Correlation Dimension Wonderland Theorems

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Existence of generic sets of self-adjoint operators, related to correlation dimensions of spectral measures, is investigated in separable Hilbert spaces. Typical results say that, given an orthonormal basis, the set of operators whose corresponding spectral measures are both 0-lower and 1-upper correlation dimensional is generic. The proofs rely on details of the relations among Fourier transform of spectral measures and Hausdorff and packing measures on the real line. Then such results are naturally combined with the Wonderland Theorem. Applications are to classes of discrete one-dimensional Schrödinger operators and general (bounded) self-adjoint operators as well. Physical consequences include a proof of exotic dynamical behavior of singular continuous spectrum in some settings.

1 Introduction

The outstanding Wonderland Theorem (WT), by Simon [18], is used to show that certain sets of self-adjoint operators with purely singular continuous spectrum are generic, that is, are dense G_δ sets. More precisely, the WT states the following: let (X, d) be a complete metric space of self-adjoint operators acting in the separable Hilbert space $\mathcal{H}$, for which convergence in the metric d implies strong resolvent convergence; if each of the sets $\{T \in X \mid T$ has purely absolutely continuous spectrum$\}$ and $\{T \in X \mid T$ has pure point spectrum$\}$ is dense in X, then $C_{\rm sc} := \{T \in X \mid T$ has purely singular continuous spectrum$\}$ is generic.

In this work, we present a version on generic sets related to correlation dimensions $D_2^{\pm}(\mu_\psi^T)$ of spectral measures μ_ψ^T of self-adjoint operators T, given some $\psi \in \mathcal{H}$ (see Definition 2.12). In fact, we have two versions, one for the lower correlation dimension $D_2^-(\mu_\psi^T)$, which is connected to Hausdorff dimensional properties of the corresponding spectral measures, discussed in Section 3, and another one for the upper correlation dimension $D_2^+(\mu_\psi^T)$, which is connected to packing dimensional properties and discussed in Section 4. From the

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beginning, we try to combine both versions with WT in order to give more precise information on some generic sets of operators with singular continuous spectra, that is, additional dimensional information about the spectra of suitable generic subsets of $C_{\rm sc}$.

This is not the first result which refines the WT; in fact, the present authors have shown in [3], under different hypotheses, that for some metric spaces of Schrödinger operators, the set of operators whose upper and lower dynamical exponents are simultaneously one and zero, respectively, is generic (see Subsection 1.1 for a discussion about dynamical exponents). It is also possible to show that both results are related; more precisely, the result about the upper correlation dimension also gives sufficient conditions for quasi-ballistic dynamics. The hypotheses, however, as mentioned above, are stronger here. Nonetheless, in spite of the dynamical consequences, we are also interested in the spectral properties described in this paper. In this regard, we also note that a third work that extend the WT for one-packing and zero-Hausdorff dimension spectra is being prepared [5].

Let $\alpha, \alpha' \in [0,1]$ and fix $0 \neq \psi \in \mathcal{H}$. Typically, our abstract results give conditions to guarantee that the subset of operators $T \in X$ for which the spectral measures μ_ψ^T are purely singular continuous and $D_2^-(\mu_\psi^T) \leq \alpha$ is generic (Corollary 3.6), and also conditions so that the set of T with $D_2^+(\mu_\psi^T) \geq \alpha'$ is generic (Corollary 4.2). Although we state such results as corollaries, we refer to them as *Correlation Dimension Wonderland Theorems*. Since the Hilbert spaces are separable, the results immediately generalize to simultaneously include all vectors in a given orthonormal basis.

With respect to the abstract results, one main technical step is to prove that both $\{T \in X \mid D_2^-(\mu_\psi^T) \leq \alpha\}$ and $\{T \in X \mid D_2^+(\mu_\psi^T) \geq \alpha\}$ are G_δ subsets of X (see Theorems 3.5 and 4.1), and for this we shall employ dynamical arguments, as (in particular) in Lemma 2.15. Each application asks for specific arguments to present suitable dense subsets.

Given a real Borel measure μ and a Borel set $A \subset \mathbb{R}$, write $\mu_{;A}(\cdot) := \mu(A \cap \cdot)$ for its restriction to A.

Definition 1.1 Let $\alpha \in [0,1]$, I be a non-degenerated real interval, T a self-adjoint operator acting in $\mathcal{H}$ and $\psi \in \mathcal{H}$. Then, the spectral measure μ_ψ^T is called:

- α*-lower correlation dimensional* (α-LCD) on I if $D_2^-(\mu_{\psi;I}^T) \leq \alpha$;
- α*-upper correlation dimensional* (α-UCD) on I if $D_2^+(\mu_{\psi;I}^T) \geq \alpha$.

In case $I \supset \sigma(T)$ we just say that μ_ψ^T is α-LCD and α-UCD.

Now, we describe the applications we have gotten for a class of bounded discrete Schrödinger operators acting on $l^2(\mathbb{N})$ ($\mathbb{N} = \{0, 1, 2, 3, \ldots\}$), and denote its canonical orthonormal basis by $(\delta_j)_{j\in\mathbb{N}}$, that is, δ_j is the vector which is equal to 1 at $n = j$ and zero otherwise. For fixed $r > 0$ and $\phi \in [0, \pi)$, let X_ϕ^r be the set of operators H_ϕ^v with action

$$(H_\phi^v \psi)_n = \psi_{n+1} + \psi_{n-1} + V_n \psi_n \,, \tag{1.1}$$

and satisfying the boundary condition

$$\psi_{-1} \cos\phi - \psi_0 \sin\phi = 0 \,, \tag{1.2}$$

where the potential $v = (V_n)$ is an arbitrary real sequence with $|V_n| \le r$, for all $n \in \mathbb{N}$. We equip X_ϕ^r with the topology of pointwise convergence of sequences. By combining both correlation dimension WT's with a specific construction, we will prove the rather unexpected

Theorem 1.2 *Fix $r > 0$. Then, for every $\phi \in [0, \pi)$, the set $\{T \in X_\phi^r \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T = [-2 - r, 2 + r]$, it is purely singular continuous, and $\mu_{\delta_j}^T$ is 0-lower and 1-upper correlation dimensional on $(-2, 2)$ for all $j\}$ is generic in X_ϕ^r.*

Next, we present an application to a space of unbounded discrete Schrödinger operators. Consider the space of real sequences $\tilde{X} := \{v = (v_n) \mid v_n \in \mathbb{R}\}$, with the metric

$$\tilde{d}(u, v) := \sum_{n=0}^{\infty} 2^{-n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}, \quad u, v \in \tilde{X}. \tag{1.3}$$

For each $v \in \tilde{X}$, one associates the (unique) self-adjoint discrete Schrödinger operator H_ϕ^v with domain $\operatorname{dom} H_\phi^v = \{\psi \in l^2(\mathbb{N}) \mid \sum_n v_n^2 |\psi_n|^2 < \infty\}$, whose action is given by (1.1) and which is subjected to the ϕ-boundary condition (1.2). By denoting, for each fixed $\phi \in [0, \pi)$,

$$X_\phi := \big\{ H_\phi^v \mid v \in \tilde{X} \big\}, \tag{1.4}$$

we consider the metric space (X_ϕ, d), with $d(H_\phi^u, H_\phi^v) := \tilde{d}(u, v)$, $u, v \in \tilde{X}$. We will prove the following

Theorem 1.3 *For each $\phi \in [0, \pi)$, the set $\{T \in X_\phi \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T \supset (-2, 2)$, it is purely singular continuous, and $\mu_{\delta_j}^T$ is 0-lower and 1-upper correlation dimensional on $(-2, 2)$ for all $j\}$ is generic in X_ϕ.*

We also apply our dimensional Wonderland Theorems to general bounded self-adjoint operators, defined on a separable Hilbert space $\mathcal{H}$, i.e., not necessarily Schrödinger-like operators. Our next results give additional information to Theorems 3.1 and 3.2 in [18].

Fix $r > 0$. Let $X_r = \{T \mid T \text{ is self-adjoint}, \|T\| \le r\}$, and define

$$d(T, T') = \sum_{j=0}^{\infty} \min(2^{-j}, \|(T - T')\tilde{\xi}_j\|) \,,$$

where $(\tilde{\xi})_{j \ge 0}$ is an orthonormal basis of $\mathcal{H}$, for which convergence corresponds to strong operator convergence.

Theorem 1.4 *Fix $r > 0$ and an orthonormal basis (ξ_j) of $\mathcal{H}$. The set $\{T \in X_r \mid \sigma(T) = [-r, r]$, it is purely singular continuous, and $\mu^T_{\xi_j}$ is 0-lower and 1-upper correlation dimensional for all $j\}$ is generic in X_r.*

We also say something about the set of general bounded self-adjoint operators on $\mathcal{H}$ with the usual operator norm. Fix $a < b$, and let $Y_{a,b} := \{T \mid T$ is self-adjoint with spectrum $[a, b]\}$ be equipped with the metric $d(T, T') = \|T - T'\|$.

Theorem 1.5 *Fix $a < b$ and an orthonormal basis (ξ_j) of $\mathcal{H}$. The set $\{T \in Y_{a,b} \mid \sigma(T)$ is purely singular continuous, and $\mu^T_{\xi_j}$ is 0-lower and 1-upper correlation dimensional for all $j\}$ is generic in $Y_{a,b}$.*

Recall that a positive Borel measure μ on $\mathbb{R}$ is called uniformly α-Hölder continuous, with $\alpha \in [0, 1]$, if there exists $C > 0$ so that $\mu(I) \leq C\ell(I)^\alpha$ for every interval I with Lebesgue measure $\ell(I) < 1$. Such measures have been playing an important role in quantum dynamics, as discussed in [14] and references therein. Our next result indicates why it is not an easy task to prove that spectral measures are uniformly α-Hölder continuous ($\alpha > 0$); although we state it for $Y_{a,b}$, it has analogous versions for the above spaces X^r_ϕ, X_ϕ and X_r as well, which are obtained similarly.

Theorem 1.6 *Let $0 \neq \psi \in \mathcal{H}$. Then, the set of operators $\{T \in Y_{a,b} \mid \mu^T_\psi$ is uniformly α-Hölder continuous for some $\alpha > 0\}$ is meager (i.e., first category) in $Y_{a,b}$.*

Our strategy to obtain the Correlation Dimensional Wonderland Theorems is to use dynamical characterizations of measures μ, i.e., the behavior of their Fourier transforms $\hat{\mu}(t)$ as "time" $t \to \infty$, in contrast with the (different) arguments in the original version of the WT [18]. This strategy is motivated by the well-known equivalence between strong resolvent and strong dynamical convergences (see [6], Chapter 10, for a discussion). Our main abstract results are presented in Corollaries 3.6, 3.7, 4.2, and 4.3. The usual "duality" between Hausdorff and packing measures is also reflected here in the occurrences of $D_2^- \leq \alpha$ and $D_2^+ \geq \alpha$, related to the Hausdorff and packing settings, respectively.

