GLOBAL L^q-GEVREY FUNCTIONS AND THEIR APPLICATIONS

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Abstract. The goal of this paper is to introduce a class of C^∞ functions whose derivatives satisfy quantitative size estimates. The estimates, called global L^q Gevrey estimates, first arose in the work of Boggess and Raich [BR13] when they investigated how to capture a particular type of exponential decay through estimates on the Fourier transform. In the present work, we refine the notion of global L^q-Gevrey functions and include a discussion of the function theory as well as the relationship to Gevrey classes and known function spaces. Additionally, we present explicit examples of global L^q-Gevrey functions and ways to generate new global L^q-Gevrey functions from old ones. We conclude with three applications: The first is solving a Carleman-type problem for constructing functions whose derivatives are a given sequence of global L^q-Gevrey functions. The other two applications concern extensions of a given global L^q-Gevrey function: the first is constructing an almost analytic extension, and the second is building an approximate solution to a first-order complex vector field whose coefficients are global L^q-Gevrey functions.

1. Introduction

The purpose of this paper is to introduce classes of functions that we call global L^q-Gevrey functions. We develop their function theory as well as find applications to almost analytic extensions and approximate solutions of certain first order linear partial differential operators.

An early version of these functions first arose in the work of Boggess and Raich [BR13] when they bounded the $\Box_b$-heat kernel on decoupled polynomial models in $\mathbb{C}^n$. In $\mathbb{C}^2$, a polynomial model is a CR manifold of the form

$$M = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = p(z)\}$$

where $p : \mathbb{C} \to \mathbb{R}$ is a polynomial so that $\triangle p \geq 0$. Such models serve as local models of CR manifolds of finite type as well as providing a model case of an unbounded domain. If $p(z) = |z|^2$, then M is the Heisenberg group. M is isomorphic to $\mathbb{C} \times \mathbb{R}$ and under this isomorphism, the Kohn Laplacian can be identified with an operator that is translation invariant in $\operatorname{Re} w$. As such, it is natural to investigate the Kohn Laplacian via a partial Fourier transform in $\operatorname{Re} w$. This has been done by several authors [Nag86, Chr91, Has94, Has98, Ber96, Rai06b, Rai06a, Rai07, Rai10, Rai12, BR13, HNW10, RT15, Pet]. In [BR13], Boggess and Raich proved that the $\Box_b$-heat kernel on polynomial models in $\mathbb{C}^2$ satisfies Gaussian decay estimates when the decay is phrased in terms of the nonisotropic control

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metric. It turns out that at points when the manifold is flat in the sense that the point is of higher type, the decay is essentially of the form $e^{-a|t|^{1/\beta}}$ for some $\beta > 1$. Since Boggess and Raich analyzed the kernel using a partial Fourier transform, they needed to determine how to recover exponential decay via a Fourier transform. Had it been the case that $\beta = 1$, then the answer is classical: the heat kernel would be holomorphic in a strip.

Taking a hint from Gelfand and Shilov [GS67], Boggess and Raich show that the estimate

$$\|\varphi^{(\ell)}\|_{L^q(\mathbb{R})} \leq CA^\ell \ell^{\ell\beta} \tag{1}$$

is *sufficient* to prove that $|\hat{\varphi}(t)| \leq C' e^{-a|t|^{1/\beta}}$ where $A = (\frac{\beta}{ae})^\beta$ when $q = 1$ and *necessary* when $q = \infty$. They called such estimates quantitative smoothness estimates and they seeded the idea for the global L^q-Gevrey functions. In the $q = \infty$ case, it follows from the Denjoy-Carleman Theorem that the class of functions that satisfy (1) for some $\beta > 1$ cannot reside in any quasianalytic class (see Proposition 2.6 below). Furthermore, they are more special than the standard Gevrey class of order β because the constants C and A depend on φ and Ω but not any compact set $K \subset \Omega$.

The rest of the paper is organized as follows. In Section 2, we define spaces of global L^q-Gevrey functions and investigate their function theory. In Section 3, we use the Fourier transform to explore the relationship between the global L^q-Gevrey function classes and exponential decay of the Fourier transform. In Section 4, we build explicit examples of global L^q-Gevrey functions for all $q \in [1, \infty]$ and $\beta > 1$. In Section 5, we show that global L^q-Gevrey function classes are preserved under appropriate diffeomorphisms. Finally, in Section 6, we solve a Carleman-type problem for global L^q-Gevrey functions, namely, that given a sequence of global L^q-Gevrey functions $\{v_k(x)\}$ on a domain $\Omega \subset \mathbb{R}^d$, there exists a global L^q-Gevrey function $f(x,t)$ on $\Omega \times (-1, 1)$ so that the $\frac{\partial^k f}{\partial t}|_{t=0} = v_k$. We then use this solution to build almost analytic extensions of a given global L^q-Gevrey function on a domain $\Omega \subset \mathbb{R}^d$, and, more generally, an approximate solution to any appropriate first order differential operator. The proofs in section 6 are inspired by similar proofs in [AH1].

2. Definition and basic properties of $\mathcal{G}^{q,\beta}(\Omega)$

2.1. **Definitions.** Let $\Omega \subset \mathbb{R}^d$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers, positive constants $A, \beta > 0$ and $1 \leq q \leq \infty$, define the seminorm $\varrho_{\alpha,A,\Omega,q,\beta} : C^\infty(\Omega) \to [0, \infty)$

$$\varrho_{\alpha,A,\Omega,q,\beta}(g) = \varrho_\alpha(g) = \frac{\|D^\alpha g\|_{L^q(\Omega)}}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}.$$

We suppress as many indices for the ϱ_α as possible.

Definition 2.1. Let $1 \leq q \leq \infty$ and $\beta \geq 0$. A function $g \in C^\infty(\Omega)$ is said to satisfy *global L^q-Gevrey estimates of order β* if there exist constants $A, C > 0$ so that for every d-tuple of nonnegative integers α

$$\|D^\alpha g\|_{L^q(\Omega)} \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta}.$$

We say that a function that satisfies global L^q-Gevrey estimates of order β is a *global L^q-Gevrey function of order β*. For a fixed $A > 0$, we set

$$\mathcal{G}^{q,\beta}_A(\Omega) = \left\{ g \in C^\infty(\Omega) : \{\varrho_{\alpha,A,\Omega,q,\beta}(g)\}_{|\alpha|\geq 0} \in \ell^q(\mathbb{Z}^d_{\geq 0}) \right\}$$

and

$$\mathcal{G}^{q,\beta}(\Omega) = \bigcup_{A>0} \mathcal{G}^{q,\beta}_A(\Omega).$$

We recall Stirling's Approximation which says that $\sqrt{2\pi} n^n \sqrt{n} \leq e^n n! \leq e n^n \sqrt{n}$. This inequality in turn implies

$$n^n \leq e^n n! \leq e^n n^n. \tag{2}$$

Proposition 2.2. *$\mathcal{G}^{q,\beta}_A(\Omega)$ is a Banach space with norm $\|\cdot\|_{\mathcal{G}^{q,\beta}_A(\Omega)}$ defined by*

$$\|g\|_{\mathcal{G}^{q,\beta}_A(\Omega)} = \Big(\sum_{|\alpha|\geq 0} \varrho_\alpha(g)^q\Big)^{1/q}.$$

Proof. That $\|\cdot\|_{\mathcal{G}^{q,\beta}_A(\Omega)}$ is a norm follows immediately from the fact that $L^q(\Omega)$ and $\ell^q(\mathbb{Z}^d_{\geq 0})$ are both normed spaces. We now show completeness. Let $\{g_k\}$ be a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{G}^{q,\beta}_A(\Omega)}$. Observe that $\mathcal{G}^{q,\beta}_A$ embeds continuously in $W^{m,q}(\Omega)$ for every $m \geq 0$, so $\{g_k\}$ is Cauchy in $W^{m,q}(\Omega)$ for every $m \geq 0$. Thus, there exists $g \in W^{m,q}(\Omega)$ so that $g_k \to g$ in $W^{m,q}(\Omega)$ for every m. The limit g is smooth by the Sobolev Embedding Theorem.

We prove by contradiction that $g \in \mathcal{G}^{q,\beta}_A(\Omega)$. If $\limsup_{k\to\infty} \|g_k - g\|_{\mathcal{G}^{q,\beta}_A} = 100\delta > 0$, then there exists a $k \in \mathbb{N}$ so that

$$99\delta < \|g_k - g\|_{\mathcal{G}^{q,\beta}_A(\Omega)} < 101\delta \quad \text{and} \quad \|g_k - g_n\|_{\mathcal{G}^{q,\beta}_A(\Omega)} < \delta$$

for all $n \geq k$.

Since k is fixed, there exists $m \in \mathbb{N}$ so that

$$\Big(\sum_{|\alpha|\leq m} \varrho_{\alpha,A,\Omega}(g_k - g)^q\Big)^{1/q} > 98\delta.$$

We know that $g_n \to g$ in $W^{m,q}(\Omega)$, so choose $n \geq k$ sufficiently large that

$$\Big(\sum_{|\alpha|\leq m} \varrho_{\alpha,A,\Omega}(g_n - g)^q\Big)^{1/q} < \delta.$$

Then

$$\|g_k - g\|_{\mathcal{G}^{q,\beta}_A(\Omega)} \leq 2\Big(\sum_{|\alpha|\leq m} \varrho_\alpha(g_k - g)^q\Big)^{1/q}$$
$$\leq 2\|g_k - g_n\|_{\mathcal{G}^{q,\beta}_A(\Omega)} + 2\Big(\sum_{|\alpha|\leq m} \varrho_\alpha(g_n - g)^q\Big)^{1/q} < 4\delta.$$

This is a contradiction. Hence $g_k \to g$ in $\mathcal{G}^{q,\beta}_A(\Omega)$. □

Proposition 2.3. *Let $g \in \mathcal{G}^{q,\beta}_A(\Omega)$. If J is a multiindex and $A' > A$, then $D^J g \in \mathcal{G}^{q,\beta}_{A'}(\Omega)$. In particular, $\mathcal{G}^{q,\beta}(\Omega)$ is closed under differentiation.*

Proof. The proof follows from a computation.

$$\|D^\alpha D^J g\|_{L^q(\Omega)} \leq C A^{|\alpha|+|J|}(|\alpha| + |J|)^{(|\alpha|+|J|)\beta} = C A^{|\alpha|+|J|}\Big(1 + \frac{|J|}{|\alpha|}\Big)^{(|\alpha|+|J|)\beta} |\alpha|^{|J|\beta} |\alpha|^{|\alpha|\beta}.$$

Since

$$\Big(1+\frac{|J|}{|\alpha|}\Big)^{(|\alpha|+|J|)\beta} \le \Big(1+\frac{|J|}{|\alpha|}\Big)^{|\alpha|\beta}\Big(1+\frac{|J|}{|\alpha|}\Big)^{|J|\beta} \le C_{J,\alpha} e^{|J|\beta} 2^{|J|\beta}$$

where $C_{J,\alpha} = \max\{(1+\frac{|J|}{|\alpha|})^{|J|\beta} : |\alpha| \le |J|\}$. Also, given any $\epsilon > 0$, there exists $C' > 0$ so that $|\alpha|^{|J|} \le C'(1+\epsilon)^{|\alpha|}$, and the result follows. □

Lemma 2.4. *Let $1 \le q \le \infty$ and $f \in \mathcal{G}^{q,\beta}(\Omega)$. If $g \in \mathcal{G}^{\infty,\beta}(\Omega)$, then $h = fg \in \mathcal{G}^{q,\beta}(\Omega)$. Similarly, if $f \in \mathcal{G}^{q,\beta}(\Omega)$ and $g \in \mathcal{G}^{q',\beta}(\Omega)$ then $fg \in \mathcal{G}^{1,\beta}(\Omega)$.*

Proof. There exists A' so that $f \in \mathcal{G}^{q,\beta}_{A'}(\Omega)$ and $g \in \mathcal{G}^{\infty,\beta}_{A'}(\Omega)$. We now compute

$$\begin{aligned}\sum_{|\alpha|\ge 0} \rho_{\alpha,A}(h)^q &\le \sum_{|\alpha|\ge 0} \frac{\left(\sum_{J\subset\alpha}\binom{\alpha}{J}\|D^J f\|_{L^q(\Omega)}\|D^{\alpha-J}g\|_{L^\infty(\Omega)}\right)^q}{A^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} \\ &\le C\sum_{|\alpha|\ge 0}\Big(\frac{A'}{A}\Big)^{q|\alpha|}\left(\sum_{J\subset\alpha}\binom{\alpha}{J}\frac{|J|^{|J|\beta}(|\alpha|-|J|)^{(|\alpha|-|J|)\beta}}{|\alpha|^{|\alpha|\beta}}\right)^q\end{aligned}$$

For a fixed α, $\sum_{J\subset\alpha}\binom{\alpha}{J} = 2^{|\alpha|}$, so it follows that if A is suitably larger than A', (e.g., $A \ge 2A'$) then the above sum is finite. By Hölder's inequality, the second part of the lemma follows analogously. □

In a similar fashion, we can also prove the following result with multiplication replaced by convolution.

Lemma 2.5. *Let $1 \le p, q, r \le \infty$ and p, q, r satisfy*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in \mathcal{G}^{q,\beta}(\Omega)$ and $g \in L^p(\Omega)$, then $h = f * g \in \mathcal{G}^{r,\beta}(\Omega)$.

Proof. The proof follows immediately from Young's inequality and the fact that $D^\alpha(f*g) = (D^\alpha f) * g$. □

Lemma 2.5 provides a way to generate many functions in $\mathcal{G}^{q,\beta}(\Omega)$.

2.2. **Functions with compact support.**

Proposition 2.6. *If $\beta > 1$, then $\mathcal{G}^{\infty,\beta}$ is not contained in any quasianalytic class.*

Proof. From [Rud87, p.377], for a positive sequence $M_0, M_1, \ldots$, the class $C\{M_\ell\}$ is the set of $f \in C^\infty$ such that there exist constants β_f, A_f such that

$$\|D^\ell f\|_{L^\infty(\mathbb{R})} \le \beta_f A_f^\ell M_\ell.$$

The Denjoy-Carleman theorem says that for $C\{M_\ell\}$ to be a quasianalytic class, $\{M_\ell\}$ must satisfy

$$\sum_{\ell=1}^{\infty} \inf_{k\ge\ell} M_k^{-1/k} = \infty.$$

In the language of quasianalytic classes, we have $M_\ell = \ell^{\ell\beta}$. However, $M_k^{-1/k} = k^{-\beta}$, so if $\beta > 1$,

$$\sum_{\ell=2}^{\infty} \inf_{k\geq\ell} M_k^{-1/k} = \sum_{\ell=2}^{\infty} \ell^{-\beta} < \infty$$

while if $\beta \leq 1$,

$$\sum_{\ell=2}^{\infty} \inf_{k\geq\ell} M_k^{-1/k} = \sum_{\ell=2}^{\infty} \ell^{-\beta} = \infty.$$

□

Corollary 2.7. *If $\beta > 1$, then $\mathcal{G}^{q,\beta}(\Omega)$ contains a nontrivial function of compact support.*

Proof. By taking products, it is enough to assume $\Omega \subset \mathbb{R}$. Since $\mathcal{G}^{\infty,\beta}(\mathbb{R})$ is the class $C\{\ell^{\ell\beta}\}$, and Proposition 2.6 shows that $\mathcal{G}^{\infty,\beta}(\mathbb{R})$ is not a quasianalytic class, [Rud87, Theorem 19.10] shows that $\mathcal{G}^{\infty,\beta}(\mathbb{R})$ contains a nontrivial function with compact support g. By replacing $g(x)$ with $g(\lambda(x - x_0))$ for appropriate $\lambda > 0$, we have $\operatorname{supp} g \subset \Omega$. Bounded functions with compact support are elements of $L^q(\Omega)$ by Hölder's inequality. □

A corollary to Corollary 2.7 is that $\mathcal{G}^{q,\beta}(\Omega)$ contains bump functions.

