ZERO SETS OF EQUIVARIANT MAPS FROM PRODUCTS OF SPHERES TO EUCLIDEAN SPACES

DENISE DE MATTOS, PEDRO L. Q. PERGHER, EDIVALDO L. DOS SANTOS AND MAHENDER SINGH

ABSTRACT. Let $E \to B$ be a fiber bundle and $E' \to B$ be a vector bundle. Let G be a compact group acting fiber preservingly and freely on both E and $E' - 0$, where 0 is the zero section of $E' \to B$. Let $f : E \to E'$ be a fiber preserving G-equivariant map, and let $Z_f = \{x \in E \mid f(x) = 0\}$ be the zero set of f. It is an interesting problem to estimate the dimension of the set Z_f. In 1988, Dold [5] obtained a lower bound for the cohomological dimension of the zero set Z_f when $E \to B$ is the sphere bundle associated to a vector bundle which is equipped with the antipodal action of $G = \mathbb{Z}/2$. In this paper, we generalize this result to products of finitely many spheres equipped with the diagonal antipodal action of $\mathbb{Z}/2$. We also prove a Bourgin-Yang type theorem for products of spheres equipped with the diagonal antipodal action of $\mathbb{Z}/2$.

1. INTRODUCTION

In 1955, C. T. Yang [27] and D. G. Bourgin [2] independently proved the following generalization of the classical Borsuk-Ulam theorem.

Theorem 1.1. *Let T be a free involution on $\mathbb{S}^n$, and $f : \mathbb{S}^n \to \mathbb{R}^m$ be a continuous map. Let $A_f = \{x \in \mathbb{S}^n \mid f(x) = f(T(x))\}$ be the coincidence point set of f. If $n \geq m$, then $cohom.dim(A_f) \geq (n - m)$.*

The theorem was later extended to the setting of fiber bundles by Dold [5], Fadell [7], Izydorek [8, 9, 10], Jaworowski [13] and Nakaoka [21]. Specifically, Dold [5] proved the following extension of the above theorem.

Theorem 1.2. *Let B be a paracompact space. Let $E \to B$ and $E' \to B$ be vector bundles of dimensions n and m, respectively. Let $f : S(E) \to E'$ be a fiber preserving map such that $f(-x) = -f(x)$ for all $x \in S(E)$, where $S(E)$ is the sphere bundle associated to $E \to B$. If $n > m$ and $Z_f = \{x \in S(E) \mid f(x) = 0\}$, then*

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n - m - 1).$$

It is a classical problem to determine the dimension of zero sets of maps from manifolds to Euclidean spaces. The problem becomes more challenging when there is an action of a compact group on the manifolds under consideration, and even more, when the manifolds and Euclidean

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spaces are replaced by fiber bundles and vector bundles, respectively. Recently, many interesting Borsuk-Ulam kind results have been proved in the setting of fiber bundles. de Mattos and dos Santos [17] proved parametrized Borsuk-Ulam theorems for bundles whose fiber has the mod p cohomology algebra (p odd prime) of a product of two spheres with any free $\mathbb{Z}/p$-action and for bundles whose fiber has the rational cohomology algebra of a product of two spheres with any free $\mathbb{S}^1$-action. Jaworowski obtained parametrized Borsuk-Ulam theorems for lens space bundles in [16] and for sphere bundles in [13, 14, 15]. Singh [24] proved parametrized Borsuk-Ulam theorems for fiber bundles whose fiber has the mod 2 cohomology algebra of a real or a complex projective space with any free involution. This, in particular, included the case when the fiber bundle is projectivisation of a vector bundle. In a very recent paper, de Mattos, Pergher and dos Santos [18] have proved results of this kind for fiber bundles with fiber a space of type (a, b). Here a space of type (a, b) is a certain product or wedge of spheres and projective spaces depending on the parity of the integers a and b. These spaces were introduced independently by Toda [25] and James [12]. In a recent paper, Singh [23] has proved an extension of Dold's theorem by replacing spheres by Stiefel manifolds.

The purpose of this paper is to generalize Dold's theorem in another direction by considering fiber bundles whose fiber is a product of finitely many spheres equipped with the diagonal antipodal action of $\mathbb{Z}/2$. We refer to such bundles as product sphere bundles. The difficulty lies in the fact that a product of spheres $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ has non-trivial cohomology in many dimensions less than $n_1 + \cdots + n_r$, the dimension of the manifold. A more ambitious problem would consider arbitrary free involutions on products of spheres. However, essentially nothing is known about the classification of free involutions (and hence cohomology of their orbit spaces) on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ when $r \geq 3$. When $r = 1$, it is known that $\mathbb{S}^n/\mathbb{Z}_2$ is homotopy equivalent to the real projective space $\mathbb{R}P^n$. When $r = 2$, a complete classification of free involutions on $\mathbb{S}^1 \times \mathbb{S}^n$ was given in a recent paper by Jahren and Kwasik [11]. In a very recent paper [4], Donald Davis studied the diagonal antipodal involution on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ with no restriction on r. He computed the integral and mod 2 cohomology algebra of the quotient of this action as well as its complex K-theory besides investigating other properties such as span and stable span. The proof of our main theorem relies on the mod 2 cohomology algebra of the quotient of products of spheres by the diagonal antipodal involution. Our main result is the following theorem.