1.1 Physical consequences

There are important spin-offs from our estimates of correlation dimensions when certain classes of Schrödinger operators H are considered; the quantities ahead give a flavor of *delocalization* or *quantum transport*. The first one is that $D_2^\pm(\mu^H_\psi)$ rule the long time behavior of the average quantum return probability $\langle p^H_\psi \rangle_t$ to the initial state ψ. The point here is that (see also Subsection 2.2)

$$\langle p^H_\psi \rangle_t := \frac{1}{t} \int_0^t \left| \langle \psi, e^{-isH} \psi \rangle \right|^2 \, \mathrm{d}s = \langle |\hat{\mu}^H_\psi|^2 \rangle_t,$$

together with the important relations (Equations (16) and (17) in [1]; see also [17])

$$\limsup_{t\to\infty} \frac{\ln\langle p_\psi^H\rangle_t}{\ln t} = -D_2^-(\mu_\psi^H) \quad \text{and} \quad \liminf_{t\to\infty} \frac{\ln\langle p_\psi^H\rangle_t}{\ln t} = -D_2^+(\mu_\psi^H). \tag{1.5}$$

The faster $\langle p_\psi^H\rangle_t$ decays as $t \to \infty$, the more delocalized is the physical state. Taking into account these relations, given H, we say that a vector ψ has *ballistic escaping* if $D_2^-(\mu_\psi^H) = 1$ and *quasi-ballistic escaping* if $D_2^+(\mu_\psi^H) = 1$; ballistic escaping implies the quasi-ballistic spreading property, discussed in the next paragraph. On the other hand, the slower $\langle p_\psi^H\rangle_t$ decays, the more ψ "survives," so we say that ψ has a *strong survival* if $D_2^+(\mu_\psi^H) = 0$ and *weak survival* if $D_2^-(\mu_\psi^H) = 0$.

Another quantity of physical interest is the algebraic growth of the width of quantum wave packets, usually associated with the (time-averaged) q-moments $\langle M_{\psi,H}^q\rangle(t)$ of the position operator at time $t > 0$; here, we restrict ourselves to the Hilbert space $l^2(\mathbb{N})$. For an interval $I \subset \mathbb{R}$, let $P^H(I)$ be the corresponding spectral projections of H, and note that, for spectral measures, one has $\mu_{P^H(I)\psi}^H(\cdot) = \mu_{\psi;I}^H(\cdot) = \mu_\psi^H(I \cap \cdot)$. The dynamics generated by H is called *quasi-ballistic spreading in the interval I* if the *upper dynamical exponent* $\beta_\psi^+(q, H)$ satisfies $\beta_\psi^+(q, H) \geq 1$ for all meaningful $\psi \in \operatorname{dom} H \cap P^H(I)\mathcal{H}$ and all $q > 0$. Precisely, one defines

$$\beta_\psi^+(q, H) := \limsup_{t\to\infty} \frac{\ln\langle M_{\psi,H}^q\rangle(t)}{q \ln t}, \tag{1.6}$$

with the q-moment related to the initial state ψ given by

$$\langle M_{\psi,H}^q\rangle(t) := \frac{1}{t}\int_0^t \sum_{n=0}^{\infty} n^q |\langle e^{-isH}\psi, \delta_n\rangle|^2 \, \mathrm{d}s;$$

it is tacitly assumed that the initial states ψ are such that the moments are meaningful for all $t > 0$.

Analogously, one defines

$$\beta_\psi^-(q, H) := \liminf_{t\to\infty} \frac{\ln\langle M_{\psi,H}^q\rangle(t)}{q \ln t},$$

the so-called *lower dynamical exponent* of $\langle M_{\psi,H}^q\rangle(t)$ for all meaningful $\psi \in \operatorname{dom} H \cap P^H(I)\mathcal{H}$ and all $q > 0$. The dynamics generated by H is, in this case, called *ballistic spreading in the interval I*.

Denote the upper packing dimension of the spectral measure μ_ψ^H by $\dim_{\mathrm{P}}^+(\mu_\psi^H)$, as defined in Subsection 2.1, which is related to the upper dynamical exponents by inequality

$$\beta_\psi^+(q, H) \geq \dim_{\mathrm{P}}^+(\mu_\psi^H), \tag{1.7}$$

originally presented in [12]. Hence, if one obtains $\dim_{\mathrm{P}}^+(\mu_\psi^H) = 1$, then the quasi-ballistic spreading follows for such initial condition ψ. Thus, by combining Theorems 1.2 and 1.3

with Proposition 2.13, we have quasi-ballistic dynamics for suitable generic subsets in the spaces of Schrödinger operators X^r_ϕ and X_ϕ (see item *(ii)* of Theorem 1.7).

A (multifractal) characterization of the structure of wave packets in quantum mechanics has been proposed by Evangelou and Katsanos [8], which led to another set of interesting growth exponents $\mathcal{D}^\pm_q$, $q > 0$, that were called *dynamical dimensions* in [11]. The construction resembles a thermodynamic formalism for the measure given by the average return probability to basis vectors as time $t \to \infty$. Let H be a Schrödinger operator acting in the Hilbert space $\mathcal{H}$, $\psi \in \mathcal{H}$ a normalized vector, and $(\eta_j)_{j\geq 0}$ an orthonormal basis of $\mathcal{H}$, with $\eta_0 = \psi$. Write

$$\langle p^H_{\psi,j}\rangle_t := \frac{1}{t}\int_0^t \left|\langle \eta_j, e^{-isH}\psi\rangle\right|^2 \,\mathrm{d}s,$$

for the average sojourn of ψ in η_j up to time t; note that $\langle p^H_{\psi,0}\rangle_t = \langle p^H_\psi\rangle_t = \langle |\hat{\mu}^H_\psi|^2\rangle_t$. Shortly, consider the partition function-like quantity [8, 11]

$$Z_{q,H}(\psi,t) := \sum_j \langle p^H_{\psi,j}\rangle^q_t,$$

which converges (at least) for any $q \geq 1$, and then the number of states

$$N_{q,H}(\psi,t) := Z^{1/(1-q)}_{q,H}(t), \quad q \neq 1,$$

which leads to the upper and lower dynamical dimensions

$$\mathcal{D}^+_q(\psi,H) := \limsup_{t\to\infty} \frac{\ln N_{q,H}(\psi,t)}{\ln t}, \qquad \mathcal{D}^-_q(\psi) := \liminf_{t\to\infty} \frac{\ln N_{q,H}(\psi,t)}{\ln t}.$$

A simple calculation (see the short proof of Proposition 8, page 5205 of [11]) gives

$$\mathcal{D}^-_q(\psi,H) \leq \frac{q}{q-1} D^-_2(\mu^H_\psi), \qquad \forall q > 1. \tag{1.8}$$

Therefore, for all operators that we have gotten 0-LCD for ψ, it automatically follows that the lower dynamical dimensions $\mathcal{D}^-_q(\psi,H)$ vanish for all $q > 1$. Note that although $\mathcal{D}^-_q(\psi,H)$ is basis dependent, the inequality (1.8) always holds true if ψ is an element of the (orthonormal) basis; from this point of view, item *(iii)* in Theorem 1.7 is robust.

For the above space X^r_ϕ of some discrete Schrödinger operators, by taking into account Theorem 1.2, such set of physical appealing results are gathered in the next theorem, and they rigorously illustrate the "exotic" dynamical behavior expected for Schrödinger operators with singular continuous spectra. For example, given $\psi \in \mathcal{H}$, item *(i)* of Theorem 1.7 says that quasi-ballistic escaping or weak survival are generic properties, whereas ballistic escaping and strong survival may occur only in a meager set of operators. Of course, by Theorem 1.3, a very similar theorem holds true for the space X_ϕ.

Theorem 1.7 *Let $\mathcal{H} = l^2(\mathbb{N})$, X^r_ϕ as above, and fix a normalized $\psi \in \mathcal{H}$. Then, for each $\phi \in [0,\pi)$, there is a generic set of operators H in X^r_ϕ so that:*

(i) $\liminf_{t\to\infty} \dfrac{\ln\langle p_\psi^H\rangle_t}{\ln 1/t} = 0$ and $\limsup_{t\to\infty} \dfrac{\ln\langle p_\psi^H\rangle_t}{\ln 1/t} = 1$.

(ii) For all $q > 0$, one has $\beta_\psi^+(q,H) \geq 1$, so that the corresponding dynamics of ψ is quasi-ballistic spreading (of course, if $\beta_\psi^+(q)$ is meaningful).

(iii) The lower dynamical dimensions vanish, that is, $\mathcal{D}_q^-(\psi,H) = 0$, for all $q > 1$.

1.2 Organization

In order to properly formulate our results, it was necessary to describe some decompositions of Borel measures with respect to Hausdorff and packing measures and dimensions, as well as the corresponding spectral decompositions of self-adjoint operators. In Subsections 2.1, we recall such Hausdorff decompositions, by mostly following [12, 14] and references therein, and joint the corresponding packing decompositions. In Subsection 2.2 we recall important relations between correlation dimensions and dynamical properties of self-adjoint operators.

Our lower and upper (correlation dynamical) generic results are discussed in Sections 3 and 4, respectively. Section 5 presents applications to subclasses of bounded and unbounded discrete Schrödinger operators, along with the proofs of Theorems 1.2 and 1.3. Applications to general bounded operators, including proofs of Theorems 1.4, 1.5 and 1.6, appear in Section 6. We have tried to present detailed proofs of such results.

We close this Introduction with some words about notation. $\mathcal{H}$ will always denote a complex separable Hilbert space. The spectrum of a self-adjoint operator T will be denoted by $\sigma(T)$ and its point spectrum is the closure of the set of its eigenvalues, whose corresponding spectral projection is denoted by P_{p}^T; similarly, P_{ac}^T is the projection onto the corresponding absolutely continuous subspace, and so on. Herein, μ will always indicate a finite nonnegative Borel measure on $\mathbb{R}$, and, as already mentioned, its restriction to a set A will be $\mu_{;A}(\cdot) := \mu(A\cap\cdot)$. The adjective *absolutely continuous* without specification means that μ is absolutely continuous with respect to Lebesgue measure, and the latter will be denoted by ℓ. The measure μ is *massless* if it has no atoms (i.e., μ is a continuous measure), and it is *singular* if μ and ℓ are mutually singular. A Borel measure ν on $\mathbb{R}$ is *supported* on a Borel set S if $\nu(\mathbb{R}\setminus S) = 0$; $\mathrm{supp}(\nu)$ denotes the *support* of ν, that is, the complement of the largest open set B with $\nu(B) = 0$. Finally, it will also be convenient to use the symbol K to refer to either H or P, which stands for Hausdorff and packing properties, respectively.

2 Preliminaries

For the sake of completeness, in this section we recall important decompositions of Borel measures on $\mathbb{R}$ with respect to Hausdorff and packing measures and dimensions, along with the corresponding spectral decompositions of self-adjoint operators. The notion of correlation dimension is also recalled. This section also fixes some notation.