Corollary 2.8. *If $\beta > 1$ and $r > 0$, then there exists $\varphi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ so that:*

(1) $\varphi \in C_c^\infty(B(0, 2r))$;
(2) $\varphi \equiv 1$ on $B(0, r)$.

Proof. By taking products and scaling, it suffices to prove this for $r = 1$ and $d = 1$. By Corollary 2.7, there exists a function $g \in \mathcal{G}^{q,\beta}(\mathbb{R}) \cap C_c(\mathbb{R})$. By Lemma 2.4, it follows that $f = g\bar{g} \in \mathcal{G}^{q,\beta}(\mathbb{R}) \cap C_c(\mathbb{R})$. By translating and dilating, we may assume that $\operatorname{supp} f \in [-1, 0]$. Since $f \geq 0$, the function

$$g_1(x) = \frac{\int_{-1}^{x} f(y)\, dy}{\int_{-1}^{1} f(t)\, dt}$$

has the properties

(i) $g_1' \in \mathcal{G}^{q,\beta}(\mathbb{R})$;
(ii) $g_1(x) = 0$ if $x \leq -1$;
(iii) $0 \leq g_1(x) \leq 1$ if $-1 \leq x \leq 0$; and
(iv) $g_1(x) = 1$ if $x \geq 0$.

Finally, the function $\varphi(x) = g_1(x + 1)g_1(1 - x)$ satisfies (1) and (2) from the statement of the corollary, when $r = 1$. □

Remark 2.9. Although we can construct "good" bump functions, the scaling that decreases r causes an increase of A to offset the scaling of the derivatives. On the flip side, increasing r decreases A. To maintain control over A (but possibly lose control over C), we could take a good bump function from $\mathcal{G}^{q,\beta'}(\Omega)$ for some β' satisfying $1 < \beta' < \beta$.

Corollary 2.10. *Let k be a nonnegative integer, then there exists a compactly supported function $f \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ with $1 \leq q \leq \infty$ and $\beta > 1$, such that*

$$\int x^\alpha f(x) dx = 0, \qquad |\alpha| \leq k. \tag{3}$$

Proof. Assume, without loss of generality, that $d = 1$. By Corollary 2.8, there exists $\varphi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ so that $0 \leq \varphi \in C_c^\infty([0,1])$. Define

$$\psi_0(x) = \varphi(x) - \varphi(-x)$$

then ψ_0 satisfies (3) for $k = 0$.

The function $\psi_0(x-1)$ is supported in $\mathbb{R}^+$ and satisfies (3) for $k = 0$ and if we extend it in a even fashion to $x \leq 0$,

$$\psi_1(x) = \psi_0(x-1) + \psi_0(-x+1).$$

we will also have $\int x\psi_1(x)dx = 0$.

For $k = 2$, we note that $\psi_1(x-2)$ is supported in $\mathbb{R}^+$ and satisfies (3) for $k = 1$, therefore, if we extend it in an odd fashion

$$\psi_2(x) = \psi_1(x-2) - \psi_1(-x+2).$$

clearly $\int x^2\psi_2(x)dx = 0$.

This procedure can be further continued as we will now show. If ψ_k is constructed to satisfy (3) and is supported in $[-2^{k-1}, 2^{k-1}]$ then $\psi_k(x - 2^{k-1})$ will be supported in $[0, 2^k]$. If we extend it either in an odd fashion or in an even fashion accordingly if k is odd or even respectively we obtain a function ψ_{k+1} that will satisfy (3) for $k+1$. To be more precise, define

$$\psi_{k+1} = \begin{cases} \psi_k(x-2^k) + \psi_k(-x+2^k) & \text{if} \quad k \text{ is even} \\ \psi_k(x-2^k) - \psi_k(-x+2^k) & \text{if} \quad k \text{ is odd.} \end{cases}$$

This concludes the proof of the lemma. ☐

2.3. $\mathcal{G}^{q,\beta}(\Omega)$ **as DF-spaces.** We will start with the following density result. We say that a function f has *bounded support in* Ω if there exists a function $\phi : \mathbb{R}^d \to \mathbb{R}$ with compact support in $\mathbb{R}^d$ so that $f\phi = f$ on Ω.

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^d$. For every $1 \leq q \leq \infty$, $\beta > 0$, and $A > 0$, and for every $A' > 2A$ and $u \in \mathcal{G}_A^{q,\beta}(\Omega)$, there exists a sequence $\{u_m\} \subset \mathcal{G}_{A'}^{q,\beta}(\Omega)$ of functions with bounded support in Ω such that $u_m \to u \in \mathcal{G}_{A'}^{q,\beta}(\Omega)$.*

Proof. Suppose that $u \in \mathcal{G}_A^{q,\beta}(\Omega)$ is given and $\epsilon > 0$. Let $\phi \in C_c^\infty(\mathbb{R}^d) \cap \mathcal{G}_A^{\infty,\beta}(\mathbb{R}^d)$ and satisfy $\phi \equiv 1$ on $B(0,1)$ Set $\phi_m(x) = \phi(x/m)$. Define u_m by $u_m(x) = u(x)\phi_m(x)$. It is immediate that u_m has bounded support in Ω. We claim that $u_m \to u$ in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$ for any $A' > 2A$. Using the argument of Lemma 2.4, we estimate that

$$\begin{aligned}\sum_{|\alpha|\geq 0} \rho_{\alpha,A'}(u(1-\phi_m))^q &\leq \sum_{|\alpha|\geq 0} \frac{\|(1-\phi_m)D^\alpha u\|_{L^q(\Omega)}^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} \\ &\quad + \sum_{|\alpha|\geq 0} \frac{(\sum_{\substack{J\subset\alpha \\ |J|\geq 1}} \binom{\alpha}{J} \|D^{\alpha-J}u\|_{L^q(\Omega)} \|D^J\phi_m\|_{L^\infty(\Omega)})^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} \\ &\leq \sum_{|\alpha|\geq 0} \frac{\|(1-\phi_m)D^\alpha u\|_{L^q(\Omega)}^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} + \frac{C}{m} \sum_{|\alpha|\geq 0} \left(\frac{A}{A'}\right)^{q|\alpha|} \left(\sum_{\substack{J\subset\alpha \\ |J|\geq 1}} \binom{\alpha}{J} \frac{|J|^{|J|\beta}(|\alpha|-|J|)^{(|\alpha|-|J|)\beta}}{|\alpha|^{|\alpha|\beta}} \right)^q\end{aligned}$$

The second sum tends to 0 as $m \to \infty$. To estimate the first sum, since $u \in \mathcal{G}_A^{q,\beta}(\Omega)$, there exists K so that

$$\sum_{|\alpha|\geq K} \rho_{\alpha,A'}(u)^q < \epsilon/2.$$

Also, since the operator given by multiplication by $1-\varphi_m$ tends to 0 in $L^q(\Omega)$, it follows that

$$\sum_{|\alpha|\geq 0} \frac{\|(1-\phi_m)D^\alpha u\|_{L^q(\Omega)}^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} = \sum_{|\alpha|\leq K} \frac{\|(1-\phi_m)D^\alpha u\|_{L^q(\Omega)}^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} + \sum_{|\alpha|\geq K} \frac{\|D^\alpha u\|_{L^q(\mathbb{R}^d)}^q}{(A')^{q|\alpha|}|\alpha|^{q|\alpha|\beta}} < \epsilon$$

for m suitably large. □

Remark 2.12. The Lemma above shows that the class of functions in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$ with bounded support in Ω is dense in $\mathcal{G}_A^{q,\beta}(\Omega)$ under the continuous inclusion $\mathcal{G}_A^{q,\beta}(\Omega) \hookrightarrow \mathcal{G}_{A'}^{q,\beta}(\Omega)$,

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^d$, $1 \leq q \leq \infty$ and $\beta > 0$. If $2A < A'$ then the inclusion*

$$\mathcal{G}_A^{q,\beta}(\Omega) \hookrightarrow \mathcal{G}_{A'}^{q,\beta}(\Omega)$$

is compact.

Proof. The inclusion is continuous. To prove that it is also compact, we will show that if $\{u_m\}_{m\in\mathbb{N}}$ is a sequence in the (closed) unit ball of $\mathcal{G}_A^{q,\beta}(\Omega)$ then there exist a subsequence $\{u_{m_j}\}_{j\in\mathbb{N}}$ which converges in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$. Consider the approximations $u_{m,M}(x) = u_m(x)\phi_M(x)$ where ϕ is an in Lemma 2.11. It will be enough to show that the double sequence $\{u_{m,M}\}_{(m,M)\in\mathbb{N}^2}$ has the following two properties:

(i) For any fixed $M \in \mathbb{N}$, the sequence $\{u_{m,M}\}_{m\in\mathbb{N}}$ is a precompact subset of $\mathcal{G}_{A'}^{q,\beta}(\Omega)$.
(ii) $u_{m,M} \to u_m$ in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$ uniformly in m as $M \to \infty$.

Note that $\{u_{m,M}\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$, uniformly in M as a consequence of (the proof of) Lemma 2.4 and the uniform boundedness of $\{u_m\} \subset \mathcal{G}_A(\Omega)$. Moreover, for each fixed M, $\operatorname{supp}(u_{m,M}) \subset B(0, Mr)$, for some fixed $r > 0$ and for every $m \in \mathbb{N}$.

Let $\delta > 0$ and $K \in \mathbb{N}$ with $K - \frac{d}{q} > 0$ and such that

$$\left\{\sum_{|\alpha|\geq K} (A/A')^{|\alpha|q}\right\}^{1/q} \leq \delta/2C$$

where C satisfies $\|u_m\|_{\mathcal{G}_A^{q,\beta}(\Omega)} \leq C$. Also, let q be a norm on $\mathcal{G}_{A'}^{q,\beta}(\Omega)$ defined by

$$q(f) = \left\{\sum_{|\alpha|\leq K+d} \left(\varrho_{\alpha,A'}(g)\right)^q\right\}^{1/q} = \left\{\sum_{|\alpha|\leq K+d} \left(\frac{\|D^\alpha g\|_{L^q}}{A'^{|\alpha|}|\alpha|^{|\alpha|\beta}}\right)^q\right\}^{1/q}.$$

Since $\{u_{m,M}\}_m$ is uniformly bounded in $\mathcal{G}_{A'}^{q,\beta}(\Omega)$, it is also uniformly bounded under q. Since q is equivalent to the standard norm on $W^{K+d,q}(\Omega)$, it follows from the Rellich-Kondrachov compactness theorem that $\mathcal{G}_{A'}^{q,\beta}(\Omega) \hookrightarrow C^K(\Omega)$ compactly. We have therefore shown that for each $M \in \mathbb{N}$, $\{u_{m,M}\}_{m\in\mathbb{N}}$ is a precompact subset of $\mathcal{G}_{A'}^{q,\beta}(\Omega)$ with the topology associated with the norm q; in other words, there exists a subsequence $\{u_{m_j,M}\}_{j\in\mathbb{N}}$ of $\{u_{m,M}\}_{m\in\mathbb{N}}$ such that for j,ℓ big enough

$$q(u_{m_j,M} - u_{m_\ell,M}) \le \delta.$$

Finally, if $|\alpha| > K$, then

$$\left\{\sum_{|\alpha|\ge K}\left(\frac{\|D^\alpha u_{m_j,M} - D^\alpha u_{m_\ell,M}\|_{L^q}}{A'^{|\alpha|}|\alpha|^{|\alpha|\beta}}\right)^q\right\}^{1/q}$$
$$\le \left\{\sum_{|\alpha|\ge K}\left(\frac{\|D^\alpha u_{m_j,M}\|_{L^q}}{A'^{|\alpha|}|\alpha|^{|\alpha|\beta}}\right)^q\right\}^{1/q} + \left\{\sum_{|\alpha|\ge K}\left(\frac{\|D^\alpha u_{m_\ell,M}\|_{L^q}}{A'^{|\alpha|}|\alpha|^{|\alpha|\beta}}\right)^q\right\}^{1/q}$$
$$\le 2C\left\{\sum_{|\alpha|\ge K}\left(\frac{A}{A'}\right)^{|\alpha|q}\right\}^{1/q} \le \delta.$$

Moreover

$$\|u_{m_j,M} - u_{m_\ell,M}\|_{\mathcal{G}^{q,\beta}_{A'}} = \left(\sum_{|\alpha|\ge 0}\varrho_{\alpha,A'}(u_{m_j,M} - u_{m_\ell,M})^q\right)^{1/q} \le 2\delta.$$

The proof (ii) is included in the proof of Lemma 2.11. ☐

Corollary 2.14. *Let $\Omega \subset \mathbb{R}^d$ and $1 \le q \le \infty$. If $A' > 2A$, then $\mathcal{G}^{q,\beta}_A(\Omega)$ is a proper subset of $\mathcal{G}^{q,\beta}_{A'}(\Omega)$.*

Corollary 2.15. *For every $1 \le q \le \infty$ and $\beta > 0$, the spaces $\mathcal{G}^{q,\beta}(\Omega)$ are DF-spaces.*

Proof. It follows from Komatsu [Kom67, Theorem 6′]. See also Grothendieck [Gro54]. ☐

2.4. **The dual space.** Recall that a *Baire space* X is one with the property that the union of any countable collection of closed sets with empty interior has empty interior. This definition is equivalent to the statement that whenever the union of countably many closed subsets of X has an interior point, then one of the closed subsets must have an interior point.

Proposition 2.16. *$\mathcal{G}^{q,\beta}(\Omega)$ is not a Baire space.*

Proof. The proof of this proposition is implicit in [Trè67, Remark 13.1]. $\mathcal{G}^{q,\beta}_n(\Omega)$ is a complete (and therefore closed) subspace of $\mathcal{G}^{q,\beta}(\Omega)$ since the topology induced on $\mathcal{G}^{q,\beta}_n(\Omega)$ agrees with the norm topology on $\mathcal{G}^{q,\beta}_n(\Omega)$ (see [Trè67, Lemma 13.1]). Thus, as $\mathcal{G}^{q,\beta}(\Omega)$ is the countable union of closed subsets, at least one of the $\mathcal{G}^{q,\beta}_n(\Omega)$ would contain an interior point, $\mathcal{G}^{q,\beta}_{n_0}(\Omega)$, if $\mathcal{G}^{q,\beta}(\Omega)$ were Baire. Let $g_0 \in \mathcal{G}^{q,\beta}_{n_0}(\Omega)$ be that interior point, i.e., there is a neighborhood $U \subset \mathcal{G}^{q,\beta}(\Omega)$ of g_0 so that $U \subset \mathcal{G}^{q,\beta}_{n_0}(\Omega)$.