Theorem 1.3. *Let $E \to B$ be a $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ bundle, where $n_1 \leq \cdots \leq n_r$. Suppose that $E \to B$ is equipped with a fiber preserving free $\mathbb{Z}/2$-action such that the induced action on each fiber is equivalent to the diagonal antipodal action, and that the quotient bundle $\overline{E} \to B$ admits a cohomology extension of the fiber with respect to $\mathbb{Z}/2$. Let $E' \to B$ be a m-dimensional vector bundle with a fiber preserving $\mathbb{Z}/2$-action which is free outside the zero section 0. Let $f : E \to E'$ be a fiber preserving $\mathbb{Z}/2$-equivariant map, and $Z_f = \{x \in E \mid f(x) = 0\}$. If $n_1 \geq m$, then*

$$\textit{cohom.dim}(Z_f) \geq \textit{cohom.dim}(B) + (n_1 - m).$$

We also prove a Bourgin-Yang type theorem for products of spheres equipped with the diagonal antipodal action of $\mathbb{Z}/2$. Again, the difficulty lies in the fact that products of spheres have non-trivial cohomology in many dimensions less than the top dimension.

Theorem 1.4. *Let $\mathbb{Z}/2$ act by the diagonal antipodal action on $M = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$, where $n_1 \leq \cdots \leq n_r$. Let $f : M \to \mathbb{R}^m$ be a continuous map and $A_f = \{x \in M \mid f(x) = f(-x)\}$ be the coincidence point set. If $n_1 \geq m$, then*

$$cohom.dim(A_f) \geq \dim(M) - m.$$

The notations in the theorem are explained in § 1 and § 2. In § 2, we also recorded some well known results that we will use later. In § 3, we discuss Davis's work on cohomology algebra of orbit space of $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ under the diagonal antipodal involution. In § 4, we construct characteristic polynomials for product sphere bundles following Dold's technique [5], and use them to prove our main theorem in § 5. Finally, in § 6, we prove a Bourgin-Yang type theorem for products of spheres.

2. Notation and Basic Results

All spaces under consideration will be paracompact Hausdorff spaces, and the cohomology used will be the Čech cohomology. Throughout we will use $\mathbb{Z}/2$ coefficients unless otherwise stated. The Čech cohomology theory satisfies the continuity property, in the sense that, if a cohomology class vanishes on a closed set, then it also vanishes on a neighbourhood of that set [6, Chapter X].

The cohomological dimension, denoted *cohom.dim*(X, A), of a paracompact Hausdorff space X with respect to an abelian group A is the largest positive integer n such that $H^n(X, Y; A) \neq 0$ for some closed subspace Y of X. See, for example, Nagami [20] for basic results on dimension theory. We will use the following well known result of Quillen [22] relating the cohomological dimension of a space with the cohomological dimension of its orbit space under a compact group action.

Theorem 2.1. [22, Proposition A.11] *Let G be a compact group acting on a paracompact Hausdorff space X and X/G be the orbit space. Then*

$$cohom.dim(X/G, \mathbb{Z}/2) \leq cohom.dim(X, \mathbb{Z}/2).$$

Let G be a compact group acting continuously and freely on a space X. Then

$$X \to X/G$$

is a principal G-bundle. Let

$$G \hookrightarrow E_G \to B_G$$

be the universal principal G-bundle, where B_G is the classifying space of the group G. Then we can take a classifying map

$$X/G \to B_G$$

for the principal G-bundle $X \to X/G$. The group G acts diagonally on $X \times E_G$ with orbit space

$$X_G = (X \times E_G)/G.$$

The projection $X \times E_G \to E_G$ is G-equivariant, and gives a fibration

$$X \hookrightarrow X_G \longrightarrow B_G.$$

This construction is originally due to Borel [1, Chapter IV]. We recall that $B_{\mathbb{Z}/2} = \mathbb{R}P^\infty$ and

$$H^*(B_{\mathbb{Z}/2}) \cong \mathbb{Z}/2[s],$$

where s is a homogeneous element of degree one.