2.1 Hausdorff and packing decompositions

Definition 2.1 Given an interval $I \subset \mathbb{R}$, denote by $|I|$ its diameter. For each $\alpha \in [0,1]$, $S \subset \mathbb{R}$, consider the number

$$h^\alpha_\delta(S) = \inf\left\{ \sum_{k=1}^{\infty} |I_k|^\alpha \mid |I_k| < \delta,\ \forall k;\ S \subset \bigcup_{k=1}^{\infty} I_k \right\},$$

that is, the infimum is taken over all covers of S by intervals I_k of size at most δ. The limit

$$h^\alpha(S) = \lim_{\delta \downarrow 0} h^\alpha_\delta(S)$$

is called the *α-dimensional (exterior) Hausdorff measure* of S.

The counting measure (which assigns to each set S the number of elements it has), at $\alpha = 0$, and the Lebesgue measure, at $\alpha = 1$, are particular cases. The *Hausdorff dimension* of the set S, here denoted by $\dim_{\mathrm{H}}(S)$, is defined as the infimum of all α such that $h^\alpha(S) = 0$; note that $h^\alpha(S) = \infty$ if $\alpha < \dim_{\mathrm{H}}(S)$.

A δ-packing of an arbitrary set $S \subset \mathbb{R}$ is a countable disjoint collection $(\bar{B}(x_k; r_k))_{k\in\mathbb{N}}$ of closed intervals centered at $x_k \in S$ and radii $r_k \le \delta/2$, so with diameters at most of δ. Define $P^\alpha_\delta(S)$, $0 \le \alpha \le 1$, as

$$P^\alpha_\delta(S) = \sup\left\{ \sum_{k=1}^{\infty} (2r_k)^\alpha \mid (\bar{B}(x_k; r_k))_k \text{ is a } \delta\text{-packing of } S \right\},$$

that is, the supremum is taken over all δ-packings of S. Then, take the decreasing limit

$$P^\alpha_0(S) = \lim_{\delta \downarrow 0} P^\alpha_\delta(S)$$

as a pre-measure.

Definition 2.2 The *α-dimensional (exterior) packing measure* $P^\alpha(S)$ of S is given by

$$P^\alpha(S) := \inf\left\{ \sum_{k=1}^{\infty} P^\alpha_0(S_k) \mid S \subset \bigcup_{k=1}^{\infty} S_k \right\}.$$

The *packing dimension* of the set S, here denoted by $\dim_{\mathrm{P}}(S)$, is defined (in analogy to $\dim_{\mathrm{H}}(S)$) as the infimum of all α such that $P^\alpha(S) = 0$, which coincides with the supremum of all α so that $P^\alpha(S) = \infty$.

It is possible to show [9, 16] that the Hausdorff and packing dimensions are related by the inequality $\dim_{\mathrm{H}}(S) \le \dim_{\mathrm{P}}(S)$, and this inequality is in general strict. It is also important to mention that P^α and h^α are Borel (regular) measures and, for $0 \le \alpha < 1$, they are not σ-finite; furthermore, $P^0 \equiv h^0$, $P^1 \equiv h^1$ and they are equivalent, respectively, to counting and Lebesgue measures.

The notions of Hausdorff packing measures and dimensions presented above lead to a series of notions of continuity and singularity of Borel measures with respect to them [3, 14]. In the following, K stands for either Hausdorff or packing properties; when we need to refer to a specific one, we use H for Hausdorff and P for packing. Recall that here μ denotes a finite (nonnegative) Borel measure on $\mathbb{R}$.

Definition 2.3 Let $\alpha \in [0, 1]$. A finite nonnegative Borel measure μ on $\mathbb{R}$ is called:

1. α-K *continuous*, denoted αKc, if $\mu(S) = 0$ for every Borel set S with $\mathrm{K}^\alpha(S) = 0$ (for instance, if K = H, then μ is called *α-Hausdorff continuous*, denoted αHc, if $\mu(S) = 0$ for every Borel set S with $h^\alpha(S) = 0$).

2. *strongly* α-K *continuous*, denoted sαKc, if $\mu(S) = 0$ for every Borel set S which has σ-finite K^α measure (i.e., $S = \cup_{j=1}^\infty S_j$ and for all j, one has $\mathrm{K}^\alpha(S_j) < \infty$).

3. α-K *singular*, denoted αKs, if it is supported on some Borel set S with $\mathrm{K}^\alpha(S) = 0$.

4. *almost* α-K *singular*, denoted aαKs, if it is supported on some Borel set S which has σ-finite K^α measure.

5. to have the α-K *Radon-Nikodym property*, denoted αKRN, if $\mathrm{d}\mu = f\mathrm{d}\mathrm{K}^\alpha$ for some K^α-measurable function $f : \mathbb{R} \to [0, \infty)$.

Definition 2.4 Let $\alpha \in [0, 1]$. A finite nonnegative Borel measure μ on $\mathbb{R}$ is called:

1. α-K *dimension continuous*, denoted αKdc, if $\mu(S) = 0$ for every Borel set S with $\dim_{\mathrm{K}}(S) < \alpha$.

2. *strongly* α-K *dimension continuous*, denoted sαKdc, if $\mu(S) = 0$ for every Borel set S with $\dim_{\mathrm{K}}(S) \le \alpha$.

3. α-K *dimension singular*, denoted αKds, if it is supported on some Borel set $S = \cup_{j=1}^\infty S_j$, with $\dim_{\mathrm{K}}(S_j) < \alpha$ for each j.

4. *almost* α-K *dimension singular*, denoted aαKds, if it is supported on some Borel set S with $\dim_{\mathrm{K}}(S) \le \alpha$.

5. μ is said to have *exact* K *dimension* α, denoted eKdα, if it is both αKdc and aαKds.

Definition 2.5 A finite nonnegative Borel measure μ on $\mathbb{R}$ is called:

1. 0-K *dimensional*, denoted 0Kd, if it is supported on a Borel set S with $\dim_{\mathrm{K}}(S) = 0$.
2. 1-K *dimensional*, denoted 1Kd, if $\mu(S) = 0$ for any Borel set S with $\dim_{\mathrm{K}}(S) < 1$.

Note that the concepts of 0Kd and 1Kd measures are equivalent to a measure of eKd0 and eKd1, respectively. In these cases, we keep both nomenclatures given the importance of such boundary cases. The next result is simple to prove, and it relates continuity properties of measures to dimensional ones.

Proposition 2.6 *Consider a finite nonnegative Borel measure μ on $\mathbb{R}$ and $\alpha \in (0,1)$. Then:*

1. *μ is* αKdc *if, and only if, it is* $(\alpha - \varepsilon)$Kc *for every* $0 < \varepsilon \le \alpha$*;*
2. *μ is* aαKds *if, and only if, it is* $(\alpha + \varepsilon)$Ks *for every* $0 < \varepsilon \le 1 - \alpha$*.*

Remark 2.7 The results of Proposition 2.6 can be extended to $\alpha \in \{0,1\}$; namely

1. μ is 0Kd if, and only if, it is εKs for every $0 < \varepsilon \le 1$;
2. μ is 1Kd if, and only if, it is $(1 - \varepsilon)$Kc for every $0 \le \varepsilon < 1$.

They can also be strengthened, that is,

3. for every $\alpha \in [0,1]$, if μ is αKc, then it is αKdc;
4. for every $\alpha \in (0,1]$, if μ is αKs, then it is aαKds.

The converse, nevertheless, is not valid in general.

Proposition 2.8 *Let $T : \operatorname{dom} T \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator in the Hilbert space $\mathcal{H}$, and μ_ψ^T the spectral measure of T associated with the vector $\psi \in \mathcal{H}$. Given $\alpha \in (0,1)$, consider*

$$\mathcal{H}_{\alpha\mathrm{Kc}}^T := \left\{\psi \mid \mu_\psi^T \text{ is } \alpha\mathrm{Kc}\right\}, \qquad \mathcal{H}_{\alpha\mathrm{Ks}}^T := \left\{\psi \mid \mu_\psi^T \text{ is } \alpha\mathrm{Ks}\right\}.$$

Then, $\mathcal{H}_{\alpha\mathrm{Kc}}^T$ and $\mathcal{H}_{\alpha\mathrm{Ks}}^T$ are closed and mutually orthogonal subspaces of $\mathcal{H}$, which are invariant under T, and $\mathcal{H} = \mathcal{H}_{\alpha\mathrm{Kc}}^T \oplus \mathcal{H}_{\alpha\mathrm{Ks}}^T$.

Proof. The statements involving the subsets $\mathcal{H}_{\alpha\mathrm{Hc}}^T$ and $\mathcal{H}_{\alpha\mathrm{Hs}}^T$ correspond to Theorem 5.1 in [14]. The statements regarding packing measures are similarly obtained. □

Remark 2.9 By using a self-explained notation, the following decompositions are also valid: $\mathcal{H} = \mathcal{H}_{\alpha\mathrm{Kdc}}^T \oplus \mathcal{H}_{\alpha\mathrm{Kds}}^T$ and $\mathcal{H} = \mathcal{H}_{\mathrm{s}\alpha\mathrm{Kdc}}^T \oplus \mathcal{H}_{\mathrm{a}\alpha\mathrm{Kds}}^T$. These are known for the Hausdorff case [14]; the packing versions follow by similar arguments.

One also introduces the fractal upper and lower dimensions of a nonnegative finite Borel measure μ; see, e.g., [12, 21]. Recall the notation $\mu_{;I}(\cdot) = \mu(I \cap \cdot)$.

Definition 2.10 Let $I \subset \mathbb{R}$ be a Borel set. The K *upper dimension* of μ restricted to I, denoted by $\dim_{\mathrm{K}}^{+}(\mu_{;I})$, is defined as

$$\dim_{\mathrm{K}}^{+}(\mu_{;I}) := \inf\{\dim_{\mathrm{K}}(S) \mid \mu(S) = \mu(I),\, S \text{ a Borel subset of } I\},$$

and the K *lower dimension* of μ restricted to I, denoted by $\dim_{\mathrm{K}}^{-}(\mu_{;I})$, as

$$\dim_{\mathrm{K}}^{-}(\mu_{;I}) := \sup\{\alpha \mid \mu(S) = 0 \text{ if } \dim_{\mathrm{K}}(S) < \alpha,\, S \text{ a Borel subset of } I\}\,,$$

if $\mu(I) > 0$, and $\dim_{\mathrm{K}}^{-}(\mu_{;I}) := 1$ otherwise.