Since $g \mapsto g - g_0$ is a homeomorphism of $\mathcal{G}^{q,\beta}(\Omega)$ onto itself, it follows that the origin is an interior point of $\mathcal{G}^{q,\beta}(\Omega)$. This means that $\mathcal{G}^{q,\beta}_{n_0}(\Omega)$ is a neighborhood of the origin. As a neighborhood of zero is absorbing, $\mathcal{G}^{q,\beta}_{n_0}(\Omega)$ is absorbing. Since $\mathcal{G}^{q,\beta}_{n_0}(\Omega)$ is a linear space, this forces $\mathcal{G}^{q,\beta}(\Omega) = \mathcal{G}^{q,\beta}_{n_0}(\Omega)$. However, $\mathcal{G}^{q,\beta}_{n_0}(\Omega)$ is strictly contained in $\mathcal{G}^{q,\beta}_{2n_0}(\Omega)$ by Corollary 2.14, hence $\mathcal{G}^{q,\beta}(\Omega) \ne \mathcal{G}^{q,\beta}_{n_0}(\Omega)$. ☐

2.5. **The dual spaces $\mathcal{G}^{q,\beta}(\Omega)'$ and $\mathcal{G}^{q,\beta}_A(\Omega)'$.** It is immediate that if $A \leq A'$, then $\mathcal{G}^{q,\beta}_A \hookrightarrow \mathcal{G}^{q,\beta}_{A'}$. Since Banach spaces are F-spaces, the space

$$\mathcal{G}^{q,\beta}(\Omega) = \bigcup_{A>0} \mathcal{G}^{q,\beta}_A(\Omega) = \bigcup_{n=1}^{\infty} \mathcal{G}^{q,\beta}_n(\Omega)$$

has a well-understood topology. Recall the following discussion from [Trè67]. Since $\mathcal{G}^{q,\beta}(\Omega)$ is a complex vector space and a union of an increasing sequence of F-spaces $\mathcal{G}^{q,\beta}_n(\Omega)$, $\mathcal{G}^{q,\beta}(\Omega)$ is an *LF space*. LF spaces have the structure of a Hausdorff locally convex space that is complete. A convex set V is a neighborhood of $0 \in \mathcal{G}^{q,\beta}(\Omega)$ if and only if $V \cap \mathcal{G}^{q,\beta}_n(\Omega)$ is a neighborhood of $0 \in \mathcal{G}^{q,\beta}_n(\Omega)$. That $\mathcal{G}^{q,\beta}(\Omega)$ is an LF space will be critical in determining its dual space. In particular, we know from [Trè67] that

Proposition 2.17. *A linear map on $\mathcal{G}^{q,\beta}(\Omega)$ is continuous if and only if its restriction to every $\mathcal{G}^{q,\beta}_A(\Omega)$ is continuous. Formally, this means*

$$\mathcal{G}^{q,\beta}(\Omega)' = \Big(\bigcup_{A>0} \mathcal{G}^{q,\beta}_A(\Omega)\Big)' = \bigcap_{A>0} \mathcal{G}^{q,\beta}_A(\Omega)'.$$

We turn to finding the dual space of $\mathcal{G}^{q,\beta}_A(\Omega)$. Our discussion uses the Hahn-Banach Theorem and the embedding of $\mathcal{G}^{q,\beta}_A(\Omega)$ into $\ell^q(L^q(\Omega))$ where

$$\ell^q(L^q(\Omega)) = \{g = (g_\alpha) \in L^q(\Omega)^\omega : \sum_{|\alpha|\geq 0} \|g_\alpha\|^q_{L^q(\Omega)} < \infty\}.$$

Lemma 2.18. *If $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The dual space*

$$\ell^q(L^q(\Omega))' = \ell^p(L^p(\Omega)).$$

If $q = \infty$, then

$$\ell^\infty(L^\infty(\Omega))' = (\ell^\infty)'(L^\infty(\Omega)').$$

Proof. First assume that $1 \leq q < \infty$. Let $f = (f_\alpha) \in \ell^p(L^p(\Omega))$. Then by two uses of Hölder's inequality

$$\begin{aligned}\big|\langle f, g\rangle\big| = \Big|\sum_{|\alpha|\geq 0} \int_\Omega g_\alpha \overline{f_\alpha}\, dx\Big| &\leq \sum_{|\alpha|\geq 0} \|f_\alpha\|_{L^p(\Omega)} \|g_\alpha\|_{L^q(\Omega)} \\ &\leq \Big(\sum_{|\alpha|\geq 0} \|f_\alpha\|^p_{L^p(\Omega)}\Big)^{1/p} \Big(\sum_{|\alpha|\geq 0} \|g_\alpha\|^q_{L^q(\Omega)}\Big)^{1/q} = \|f\|_{\ell^p(L^p(\Omega))} \|g\|_{\ell^q(L^q(\Omega))}.\end{aligned}$$

Conversely, let $T \in \ell^q(L^q(\Omega))'$. Set $g = (0, \dots, 0, g_\alpha, 0, \dots)$. Then

$$|T(g)| \leq C \|g_\alpha\|_{L^q(\Omega)}.$$

Define the map $T_\alpha : L^q(\Omega) \to \mathbb{C}$ by $T_\alpha(g_\alpha) = T(0, \dots, 0, g_\alpha, 0 \dots)$. Then $T_\alpha \in L^q(\Omega)'$ and hence is given by integration against a function in $L^p(\Omega)$, say f_α. Let $f = (f_\alpha)$. We can now identify T with the vector $f = (f_\alpha)$. We claim that $f \in \ell^p(L^p(\Omega))$. We know that $f_\alpha \in L^p(\Omega)$ for each α. Our progress thus far is the statement

$$\big(L^q(\Omega)^N\big)' = L^p(\Omega)^N.$$

and $T = (T_\alpha)$ where $(T_\alpha)_{|\alpha|\le N} = (f_\alpha)_{|\alpha|\le N}$. Note, however, that $\|(T_\alpha)_{|\alpha|\le N}\| \le \|T\|$, so by the Monotone Convergence Theorem

$$\|f\|_{\ell^p(L^p(\Omega))} = \lim_{N\to\infty} \|(f_\alpha)_{|\alpha|\le N}\|_{L^p(\Omega)^{N'}}$$

where $N' = |\{\alpha : |\alpha| \le N\}|$.

The proof for $q = \infty$ is similar. $\square$

The reason that $\ell^q(L^q(\Omega))$ is relevant with regards to the spaces $\mathcal{G}^{q,\beta}_A(\Omega)$ is that there is a natural embedding $\mathcal{G}^{q,\beta}_A(\Omega) \hookrightarrow \ell^q(L^q(\Omega))$ given by

$$g \mapsto \Big(\frac{D^\alpha g}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}\Big).$$

Let the embedding be given by $j_{q,A} = j_{q,A,\Omega,\beta}$. We need to show that $j_{q,A}(\mathcal{G}^{q,\beta}_A(\Omega))$ is closed in $\ell^q(L^q(\Omega))$. Note, however, that if $g \in \mathcal{G}^{q,\beta}_A(\Omega)$, then

$$\|j_{q,A}(g)\|^q_{\ell^q(L^q(\Omega))} = \sum_{|\alpha|\ge 0} \Big\|\frac{D^\alpha g}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}\Big\|^q_{L^q(\Omega)} = \sum_{|\alpha|\ge 0} \rho_\alpha(g)^q = \|g\|^q_{\mathcal{G}^{q,\beta}_A(\Omega)}.$$

Since $\mathcal{G}^{q,\beta}_A(\Omega)$ is a Banach space, it is closed under this norm. Hence $j_{q,A}(\mathcal{G}^{q,\beta}_A(\Omega))$ is closed as well. Standard functional analysis now allows us to find $\mathcal{G}^{q,\beta}_A(\Omega)'$, e.g., see [Rud91, Theorem 4.9]. To write down the dual, recall that if X is a Banach space and M is a subset of X, then

$$M^\perp = \{x' \in X' : x'(x) = 0 \text{ for all } x \in M\}.$$

The Hahn-Banach Theorem extends each $f \in (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))'$ to a functional $F \in \ell^q(L^q(\Omega))' = \ell^p(L^p(\Omega))$. Define

$$\sigma : (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))' \to \ell^q(L^q(\Omega))'/(j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp$$

by

$$\sigma f = F + (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp.$$

A consequence of [Rud91, Theorem 4.9] is that the map σ is an isometric isomorphism of $(j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))'$ onto $\ell^q(L^q(\Omega))'/\mathcal{G}^{q,\beta}_A(\Omega)^\perp$. We now have the pieces to find $\mathcal{G}^{q,\beta}_A(\Omega)'$. Let $\pi_\alpha : \ell^q(L^q(\Omega))$ be the projection $\pi_\alpha(g) = g_\alpha$. Then $j^{-1}_{q,A} = \pi_0$. Consequently, we can view

$$\pi_0 : j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega) \to \mathcal{G}^{q,\beta}_A(\Omega)$$

so that the pullback

$$\pi_0^* : \mathcal{G}^{q,\beta}_A(\Omega)' \to \big(j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega)\big)'.$$

Specifically, if $f \in \mathcal{G}^{q,\beta}_A(\Omega)'$ and $g \in j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega)$, then

$$\pi_0^* f(g) = f(\pi_0 g).$$

Since π_0 is an isometric isomorphism of $j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega)$ onto $\mathcal{G}^{q,\beta}_A(\Omega)$, it follows that every element of $\mathcal{G}^{q,\beta}_A(\Omega)'$ is of the form $\pi_0^* f$ for $f \in (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))'$. Summarizing our results, we have shown the following.

Proposition 2.19. *The Banach spaces $\mathcal{G}^{q,\beta}_A(\Omega)'$ and $(\ell^q(L^q(\Omega))'/(j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))'$ are isomorphic. In particular, if $f \in \mathcal{G}^{q,\beta}_A(\Omega)'$, then there exists an element $F + (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp \in (\ell^q(L^q(\Omega))'/(j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp$ so that*

$$f = j^*_{q,A}\sigma^{-1}\Big(F + (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp\Big).$$

We can write the operator norm of f by

$$\|f\| = \sup_{\substack{g\in\mathcal{G}^{q,\beta}_A(\Omega)\\ \|g\|_{\mathcal{G}^{q,\beta}_A(\Omega)}=1}} f(g) = \sup_{\substack{g\in\mathcal{G}^{q,\beta}_A(\Omega)\\ \|g\|_{\mathcal{G}^{q,\beta}_A(\Omega)}=1}} F(j_{q,A}g) = \inf\{\|F\| : \sigma\pi_0^* f = F + (j_{q,A}\mathcal{G}^{q,\beta}_A(\Omega))^\perp\}.$$

Combining Proposition 2.17 and Proposition 2.19, we now have a description of the $\mathcal{G}^{q,\beta}(\Omega)'$.

3. Global L^q-Gevrey estimates and exponential decay

3.1. **Global L^q-Gevrey estimates and the Fourier transform.** We start our discussion by recalling the following results from [BR13].

Proposition 3.1. *Let $a, \beta > 0$ and $\gamma \in \mathbb{R}$. Then*

$$(4)\quad \frac{1}{|t|^\gamma}e^{-a|t|^{1/|\beta|}} = \inf_{r\geq\gamma}\left\{\Big(\frac{(r-\gamma)\beta}{ae}\Big)^{(r-\gamma)\beta}\frac{1}{|t|^r}\right\}$$
$$\leq \inf_{\substack{m\in\mathbb{Z}\\ m\geq\lceil\gamma\rceil}}\left\{\frac{1}{|t|^\gamma}, \Big(\frac{(m-\gamma)\beta}{ae}\Big)^{(m-\gamma)\beta}\frac{1}{|t|^m}\right\} \leq \frac{e^{e\beta/2}}{|t|^\gamma}e^{-a|t|^{1/|\beta|}}.$$

Corollary 3.2. *Let $\beta, A, C > 0$ and $\varphi : \mathbb{R}\to\mathbb{C}$ be a function that satisfies*

$$\left\|t^\ell\varphi\right\|_{L^\infty(\mathbb{R})} \leq C\Big(\frac{\ell\beta}{ae}\Big)^{\ell\beta}$$

for all integers $\ell \geq 0$. Then

$$|\varphi(t)| \leq Ce^{-a|t|^{1/\beta}}.$$

Corollary 3.3. *Let the function φ satisfy*

$$|t^m\varphi(t)| \leq CA^m m^{m\beta}$$

for some constants $C, A > 0$ when $m \geq \ell$. Then whenever $|t| \geq A(\ell e)^\beta$,

$$|\varphi(t)| \leq Ce^{-a|t|^{1/\beta}}$$

where $A = (\frac{\beta}{ae})^\beta$

We define the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ by

$$\hat{f}(\tau) = \int_{\mathbb{R}^d} e^{-it\tau}f(t)\,dt$$

and the inverse Fourier transform is given by

$$\check{f}(\tau) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{it\tau}f(t)\,dt = \frac{1}{(2\pi)^d}\hat{f}(-\tau).$$

We now generalize [BR13, Theorem 2.2].

Theorem 3.4. *Let $\varphi : \mathbb{R}^d \to \mathbb{C}$.*

(1) *Suppose there exist constants $a, \beta > 0$ so that $|\varphi(t)| \le Ce^{-a|t|^{1/\beta}}$. If $A = (\frac{\beta}{ae})^\beta$ and $A' > A$, then it follows that*

$$\big||t|^\ell \varphi(t)\big| \le CA^\ell \ell^{\ell\beta}$$

for all integers $\ell \ge 0$ and $\hat{\varphi} \in \mathcal{G}^{\infty,\beta}_{A'}(\mathbb{R}^d)$.

(2) *Suppose that $\hat{\varphi} \in \mathcal{G}^{1,\beta}_{A}(\mathbb{R}^d)$. Then there exists $C > 0$ so that*

$$|t^\ell \varphi(t)| \le CA^\ell \ell^{\ell\beta}$$

for all integers $\ell \ge 0$, i.e.,

$$|\varphi(t)| \le Ce^{-a|t|^{1/\beta}}$$

where $A = (\frac{\beta}{ae})^\beta$.

Proof. Proof of (1).

$$\begin{aligned}\|D^\alpha \hat{\varphi}\|_{L^\infty(\mathbb{R}^d)} \le \|t^\alpha \varphi\|_{L^1(\mathbb{R}^d)} &= \int_{B(0,1)} |t|^{|\alpha|}|\varphi(t)|\,dt + \int_{\mathbb{R}^d\setminus B(0,1)} |t|^{-d+1}\big||t|^{|\alpha|+d-1}\varphi(t)\big|\,dt \\ &\le C|B(0,1)|A^{|\alpha|}|\alpha|^{|\alpha|\beta} + C|\partial B(0,1)|A^{|\alpha|+d-1}(|\alpha|+d-1)^{(|\alpha|+d-1)\beta}\end{aligned}$$

Let $A' > A$. Then for some $\epsilon > 0$, $A' = A(1+\epsilon)$. For suitably large C and $|\alpha| \ge 1$, it follows that

$$(|\alpha|+d-1)^{|\alpha|+d-1} = |\alpha|^{|\alpha|\beta}\left[(|\alpha|+d-1)^{(d-1)\beta}\frac{(|\alpha|+d-1)^{|\alpha|\beta}}{|\alpha|^{|\alpha|\beta}}\right] \le C(1+\epsilon)^{|\alpha|}|\alpha|^{|\alpha|\beta}$$

since polynomial growth is slower than exponential growth. The proof of (1) is therefore complete.

The proof of (2) is immediate from the equality $|t^\alpha \varphi(t)| = \big|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{it\tau}D^\alpha\hat{\varphi}(\tau)\,d\tau\big|$. □

Proving generalizations of Theorem 3.4 to other q is not immediate. For example, constant functions are elements of $\mathcal{G}^{\infty,\beta}_A(\mathbb{R}^d)$ for every A but their Fourier transforms are not functions. In particular, we have the following result.

