and since $m-1$ is odd, $A_1 = [(m-1)\lambda_1 \to$

$RP^1] = [\lambda_1 \oplus (m-2)R \to RP^1]$ is nonzero. Now, in general, suppose $\eta \to N$ is an $2r$-dimensional vector bundle over an n-dimensional smooth closed manifold N. Suppose that η admits a complex structure as a complex vector bundle of dimension r. Consider $CP(\eta)$ the total space of the complex projective bundle associated to η, with real dimension $n+2r-2$, and $\mu^2 \to CP(\eta)$ the complex Hopf line bundle, with real dimension 2. Denote by $C \to N$ the one dimensional trivial complex vector bundle over N. Consider the involution $CP(\eta \oplus C) \to CP(\eta \oplus C)$ given by $[(v,z)] \to [(v,-z)]$. The fixed point set with normal bundles of this involution is $(\eta \to N) \cup (\mu^2 \to CP(\eta))$, so $\partial([\eta \to N]) = \partial([\mu^2 \to CP(\eta)])$.

With these ingredients on hand, we are ready to present the desired example. Write $m = 2n$, and consider $0 \leq i \leq n-2$. Then the vector bundle $(n-i)(\lambda_{2i} \otimes C) \to RP^{2i}$ has a structure of complex vector bundle, with complex dimension $n-i$ (for $i=0$, it is $nC \to \{point\}$). Viewed as a real vector bundle, it represents an element of $\mathcal{N}_{2i}(BO(m-2i))$, $0 \leq i \leq n-2$, which is nonzero because RP^{2i} does not bound. Write $\mu_i^2 \to CP((n-i)(\lambda_{2i} \otimes C))$ for the corresponding complex Hopf line bundle, which viewed as a real vector bundle represents an element of $\mathcal{N}_{m-2}(BO(2))$. Set $X = \sum_{i=0}^{n-2}[(n-i)(\lambda_{2i} \otimes C) \to RP^{2i}]$ and Y$=\sum_{i=0}^{n-2}[\mu_i^2 \to CP((n-i)(\lambda_{2i} \otimes C))]$. Then $\partial(X) = \partial(Y)$, and thus $\partial(X+Y+A_1+A_{m-2}) = 0$. It follows that $(\lambda_1 \oplus (m-2)R \to RP^1) \cup (\cup_{i=0}^{n-2}(n-i)(\lambda_{2i} \otimes C) \to RP^{2i}) \cup (\eta^2 \to V^{m-2})$, where $\eta^2 \to V^{m-2}$ is a representative of $Y + A_{m-2}$, is the fixed point set with normal bundles of an involution (M,T), where $dim(M) = m$. From Section 2, one has $P(\partial([\eta^2 \to V^{m-2}])) = 1$, where P is the homomorphism corresponding to the polynomial $P = x_1^{m-2}t$. Then $[\eta^2 \to V^{m-2}] \neq 0$ and (M,T) is the desired example.

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