By using known results for Hausdorff and packing measures [12], it is found that

Proposition 2.11 *Let μ be finite nonnegative Borel measure on $\mathbb{R}$, and I a Borel subset of $\mathbb{R}$. Then,*

1. *$\mu_{;I}$ is α*Kdc *if, and only if, $\alpha \le \dim_{\mathrm{K}}^{-}(\mu_{;I})$;*
2. *$\mu_{;I}$ is* aαKds *if, and only if, $\dim_{\mathrm{K}}^{+}(\mu_{;I}) \le \alpha$;*
3. *$\mu_{;I}$ is* sαKdc *if, and only if, $\alpha < \dim_{\mathrm{K}}^{-}(\mu_{;I})$;*
4. *$\mu_{;I}$ is α*Kds *if, and only if, $\dim_{\mathrm{K}}^{+}(\mu_{;I}) < \alpha$.*

Note that μ has eKdα *on I if, and only if, $\alpha = \dim_{\mathrm{K}}^{-}(\mu_{;I}) = \dim_{\mathrm{K}}^{+}(\mu_{;I})$.*

2.2 Dynamics and correlation dimensions

Now we recall the definition of correlation dimension of measures, a pivotal concept for this work.

Definition 2.12 Let μ be a finite nonnegative Borel measure on $\mathbb{R}$. The *lower and upper correlation dimensions* of μ are, respectively, defined as

$$D_2^{-}(\mu) := \liminf_{\varepsilon\to 0} \frac{\ln\left[\int \mu(B(x;\varepsilon))\,\mathrm{d}\mu(x)\right]}{\ln\varepsilon}\,, \qquad D_2^{+}(\mu) := \limsup_{\varepsilon\to 0} \frac{\ln\left[\int \mu(B(x;\varepsilon))\,\mathrm{d}\mu(x)\right]}{\ln\varepsilon}\,,$$

if, for every $\varepsilon > 0$, $\int \mu(B(x;\varepsilon))\,\mathrm{d}\mu(x) > 0$, and $D_2^{-}(\mu) = D_2^{+}(\mu) := 1$ otherwise.

Proposition 2.13 *If μ is a finite nonnegative Borel measure on $\mathbb{R}$, then*

$$D_2^{-}(\mu) \le \dim_{\mathrm{H}}^{-}(\mu)\,, \qquad D_2^{+}(\mu) \le \dim_{\mathrm{P}}^{-}(\mu).$$

Proof. See Proposition 4.1 in [2] for a proof when μ is positive. The case $\mu = 0$ is a direct consequence of Definitions 2.10 and 2.12. □

Remark 2.14 It follows that $0 \le D_2^-(\mu) \le D_2^+(\mu) \le 1$; we also note that the inequalities above may be strict [2].

Due to Proposition 2.13, ahead we will denote $D_2^{\mathrm{H}}(\mu) = D_2^-(\mu)$ and $D_2^{\mathrm{P}}(\mu) = D_2^+(\mu)$, both encoded in the symbol $D_2^{\mathrm{K}}(\mu)$; we also set lim-K := lim sup if K = H, lim-K := lim inf if K = P. Equation (1.5) can be recast as

$$\operatorname*{lim\text{-}K}_{t\to\infty} \frac{\ln\langle|\hat{\mu}_\psi^T|^2\rangle_t}{\ln t} = -D_2^{\mathrm{K}} \, ,$$

from which we obtain the following critical remark to our analysis in Sections 3 and 4 (the trivial case $\mu_\psi^T(\cdot) = 0$ follows readily).

Lemma 2.15 *Let T be a self-adjoint operator in $\mathcal{H}$ and μ_ψ^T its spectral measure associated with some $\psi \in \mathcal{H}$. Let $\alpha \in (0,1)$. Then, for every $0 < \varepsilon \le \min\{\alpha, 1-\alpha\}$, one has*

$$\{\psi \mid \operatorname*{lim\text{-}K}_{t\to\infty} t^{\alpha+\varepsilon}\langle|\hat{\mu}_\psi^T|^2\rangle_t < \infty\} \subset \{\psi \mid D_2^{\mathrm{K}}(\mu_\psi^T) \ge \alpha+\varepsilon\} \subset \{\psi \mid \operatorname*{lim\text{-}K}_{t\to\infty} t^{\alpha-\varepsilon}\langle|\hat{\mu}_\psi^T|^2\rangle_t < \infty\}. \tag{2.1}$$

Lemma 2.16 *Let $0 \ne \psi \in \mathcal{H}$ be a cyclic vector of the self-adjoint operator T, and suppose that $\dim_{\mathrm{H}}^+(\mu_\psi^T) \le \alpha$ for some $\alpha \in [0,1]$. Then, for every $0 \ne \varphi \in \mathcal{H}$, $D_2^-(\mu_\varphi^T) \le \alpha$.*

Proof. Since ψ is a cyclic vector of T, there is, for each fixed $0 \ne \varphi \in \mathcal{H}$, an $f \in \mathrm{L}^2(\mathbb{R}, d\mu_\psi^T)$ such that, for every Borel set $S \subset \mathbb{R}$,

$$\mu_\varphi^T(S) = \int_S |f(x)|^2 d\mu_\psi^T(x) \, .$$

Suppose now that $\dim_{\mathrm{H}}^+(\mu_\psi^T) \le \alpha$, for some $\alpha \in [0,1]$; then, by Definition 2.10, for each $\varepsilon > 0$ there is a Borel set A_ε such that $\mu_\psi^T(A_\varepsilon) = \mu_\psi^T(\mathbb{R})$, and $\dim_{\mathrm{H}}(A_\varepsilon) \le \alpha + \varepsilon$; but then, since

$$\mu_\varphi^T(\mathbb{R}) = \int_{\mathbb{R}} |f(x)|^2 d\mu_\psi^T(x) = \int_{A_\varepsilon} |f(x)|^2 d\mu_\psi^T(x) = \mu_\varphi^T(A_\varepsilon) \, ,$$

it follows that μ_φ^T is supported on a set of Hausdorff dimension less than or equal to $\alpha+\varepsilon$, which implies, once again by Definition 2.10, that $\dim_{\mathrm{H}}^+(\mu_\varphi^T) \le \alpha$. An application of Proposition 2.13 concludes the proof. □

3 Lower correlation dimension Wonderland Theorem

Let (X, d) be a complete metric space of self-adjoint operators, acting in the infinite-dimensional separable Hilbert space $\mathcal{H}$, such that the metric d convergence implies strong resolvent convergence. As in previous sections, T denotes a self-adjoint operator acting in $\mathcal{H}$. In order to obtain the main results of this section (Corollaries 3.6 and 3.7), we will prove, for some fixed vector $\psi \in \mathcal{H}$ and each $\alpha \in [0, 1]$, that

$$C^{\psi}_{\alpha \mathrm{ld}} := \{T \in X \mid D_2^-(\mu^T_{\psi;(a,b)}) \le \alpha\}$$

is a G_δ set in X. We begin with two known results; the first one is Proposition 2.6 in [15] and the second one is obtained by following the steps of the proof of Theorem 1.3 in [18]. They will then be followed by a technical lemma.

Proposition 3.1 *Fix $(a, b) \subset \mathbb{R}$, $b > a$, $\psi \in \mathcal{H}$. Then, $Y^\psi := \{T \in X \mid \mu^T_{\psi;(a,b)}$ is massless$\}$ is a G_δ set in X.*

Proposition 3.2 *Fix $(a, b) \subset \mathbb{R}$, $b > a$, and $0 \ne \psi \in \mathcal{H}$. Then, $U^\psi := \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a, b)\}$ is a G_δ set in X.*

Lemma 3.3 *Let $\alpha \in (0, 1)$, $(a, b) \subset \mathbb{R}$, and $\psi \in \mathcal{H}$. Then,*

$$C^{\psi}_{\alpha \mathrm{ld}} = \bigcap_{k=k_0}^{\infty} A^{\psi}_{\alpha+1/k}\,, \tag{3.1}$$

where

$$A^{\psi}_{\alpha} := \bigcap_{n=0}^{\infty} \left\{T \in X \mid \text{ for each } m,\ \exists\, t > m \text{ with } t^{\alpha} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t > n \right\},$$

and $k_0 := [3/(\min\{\alpha, 1-\alpha\})] + 1$.

Proof. By Lemma 2.15, we have, for each $T \in X$ and each $0 < \varepsilon \le \min\{\alpha, 1-\alpha\}$,

$$\limsup_{t\to\infty} t^{\alpha+\varepsilon} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t < \infty \Longrightarrow D_2^-(\mu^T_{\psi;(a,b)}) \ge \alpha + \varepsilon \Longrightarrow \limsup_{t\to\infty} t^{\alpha-\varepsilon} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t < \infty\,;$$

this implies, for a fixed $\psi \in \mathcal{H}$ and a fixed $0 < \varepsilon \le \min\{\alpha, 1-\alpha\}$, that

$$\bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcup_{t>m} \left\{T \in X \mid t^{\alpha-\varepsilon} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t > n \right\} \subset \left\{T \in X \mid D_2^-(\mu^T_{\psi;(a,b)}) < \alpha + \varepsilon \right\}$$

$$\subset \bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcup_{t>m} \left\{T \in X \mid t^{\alpha+\varepsilon} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t > n \right\},$$

that is,

$$A^{\psi}_{\alpha-\varepsilon} \subset \{T \in X \mid D_2^-(\mu^T_{\psi;(a,b)}) < \alpha + \varepsilon\} \subset A^{\psi}_{\alpha+\varepsilon} \, .$$

Therefore, by replacing α by $\alpha + 2\varepsilon$ and taking $\varepsilon = 1/k$, $k \geq k_0$, it follows that

$$\bigcap_{k=k_0}^{\infty} A^{\psi}_{\alpha+1/k} \subset C^{\psi}_{\alpha \mathrm{ld}} \subset \bigcap_{k=k_0}^{\infty} A^{\psi}_{\alpha+3/k} \, , \tag{3.2}$$

since $C^{\psi}_{\alpha \mathrm{ld}} = \bigcap_{k=k_0}^{\infty} \{T \in X \mid D_2^-(\mu^T_{\psi;(a,b)}) < \alpha + 3/k\}$. □

Remark 3.4 The case $\alpha = 0$ must be treated separately. The relation (3.1) leads us to $\bigcap_{l \geq 2} \bigcap_{k \geq k_0(l)} A^{\psi}_{1/l+1/k} = \bigcap_{l \geq 2} \bigcap_{k \geq k_0(l)} \{T \in X \mid D_2^-(\mu^T_{\psi;(a,b)}) \leq 1/l + 3/k\} = C^{\psi}_{0 \mathrm{ld}}$.

Theorem 3.5 *Fix $\alpha \in [0,1]$ and $\psi \in \mathcal{H}$. Then, $C^{\psi}_{\alpha \mathrm{ld}}$ is a G_δ set in X.*

Proof. We discuss the case $0 < \alpha < 1$; the case $\alpha = 1$ is trivial, and similar arguments, taking into account Remark 3.4, cover the case $\alpha = 0$.

So, fix $\alpha \in (0,1)$, $\psi \in \mathcal{H}$. If, for every $T \in X$, $\mu^T_{\psi;(a,b)}(\cdot) = 0$ (which is the case when, for every $T \in X$, $\mathrm{supp}(\mu^T_\psi) \cap (a,b) = \emptyset$), then $D_2^-(\mu^T_{\psi;(a,b)}) = 1$, and consequently, $C^{\psi}_{\alpha \mathrm{ld}} = \emptyset$ is a G_δ set in X.