Proposition 3.5. *Let $\mathrm{Exp}_\beta(\Omega)$ be the set of measurable functions on Ω for which $f \in \mathrm{Exp}_\beta(\Omega)$ means there exist $C, a > 0$ so that $|f(x)| \le Ce^{-a|x|^{1/\beta}}$ a.e.*

(1) *$\mathcal{F}\big(\mathcal{G}^{\infty,\beta}(\mathbb{R}^d)\big) \not\subset \mathrm{Exp}_\beta(\Omega)$ for any $\beta > 0$.*

(2) *$\mathcal{F}\big(\mathrm{Exp}_\beta(\Omega)\big) \not\subset \mathcal{G}^{1,\beta}(\mathbb{R}^d)$ for any $\beta > 0$.*

Proof. Part (1) is simple. Take $f(x) = 1$. Then $\hat{f} = (2\pi)^d\delta_0$ a tempered distribution which is not even a function. Part (2) is also straight forward. In particular, the Fourier transform of an L^1 function is continuous, so we simply need to take a function that decays suitably fast but is not continuous. It cannot be the transform of an L^1 function. In particular, take

$$f(x) = \begin{cases} e^{-|x|^{1/\beta}} & |x| \ge 1 \\ 0 & |x| < 1. \end{cases}$$

□

3.2. **Exponential decay and distributions with decay.** Proposition 3.5 raises an interesting question. Although the Fourier transform of very smooth functions do not have to decay in the pointwise sense, they ought to exhibit decay. The challenge, of course, is measuring the decay. Even the example used in the proof of Proposition 3.5, $f(x) = 1$ has a Fourier transform $\hat{f} = (2\pi)^d\delta_0$, a distribution with compact support. Compact support is the most extreme version of decay, so even in this example, we have the heuristic that smooth objects should have Fourier transforms with decay.

Let $g \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$. This means $\hat{g} \in \mathcal{S}'(\mathbb{R}^d)$. Additionally, smoothness for g will yield decay for $\hat{g}$. In particular, if $\psi \in \mathcal{S}(\mathbb{R}^d)$, then so is $x^\alpha\psi$. This means

$$(5)\quad \big|\langle \hat{g}, x^\alpha\psi\rangle\big| = \big|\langle g, D^\alpha\hat{\psi}\rangle\big| = \big|\langle D^\alpha g, \hat{\psi}\rangle\big| \le \|D^\alpha g\|_{L^q(\mathbb{R}^d)}\|\hat{\psi}\|_{L^p(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\hat{\psi}\|_{L^p(\mathbb{R}^d)}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$. As a consequence of this calculation and the fact that $\mathcal{F}$ is of type (q,p) for $1 \le q \le 2$, we see that the tempered distribution $x^\alpha\hat{g} \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$$\big|\langle x^\alpha\hat{g}, \psi\rangle\big| \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\hat{\psi}\|_{L^p(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\psi\|_{L^q(\mathbb{R}^d)}$$

if $1 \le q \le 2$. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$ for any $1 \le q < \infty$, it follows that if $1 \le q \le 2$ and $g \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$, then $x^\alpha\hat{g} \in L^p(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(6)\qquad \|x^\alpha\hat{g}\|_{L^p(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}.$$

In the reverse direction, let us now assume that (6) holds for some $1 \le p \le \infty$ and $\hat{g} \in \mathcal{S}'(\mathbb{R}^d)$. We would like to see if this is enough to establish that $g \in \mathcal{G}_{A'}^{q,\beta}(\mathbb{R}^d)$ for some $A' \ge A$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$(7)\quad \big|\langle D^\alpha g, \psi\rangle\big| = \big|\langle \widehat{D^\alpha g}, \check{\psi}\rangle\big| = \big|\langle x^\alpha\hat{g}, \check{\psi}\rangle\big| \le \|x^\alpha\hat{g}\|_{L^p(\mathbb{R}^d)}\|\check{\psi}\|_{L^q(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\hat{\psi}\|_{L^q(\mathbb{R}^d)}.$$

As above, we know that $\|\hat{\psi}\|_{L^q(\mathbb{R}^d)} \le \|\psi\|_{L^p(\mathbb{R}^d)}$ if $2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, for such q,

$$\big|\langle D^\alpha g, \psi\rangle\big| \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\psi\|_{L^p(\mathbb{R}^d)}.$$

By the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ for any p, $1 \le p < \infty$, it then follows that $D^\alpha g \in L^q(\mathbb{R}^d)$ and moreover

$$\|D^\alpha g\|_{L^q(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}.$$

This proves that for any $A' > A$, $g \in \mathcal{G}_{A'}^{q,\beta}(\mathbb{R}^d)$. Summarizing our results, we have shown the following.

Proposition 3.6. *Let $A, \beta > 0$ and assume that $1 \le p, q \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.*

i. If $1 \le q \le 2$ and $g \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$, then for all multiindices α, $x^\alpha\hat{g} \in L^p(\mathbb{R}^d)$ and satisfies (6).

ii. If $2 \le q \le \infty$ and $x^\alpha\hat{g} \in L^p(\mathbb{R}^d)$ and satisfies (6) *for all multiindices α, then $g \in \mathcal{G}_{A'}^{q,\beta}$ for any $A' > A$.*

By repeating the calculations above, replacing x^α and D^α with $\frac{x^\alpha}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}$ and $\frac{D^\alpha}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}$, respectively, and summing in α, we can establish the following corollary (as well as sharpening slightly the statement of Proposition 3.6)

Corollary 3.7. *Let $A, \beta > 0$. A tempered distribution g is an element of $\mathcal{G}^{2,\beta}_A(\mathbb{R}^d)$ if and only if*

$$\left\{\frac{1}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}\|x^\alpha \hat{g}\|_{L^2(\mathbb{R}^d)}\right\} \in \ell^2(\mathbb{Z}^d_{\geq 0})$$

where the summation occurs over $\{\alpha \in \mathbb{Z}^d_{\geq 0}\}$.

3.3. **A nonexample of exponential decay.** Fix $2 \leq q \leq \infty$ and choose p so that $\frac{1}{p}+\frac{1}{q} = 1$. We construct a function φ in $L^p(\mathbb{R})$ so that $\hat{\varphi} \in \mathcal{G}^{q,\beta}(\mathbb{R})$ but φ does not decay. In fact, φ will satisfy $\|\varphi\|_{L^\infty([a,a+1))} = 1$. By Hausdorff-Young,

$$\|\hat{\varphi}^{(\ell)}\|_{L^q(\mathbb{R})} \leq \|t^\ell \varphi\|_{L^p(\mathbb{R})},$$

so we only need to construct φ that does not decay and satisfies $\|t^\ell\varphi\|_{L^p(\mathbb{R})} \leq CA^\ell \ell^{\ell\beta}$ for some $C, A > 0$. Set

$$\varphi(t) = \sum_{k=1}^{\infty} \chi_{[k-\delta_k,k]}(t)$$

where $0 < \delta_k < 1$.

Case 1: $q = 2$ or $q = \infty$. In this case,

$$\|t^\ell\varphi\|^p_{L^p(\mathbb{R})} \leq \sum_{k=0}^{\infty} k^{p\ell}\delta_k.$$

The difficulty is to choose δ_k so that the sum is estimable. If we choose $\delta_k = \frac{1}{k!}$ and set $f(t) = e^t$, then

$$\sum_{k=0}^{\infty} k^{p\ell}\delta_k = \sum_{k=0}^{\infty} \frac{k^{p\ell}}{k!} = \left[\left(t\frac{d}{dt}\right)^{p\ell} f\right](1). \tag{8}$$

Indeed, if $g(t) = \sum_{k=0}^{\infty} a_k t^k$ is real analytic, then $\frac{dg}{dt} = \sum_{k=1}^{\infty} k a_k t^{k-1}$. This means

$$t\frac{dg}{dt} = \sum_{k=0}^{\infty} k a_k t^k.$$

Repeating this argument $p\ell - 1$ times shows

$$\left(t\frac{d}{dt}\right)^{p\ell} g = \sum_{k=0}^{\infty} k^{p\ell} a_k t^k.$$

The equality (8) follows. If the coefficients a_{jk} are defined so that

$$\left[\left(t\frac{d}{dt}\right)^{j} f\right](t) = \sum_{k=1}^{j} a_{jk} t^k f^{(k)}(t), \tag{9}$$

then it is easy to show that $a_{j1} = a_{jj} = 1$ and if $1 < k < j$, then

$$a_{jk} = a_{j-1,k-1} + k a_{j-1,k}. \tag{10}$$

We claim that

$$\sum_{k=1}^{j} a_{jk} \leq j!$$

We prove the claim via induction. The $j=1$ case is clear as $a_{11}=1$. Assume the claim for $j=1$. Then by the recursion formula (10) and the fact that $a_{kk}=a_{k1}=1$ for all k,

$$\begin{aligned}\sum_{k=1}^{j} a_{jk} &= a_{j1} + \sum_{k=2}^{j-1}\left(a_{j-1,k-1} + k a_{j-1,k}\right) + a_{jj} \\ &= 1a_{j-1,1} + \sum_{k=2}^{j-1}\left(a_{j-1,k-1} + k a_{j-1,k}\right) + a_{j-1,j-1} \\ &= \sum_{k=1}^{j-1} k a_{j-1,k} + \sum_{k=1}^{j-1} a_{j-1,k} \\ &= \sum_{k=1}^{j-1} a_{j-1,k}(k+1) \leq j \sum_{k=1}^{j-1} a_{j-1,k} = j!\end{aligned}$$

Therefore, since $f^{(k)}=e$ for all k, it follows from (9) that

$$\left[\left(t\frac{d}{dt}\right)^j f\right](1) \leq \sum_{k=1}^{j} a_{jk} \leq j!e < j^j.$$

Thus,

$$\|t^\ell \varphi\|_{L^p(\mathbb{R})}^p \leq (p\ell)^{p\ell} = p^{p\ell}\ell^{p\ell}.$$

Taking pth roots shows that φ provides a nonexample for $q=2$ and $q=\infty$). Of course, we already had an example for $q=\infty$, namely, $\varphi(t)=1$.

Case 2: $2<q<\infty$. Given $1<p<2$, there exists $\gamma\in(0,1)$ so that $p=\gamma+(1-\gamma)2$. By Hölder's inequality with $p'=\frac{1}{\gamma}$ and $q'=\frac{1}{1-\gamma}$, it follows that for any function ψ, writing $|\psi|^p=|\psi|^\gamma|\psi|^{2(1-\gamma)}$, we have

$$\|\psi\|_{L^p(\mathbb{R})}^p \leq \|\psi\|_{L^1(\mathbb{R})}^\gamma \|\psi\|_{L^2(\mathbb{R})}^{2(1-\gamma)}$$

which means that

$$\|t^\ell\varphi\|_{L^p(\mathbb{R})}^p \leq \|t^\ell\varphi\|_{L^1(\mathbb{R})}^\gamma \|t^\ell\varphi\|_{L^2(\mathbb{R})}^{2(1-\gamma)} \leq C\ell^{\gamma\ell}2^{2\gamma\ell}\ell^{2\ell(1-\gamma)} = C2^{2\gamma\ell}\ell^{\ell(\gamma+(1-\gamma)2)} = C2^{2\gamma\ell}\ell^{\ell p}$$

By Hausdorff-Young, this concludes the $2<q<\infty$ case.

If, however, we know more about φ, then φ may exhibit exponential decay. In the next proof, we use the elementary fact that if $a,b>0$, then

$$(a+b)^\ell \leq (2\max\{a,b\})^\ell = 2^\ell \max\{a^\ell,b^\ell\} \leq 2^\ell(a^\ell+b^\ell).$$

Lemma 3.8. *Let $\varphi\in L^p(\mathbb{R})$ for some $2\leq p\leq\infty$. If $\frac{1}{p}+\frac{1}{q}=1$ and $\hat\varphi\in\mathcal{G}^{q,\beta}(\mathbb{R})$ and φ is a decreasing function as $|t|\to\infty$, then there exist constants $a,C>0$ so that*

$$|\varphi(t)| \leq Ce^{-a|t|^{1/\beta}}.$$

Proof. Assume that $t>0$. Then since φ is decreasing and $\hat\varphi\in\mathcal{G}_A^{2,\beta}(\mathbb{R})$ for some $A>0$, there exists a constant $C>0$ so that

$$\begin{aligned}t^\ell|\varphi(t)| \leq t^\ell\|\varphi\|_{L^p(t-1,\infty))} &\leq \|(|s|+1)^\ell\varphi\|_{L^p([t-1,\infty))} \leq 2^\ell(\||s|^\ell\varphi\|_{L^p(\mathbb{R})} + \|\varphi\|_{L^p(\mathbb{R})}) \\ &\leq 2^\ell\left(\|\hat\varphi^{(\ell)}\|_{L^q(\mathbb{R}^d)} + \|\hat\varphi\|_{L^q(\mathbb{R}^d)}\right) \leq 2C(2A)^\ell\ell^{\ell\beta}.\end{aligned}$$

Divide both sides by t^ℓ and use Corollary 3.2 to finish the argument for $t > 0$. An analogous argument works for $t < 0$. $\square$

4. Construction of functions in $\mathcal{G}^{q,\beta}(\Omega)$

4.1. **Construction of $\mathcal{G}^{2,\beta}(\mathbb{R}^d)$ functions.** As a warmup, we start with the $q = 2$ case because it suffices to consider functions on the transform side by Corollary 3.7.