Otherwise, since strong resolvent convergence is equivalent to strong dynamical convergence (see Theorem 10.1.15 in [6]), it follows, for each $(a,b) \subset \mathbb{R}$, each $t > 0$, and each $\psi \in \mathcal{H}$, that the mapping $X \ni T \mapsto \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t$ is continuous; namely, it is the composition of two continuous mappings, $X \ni T \mapsto \langle |\hat{\mu}^T_\psi|^2 \rangle_t$, and $\mathbb{R} \ni \langle |\hat{\mu}^T_\psi|^2 \rangle_t \mapsto \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t$ (the latter is continuous since $\mathcal{M}_+(\mathbb{R}) \ni \mu(\cdot) \mapsto \mu((a,b) \cap \cdot) \in \mathcal{M}_+((a,b))$ is continuous; see [15] for details). This implies, for every $n \geq 0$, $k \geq k_0$ and $t > 0$, that $\{T \in X \mid t^{\alpha+1/k} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t \leq n\}$ is a closed set in X. Now, the relation

$$C^{\psi}_{\alpha \mathrm{ld}} = \bigcap_{k=k_0}^{\infty} \bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcup_{t > m} \{T \in X \mid t^{\alpha+1/k} \langle |\hat{\mu}^T_{\psi;(a,b)}|^2 \rangle_t > n\}$$

completes the proof that $C^{\psi}_{\alpha \mathrm{ld}}$ is a G_δ. □

We are finally able to present our main results in this section.

Corollary 3.6 *(Lower correlation dimension Wonderland) Let $\alpha \in (0,1)$, $(a,b) \subset \mathbb{R}$, $\psi \in \mathcal{H}$, and denote by $h^{\alpha,T}_{\psi;(a,b)}$ the α-Hausdorff singular component of $\mu^T_{\psi;(a,b)}$. Suppose that each one of the sets*

- $C^{\psi}_{\alpha \mathrm{lHd}} := \{T \in X \mid h^{\alpha,T}_{\psi;(a,b)} \neq 0\}$,

- $Y^\psi = \{T \in X \mid \mu^T_{\psi;(a,b)}$ *is massless*$\}$,
- $U^\psi = \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)\}$,

is dense in X*. Then, the set* $C^\psi_{\alpha\text{ld-c}} := \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)$, $\mu^T_{\psi;(a,b)}$ *is massless and* α*-LCD on* $(a,b)\}$ *is generic in* X*.*

Proof. Since $\mu^T_{\psi;(a,b)}$ has a nontrivial almost α-Hausdorff dimension singular part if, and only if, it is not purely strongly α-Hausdorff dimension continuous (see Remark 2.9), one has, by item (3) of Proposition 2.11, that $\dim^-_{\mathrm{H}}(\mu^T_{\psi;(a,b)}) \le \alpha$. Now, by Proposition 2.13, $\dim^-_{\mathrm{H}}(\mu^T_{\psi;(a,b)}) \le \alpha$ implies that $D^-_2(\mu^T_{\psi;(a,b)}) \le \alpha$; therefore, $C^\psi_{\alpha\text{ld}} \supset \{T \in X \mid \dim^-_{\mathrm{H}}(\mu^T_{\psi;(a,b)}) \le \alpha\} = \{T \in X \mid \mu^T_{\psi;(a,b)}$ has a nontrivial aαHds part$\}$, again by Proposition 2.11.

On the other hand, since $\{T \in X \mid \mu^T_{\psi;(a,b)}$ has a nontrivial aαHds part$\} \supset C^\psi_{\alpha\text{lHd}}$ (see item (4) in Remark 2.7), the first hypothesis in this corollary and the previous results imply that $C^\psi_{\alpha\text{ld}}$ is dense, and therefore generic, by Theorem 3.5. Now it is enough to observe that $C^\psi_{\alpha\text{ld-c}} = Y^\psi \cap U^\psi \cap C^\psi_{\alpha\text{ld}}$ is also generic, by the other hypotheses of the corollary and Propositions 3.1 and 3.2. □

We can briefly restate the conclusions of Corollary 3.6 as follows: under suitable hypotheses, the set of operators in X, whose spectral measures associated with the vector ψ and restricted to (a,b) are only continuous and of lower correlation dimension less than or equal to α, is generic in X.

Corollary 3.7 *(Lower correlation dimension Wonderland for* $\alpha = 0$*) Let* $(a,b) \subset \mathbb{R}$, $\psi \in \mathcal{H}$*, and denote by* $h^{0,T}_{\psi;(a,b)}$ *the* 0*-Hausdorff dimensional component of* $\mu^T_{\psi;(a,b)}$*. Suppose that each one of the sets*

- $C^\psi_{0\text{lHd}} := \{T \in X \mid h^{0,T}_{\psi;(a,b)} \ne 0\}$,
- $Y^\psi = \{T \in X \mid \mu^T_{\psi;(a,b)}$ *is massless*$\}$,
- $U^\psi = \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)\}$,

is dense in X*. Then, the set* $C^\psi_{0\text{ld-c}} := \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)$, $\mu^T_{\psi;(a,b)}$ *is massless and* $D^-_2(\mu^T_{\psi;(a,b)}) = 0\}$ *is generic in* X*;*

Proof. Suppose that $h^{0,T}_{\psi;(a,b)} \ne 0$; thus, by Definitions 2.5 and 2.10, $\dim^-_{\mathrm{H}}(\mu^T_{\psi;(a,b)}) = 0$. Now, an application of Proposition 2.13 implies that $D^-_2(\mu^T_{\psi;(a,b)}) = 0$; therefore, one has $C^\psi_{0\text{ld}} \supset C^\psi_{0\text{lHd}}$. Since $C^\psi_{0\text{lHd}}$ is dense, it follows, from the previous result and Theorem 3.5, that $C^\psi_{0\text{ld}}$ is generic, as well as $C^\psi_{0\text{ld-c}} = Y^\psi \cap U^\psi \cap C^\psi_{0\text{ld}}$, by the hypotheses of the corollary, Propositions 3.1 and 3.2. □

Remark 3.8 In some applications, the set U^ψ is not considered, so that, for each $0 \le \alpha < 1$, the conclusion is that the set $\{T \in X \mid \mu^T_{\psi;(a,b)}$ is massless and α-LCD on $(a,b)\}$ is generic in X.

4 Upper correlation dimension Wonderland Theorem

Our main purpose in this section is to state and prove the upper correlation dimension versions of Corollaries 3.6 and 3.7. Recall that (X, d) is a complete metric space of self-adjoint operators T, acting in the separable Hilbert space $\mathcal{H}$, such that the metric d convergence implies strong resolvent convergence. For $0 \le \alpha \le 1$, put

$$C^\psi_{\alpha\mathrm{ud}} := \{T \in X \mid D_2^+(\mu^T_{\psi;(a,b)}) \ge \alpha\}.$$

Theorem 4.1 *Fix $\alpha \in [0,1]$ and $\psi \in \mathcal{H}$. Then, $C^\psi_{\alpha\mathrm{ud}}$ is a G_δ set in X.*

Corollary 4.2 *(Upper correlation dimension Wonderland) Let $\alpha \in (0,1)$, $(a,b) \subset \mathbb{R}$, $\psi \in \mathcal{H}$, and suppose that each of the sets*

- $C^\psi_{\alpha\mathrm{ud}} = \{T \in X \mid D_2^+(\mu^T_{\psi;(a,b)}) \ge \alpha\}$,
- $Z^\psi = \{T \in X \mid \mu^T_{\psi;(a,b)}$ *is singular*$\}$,
- $U^\psi = \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)\}$,

is dense in X. Then, the set $C^\psi_{\alpha\mathrm{ud\text{-}s}} := \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)$, $\mu^T_{\psi;(a,b)}$ is singular and α-UCD on $(a,b)\}$ is generic in X.

We can briefly restate the result in Corollary 4.2 as follows: under suitable hypotheses, the set of operators in X, whose spectral measures associated with the vector ψ and restricted to (a,b) are singular and of upper correlation dimension greater than or equal to α, is generic in X.

Corollary 4.3 *(Upper correlation dimension Wonderland for $\alpha = 1$) Fix $(a,b) \subset \mathbb{R}$, $\psi \in \mathcal{H}$, and suppose that each of the sets*

- $C^\psi_{1\mathrm{ud}} = \{T \in X \mid D_2^+(\mu^T_{\psi;(a,b)}) = 1\}$,
- $Z^\psi = \{T \in X \mid \mu^T_{\psi;(a,b)}$ *is singular*$\}$,
- $U^\psi = \{T \in X \mid \mathrm{supp}(\mu^T_\psi) \supset (a,b)\}$,

is dense in X. Then, the set $C^{\psi}_{1\text{ud-s}} := \{T \in X \mid \operatorname{supp}(\mu^T_\psi) \supset (a,b),\ \mu^T_{\psi;(a,b)}$ is singular and $D^+_2(\mu^T_{\psi;(a,b)}) = 1\}$ is generic in X.

Note that Corollary 4.2 can be viewed somewhat as a dual version to Corollary 3.6, in the sense that now we guarantee the generic property for a set of operators whose spectral measures restricted to (a,b) are singular and have upper correlation dimensions greater than or equal to α. Similarly, one has a duality between Corollaries 3.7 and 4.3.

Remark 4.4 As in the previous section, in some applications the set U^ψ is not considered, so that, for each $0 < \alpha \le 1$, the conclusion is that the set $\{T \in X \mid \mu^T_{\psi;(a,b)}$ is singular and α-UCD on $(a,b)\}$ is generic in X.

In order to prove Corollaries 4.2 and 4.3, we present some corresponding results to those of the previous section. We begin with another result from [15] (see the proof of Proposition 2.6 therein):

Proposition 4.5 *Let $(a,b) \subset \mathbb{R}$ and $\psi \in \mathcal{H}$. Then, $Z^\psi = \{T \in X \mid \mu^T_{\psi;(a,b)}$ is singular$\}$ is a G_δ set in X.*

Lemma 4.6 *Let $\alpha \in (0,1)$, $(a,b) \subset \mathbb{R}$ and $\psi \in \mathcal{H}$. Then,*

$$C^\psi_{\alpha\text{ud}} = \bigcap_{l=l_0}^{\infty} B^\psi_{\alpha - 1/l}\,, \tag{4.1}$$

where

$$B^\psi_\alpha := \left\{T \in X \mid \text{for each } k > 0,\ \exists\, t > k \text{ with } t^\alpha \langle |\hat{\mu}^T_{\psi;(a,b)}|^2\rangle_t < c\right\},$$

with $l_0 := [3/\alpha] + 1$, and c a positive constant.