We build examples by first constructing functions in $\mathcal{G}^{2,\beta}(\mathbb{R})$. Let $f_a(x) = e^{-a|x|^{1/\beta}}$ for a fixed $\beta > 1$. Note then

$$\hat{f}_a(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-a|x|^{1/\beta}}\,dx = \int_{\mathbb{R}} e^{-iy(\xi/a^\beta)} e^{-|y|^{1/\beta}}\,dy = \frac{1}{a^\beta}\hat{f}_1(\xi/a^\beta).$$

We will therefore investigate $\hat{f}_1(\xi)$. By Plancherel's Theorem,

$$\|\hat{f}_1^{(\ell)}\|^2_{L^2(\mathbb{R})} = \|x^\ell f_1\|^2_{L^2(\mathbb{R})} = 2\int_0^\infty x^{2\ell} e^{-2x^{1/\beta}}\,dx.$$

Using the change of variables $y = 2x^{1/\beta}$ (so that $x = (y/2)^\beta$), we compute

$$\|\hat{f}_1^{(\ell)}\|^2_{L^2(\mathbb{R})} = 2\int_0^\infty x^{2\ell} e^{-2x^{1/\beta}}\,dx = \frac{\beta}{2^{\beta(2\ell+1)-1}}\int_0^\infty y^{\beta(2\ell+1)-1} e^{-y}\,dy = \frac{\beta}{2^{\beta(2\ell+1)-1}}\Gamma\big(\beta(2\ell+1)\big).$$

We can now investigate the $\mathcal{G}^{2,\beta}_A$ for $\hat{f}_1$ (and consequently $\hat{f}_a$ as well).

$$\|\hat{f}_1\|^2_{\mathcal{G}^{2,\beta'}_A} = \sum_{\ell=0}^\infty \varrho_{\ell,\beta'}(\hat{f}_1)^2 = \sum_{\ell=0}^\infty \frac{\beta}{2^{\beta(2\ell+1)-1}}\frac{\Gamma(\beta(2\ell+1))}{A^{2\ell}\ell^{2\ell\beta'}} = \sum_{\ell=0}^\infty a_\ell.$$

We use the Root Test to determine when $\sum a_\ell$ converges. Applying Stirling's formula to simplify the Γ term,

$$\begin{aligned}
\lim_{\ell\to\infty} a_\ell^{1/\ell} &= \lim_{\ell\to\infty}\left[\frac{\beta}{2^{\beta(2\ell+1)-1}}\frac{\Gamma(\beta(2\ell+1))}{A^{2\ell}\ell^{2\ell\beta'}}\right]^{1/\ell} \\
&= \lim_{\ell\to\infty}\frac{1}{2^{2\beta}A^2\ell^{2\beta'}}\left[\sqrt{2\pi}\Big(\beta(2\ell+1)-1\Big)^{\beta(2\ell+1)-1/2} e^{-\beta(2\ell+1)-1}\right]^{1/\ell} \\
&= \lim_{\ell\to\infty}\frac{(2\ell\beta)^{2\beta}}{2^{2\beta}A^2\ell^{2\beta'}e^{2\beta}} = \lim_{\ell\to\infty}\frac{\ell^{2(\beta-\beta')}\beta^{2\beta}}{A^2e^{2\beta}} = \begin{cases}\infty & \beta>\beta' \\ 0 & \beta<\beta' \\ \frac{\beta^{2\beta}}{A^2e^{2\beta}} & \beta=\beta'.\end{cases}
\end{aligned}$$

Thus the sum converges when $\beta < \beta'$ or $\beta = \beta'$ and $A > (\frac{\beta}{e})^\beta$. The Root Test yields no information if $A = (\frac{\beta}{e})^\beta$. Note the form of A in context of Theorem 3.4 with $a = 1$. This computation yields that $\mathcal{G}^{2,\beta}(\mathbb{R})$ is a proper subset of $\mathcal{G}^{2,\beta'}(\mathbb{R})$ is $\beta < \beta'$. Also, combining the scaling for $\hat{f}_a$ with $\hat{f}_1$ above, we also may conclude that $\mathcal{G}^{2,\beta}_A(\mathbb{R})$ is a proper subset of $\mathcal{G}^{2,\beta}_{A'}(\mathbb{R})$ when $A > A'$ (and both are nonempty!!).

This calculation generalizes to $\mathbb{R}^d$. If $g_a(x) = \prod_{j=1}^d e^{-a|x_j|^{1/\beta}}$, then

$$\hat{g}_a(\xi) = \frac{1}{a^{\beta d}}\hat{g}_1(\xi/a)$$

and if $\alpha = (\alpha_1, \dots, \alpha_d)$, then

$$\|D^\alpha \hat{g}_1\|^2_{L^2(\mathbb{R}^d)} = \|x^\alpha g_1\|^2_{L^2(\mathbb{R}^d)} = \prod_{j=1}^d \|x_j^{\alpha_j} e^{-|x_j|^{1/\beta}}\|_{L^2(\mathbb{R})} = \prod_{j=1}^d \frac{\beta}{2^{\beta(2\alpha_j+1)-1}} \Gamma\big(\beta(2\alpha_j+1)\big).$$

The $\mathcal{G}_A^{2,\beta'}$ norm is given by

$$\begin{aligned}\|\hat{g}_1\|^2_{\mathcal{G}_A^{2,\beta'}} &= \sum_{|\alpha|\geq 0} \varrho_{\alpha,\beta'}(\hat{g}_1)^2 = \sum_{|\alpha|\geq 0} \frac{\beta^d}{A^{2|\alpha|} 2^{\beta(2|\alpha|+d)-d} |\alpha|^{2|\alpha|\beta'}} \left[\prod_{j=1}^d \Gamma\big(\beta(2\alpha_j+1)\big)\right] \\ &= \sum_{\ell=0}^\infty \left[\frac{\beta^d}{A^{2\ell} 2^{\beta(2\ell+d)-d} \ell^{2\ell\beta'}} \sum_{|\alpha|=\ell} \prod_{j=1}^d \Gamma\big(\beta(2\alpha_j+1)\big)\right] = \sum_{\ell=0}^\infty a_\ell.\end{aligned}$$

To find when this sum converges, we again use the Root Test and Stirling's Formula. We estimate

$$\begin{aligned}\limsup_{\ell\to\infty} a_\ell^{1/\ell} &\sim \limsup_{\ell\to\infty} \frac{1}{A^2 2^{2\beta} \ell^{2\beta'}} \sum_{|\alpha|=\ell} \left[\prod_{j=1}^d \big(\beta(2\alpha_j+1)\big)^{\beta(2\alpha_j+1)-1/2} e^{-\beta(2\alpha_j+1)+1}\right]^{1/\ell} \\ &\sim \limsup_{\ell\to\infty} \frac{\beta^{2\beta}}{A^2 2^{2\beta} \ell^{2\beta'} e^{2\beta}} \sum_{|\alpha|=\ell} \left[\prod_{j=1}^d (2\alpha_j+1)^{\beta(2\alpha_j+1)}\right]^{1/\ell}\end{aligned}$$

Since the partition function is bounded by the $(n+1)$st Fibonacci number, denoted f_{n+1}, it follows that

$$\limsup_{\ell\to\infty} a_\ell^{1/\ell} \leq \lim_{\ell\to\infty} \frac{\beta^{2\beta}}{A^2 2^{2\beta} \ell^{2\beta'} e^{2\beta}} f_{\ell+1}^{1/\ell} (2\ell+1)^\beta = \lim_{\ell\to\infty} \frac{\beta^{2\beta} \ell^{2(\beta-\beta')}}{A^2 e^{2\beta}} \frac{1+\sqrt{5}}{2}.$$

By choosing $\alpha = (\ell, 0, \dots, 0)$, we also have the lower bound

$$\liminf_{\ell\to\infty} a_\ell^{1/\ell} \gtrsim \lim_{\ell\to\infty} \frac{\beta^{2\beta} \ell^{2(\beta-\beta')}}{A^2 e^{2\beta}}.$$

As with the $d=1$ case, $\mathcal{G}^{2,\beta}(\mathbb{R}^d)$ is a proper subset of $\mathcal{G}^{2,\beta'}(\mathbb{R}^d)$ if $\beta < \beta'$. Also, $\mathcal{G}_A^{2,\beta}(\mathbb{R}^d)$ is a proper subset of $\mathcal{G}_{2A}^{2,\beta}(\mathbb{R}^d)$.

4.2. **Construction of functions in $\mathcal{G}^{q,\beta}(\Omega)$, $q \neq 2$, built by the Fourier transform.** We continue to use the notation of §4.1. We start by first constructing functions in $\mathcal{G}^{q,\beta}(\mathbb{R})$. By the Hausdorff-Young inequality, when $2 \leq q \leq \infty$,

$$\|\hat{f}_1^{(\ell)}\|^q_{L^q(\mathbb{R})} \leq C\|x^\ell f_1\|^q_{L^p(\mathbb{R})} = C\left(2\int_0^\infty x^{p\ell} e^{-px^{1/\beta}}\,dx\right)^{q/p}.$$

Using the change of variables $y = px^{1/\beta}$ (so that $x = (y/p)^\beta$) and Stirling's formula, we compute

$$\|\hat{f}_1^{(\ell)}\|^q_{L^q(\mathbb{R})} \leq C\left(\frac{2\beta}{p^{\beta(p\ell+1)}} \int_0^\infty y^{\beta(p\ell+1)-1} e^{-y}\,dx\right)^{q/p} = C\left(\frac{2\beta}{p^{\beta(p\ell+1)}} \Gamma\big(\beta(p\ell+1)\big)\right)^{q/p}.$$

Therefore

$$\|\hat{f}_1\|^2_{\mathcal{G}_A^{q,\beta'}} = \sum_{\ell=0}^{\infty} \varrho_{\ell,\beta'}(\hat{f}_1)^q \le C \sum_{\ell=0}^{\infty} \left(\frac{2\beta}{p^{\beta(p\ell+1)}} \frac{\Gamma\big(\beta(p\ell+1)\big)}{A^{p\ell}\ell^{p\ell\beta'}} \right)^{q/p} = \sum_{\ell=0}^{\infty} a_\ell^{q/p}.$$

We use the Root Test again to determine when $\sum a_\ell^{q/p}$ converges. Applying Stirling's formula to simplify the Γ term,

$$\begin{aligned} \lim_{\ell\to\infty} a_\ell^{1/\ell} &= \lim_{\ell\to\infty} \left[\frac{2\beta}{p^{\beta(p\ell+1)}} \frac{\Gamma\big(\beta(p\ell+1)\big)}{A^{p\ell}\ell^{p\ell\beta'}} \right]^{1/\ell} \\ &= \lim_{\ell\to\infty} \frac{1}{p^{p\beta}A^p\ell^{p\beta'}} \frac{(\beta p\ell)^{\beta p}}{e^{\beta p}} \\ &= \lim_{\ell\to\infty} \frac{\beta^{\beta p}\ell^{p(\beta-\beta')}}{A^p e^{\beta p}} = \begin{cases} \infty & \beta > \beta' \\ 0 & \beta < \beta' \\ \frac{\beta^{2\beta}}{A^p e^{p\beta}} & \beta = \beta'. \end{cases} \end{aligned}$$

Thus the sum converges when $\beta < \beta'$ or $\beta = \beta'$ and $A > (\frac{\beta}{e})^\beta$.

This calculation generalizes to $\mathbb{R}^d$ by taking products.

4.3. **Constructions built by direct calculation.** In this section, we generalize Lemma 2.4 and recognize that functions which are not necessarily in $\mathcal{G}_A^{q,\beta}(\Omega)$ may be used to generate functions in $\mathcal{G}_A^{q,\beta}(\Omega)$.

Example 4.1. Let $\Omega = \mathbb{R}$,

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = f(x)f(1-x) = \begin{cases} e^{-\frac{1}{x}}e^{-\frac{1}{1-x}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Although f is not a smooth function, g is. Moreover, $g \in C_c^\infty([0,1])$, so this example effectively passes the construction from a problem on $\mathbb{R}$ to a problem on $(0,1)$. We will see that $f \in \mathcal{G}^{q,2}(0,1)$ for $1 \le q \le \infty$. An induction argument (see, for example, (11) below with $\alpha = 1$) shows that

$$f^{(k)}(x) = e^{-\frac{1}{x}} \Big(\frac{1}{x^{2k}} - \frac{a_{k,1}}{x^{2k-1}} + \cdots + (-1)^{k+1} \frac{a_{k,k-1}}{x^{k+1}} \Big) = e^{-\frac{1}{x}} \sum_{j=0}^{k-1} \frac{a_{k,j}}{x^{2k-j}}$$

where

$$a_{k,j} = \begin{cases} \frac{1}{j!}k(k-1)^2(k-2)^2\cdots(k-j+1)^2(k-j) = j!\binom{k}{j}\binom{k-1}{j} & 0 \le j \le k-1 \\ 0 & j < 0 \text{ or } j \ge k. \end{cases}$$

Observe that $j! \le a_{k,j} \le 2^{2k} j!$. Also,

$$\int_0^1 \frac{1}{x^j} e^{-\frac{1}{x}}\, dx = \int_1^\infty y^{j-2} e^{-y}\, dy.$$

If b_k is defined by

$$b_k = e \int_1^\infty y^k e^{-y}\, dy,$$

then

$$b_k \le e \int_0^\infty y^k e^{-y}\, dy = e\Gamma(k+1) = k!e$$

while an integration by parts shows that

$$b_0 = 1 \quad \text{and} \quad b_k = 1 + kb_{k-1},\ k \ge 1.$$

Since $kb_{k-1} \le b_k$, it is immediate that

$$k! \le b_k \le k!e.$$

Next, $\frac{e^{-\frac{1}{x}}}{x^k}$ attains its maximum $k^k e^{-k}$ at $x = \frac{1}{k}$. Consequently, up to sub-geometric factors,

$$\begin{aligned}\|f^{(k)}\|_{L^\infty(0,1)} &\le \sum_{j=0}^{k-1} 2^{2k} j!(2k-j)^{2k-j} e^{-(2k-j)} \sim \sum_{j=0}^{k-1} 2^{2k} j!(2k-j)! \\ &= 2^{2k}(2k)! \sum_{j=0}^{k-1} \frac{1}{\binom{2k}{j}} \sim 2^{2k}(2k)! \sim 2^{2k}(2k)^{2k}e^{-2k} = \left(\frac{2^2}{e}\right)^{2k} k^{2k}.\end{aligned}$$

This means $f \in \mathcal{G}^{\infty,2}_{8/e+\epsilon}(0,1)$ for any $\epsilon > 0$.

Next,

$$\begin{aligned}\int_0^1 |f^{(k)}(x)|\, dx &\le \sum_{j=0}^{k-1} a_{k,j} \int_0^1 \frac{e^{-\frac{1}{x}}}{x^{2k-j}}\, dx \lesssim e \sum_{j=0}^{k-1} 2^{2k} j!(2k-j-2)! \\ &\sim 2^{2k}(2k-2)! \sum_{j=0}^{k-1} \frac{1}{\binom{2k-2}{j}} \le 2^{2k}(2k-2)! \sum_{j=0}^{k-1} 1 \sim 2^{2k}(2k-2)^{2k-2}e^{2k-2} \\ &= \left(\frac{2^2}{e}\right)^{2k} (k-1)^{2(k-1)} \sim \left(\frac{2^2}{e}\right)^{2k} k^{2k}.\end{aligned}$$

Consequently, $f \in \mathcal{G}^{1,2}(0,1) \cap \mathcal{G}^{\infty,2}(0,1)$ so that by Hölder's inequality, $f \in \mathcal{G}^{q,2}(0,1)$ for all $1 \le q \le \infty$ and $g \in \mathcal{G}^{q,2}(\mathbb{R})$ for $1 \le q \le \infty$.

Example 4.2. We present a more general version of Example 4.1. For $\alpha > 0$, set

$$f_\alpha(x) = \begin{cases} e^{-\frac{1}{x^\alpha}} & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_\alpha(x) = f_\alpha(x) f_\alpha(1-x) = \begin{cases} e^{-\frac{1}{x^\alpha}} e^{-\frac{1}{(1-x)^\alpha}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

In this case, a recursion quickly shows that

$$f_\alpha^{(k)}(x) = e^{-\frac{1}{x^\alpha}} \sum_{j=1}^{k} (-1)^{k-j} \frac{a_{k,j}}{x^{j\alpha+k}}$$

where

(11)

$$a_{k,1} = \alpha(\alpha+1)\cdots(\alpha+k-1),\ a_{k,k} = \alpha^k, \quad \text{and} \quad a_{k+1,j} = \begin{cases} \alpha a_{k,j-1} + (j\alpha+k) a_{k,j} & 1 \le j \le k \\ 0 & \text{otherwise} \end{cases}.$$

The problem is that computing $a_{k,j}$ is more difficult. Instead, we can use the Cauchy Integral Formula to obtain an estimate. We are interested in the case $\alpha \le 1$, but the computations hold for all $\alpha > 0$. The kind of construction we envision with $f_\alpha(x)$ is the function $f(x) = f_\alpha(x) f_\alpha(1-x)$ and we want to see in which global L^q-Gevrey classes f resides.

Using the standard branch cut, the function

$$f_\alpha(z) = e^{-\frac{1}{z^\alpha}}$$

is holomorphic on $\mathbb{C} \setminus \{z = x : x \le 0\}$. Let $0 < x < 1$ and $0 < r < 1$ be fixed. Then $b \in H(D(x, rx))$ so that by the Cauchy Integral Formula,

$$f_\alpha^{(n)}(x) = \frac{n!}{2\pi (rx)^n} \int_0^{2\pi} f_\alpha(x + rxe^{i\theta})\, d\theta.$$

Then

$$f_\alpha(x + rxe^{i\theta}) = \exp\Big[-\frac{1}{x^\alpha} \Big(\frac{1}{1 + re^{i\theta}} \Big)^\alpha \Big] = \exp\Big[-\frac{(1 + re^{-i\theta})^\alpha}{x^\alpha (1 + 2r\cos\theta + r^2)^\alpha} \Big]$$

In the case $r = \frac{1}{2}$, the fact that $\mathrm{Re}(1 + re^{-i\theta}) \ge \frac{1}{2}$ means

$$\begin{aligned} \frac{1}{2\pi} \Big| \int_0^{2\pi} f_\alpha(x + rxe^{i\theta})\, d\theta \Big| &\le \frac{1}{2\pi} \int_0^{2\pi} |f_\alpha(x + rxe^{i\theta})|\, d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \exp\Big[-\frac{1}{2^\alpha x^\alpha} \Big]\, d\theta = e^{-\frac{1}{(2x)^\alpha}}. \end{aligned}$$

This means

$$|f_\alpha^{(n)}(x)| \le \frac{2^n n!}{x^n} e^{-\frac{1}{(2x)^\alpha}}.$$

Elementary calculus shows that the max of $f(x) = \frac{2^n n!}{x^n} e^{-\frac{1}{(2x)^\alpha}}$ occurs at $x_{\max} = \frac{1}{2}(\frac{\alpha}{n})^{1/\alpha}$ and by Stirling's formula (and up to sub-geometric factors in the last line)

$$f(x_{\max}) = 2^{2n} n! \Big(\frac{n}{\alpha} \Big)^{\frac{n}{\alpha}} e^{-\frac{n}{\alpha}} = \Big(\frac{4}{(\alpha e)^{1/\alpha}} \Big)^n n! n^{\frac{n}{\alpha}} \sim \Big(\frac{4}{e(\alpha e)^{1/\alpha}} \Big)^n n^{n(1 + \frac{1}{\alpha})}.$$

Observe that $x_{\max} \to 0$ as $n \to \infty$.

Consequently, if $\beta > 0$ and

$$g_\beta(t) = e^{-|t|^{-\beta}},$$

then up to sub-geometric factors

$$|g_\beta^{(n)}(t)| \lesssim \Big(\frac{4\beta}{e^{\beta+1}} \Big)^n n^{n(\beta+1)}.$$

Note that this computation agrees with the $\beta = 1$ calculation that we computed directly in Example 4.1. It follows easily that $g_\beta(t) g_\beta(1-t) \in \mathcal{G}^{q, 1+\beta}(\mathbb{R})$.

4.4. **Comparison with (local - L^∞) Gevrey functions.** The (local - L^∞) Gevrey spaces in $\Omega \subset \mathbb{R}^d$, $G^\beta(\Omega)$ with $\beta > 1$, are defined as follows: $f \in G^\beta(\Omega)$ if for every compact set $K \subset \Omega$, there exists a constant $C = C(K)$ such that

$$\sup_{x \in K} |D^\alpha f(x)| \le C^{|\alpha|+1} |\alpha|^{|\alpha|\beta}, \qquad \forall\, \alpha \in \mathbb{N}^d.$$

We recall the definition of the Sobolev spaces $W^{k,q}(\mathbb{R}^n)$, $k = 0, 1, 2, \dots$ and $1 \le q$:

$$W^{k,q}(\mathbb{R}^n) = \{ f \in L^q : \partial^\alpha f \in L^q,\ |\alpha| \le k \}.$$

Proposition 4.3. *For any $k = 0, 1, 2, \ldots$, $1 \le q \le \infty$ and $\beta > 1$, we have*

$$\mathcal{G}^{q,\beta}(\mathbb{R}^d) \subsetneq W^{k,q}(\mathbb{R}^d) \cap G^\beta(\mathbb{R}^d).$$

Proof. Let k and q be fixed and let $\psi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ be a nonnegative, compactly supported function with $\operatorname{supp}\psi \subset B(0,1)$. The existence of such a function is guaranteed by Corollary 2.8. For each $\ell \in \mathbb{N}$ let

$$\psi_\ell(x) = \frac{1}{\ell^{k+2}\|\psi\|_{W^{k,q}}}\ell^{d/q}\psi(\ell(x - \mathfrak{l})), \quad \mathfrak{l} := (\ell, \ldots, \ell)$$

It is straight forward to check that

$$\Psi(x) = \sum_{\ell=1}^{\infty} \psi_\ell(x) \in W^{k,q}(\mathbb{R}^d) \cap G^\beta(\mathbb{R}^d) \setminus \mathcal{G}^{q,\beta}(\mathbb{R}^d)$$

as we wished to prove. □

Proposition 4.4. *For any $k = 0, 1, 2, \ldots$, $1 \le q \le \infty$ and $\beta > 1$, we have*

$$\mathcal{G}^{q,\beta}((0,1)^d) \subsetneq W^{k,q}((0,1)^d) \cap G^\beta((0,1)^d).$$

Proof. Let k and q be fixed and let $0 \le \psi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ given by Corollary 2.8, supported in $B(0, 1/2)$. For each $\ell \in \mathbb{N}$ let

$$\psi_\ell(x) = \frac{1}{\ell^2 2^{\ell k}\|\psi\|_{W^{k,q}}}2^{d\ell/q}\psi(2^\ell(x - \mathfrak{l})), \quad \mathfrak{l} := \Big(\frac{2^\ell - 1}{2^\ell}, \ldots, \frac{2^\ell - 1}{2^\ell}\Big)$$

Then, it is easy to check that

$$\Psi(x) = \sum_{\ell=1}^{\infty} \psi_\ell(x) \in W^{k,q}((0,1)^d) \cap G^\beta((0,1)^d) \setminus \mathcal{G}^{q,\beta}((0,1)^d)$$

as we wished to prove. □

As a corollary of these two propositions, we have

Corollary 4.5. *For any $k = 0, 1, 2, \ldots$, $1 \le q \le \infty$ and $\beta > 1$, If $\Omega^o = (\bar{\Omega})^o$, then $\mathcal{G}^{q,\beta}(\Omega) = G^\beta(\Omega)$ if and only if Ω is compact.*

5. Behavior under diffeomorphisms

Let $D, \Omega \subset \mathbb{R}^d$ and $F = (F_1, \ldots, F_d) : D \to \Omega$ be a diffeomorphism. We want to examine under what conditions $h = g \circ F$ is an element of $\mathcal{G}^{q,\beta}(D)$ if $g \in \mathcal{G}^{q,\beta}(\Omega)$. We need to understand the partial derivatives of a composition of functions. We will use variables $y \in D$ and $x \in \Omega$. With this distinction, we compute

$$D_{y_j}h(y) = \sum_{k=1}^{d}(D_{x_k}g) \circ F\, D_{y_j}F_k$$

$$D^2_{y_{j_1}y_{j_2}}h(y) = \sum_{k_1,k_2=1}^{d}(D^2_{x_{k_1}x_{k_2}}g) \circ F D_{y_{j_1}}F_{k_1}D_{y_{j_2}}F_{k_2} + \sum_{k=1}^{d}(D_{x_k}g) \circ F\, D^2_{y_{j_1}y_{j_2}}F_k$$

and

$$D^2_{y_{j_1}y_{j_2}y_{j_3}}h(y) = \sum_{k_1,k_2,k_3=1}^{d} (D^3_{x_{k_1}x_{k_2}x_{k_3}}g)\circ F D_{y_{j_1}}F_{k_1}D_{y_{j_2}}F_{k_2}D_{y_{j_3}}F_{k_3}$$
$$+ \sum_{k_1,k_2=1}^{d} (D^2_{x_{k_1}x_{k_2}}g)\circ F\Big(D^2_{y_{j_1}y_{j_2}}F_{k_1}D_{y_{j_3}}F_{k_2}D^2_{y_{j_1}y_{j_3}}F_{k_1}D_{y_{j_2}}F_{k_2} + D^2_{y_{j_2}y_{j_3}}F_{k_1}D_{y_{j_2}}F_{k_2}\Big)$$
$$+ \sum_{k=1}^{d}(D_{x_k}g)\circ F\, D^3_{y_{j_1}y_{j_2}y_{j_3}}F_k.$$

From these calculations, we can see that

$$D^\alpha_y h(y) = \sum_{k=1}^{|\alpha|}\Bigg[\sum_{|K|=k}(D^K_x g)\circ F\Big(\sum_{J_1+\cdots+J_k=\alpha} D^{J_1}F_{K_1}\cdots D^{J_k}F_{K_k}\Big)\Bigg]. \tag{12}$$

To estimate $\|D^\alpha_y h\|_{L^q(D)}$, we need some bounds on the components of F. It seems unfeasible to assume that F satisfies anything but some type of L^∞ bound. Moreover, in the formula for $D^\alpha h$, only derivatives of components of ∇F_j appear. To prove an invariance result under certain types of diffeomorphisms, we need the following result.

Lemma 5.1. *If $1 \le \ell_1 \le \ell_2$, then*

$$\ell_1^{\ell_1}\ell_2^{\ell_2} \le (\ell_1-1)^{\ell_1-1}(\ell_2+1)^{\ell_2+1}.$$

Proof. Let $\ell = \ell_1+\ell_2$ and

$$f(x) = (\ell-x)^{\ell-x}x^x.$$

Since

$$f'(x) = f(x)\big(\log x - \log(\ell-x)\big),$$

the function $f(x)$ is increasing once $x > \frac{1}{2}\ell$ and the result follows. □

We have the following.

Proposition 5.2. *Let $D,\Omega \subset \mathbb{R}^d$ be domains and $F = (F_1,\dots,F_d) : D \to \Omega$ be a smooth diffeomorphism whose Jacobian determinant is bounded and bounded away from 0. If the components of ∇F_ℓ are elements of $\mathcal{G}^{\infty,\beta}(D)$ for $1 \le \ell \le d$, then the pullback*

$$F^* : \mathcal{G}^{q,\beta}(\Omega) \to \mathcal{G}^{q,\beta}(D)$$

is continuous.

Proof. From (12), it is apparent that we need bounds on

$$I := \big\|(D^K_x g)\circ F D^{J_1}F_{K_1}\cdots D^{J_k}F_{K_k}\big\|_{L^q(D)}$$

where $g \in \mathcal{G}^{q,\beta}(\Omega)$, $|K| = k$, $k \le \alpha$, and $J_1+\cdots+J_k = \alpha$. By hypothesis, there exists $A,C>0$ so that

$$\|D^K g\|_{L^q(\Omega)} \le CA^k k^{k\beta} \quad\text{and}\quad \|D^{J_\ell}F_{K_\ell}\|_{L^\infty(D)} \le CA^{|J_\ell|}|J_\ell|^{|J_\ell|\beta} \text{ for } 1 \le \ell \le k.$$

If we use DF for the Jacobian of F, then

$$\begin{aligned} I &\le \|D^{J_1}F_{K_1}\|_{L^\infty(D)}\cdots\|D^{J_k}F_{K_k}\|_{L^\infty(D)}\|(D^k_x g)\circ F\|_{L^q(\Omega)} \\ &\le C_{\|DF^{-1}\|_{L^\infty}}A^{|\alpha|+k}(|J_1|^{|J_1|}\cdots|J_k|^{|J_k|})^\beta k^{k\beta}. \end{aligned} \tag{13}$$

By a repeated use of Lemma 5.1, it follows that the quantity $(|J_1|^{|J_1|}\cdots|J_k|^{|J_k|})$ achieves its maximum when $|J_1| = \cdots = |J_{k-1}| = 1$ and $|J_k| = |\alpha| - (k-1)$. Therefore, it follows that

$$I \le C_{\|DF^{-1}\|_{L^\infty}}A^{|\alpha|+k}\big(|\alpha|-(k-1)\big)^{(|\alpha|-(k-1))\beta}k^{k\beta}.$$

By a repeated use of Lemma 5.1, it follows that the quantity $\big(|\alpha|-(k-1)\big)^{(|\alpha|-(k-1))\beta}k^{k\beta}$ is maximized when $k = 1, \alpha$ since $1 \le k \le |\alpha|$. It now follows that

$$\big\|(D^K_x g)\circ F D^{J_1}F_{K_1}\cdots D^{J_k}F_{K_k}\big\|_{L^q(D)} \le C_{\|DF^{-1}\|_{L^\infty}}A^{2|\alpha|}|\alpha|^{|\alpha|\beta}.$$

We have now showed that

$$\|D^\alpha_y h(y)\|_{L^q(D)} \le C_{\|DF^{-1}\|_{L^\infty}}A^{2|\alpha|}|\alpha|^{|\alpha|\beta}\sum_{k=1}^{|\alpha|}\sum_{|K|=k}1.$$

If $p(k)$ is the partition function, then $p(k)$ is asymptotic to $\frac{1}{4n\sqrt{3}}\exp(\pi\sqrt{2k/3})$, so in particular, $p(k) \le Ce^k$. Consequently,

$$\sum_{k=1}^{|\alpha|}\sum_{|K|=k}1 \le C\sum_{k=1}^{|\alpha|}e^k \le Ce^{|\alpha|+1}.$$

Thus, $h = g\circ F \in \mathcal{G}^{q,\beta}(D)$. □

Remark 5.3. The proof of Proposition 5.2 shows that $g\circ F \in \mathcal{G}^{q,\beta}_{A'}(D)$ where $A' = eA^2$.

6. $\mathcal{G}^{q,\beta}(\Omega)$ regularity for solutions to first order linear PDE

6.1. **Carleman's problem for $\mathcal{G}^{q,\beta}(\Omega)$ functions.** We start by showing that for given a sequence of functions $\{v_k\} \subset \mathcal{G}^{q,\beta}_A(\Omega)$, there exists a function $f \in \mathcal{G}^{q,\beta}(\Omega, \mathcal{G}^{\infty,\beta}(-1,1))$ with $\frac{\partial^k f}{\partial t^k} = v_k$ on $\Omega\times\{t=0\}$. For $q=\infty$ this gives a global version of [BP09, Lemma 3.1], see also [AH15]. Like [BP09], we follow Džanašija's proof [Dža62]. We will need the function spaces in the next definition to state clearly our results.

Definition 6.1. Let $\Omega \subset \mathbb{R}^d$, $\mathcal{O} \subset \mathbb{C}^d$, and $U \subset \mathbb{R}^k$ be open sets.

1. Let $\mathcal{G}^{q,\beta}(\Omega, H(\mathcal{O}))$ denote the space of functions $f(x,\zeta) \in C^\infty(\Omega\times\mathcal{O})$ such that $f(x,\cdot)$ is holomorphic in ζ and there exists $A > 0$ so that $f(\cdot,\zeta) \in \mathcal{G}^{q,\beta}_A(\Omega)$ uniformly in ζ in the sense that
$$\|D^\alpha_x f(\cdot,\zeta)\|_{L^q(\Omega)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}$$
for some constant $C > 0$ that does not depend on $\zeta \in \mathcal{O}$.
2. Define $\mathcal{G}^{q,\beta}_A(\Omega, \mathcal{G}^{p,\beta'}_{A'}(U))$ to be the space of functions $f \in C^\infty(\Omega\times U)$ such that there exist constants $C, C', A, A' > 0$ so that
   (a) $\big\|D^{J'}_y f(x,\cdot)\big\|_{L^p(U)} \le C'(A')^{\beta'|J'|}|J'|^{|J'|\beta'}$;
   (b) $\big\|\|D^J_x D^{J'}_y f(x,\cdot)\|_{L^p(U)}\big\|_{L^q(\Omega)} \le C(A')^{\beta'|J'|}A^{\beta|J|}|J'|^{|J'|\beta'}|J|^{|J|\beta}$.