Proof. Note that (2.1) is equivalent to

$$\left\{\psi \mid \lim_{t\to\infty}\text{-K}\, t^{\alpha+\varepsilon}\langle |\hat{\mu}^T_\psi|^2\rangle_t = 0\right\} \subset \left\{\psi \mid D^{\mathrm{K}}_2(\mu^T_{\psi;(a,b)}) \ge \alpha + \varepsilon\right\} \subset \left\{\psi \mid \lim_{t\to\infty}\text{-K}\, t^{\alpha-\varepsilon}\langle |\hat{\mu}^T_\psi|^2\rangle_t = 0\right\};$$

for $\varepsilon > 0$ small enough; thus, we may work with any $c > 0$ and any fixed $\psi \in \mathcal{H}$ in the relation (take $\mathrm{K} = \mathrm{P}$)

$$\bigcap_{k=0}^{\infty}\bigcup_{t>k}\left\{T \in X \mid t^{\alpha+\varepsilon}\langle |\hat{\mu}^T_{\psi;(a,b)}|^2\rangle_t < c\right\} \subset C^\psi_{(\alpha+\varepsilon)\text{ud}} \subset \bigcap_{k=0}^{\infty}\bigcup_{t>k}\left\{T \in X \mid t^{\alpha-\varepsilon}\langle |\hat{\mu}^T_{\psi;(a,b)}|^2\rangle_t < c\right\}.$$

Finally, replacing α by $\alpha - 2\varepsilon$ and taking $\varepsilon = 1/l$, $l \ge l_0$, we obtain the inclusions

$$\bigcap_{l=l_0}^{\infty} B^\psi_{\alpha-1/l} \subset \bigcap_{l=l_0}^{\infty} C^\psi_{(\alpha-1/l)\text{ud}} \subset \bigcap_{l=l_0}^{\infty} B^\psi_{\alpha-3/l}\,, \tag{4.2}$$

and (4.1) follows directly from (4.2). □

Remark 4.7 The case $\alpha = 1$ must be considered separately. The relation (4.1) leads us to $\bigcap_{m\geq 2}\bigcap_{l\geq l_0(m)} B^{\psi}_{1-1/m-1/l} = \bigcap_{m\geq 2} C^{\psi}_{(1-1/m)\mathrm{ud}} = \{T \in X \mid D_2^+(\mu^T_{\psi;(a,b)}) \geq 1-\delta$ for every $0 < \delta \leq 1\} = C^{\psi}_{1\mathrm{ud}}$.

Proof of Theorem 4.1. We discuss the case $0 < \alpha < 1$; the case $\alpha = 0$ is trivial, and similar arguments, taking into account Remark 4.7, cover the case $\alpha = 1$.

So, fix $\alpha \in (0,1)$, $\psi \in \mathcal{H}$. If, for every $T \in X$, $\mu^T_{\psi;(a,b)}(\cdot) = 0$ (which is the case when, for every $T \in X$, $\mathrm{supp}(\mu^T_\psi) \cap (a,b) = \emptyset$), then $D_2^+(\mu^T_{\psi;(a,b)}) = 1$, and consequently, $C^{\psi}_{\alpha\mathrm{ud}} = \emptyset$ is a G_δ set in X. Otherwise, we will use Lemma 4.6 and arguments similar to those in the proof of Theorem 3.5. For each $(a,b) \subset \mathbb{R}$, each $\psi \in \mathcal{H}$, and each $t > 0$, the mapping $X \ni T \mapsto \langle|\hat{\mu}^T_{\psi;(a,b)}|^2\rangle_t$ is continuous, which implies, for every $c > 0$, that

$$C^{\psi}_{\alpha\mathrm{ud}} = \bigcap_{k\geq 0}\bigcup_{t>k}\left\{T \in X \mid t^\alpha\langle|\hat{\mu}^T_{\psi;(a,b)}|^2\rangle_t < c\right\}$$

is a G_δ set in X. □

Proof of Corollary 4.2. Since, by hypothesis, $C^{\psi}_{\alpha\mathrm{ud}}$ is dense, an application of Theorem 4.1 implies that $C^{\psi}_{\alpha\mathrm{ud}}$ is generic. Now, observe that $C^{\psi}_{\alpha\mathrm{ud\text{-}s}} = Z^\psi \cap U^\psi \cap C^{\psi}_{\alpha\mathrm{ud}}$, being therefore generic, by the hypotheses of the corollary, Propositions 3.2 and 4.5. □

Proof of Corollary 4.3. Since, by Theorem 4.1, $C^{\psi}_{1\mathrm{ud}}$ is a G_δ set, it follows, from the hypotheses of the corollary, that $C^{\psi}_{1\mathrm{ud\text{-}s}} = Z^\psi \cap U^\psi \cap C^{\psi}_{1\mathrm{ud}}$ is generic in X. □

5 Applications to Schrödinger operators

As mentioned in the Introduction, in this section we discuss some applications of the Correlation Dimension Wonderland Theorems to classes of one-dimensional Schrödinger discrete operators represented by the action (i.e., equation (1.1))

$$(H^v_\phi\psi)_n = \psi_{n+1} + \psi_{n-1} + v_n\psi_n ,$$

with real potentials $v = (v_n)$. The underlying Hilbert space is $\mathcal{H} = l^2(\mathbb{N})$, and the vectors ψ in the domain of the operators H^v_ϕ carries the phase boundary condition (i.e., equation (1.2))

$$\psi_{-1}\cos\phi - \psi_0\sin\phi = 0 ,$$

for fixed $\phi \in [0,\pi)$. Recall that $(\delta_j)_{j\geq 0}$ is the canonical basis of $\mathcal{H}$.

Definition 5.1 *Let μ be a Borel measure. We say that μ is (uniformly) Lipschitz continuous (or uniformly 1-Hölder continuous) if there is a constant $C > 0$ such that, for every interval $J \subset \mathbb{R}$ with $\ell(J) < 1$, $\mu(J) \le C\ell(J)$.*

Lemma 5.2 *Let T be some Schrödinger operator* (1.1)-(1.2)*, which is a finite rank perturbation of the free operator H_ϕ^0, and let $I = (a, b)$, $-2 < a < b < 2$. Then, $\mu^T_{\psi;I}$ is Lipschitz continuous for every $\psi \in \mathcal{H}$.*

Proof. We begin with the observation that δ_0 is cyclic for every Schrödinger operator (1.1)–(1.2) (see Section 2.5 in [20] for details); denote the inner product in $\mathcal{H}$ by $(\cdot,\cdot)$. Let J be an interval with $\ell(J) < 1$; given $\psi \in \mathcal{H}$, write $\psi = \sum_{i\ge0} a_i\delta_i$, with $\{\delta_i\}_{i\ge0}$ the canonical basis of $l^2(\mathbb{N})$. Then, we have

$$\mu^T_{\psi;I}(J) = (\psi, P^T(J\cap I)\psi) = \sum_{i,j\ge0} \overline{a_i}a_j(\delta_i, P^T(J\cap I)\delta_j) = \sum_{i,j\ge0} \overline{a_i}a_j \int_{J\cap I} \overline{u_i^\phi(x)}u_j^\phi(x)d\mu^T_{\delta_0}(x)\ ,$$

where $u^\phi(x) = (u_i^\phi(x))_i$ represents the solution to $(T\zeta)_i = x\zeta_i$, for $x \in \mathbb{R}$, which satisfies $u_{-1}^\phi(x) = \sin\phi$ and $u_0^\phi(x) = \cos\phi$ (i.e., orthogonal conditions to (1.2); see [20] for details). Since it is known that $u^\phi(x) = (u_i^\phi(x))_i$ is a continuous function of $x \in I$ and it is uniformly bounded on I (as discussed in Section 5.2 of [10]), it follows that

$$\mu^T_{\psi;I}(J) \le \sum_{i,j\ge0} |a_ia_j|\,|C_iC_j|\,\mu^T_{\delta_0;I}(J) \le C^2\,\mu^T_{\delta_0;I}(J)\sum_{i,j\ge0}|a_ia_j| \le C^2\,\|\psi\|^2\mu^T_{\delta_0;I}(J)\ ,$$

where $C_i := \sup_{x\in J\cap I} u_i^\phi(x) < \infty$, $C := \sup_{i\ge0}|C_i| < \infty$.

Thus, since there is an $E > 0$ such that $\mu^T_{\delta_0;I}(J) \le E\ell(J)$ ($\mu^T_{\delta_0;I}$ is Lipschitz continuous, since its Radon-Nikodym derivative is uniformly bounded on every such $I = (a, b) \subset [-2, 2] = \sigma_{\rm ess}(T)$ as in the statement of the lemma; see [10]), it follows that there is a $\kappa < \infty$, which does not depend on J, such that $\mu^T_{\psi;I}(J) \le \kappa\,\ell(J)$ (just take $\kappa = C^2\,E\|\psi\|^2$). Hence, $\mu^T_{\psi;I}$ is Lipschitz continuous, by the arbitrariness of J. □

Proposition 5.3 *Let T and I be as in the statement of Lemma 5.2. Then, for all $\psi \in \mathcal{H}$, $D_2^-(\mu^T_{\psi;I}) = D_2^+(\mu^T_{\psi;I}) = 1$.*

Proof. Fix $\psi \in \mathcal{H}$. Since $\mu^T_{\psi;I}$ is Lipschitz continuous by Lemma 5.2, the result follows directly from Definitions 2.12 and 5.1. In fact, by the Lipschitz continuity of $\mu^T_{\psi;I}$, it is immediate that, for ε small enough, $\int \mu^T_{\psi;I}(x-\varepsilon, x+\varepsilon)\,\mathrm{d}\mu^T_{\psi;I}(x) \le E\,\varepsilon$, for some $E > 0$, and so $D_2^-(\mu^T_{\psi;I}) = D_2^+(\mu^T_{\psi;I}) = 1$. □

Remark 5.4 (a) The spectrum of each Schrödinger operators H of the form (1.1)-(1.2) is simple, δ_0 is a cyclic vector for H and $\sigma(H) = \mathrm{supp}(\mu^H_{\delta_0})$. One can say even more; for each $j \ge 0$, $Z^{\delta_0} \subset Z^{\delta_j}$ and $Y^{\delta_0} \subset Y^{\delta_j}$.

(b) In the considered spaces (X, d) we are going to check that each of the sets

- $U := \{T \in X \mid \sigma(T) \supset (a,b)\}$,
- $Z := \{T \in X \mid P_{\text{ac}}^T((a,b)) = 0\} = \{T \mid \mu_\psi^T \text{ is singular on } (a,b),\ \forall \psi\}$,
- $Y := \{T \in X \mid P_{\text{p}}^T((a,b)) = 0\} = \{T \mid \mu_\psi^T \text{ is massless on } (a,b),\ \forall \psi\}$,

is generic (see [15] or [18] for the proof that these are G_δ sets). In fact, $U = U^{\delta_0}$, $Z = Z^{\delta_0}$ and $Y = Y^{\delta_0}$.

5.1 Bounded discrete Schrödinger operators

This application refers to families of uniformly bounded operators, and it is enough that the metric in X implies pointwise convergence of potentials. We have, in fact, strong convergence of operators.