Define

$$\mathcal{G}^{q,\beta}(\Omega, \mathcal{G}^{p,\beta'}(U)) = \bigcup_{A,A'>0} \mathcal{G}^{q,\beta}_A(\Omega, \mathcal{G}^{p,\beta'}_{A'}(U)).$$

Note that if $\beta = \beta'$, then $|J|^{|J|\beta}|J'|^{|J'|\beta} \sim (|J| + |J|')^{(|J|+|J'|)\beta}$ with cost of a geometric factor.

Lemma 6.2. *Let $\beta > 1$ and $\Omega \subset \mathbb{R}^d$. Assume $\{v_k\} \subset C^\infty(\Omega)$ and that there exists $C, A > 0$ so that*

$$\|D^\alpha v_k\|_{L^q(\Omega)} \leq C A^{|\alpha|+k} |\alpha|^{|\alpha|\beta} k^{k\beta} \tag{14}$$

for all $k = 0, 1, 2, \dots$. Then there exists $f \in \mathcal{G}^{q,\beta}_{A'}(\Omega, \mathcal{G}^{\infty,\beta}_{A'}(-1,1))$ so that

$$\frac{\partial^k f}{\partial t^k}(x, 0) = v_k(x), \quad n = 0, 1, 2, \dots \text{ and } x \in \Omega$$

and where $A' = A'(A, \beta, r)$ and is otherwise independent of Ω and $\{v_k\}$.

Note that $\mathcal{G}^{q,\beta}(\Omega, \mathcal{G}^{\infty,\beta}(-1,1)) \subset \mathcal{G}^{q,\beta}(\Omega \times (-1,1))$.

Proof. For $t \in [-1,1]$, set $a_0(t) = 1$, and for $k \geq 1$ set

$$b_k(t) = \begin{cases} 0 & t \in [-1, -\sigma_k] \cup [0,1] \\ \exp\left(\frac{-k\sigma_k^{4r}}{t^{2r}(\sigma_k + t)^{2r}}\right) & -\sigma_k < t < 0 \end{cases}$$

where $r \in \mathbb{Z}$ satisfies $\frac{1}{2r} < \beta - 1$ and $\sigma_k = B^{-1}k^{-(\beta-1)}$ for some $B > 0$ to be chosen later. Set

$$a_k(t) = \frac{\int_{-1}^t b_k(y)\,dy}{\int_{-1}^1 b_k(y)\,dy} \quad \text{if } -1 \leq t \leq 0$$

and

$$a_k(t) = a_k(-t) \quad \text{if } 0 \leq t \leq 1.$$

Note that $\operatorname{supp} a_k = [-\sigma_k, \sigma_k]$. Consider the formal series

$$\sum_{k=0}^\infty \frac{v_k(x)}{k!} a_k(t) t^k \quad \text{where } (x,t) \in \Omega \times [-1,1]. \tag{15}$$

We claim that

$$\sum_{k=1}^\infty \frac{\|v_k(x)\|_{L^q}}{k!} |a_k(t)||t|^k \leq 2C \sum_{k=1}^\infty \left(\frac{Ae}{B}\right)^k < \infty \tag{16}$$

if $(x,t) \in \Omega \times [-1,1]$, A, C are as in (14), and $B \geq Ae^{a+1}$. Using the facts that $\operatorname{supp} a_k = [-\sigma_k, \sigma_k]$, $0 \leq a_k(t) \leq 1$, and $\sigma_k = B^{-1}k^{-(\beta-1)}$, it follows that by Stirling's formula

$$\sum_{k=1}^\infty \frac{\|v_k(x)\|_{L^q}}{k!} |a_k(t)||t|^k \leq 2C \sum_{k=1}^\infty \frac{A^k k^{k\beta}}{k!} B^{-k} k^{-k(\beta-1)} \leq 2C \sum_{k=1}^\infty \left(\frac{Ae}{B}\right)^k.$$

This proves (16) which in turn shows that the series defined in (15) converges in $L^q(\Omega, L^\infty[-1,1])$.

Set

$$f(x,t) = \sum_{k=0}^\infty \frac{v_k(x)}{k!} a_k(t) t^k.$$

To show that $f(x,t)$ satisfies the conclusions of Lemma 6.2, it suffices to prove that $g(x,t)$ defined by

$$g(x,t)=\sum_{k=1}^{\infty}\frac{v_k(x)}{k!}a_k(t)t^k\in\mathcal{G}^{q,\beta}\big(\Omega,\mathcal{G}^{\infty,\beta}(-1,1)\big)$$

satisfies

$$\frac{\partial^n g}{\partial t^n}(x,0)=v_n(x),\ \text{ for } n=1,2,\ldots.$$

Set

$$w_k(x,t)=D_x^\alpha D_t^n\Big(\frac{v_k(x)}{k!}a_k(t)t^k\Big)=\frac{D_x^\alpha v_k(x)}{k!}\sum_{j=1}^{n}\binom{n}{j}a_k^{(n-j)}(t)\big(t^k\big)^{(j)}.$$

We recall [Dža62, Equation 6] which proves that there exists $T>1$ that is independent of n and k and satisfies

$$|a_k^{(n)}(t)|\le 2\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)T^nB^n\frac{n^n n^{(n-1)/2r}k^{n(\beta-1)}}{k^{(n-1)/2r}}.$$

Using this estimate, Džanašija's proof shows that

$$(17)\quad |w_k(x,t)|\le 2\frac{|D_x^\alpha v_k(x)|}{k!}\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)B^{-k}k^{-k(\beta-1)}\sum_{j=0}^{n}\binom{n}{j}T^nD^n\frac{n^{n-j}n^{(n-j-1)/2r}k^j}{k^{-n(\beta-1)}k^{(n-j-1)/2r}}.$$

Case 1: $k\le n$. We claim that for all $(x,t)\in\Omega\times(-1,1)$.

$$|w_k(x,t)|\le 2^{n+1}\frac{|D_x^\alpha v_k(x)|}{k!}B^{-k}k^{-k\beta}T^nB^nn^{n\beta}$$

In this case, we estimate (17) using the fact that $\frac{1}{2r}<\beta-1$ so that $k^{j+n(\beta-1)-\frac{n-j-1}{2r}}\le n^{j+n(\beta-1)-\frac{n-j-1}{2r}}$ and observe

$$\begin{aligned}|w_k(x,t)|&\le 2\frac{|D_x^\alpha v_k(x)|}{k!}\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)B^{-k}k^{-k(\beta-1)}\sum_{j=0}^{n}\binom{n}{j}T^nB^n\frac{n^{n-j}n^{(n-j-1)/2r}n^j}{n^{-n(\beta-1)}n^{(n-j-1)/2r}}\\&=2^{n+1}\frac{|D_x^\alpha v_k(x)|}{k!}\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)B^{-k}k^{-k(\beta-1)}T^nB^nn^{n\beta}.\end{aligned}$$

By Stirling's formula and choosing $B>2A\exp(\frac{(16)^{2r}}{3^{2r}}+1)$, it follows that

$$\|w_k\|_{L^q(\Omega,L^\infty[-1,1])}\le 2^{n-k+1}T^nB^{n+|\alpha|}n^{n\beta}|\alpha|^{|\alpha|\beta}.$$

Consequently,

$$\sum_{k=0}^{n}\|w_k\|_{L^q(\Omega,L^\infty[-1,1])}\le 2^{n+2}T^nB^{n+|\alpha|}n^{n\beta}|\alpha|^{|\alpha|\beta}.$$

Case 2: $k>n$. We again start with (17) and estimate

$$\begin{aligned}|w_k(x,t)|&\le 2\frac{|D_x^\alpha v_k(x)|}{k!}\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)B^{-k}k^{-k(\beta-1)}\sum_{j=0}^{n}\binom{n}{j}T^nD^n\frac{k^{n-j}k^{(n-j-1)/2r}k^j}{k^{-n(\beta-1)}k^{(n-j-1)/2r}}\\&=2^{n+1}\frac{|D_x^\alpha v_k(x)|}{k!}\exp\Big(\frac{(16)^{2r}k}{3^{2r}}\Big)B^{-k}k^{-k(\beta-1)}T^nB^nk^{n\beta}.\end{aligned}$$

Thus, using Stirling's formula, we estimate

$$\sum_{k=n+1}^{\infty} \|w_k\|_{L^q(\Omega, L^\infty[-1,1])} \le 2^{n+1} T^n B^n \sum_{k=n+1}^{\infty} \frac{A^{k+|\alpha|} k^{k\beta} |\alpha|^{|\alpha|\beta}}{k!} \exp\Big(\frac{(16)^{2r} k}{3^{2r}}\Big) B^{-k} k^{-k(\beta-1)} k^{n\beta}$$

$$\le 2^{n+1} T^n B^{n+|\alpha|} |\alpha|^{|\alpha|\beta} \sum_{k=n+1}^{\infty} 2^{-k} k^{n\beta} < C_{n,\beta} T^n B^{n+|\alpha|} |\alpha|^{|\alpha|\beta}.$$

The result now follows. □

The following is a small extension of Lemma 6.2 that is analogous to [BP09, Lemma 3.2].

Lemma 6.3. *Let $\beta > 1$, $\Omega \subset \mathbb{R}^d$, $\mathcal{O} \subset \mathbb{C}^d$ be as in Definition 6.1. Assume that $\{v_k(x,\zeta)\}$, $k = 0, 1, 2, \dots$ is a sequence of functions such that $\{v_k\} \in \mathcal{G}^{q,\beta}(\Omega, H(\mathcal{O}))$. Furthermore, we assume there exist constants $C, A > 0$ so that*

$$\|D_x^\alpha v_k(\cdot, \zeta)\|_{L^q(\Omega)} \le C A^{|\alpha|+k} |\alpha|^{|\alpha|\beta} k^{k\beta}.$$

Then there exists $f \in \mathcal{G}^{q,\beta}(\Omega \times (-1,1), H(\mathcal{O}))$ such that

$$D_t^n f(x, 0, \zeta) = v_n(x, \zeta), \quad n = 0, 1, 2, \dots \text{ for all } (x, \zeta) \in \Omega \times \mathcal{O}.$$

The next results extends Lemma 6.2 when the set of functions $\{v_J(x)\}$ is indexed by $J \in \mathbb{Z}_+^d$ and $y \in (-1,1)^d$.

Lemma 6.4. *Let $\beta > 1$ and $\{v_J(x)\}$ be set of functions indexed by $J \in \mathbb{Z}_+^d$ defined on the open set $\Omega \subset \mathbb{R}^d$ so that there exist constants $A, C > 0$ satisfying*

$$\|D_x^\alpha v_J(x)\|_{L^q(\Omega)} \le C A^{|\alpha|+|J|} |\alpha|^{|\alpha|\beta} |J|^{|J|\beta}.$$

There exists $A' = A'(A, \beta)$ and a function $F \in \mathcal{G}_{A'}^{q,\beta}(\Omega, \mathcal{G}_{A'}^{\infty,\beta}(-1,1)^d)$ such that $D_y^J F(x,0) = v_J(x)$ for all $x \in \Omega$ and $J \in \mathbb{Z}_+^d$.

Proof. We use the functions $a_k(t)$ from Lemma 6.2. Given a multiindex $I = (i_1, \dots, i_d) \in \mathbb{Z}_+^d$, set $A_I(y) = a_{i_1}(y_1) \cdots a_{i_d}(y_d)$ and define

$$F(x,y) = \sum_{I \in \mathbb{Z}_+^d} \frac{v_I(x)}{I!} A_I(y) y^I. \tag{18}$$

Similarly to the analysis of $f(x,t)$ in Lemma 6.2, we set

$$W_I(x,y) = D_x^\alpha D_y^J \frac{v_I(x)}{I!} A_I(y) y^I = \frac{D_x^\alpha v_I(x)}{I!} \sum_{J' \subset J} \binom{J}{J'} D_y^{J-J'} A_I(y) D_y^{J'}(y^I)$$

where $\binom{J}{J'} = \binom{j_1}{j_1'} \cdots \binom{j_d}{j_d'}$. The term $\sum_{J' \subset J} \binom{J}{J'} D_y^{J-J'} A_I(y) D_y^{J'}(y^I)$ factors into a product. Setting

$$G_k(J, I, y_k) = \sum_{j_k' \le j_k} \binom{j_k}{j_k'} D_{y_k}^{j_k - j_k'} a_{I_k}(y_k) D_{y_k}^{j_k'}(y_k^{i_k}),$$

it follows that

$$\|W_I\|_{L^q(\Omega,L^\infty[-1,1]^d)} = \frac{\|D_x^\alpha v_I(x)\|_{L^q(\Omega)}}{I!}\|G_1(J,I,y_1)\cdots G_d(J,I,y_d)\|_{L^\infty([-1,1]^d)}$$
$$\le C\frac{A^{|\alpha|+|I|}|\alpha|^{|\alpha|\beta}|I|^{|I|\beta}}{I!}\|G_1(J,I,y_1)\cdots G_d(J,I,y_d)\|_{L^\infty([-1,1]^d)}.$$

As in one-dimension, we can replace $\alpha!$ with powers $|\alpha|^{|\alpha|}$. Indeed, it follows from the multinomial theorem that $|\alpha|! \le d^{|\alpha|}\alpha!$, and this inequality and Stirling's inequality show that

$$|I|^{|I|} \le (de)^{|I|} I!. \tag{19}$$

Consequently, using Stirling's formula following an application of (19) for both α and I, we estimate

$$\|W_I\|_{L^q(\Omega,L^\infty[-1,1]^d)} \le Cd^{|I|+|\alpha|}\prod_{k=1}^d \frac{A^{\alpha_k+i_k}}{i_k!}\alpha_k^{\alpha_k\beta} i_k^{i_k\beta}\|G_k(J,I,y_k)\|_{L^\infty([-1,1]^d)}.$$

By the techniques of Lemma 6.2, we can find constants $B_k > 0$ independent of α_k and i_k, $k = 1,\dots,d$ so that

$$\sum_{I\in\mathbb{Z}_+^d}\|W_I\|_{L^q(\Omega,L^\infty[-1,1]^d)} \le C\prod_{k=1}^d\sum_{i_k=0}^\infty \frac{B_k^{\alpha_k+i_k}}{i_k!}\alpha_k^{\alpha_k\beta} i_k^{i_k\beta}\|G_k(J,I,y_k)\|_{L^\infty([-1,1]^d)}$$
$$\le C\prod_{k=1}^d B_k^{\alpha_k+i_k}\alpha_k^{\alpha_k\beta} j_k^{j_k\beta} \le B^{|\alpha|+|I|}|\alpha|^{|\alpha|\beta} I|^{|I|\beta}$$

for $(x,y) \in \Omega\times[-1,1]^d$ and where $B = \max\{B_1,\dots,B_d\}$. Thus, we have proven that $F \in \mathcal{G}^{q,\beta}(\Omega,\mathcal{G}^{\infty,\beta}(-1,1))$ and it easily follows that

$$D_y^J F(x,0) = v_J(x) \text{ for all } x\in\Omega \text{ and } J\in\mathbb{Z}_+^d.$$

□

Definition 6.5. Let $\Omega\subset\mathbb{R}^d$ be an open set and $\tilde\Omega$ be a neighborhood of Ω in $\mathbb{C}^d$. Let $f\in C^\infty(\Omega)$. A function $\tilde f(x,y)\in C^\infty(\tilde\Omega)$ is called an *almost analytic extension* of $f(x)$ if $\tilde f(x,0) = f(x)$ for all $x\in\Omega$ and for each $j = 1,\dots,d$,

$$\frac{\partial\tilde f}{\partial\bar z_j} = O(|y|^k)$$

for $k = 1,2,\dots$.