Pick $r > 0$, $\phi \in [0, \pi)$, and let X_ϕ^r be the set of Schrödinger operators given by (1.1), defined on $l^2(\mathbb{N})$ and which satisfy (1.2), where (V_n) is an arbitrary real sequence with $|V_n| \le r$, for all n; we endow X_ϕ with the topology of pointwise convergence. We have the following applications of the Correlation Dimension Wonderland Theorems:

Theorem 5.5 *Fix both $r > 0$ and $j \ge 0$. Then, for every $\phi \in [0, \pi)$, each of the sets*

- $C_{\text{0ld-c}}^{\delta_j} := \{T \in X_\phi^r \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T = [-2-r, 2+r],\ \mu_{\delta_j}^T$ *is massless and* 0*-LCD*$\}$,
- $C_{\text{1ud-s}}^{\delta_j} := \{T \in X_\phi^r \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T = [-2-r, 2+r],\ \mu_{\delta_j}^T$ *is singular and* 1*-UCD on* $(-2,2)\}$,

is generic in X_ϕ^r.

Proof. Fix $\phi \in [0, \pi)$ and the interval (a,b) with $-2 < a < b < 2$. By following the idea presented in Theorem 4.1 in [18], let $d\zeta$ be the product of the Lebesgue measures $(2r)^{-1} d\ell_n$, $n \in \mathbb{N}$, where $\operatorname{supp} \ell_n := [-r, r]$; thus, $\operatorname{supp}(d\zeta) = [-r, r]^{\mathbb{N}}$. Let $D = \{T \in X_\phi^r \mid \operatorname{supp} \mu_{\delta_0}^T = [-2-r, 2+r],\ T \text{ has pure point spectrum on } [-2-r, 2+r]\}$. Then, $\zeta(X_\phi^r \setminus D) = 0$, by Anderson localization (see [7] for a discussion about Anderson localization) and, therefore, D is dense in X_ϕ^r. Now, since $D \subset C_{\text{0ld}}^{\delta_0} \cap U^{\delta_0} \cap Z^{\delta_0}$, it follows that $C_{\text{0ld}}^{\delta_j}$ and Z^{δ_j} are also dense in X_ϕ^r, for every j; in fact, since δ_0 is cyclic, one has $C_{\text{0ld}}^{\delta_0} \subset C_{\text{0ld}}^{\delta_j}$ (see Lemma 2.16) and $Z^{\delta_0} \subset Z^{\delta_j}$ (see Remark 5.4).

Consider now the operators $T_k \in X_\phi^r$ with potentials $v_k = (\tilde{V}_n^k)$ given by

$$\tilde{V}_n^k = \begin{cases} V_n, & n \le k \\ 0, & n > k \end{cases}. \tag{5.1}$$

Since $v^k \to v$ pointwise, $T_k \to T$ in the strong resolvent sense. Now, given that each T_k is a finite rank perturbation of H_ϕ^0 (the free operator), one has, for each j, that the spectral measure $\mu_{\delta_j}^{T_k}$ of T_k is uniformly Lipschitz continuous on (a,b) (see Lemma 5.2). Hence, for each j, both sets Y^{δ_j} and $C_{\text{1ud}}^{\delta_j}$ are dense in X_ϕ^r. A use of Corollaries 3.7 and 4.3, and the fact that the argument holds true for all subintervals (a,b) with $-2 < a < b < 2$, conclude the proof. □

Proof of Theorem 1.2. By Theorem 5.5, the set $\{T \in X_\phi^r \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T = [-2-r, 2+r],$ it is purely singular continuous, 0-LCD and 1-UCD on $(-2,2)$ for all $j\}$ is generic for every $\phi \in [0,\pi)$, since it equals $\bigcap_{j \geq 0}(C_{\text{0ld-c}}^{\delta_j} \cap C_{\text{1ud-s}}^{\delta_j})$. □

5.2 Unbounded discrete Schrödinger operators

Now we consider unbounded Schrödinger operators as discussed in equations (1.3) and (1.4); that is, consider the space of real sequences $\tilde{X} := \{v = (V_n) \mid V_n \in \mathbb{R}\}$ with the metric

$$\tilde{d}(u,v) := \sum_{n=0}^{\infty} 2^{-n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}, \quad u, v \in \tilde{X}.$$

Convergence in $(\tilde{X}, \tilde{d})$ implies pointwise convergence and this is a complete metric space.

Then, for each $v \in \tilde{X}$, one associates the self-adjoint discrete Schrödinger operator H_ϕ^v with action (1.1) and with the ϕ-boundary condition (1.2). By denoting, for each fixed $\phi \in [0,\pi)$,

$$X_\phi := \left\{ H_\phi^v \mid v \in \tilde{X} \right\},$$

the pair (X_ϕ, d), with $d(H_\phi^u, H_\phi^v) := \tilde{d}(u,v)$, is a complete metric space, and from now on we naturally identify elements $v \in \tilde{X}$ with $H_\phi^v \in X_\phi$. Convergence in $\tilde{d}$ implies strong resolvent convergence in X_ϕ.

We have the following application of our abstract results:

Theorem 5.6 *Fix $I = (c,d)$, $-2 < c < d < 2$, $j \geq 0$. Then, for every $\phi \in [0,\pi)$, each of the sets $C_{\text{0ld-c}}^{I,j} := \{T \in X_\phi \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T \supset (-2,2),\ \mu_{\delta_j;I}^T$ is 0-LCD and massless on $I\}$ and $C_{\text{1ud-s}}^{I,j} := \{T \in X_\phi \mid \sigma(T) = \operatorname{supp} \mu_{\delta_0}^T \supset (-2,2),\ \mu_{\delta_j;I}^T$ is 1-UCD and singular on $I\}$ is generic in X_ϕ.*

We begin with a simple fact.

Lemma 5.7 *For every $\phi \in [0,\pi)$, the set $U = \{T \in X_\phi \mid \sigma(T) \supset [-2,2]\}$ is dense in X_ϕ.*

Proof. Fix $\phi \in [0,\pi)$. We simply use the fact that the spectrum of each operator $T_k \in X_\phi$, as defined with potential given by (5.1), contains the closed interval $[-2,2]$, since they are finite-rank perturbations of the free operator H_ϕ^0 and, by Weyl's criterion, $\sigma_{\rm ess}(T_k) = \sigma_{\rm ess}(H_\phi^0) = [-2,2]$. Thus, for every $\phi \in [0,\pi)$, U is dense in X_ϕ. □

For the proof of Theorem 5.6, we also need the following result from Theorem 1.3 in [13], and complemented in [19],

Theorem 5.8 ([13, 19]) *Let $\alpha \in (0,1)$ and write $v^\alpha = (V_n^\alpha)$ for*

$$V_n^\alpha = \begin{cases} G_k^{(1-\alpha)/2\alpha} & \text{if } n \in \mathcal{B}\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.2}$$

where $\mathcal{B} = (G_k)_k = \left(2^{(k^k)}\right)_k$. Then:

1. *For every $\phi \in [0,\pi)$, the spectrum of $H_\phi^{v^\alpha}$ consists of the interval $[-2,2]$, along with some discrete point spectrum outside this interval.*
2. *For every $\phi \in [0,\pi)$, the spectrum of $H_\phi^{v^\alpha}$ restricted to $(-2,2)$ is* eαHd.

Lemma 5.9 *Let I be as in the statement of Theorem 5.6. Then, for every $\alpha \in (0,1)$ and every $\phi \in [0,\pi)$, each of the sets $C^{I,0}_{{\rm a}\alpha{\rm Hds}} := \{T \in X_\phi \mid \dim_{\rm H}^+(\mu^T_{\delta_0;I}) \le \alpha\}$, Y^{δ_0} and Z^{δ_0} is dense in X_ϕ.*

Proof. For each fixed $\alpha \in (0,1)$ and $\phi \in [0,\pi)$, we consider operators $H_\phi^{v^\alpha}$. Then, we construct a sequence of potentials

$$V_n^k = \begin{cases} V_n, & n \le k \\ V_n^\alpha, & n > k \end{cases},$$

with V_n^α given by (5.2). Now, the pointwise convergence of potentials implies strong convergence of the associated operators, a result that combined with item (2) in Theorem 5.8 show that the set $C^{I,0}_{{\rm eHd}\alpha} := \{T \in X_\phi \mid \sigma(T) = \operatorname{supp}\mu^T_{\delta_0} \supset (-2,2)$ and $\mu^T_{\delta_0;(-2,2)}$ has exact Hausdorff dimension $\alpha\}$ is dense in X_ϕ; hence, since $C^{I,0}_{{\rm a}\alpha{\rm Hds}} \supset C^{I,0}_{{\rm eHd}\alpha}$, $Y^{\delta_0} \supset C^{I,0}_{{\rm eHd}\alpha}$ and $Z^{\delta_0} \supset C^{I,0}_{{\rm eHd}\alpha}$ (see item 5 in Definition 2.4), the sets $C^{I,0}_{{\rm a}\alpha{\rm Hds}}$, Y^{δ_0} and Z^{δ_0} are also dense in X_ϕ, for every $\alpha \in (0,1)$ and each $\phi \in [0,\pi)$. □

By using the same strategy that was employed in the proof of Theorem 5.5, we obtain

Lemma 5.10 *Let I be as in the statement of Theorem 5.6. Then, for every $\phi \in [0,\pi)$ and every $j \ge 0$, $C^{I,j}_{\rm 1ud} := \{T \in X_\phi \mid \mu^T_{\delta_j}$ is 1-UCD on $I\}$ is dense in X_ϕ.*

Lemma 5.11 *Let I be as in the statement of Theorem 5.6. Then, for every $\alpha \in (0,1)$, $\phi \in [0,\pi)$ and $j \geq 0$, the set $C^{I,j}_{\alpha\text{ld-c}} := \{T \in X_\phi \mid \operatorname{supp} \mu^T_{\delta_0} = \sigma(T) \supset (-2,2),\ \mu^T_{\delta_j;I}$ is massless and α-LCD on $I\}$ is generic in X_ϕ.*

Proof. The result is a direct consequence of Lemmas 2.16, 5.9 and 5.7, Corollary 3.6 and Remark 5.4. □

Proof of Theorem 5.6. The generic property of the set $C^{I,j}_{0\text{ld-c}}$ in X_ϕ, for each $\phi \in [0,\pi)$ and $j \geq 0$, is a direct consequence of Lemma 5.11, since $C^{I,j}_{0\text{ld-c}} = \cap_{n\geq 2} C^{I,j}_{(1/n)\text{ld-c}}$.