We now apply Lemma 6.4 to prove that a function in $\mathcal{G}^{q,\beta}(\Omega)$ has an an almost analytic extension to a function in $\mathcal{G}^{q,\beta}(\Omega\times(-1,1)^d)$.

Theorem 6.6. *Let $\Omega\subset\mathbb{R}^d$ and $f\in\mathcal{G}^{q,\beta}(\Omega)$. Then there exists $\tilde f\in\mathcal{G}^{q,\beta}(\Omega,\mathcal{G}^{\infty,\beta}((-1,1)^d))$ that is an almost analytic extension of f. More precisely, the function $\tilde f$ satisfies $\tilde f(x,0) = f(x)$ for all $x\in\Omega$ and there exist constants C, A independent of $k\in\mathbb{Z}_+$ so that*

$$\Big\|\frac{\partial\tilde f}{\partial\bar z_j}(\cdot,y)\Big\|_{L^q(\Omega)} \le CA^k k^{k(\beta-1)}|y|^k.$$

Proof. For $(x,y)\in\Omega\times[-1,1]^d$, we set

$$\tilde f(x,y)=\sum_{\alpha\in\mathbb{Z}^d_+}\frac{1}{\alpha!}D^\alpha f(x)A_\alpha(y)(iy)^\alpha$$

where $A_\alpha(y)$ is defined as in Lemma 6.4. Given a multiindex $\alpha=(\alpha_1,\dots,\alpha_d)$, set $\alpha^{\hat{j}}=(\alpha_1,\dots,\alpha_{j-1},0,\alpha_{j+1},\dots,\alpha_d)$. Then

$$\begin{aligned}\frac{\partial\tilde f}{\partial\bar z_j}(x,y)&=\frac12\Big(D_{x_j}+iD_{y_j}\Big)\sum_{\alpha^{\hat{j}}\in\mathbb{Z}^{d-1}_+}\sum_{\alpha_j=0}^\infty\frac{1}{\alpha^{\hat{j}}!\alpha_j!}D^{\alpha^{\hat{j}}}D^{\alpha_j}f(x)A_{\alpha^{\hat{j}}}(y)a_{\alpha_j}(y_j)(iy)^{\alpha^{\hat{j}}}(iy_j)^{\alpha_j}\\
&=\frac12\sum_{\alpha^{\hat{j}}\in\mathbb{Z}^{d-1}_+}\sum_{\alpha_j=0}^\infty\frac{1}{\alpha^{\hat{j}}!\alpha_j!}D^{\alpha^{\hat{j}}}D^{\alpha_j+1}f(x)A_{\alpha^{\hat{j}}}(y)a_{\alpha_j}(y_j)(iy)^{\alpha^{\hat{j}}}(iy_j)^{\alpha_j}\\
&-\frac12\sum_{\alpha^{\hat{j}}\in\mathbb{Z}^{d-1}_+}\sum_{\alpha_j=1}^\infty\frac{1}{\alpha^{\hat{j}}!(\alpha_j-1)!}D^{\alpha^{\hat{j}}}D^{\alpha_j}f(x)A_{\alpha^{\hat{j}}}(y)a_{\alpha_j}(y_j)(iy)^{\alpha^{\hat{j}}}(iy_j)^{\alpha_j-1}\\
&+\frac12\sum_{\alpha^{\hat{j}}\in\mathbb{Z}^{d-1}_+}\sum_{\alpha_j=1}^\infty\frac{1}{\alpha^{\hat{j}}!\alpha_j!}D^{\alpha^{\hat{j}}}D^{\alpha_j}f(x)A_{\alpha^{\hat{j}}}(y)D_{y_j}a_{\alpha_j}(y_j)(iy)^{\alpha^{\hat{j}}}(iy_j)^{\alpha_j}\\
&=\frac12\sum_{\alpha^{\hat{j}}\in\mathbb{Z}^{d-1}_+}\sum_{\alpha_j=1}^\infty\frac{1}{\alpha^{\hat{j}}!\alpha_j!}D^{\alpha^{\hat{j}}}D^{\alpha_j}f(x)A_{\alpha^{\hat{j}}}(y)D_{y_j}a_{\alpha_j}(y_j)(iy)^{\alpha^{\hat{j}}}(iy_j)^{\alpha_j}.\end{aligned}$$

The proof now follows from the construction of $a_{\alpha_j}(y_j)$ and Lemma 6.4. □

6.2. **Existence of $\mathcal{G}^{q,\beta}$-approximate solutions.** We first prove

Lemma 6.7. *Let $\Omega\subset\mathbb{R}^d$ be an open set, $f\in\mathcal{G}^{q,\beta}_A(\Omega)$, and $a_k\in\mathcal{G}^{q,\beta}_A(\Omega\times(-1,1))$ for $k=1,2,\dots,d$ be given. Set $u_0(x)=f(x)$ and for $j\ge1$,*

$$u_j(x)=-\frac1j\sum_{i=0}^{j-1}\frac1{i!}\sum_{k=1}^d D_{x_k}u_{j-1-i}D^i_t a_k(x,0).$$

Then there exist constants $A_0,C>1$ such that

$$\|D^\alpha_x u_j\|_{L^q(\Omega)}\le C\frac{A_0^{j+|\alpha|}}{j!}(|\alpha|+j)^{(|\alpha|+j)\beta}$$

for all $x\in\Omega$, $j\in\mathbb{Z}_+$, and $\alpha\in\mathbb{Z}^d_+$.

To prove Lemma 6.7, we need to recall [BP09, Lemma 4.2]. This result says that if $A>0$ is a fixed constant then there exists constants $L,G>1$ so that for any $\alpha\in\mathbb{Z}^d_+$,

$$\frac{A}{L-1}\sum_{\beta\le\alpha}G^{1-|\alpha-\beta|}\le1. \tag{20}$$

Proof of Lemma 6.7. From the hypotheses of the lemma, we know that

$$\|D^\alpha_x f\|_{L^q(\Omega)}\le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\quad\text{and}\quad\|D^\alpha_x D^n_t a_k(\cdot,0)\|_{L^q(\Omega)}\le CA^{|\alpha|+n}(|\alpha|+n)^{(|\alpha|+n)\beta}. \tag{21}$$

Choose $L, G > 1$ so that (20) holds and define $A_1 = \max\{dAL, AG\}$. We now induct on j. By (21), it follows that for $x \in \Omega$ and $\alpha \in \mathbb{Z}_+^d$ we have

$$\|D_x^\alpha u_0\|_{L^q(\Omega)} = \|D_x^\alpha f\|_{L^q(\Omega)} \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta} \leq C\frac{A_1^0}{0!}B^{|\alpha|}(|\alpha|+0)^{(|\alpha|+0)\beta}.$$

Now assume that the result holds for all $0 \leq m < j$, and we will show that the result holds for $m = j$. From the definition of u_j, the product rule implies that

$$|D_x^\alpha u_j(x)| \leq \frac{1}{j}\sum_{i=0}^{j}\frac{1}{i!}\sum_{k=1}^{d}\sum_{J\subset\alpha}\binom{\alpha}{J}|D_x^{J+e_k}u_{j-1-i}||D_x^{\alpha-J}D_t^i a_k(x,0)| \tag{22}$$

where $\{e_1, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$. By the induction hypothesis, we know

$$\|D_x^{J+e_k}u_{j-1-i}\|_{L^q(\Omega)} \leq C\frac{A_1^{j-i-1}}{(j-i-1)!}B^{|J|+1}(|J|+j-i)^{(|J|+j-i)\beta} \tag{23}$$

and from (21) we know that

$$\|D_x^{\alpha-J}D_t^i a_k(\cdot,0)\|_{L^q(\Omega)} \leq CA^{|\alpha-J|+i}(|\alpha-J|+i)^{(|\alpha-J|+i)\beta}. \tag{24}$$

Since $p^p q^q \leq (p+q)^{p+q}$ and $\binom{\alpha}{J} \leq \binom{|\alpha|}{|J|}$ if $J \subset \alpha \in \mathbb{Z}_+^d$, it follows that for $i = 0, 1, \ldots, j-1$,

$$\begin{aligned}
&\binom{\alpha}{J}\frac{(|J|+j-i)^{(|J|+j-i)\beta}(|\alpha-J|+i)^{(|\alpha-J|+i)\beta}}{i!(j-i-1)!}\\
&\leq \frac{|\alpha|!}{|J|!}\frac{(|J|+j-i)^{(|J|+j-i)\beta}(|\alpha-J|+i)^{(|\alpha-J|+i)(\beta-1)}e^{|\alpha|-|J|+i}}{(j-i-1)!}\frac{(|\alpha|-|J|+i)!}{i!(|\alpha|-|J|)!}\\
&\leq (2e)^{|\alpha|+j-1}\frac{|\alpha|!}{|J|!}\frac{(|J|+j-i)^{|J|+j-i}(|\alpha|+j)^{(|\alpha|+j)(\beta-1)}}{(j-i-1)!}\\
&\leq (2e)^{|\alpha|+j-1}e^{|J|+j-i}\frac{(|J|+j-i)!}{|J|!(j-i)!}\frac{|\alpha|!(j-i)!j!}{(j-i-1)!j!}(|\alpha|+j)^{(|\alpha|+j)(\beta-1)}\\
&\leq (2e)^{2(|\alpha|+j-1)}\frac{(|\alpha|+j)^{(|\alpha|+j)\beta}}{(j-1)!}.
\end{aligned}$$

Next, using (20) and (22)-(24) in combination with the previous inequality, we can establish

$$\begin{aligned}
\|D_x^\alpha u_j\|_{L^q(\Omega)} &\leq C\frac{1}{j}\sum_{i=0}^{j}\frac{1}{i!}\sum_{k=1}^{d}\sum_{J\subset\alpha}\binom{\alpha}{J}\frac{A_1^{|\alpha|+j}}{(j-i-1)!}(|J|+j-i)^{(|J|+j-i)\beta}(|\alpha-J|+i)^{(|\alpha-J|+i)\beta}\\
&\leq C(2e)^{2(|\alpha|+j-1)}\frac{(|\alpha|+j)^{(|\alpha|+j)\beta}}{(j-1)!}A_1^{|\alpha|+j}\frac{1}{j}\sum_{i=0}^{j}\sum_{k=1}^{d}\sum_{J\subset\alpha}1\\
&\leq C\frac{A_0^{|\alpha|+j}}{j!}(|\alpha|+j)^{(|\alpha|+j)\beta}
\end{aligned}$$

for $A_0 = 2dA_1$. The proof is now complete. □

With Lemma 6.7 in hand, we are now able to prove the existence of $\mathcal{G}^{q,\beta}(\Omega\times(-1,1))$-approximate solutions. See [BP09, Proposition 4.3].

Definition 6.8. Let $\Omega \subset \mathbb{R}^d$, $\mathcal{O} \subset \mathbb{C}^n$, and the complex vector field

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{d} a_j(x,t,\zeta) D_{x_j} + \sum_{k=1}^{n} b_k(x,t,\zeta) \frac{\partial}{\partial \zeta_k}$$

where the coefficients $a_j, b_k \in \mathcal{G}^{q,\beta}(\Omega, H(\mathcal{O}))$. A $\mathcal{G}^{q,\beta}(\Omega \times (-1,1))$-*approximate solution* or $\mathcal{G}^{q,\beta}$-*approximate solution* of the equation $Lw = 0$ is a function $u(x,t,\zeta) \in \mathcal{G}^{q,\beta}(\Omega \times (-1,1), H(\mathcal{O}))$ if there exists a constant A so that

$$\|Lu(\cdot,t,\zeta)\|_{L^q(\Omega)} \leq C A^\ell \ell^{\ell(\beta-1)} |t|^\ell$$

for all $(x,t,\zeta) \in \Omega \times (-1,1) \times V$, $\ell \geq 0$, and such that $u(x,0,\zeta) = f(x,\zeta)$.

Theorem 6.9. *Let $\Omega \subset \mathbb{R}^d$, $\mathcal{O} \subset \mathbb{C}^n$, and the complex vector field*

$$L = D_t + \sum_{j=1}^{d} a_j(x,t,\zeta) D_{x_j} + \sum_{k=1}^{n} b_k(x,t,\zeta) \frac{\partial}{\partial \zeta_k}$$

where the coefficients $a_j, b_k \in \mathcal{G}^{q,\beta}(\Omega \times (-1,1), H(\mathcal{O}))$. Then there exists an $\mathcal{G}^{q,\beta}(\Omega \times (-1,1))$-approximate solution u of the equation $Lw = 0$.

Proof. Given the above arguments, it is natural to look for a solution of the form $u(x,t,\zeta) = \sum_{\ell=0}^\infty u_\ell(x,\zeta) t^\ell$. We define u_ℓ recursively. Set $u_0(x,\zeta) = f(x,\zeta)$. Formally, if we require $D_t^{\ell-1} Lu(x,0,\zeta) = 0$ for $\ell \geq 1$, then we are forced to set

$$u_\ell(x,\zeta) = -\frac{1}{\ell} \sum_{i=0}^{\ell-1} \frac{1}{i!} \Big[\sum_{m=1}^{d} D_{x_m} u_{\ell-i-1}(x,\zeta) D_t^i a_m(x,0,\zeta) + \sum_{m'=1}^{n} \frac{\partial u_{\ell-i-1}}{\partial \zeta_{m'}}(x,\zeta) D_t^i b_{m'}(x,0,\zeta) \Big].$$

It now follows from the argument of Lemma 6.7 that there exist constants $C, A > 0$ so that

$$\|D_x^\alpha u_j(\cdot,\zeta)\|_{L^q(\Omega)} \leq C \frac{A^{|\alpha|+j}}{j!} (|\alpha| + j)^{(|\alpha|+j)\beta}.$$

The problem is that $u(x,t,\zeta)$ defined by $u(x,t,\zeta) = \sum_{\ell=0}^\infty u_\ell(x,\zeta) t^\ell$ may not converge. However, we can apply Lemma 6.3 with $v_j(x,\zeta) = j! u_j(x,\zeta)$ to show that there exists a function $u(x,t,\zeta) \in \mathcal{G}^{q,\beta}(\Omega \times (-1,1), H(\mathcal{O}))$ satisfying

$$u_j(x,\xi) = \frac{1}{j!} D_t^j u(x,0,\zeta)$$

for all $(x,\xi) \in \Omega \times V$ and $j = 0, 1, \dots$. Moreover, $u(x,0,\zeta) = u_0(x,\zeta) = f(x,\zeta)$. Finally, that u is an $\mathcal{G}^{q,\beta}(\Omega \times (-1,1))$-approximate solution of $Lw = 0$ follows immediately from the construction of $u(x,t,\zeta)$ given in Lemma 6.3. □

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