The generic property of the set $C^{I,j}_{1\text{ud-s}}$ in X_ϕ, for every $\phi \in [0,\pi)$ and $j \geq 0$, is a direct consequence of Lemmas 5.7, 5.9 and 5.10, Corollary 4.3 and Remark 5.4. □

Proof of Theorem 1.3. Let $J_k = (-2+1/k, 2-1/k)$. By Theorem 5.6, the set $\mathcal{G}_k := \{T \in X_\phi \mid \operatorname{supp} \mu^T_{\delta_0} = \sigma(T) \supset (-2,2),\ \mu^T_{\delta_j;J_k}$ is purely singular continuous, 0-LCD and 1-UCD on J_k for all $j\}$ is generic for every $\phi \in [0,\pi)$, since it equals $\bigcap_{j\geq 0}(C^{J_k,j}_{0\text{ld-c}} \cap C^{J_k,j}_{1\text{ud-s}})$. Now, the set of interest in Theorem 1.3 is generic, since it is exactly $\cap_{k\geq 1}\mathcal{G}_k$. □

6 Applications to general bounded operators

In this section, we present the proofs of Theorems 1.4, 1.5 and 1.6.

For $r > 0$, recall from the Introduction the notations $X_r = \{T \mid T$ is self-adjoint, $\|T\| \leq r\}$ and

$$d(T,T') = \sum_{j=0}^{\infty} \min(2^{-j}, \|(T-T')\tilde{\xi}_j\|) ,$$

where $(\tilde{\xi})_{j\geq 0}$ is an orthonormal basis of $\mathcal{H}$. It is clear that (X_r, d) is a complete metric space, whose metric d convergence implies strong resolvent convergence. Our results in this setting are:

Theorem 6.1 *Fix $r > 0$ and a normalized vector ψ. Then, each of the sets $C^{\psi}_{0\text{ld-c}} := \{T \in X_r \mid \sigma(T) = [-r,r],\ \mu^T_\psi$ is massless and 0-LCD$\}$ and $C^{\psi}_{1\text{ud-s}} := \{T \in X_r \mid \sigma(T) = [-r,r],\ \mu^T_\psi$ is singular and 1-UCD$\}$ is generic in X_r.*

Proof. Our strategy is analogous to the one in Theorem 3.1 in [18], but with very particular choices of auxiliary operators. Given the density in X_r of operators with pure point spectrum and of operators whose spectrum equals $[-r,r]$, we need only to prove that a self-adjoint pure point operator T, with $\|T\| \leq r-\varepsilon$, where $0 < \varepsilon \leq r$, can be approximated by operators whose spectral measures are 0-LCD and massless (similarly for 1-UCD and singular).

Suppose, therefore, that T has pure point spectrum. Let $(\gamma_n)_{n\geq 0}$ be the normalized eigenvectors of T, say $T\gamma_n = \lambda_n\gamma_n$, so that $(\lambda_n)_{n\geq 0}$ are the corresponding eigenvalues. Take an arbitrary $0 < \delta \leq \varepsilon$. Pick a sequence of numbers $\eta_N \to 0$, $0 < \eta_N < \delta/(2N)$, and introduce operators (B_N) by

$$B_N := \bigoplus_{j=0}^{N-1} (\lambda_j \mathbf{1} + \eta_N L_j) P_j^N \ ;$$

P_j^N is the projection onto the subspace $\mathcal{H}_j$ generated by $(\gamma_{j+kN})_{k\geq 0}$; L_j is an operator defined on $\mathcal{H}_j$ such that $\|L_j\| \leq 1$ and $\mu_\xi^{L_j}$, the spectral measure of L_j associated with ξ, is uniformly Lipschitz continuous for every $\xi \in \mathcal{H}$.

Then, for each $N \in \mathbb{N}$, $\|B_N\| \leq r$ (since $\eta_N \|L\| < \delta/(2N) < \varepsilon$), and therefore $B_N \in X_r$. Observe that for every $m \geq 0$, μ_ξ^m, the spectral measure of $\lambda_m \mathbf{1} + \eta_N L$ associated with $\xi \in \mathcal{H}$, is Lipschitz continuous; therefore, $\mu_\xi^{B_N}$, the spectral measure of B_N, is also Lipschitz continuous for each $N \geq 0$ and each $\xi \in \mathcal{H}$, since it equals $\sum_{m=0}^{N-1} \mu_\xi^m$.

In order to prove that $B_N \to T$ strongly, as $N \to \infty$, take the least integer N_0 such that $\sum_{j\geq N_0} 2^{-j} < \delta/2$; thus, for every $N \geq N_0$,

$$\begin{aligned} d(B_N, T) &= \sum_{m=0}^{N-1} \min(2^{-m}, \|\eta_N L_m \gamma_m\|) + \sum_{m\geq N} \min(2^{-m}, \|(B_N - T)\gamma_m\|) \\ &< \frac{N\delta}{2N} + \sum_{m\geq N} 2^{-m} < \frac{N\delta}{2N} + \delta/2 \leq \delta \ . \end{aligned}$$

Hence, for the given normalized ψ it follows, from the construction presented above, Proposition 5.3, and the density of $Z \supset \{T \in X_r \mid \sigma(T) \text{ is pure point on } [-r, r]\} = \{T \in X_r \mid P_{\mathrm{p}}^T([-r, r]) = \mathbf{1}\}$ in X_r, that $C_{\text{1ud-s}}^\psi$ is dense in X_r. On the other hand, by the density of both $Y \supset C_{\text{1ud}}^\psi$ and $\{T \in X_r \mid \sigma(T) \text{ is pure point on } [-r, r]\}$, it follows that $C_{\text{0ld-c}}^\psi$ is also dense in X_r. A direct application of the Correlation Dimension Wonderland Theorems concludes the proof. □

Proof of Theorem 1.4. It follows straightly from Theorem 6.1, since the set of interest is the intersection of $C_{\text{0ld-c}}^{\xi_j} \cap C_{\text{1ud-s}}^{\xi_j}$ for all j. □

For the next application, fix $a < b$ and recall from the Introduction that $Y_{a,b} = \{T \mid T$ is self-adjoint with $\sigma(T) = [a, b]\}$, with the metric generated by the operator norm

$$d(T, T') = \|T - T'\| \ .$$

$(Y_{a,b}, d)$ is a complete metric space, whose metric d convergence implies strong resolvent convergence.

Theorem 6.2 *Fix a normalized vector* ψ*. Each of the sets* $C^{\psi}_{\text{0ld-c}} := \{T \in Y_{a,b} \mid \sigma(T)$ *is massless and* μ^T_ψ *is* 0*-LCD*$\}$ *and* $C^{\psi}_{\text{1ud-s}} := \{T \in Y_{a,b} \mid \sigma(T)$ *is singular and* μ^T_ψ *is* 1*-UCD*$\}$ *is generic in* $Y_{a,b}$*.*

Proof. We follow the proofs of our Theorem 6.1 and Theorem 3.2 in [18]. That is, given the density in $Y_{a,b}$ of operators with pure point spectrum, we need only to prove that an operator T with pure point spectrum $[a,b]$ can be approximated by operators whose spectral measures are 0-LCD and massless (similarly for 1-UCD and singular).

Suppose, therefore, that $T \in Y_{a,b}$ has pure point spectrum. Let $c = (b-a)$, and given n, let

$$I_1 = [a, a + c/2^n),\ I_2 = [a + c/2^n, a + 2c/2^n),\ \ldots,\ I_{2^n} = [b - c/2^n, b]$$

be a partition of the interval $[a,b]$. Let ρ_j be the midpoint of I_j. Suppose that (γ_k) is the orthonormal family of eigenvectors of T, that is, $T\gamma_k = \lambda_k\gamma_k$. Define B_n by

$$B_n\gamma_k = \rho_j\gamma_k, \quad \text{if} \quad \lambda_k \in I_j\ ,$$

so that $\|B_n - T\| \le c/2^{n+1}$. Observe that B_n can be written as $B_n = \sum_{j=1}^{2^n} \rho_j P^T(I_j)$, where $P^T(I_j)$ represents the spectral projection of T on the interval I_j and also that $\sum_{j=1}^{2^n} P^T(I_j) = \mathbf{1}$. Note that the subspace $\mathcal{H}_j :=$ range of $P^T(I_j)$ is infinite dimensional, since (λ_k) is dense in $[a,b]$.

Define now the sequence of self-adjoint operators

$$A_n = \bigoplus_{j=1}^{2^n} (\rho_j\mathbf{1} + c/2^{n+1}L_j)P^T(I_j)\ ,$$

with L_j an operator defined on $\mathcal{H}_j$ such that $\sigma(L_j) = [-1,1]$, so $\|L_j\| = 1$, and $\mu^{L_j}_\xi$, the spectral measure of L_j associated with ξ, is Lipschitz continuous for every $\xi \in \mathcal{H}$.

It follows that

$$\sigma(A_n) = \bigcup_{j=0}^{2^n} [\rho_j - c/2^{n+1}, \rho_j + c/2^{n+1}] = [a,b].$$

Thus, A_n belongs to $\in Y_{a,b}$ for every $n \in \mathbb{N}$,

$$\|T - A_n\| \le (b-a)/2^n,$$

and consequently $A_n \to T$ uniformly as $n \to \infty$. Since, for each $1 \le i \le 2^n$, $\mu^{A_n}_{\varphi_i}$, the spectral measure of $\rho_i\mathbf{1} + c/2^{n+1}L$ associated with $\varphi_i = P^T(I_i)\xi$, is Lipschitz continuous for every $\xi \in \mathcal{H}$, it follows that $\mu^{A_n}_\xi = \sum_{i=1}^{2^n} \mu^{A_n}_{\varphi_i}$ is also Lipschitz continuous for each $n \ge 0$ and each $\xi \in \mathcal{H}$.

Finally, for the given vector ψ it follows, from the construction presented above, Proposition 5.3, and the density of $Z \supset \{T \in Y_{a,b} \mid \sigma(T) \text{ is pure point}\} = \{T \in Y_{a,b} \mid P^T_{\mathrm{p}} = \mathbf{1}\}$

in $Y_{a,b}$, that $C^{\psi}_{\text{1ud-s}}$ is dense in $Y_{a,b}$. On the other hand, it follows from the density of both $Y \supset C^{\psi}_{\text{1ud}}$ and $\{T \in Y_{a,b} \mid \sigma(T) \text{ is pure point}\}$ that $C^{\psi}_{\text{0ld-c}}$ is dense in $Y_{a,b}$. A direct application of the Correlation Dimension Wonderland Theorems concludes the proof. □

Proof of Theorem 1.5. By Theorem 6.2, the set of interest equals the intersection of $C^{\xi_j}_{\text{0ld-c}} \cap C^{\xi_j}_{\text{1ud-s}}$ for all j. □

Proof of Theorem 1.6. Suppose the spectral measure μ^T_ψ is uniformly α-Hölder continuous for some $\alpha > 0$; it is immediate that, for ε small enough, $\int \mu^T_\psi(x-\varepsilon, x+\varepsilon)\, \mathrm{d}\mu^T_\psi(x) \le C'\, \varepsilon^\alpha$, for some $C' > 0$, and so $D^-_2(\mu^T_\psi) \ge \alpha$. But for all operators in the corresponding generic sets we have proven that $D^-_2(\mu^T_\psi) = 0$. □

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