According to Bredon [3, p.372], a fiber bundle $X \hookrightarrow E \to B$ is said to admit a cohomology extension of the fiber with respect to $\mathbb{Z}/2$, if the inclusion of a typical fiber $X \hookrightarrow E$ induces surjection in cohomology

$$H^*(E) \twoheadrightarrow H^*(X).$$

Clearly, any trivial bundle admits a cohomology extension of the fiber with respect to any abelian group. Also, any projective space bundle admits a cohomology extension of the fiber (see § 4). For fiber bundles which admit a cohomology extension of the fiber, we have the following Leray-Hirsch Theorem.

Theorem 2.2. [3, p.372, Theorem 1.4] *Let $X \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fiber bundle admitting a cohomology extension of the fiber with respect to $\mathbb{Z}/2$. Suppose that $H^*(X)$ is a finitely generated free $\mathbb{Z}/2$-module with basis $\{c_j\}_j$. If $\{e_j\}_j$ are cohomology classes in $H^*(E)$ such that $i^*(e_j) = c_j$, then the map*

$$H^*(B) \otimes_{\mathbb{Z}/2} H^*(X) \to H^*(E)$$

given by $b \otimes c_j \mapsto \pi^*(b)e_j$ is an isomorphism of $H^*(B)$-modules.

In other words, $H^*(E)$ is a free $H^*(B)$-module with basis $\{e_j\}_j$. Here we view $H^*(E)$ as a $H^*(B)$-module by defining the scalar multiplication $be = \pi^*(b)e$ for $b \in H^*(B)$ and $e \in H^*(E)$. This result will be used in the proofs of our theorems.

3. Orbit space of diagonal antipodal involution on products of spheres

Group actions on products of spheres has been of great interest to topologists. Let $\overline{n} = (n_1, \ldots, n_r)$, where each n_i is a positive integer. Then the diagonal antipodal map

$$(X_1, \ldots, X_r) \mapsto (-X_1, \ldots, -X_r)$$

defines a free involution on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$. The corresponding orbit space $P_{\overline{n}}$ was investigated in detail in a recent paper by Davis [4], where he referred this manifold as a projective product space. Note that if $\overline{n} = (n)$, then $P_{\overline{n}} = \mathbb{R}P^n$. In the same paper, Davis determined the mod 2 cohomology algebra $H^*(P_{\overline{n}})$ and the action of the Steenrod algebra on $H^*(P_{\overline{n}})$. Let $\Lambda[-]$ denotes an exterior algebra over $\mathbb{Z}/2$. Then Davis proved the following theorem.

Theorem 3.1. [4, Theorem 2.1] *Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq \cdots \leq n_r$. If $n_1 < n_2$ or n_1 is odd, then there is an isomorphism of graded algebras*

$$H^*\big(P_{\overline{n}}\big) \cong \frac{\mathbb{Z}/2[u]}{\langle u^{n_1+1}\rangle} \otimes \Lambda[v_2, \ldots, v_r],$$

where $\deg(u) = 1$ and $\deg(v_j) = n_j$ for each $2 \leq j \leq r$. Further, if n_1 is even, then $H^*\big(P_{\overline{n}}\big)$ is as above, except that $v_j^2 = u^{n_1}v_j$ for all $j \geq 2$ such that $n_j = n_1$.

The above theorem will play a crucial role in our proofs. It is worth mentioning that almost nothing is known about the classification of free involutions on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ when $r \geq 3$. Some knowledge about the cohomology algebra of the orbit space of an arbitrary free involution on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ would be useful in proving stronger results than those we prove in this paper.

4. Characteristic polynomials for bundles

Let $X \hookrightarrow E \xrightarrow{\pi} B$ be a fiber bundle with a fiber preserving free action of a compact group G such that the quotient bundle $\overline{X} \hookrightarrow \overline{E} \to B$ admits a cohomology extension of the fiber with respect to $\mathbb{Z}/2$. With this hypothesis, we define characteristic polynomials for the bundles under consideration following Dold [5] and de Mattos-Pergher-dos Santos [17, 18].

4.1. **Characteristic polynomials for $E \to B$ with $\mathbb{Z}/2$-action.**
Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq \cdots \leq n_r$, and $\mathbb{Z}/2$ act freely on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ by the diagonal antipodal map

$$(X_1, \ldots, X_r) \mapsto (-X_1, \ldots, -X_r).$$

Then by Theorem 3.1, $H^*(P_{\overline{n}})$ is a free graded algebra over $\mathbb{Z}/2$, generated by the set

$$\Big\{u^i v_2^{\epsilon_2} \ldots v_r^{\epsilon_r} \mid 0 \leq i \leq n_1 \text{ and } \epsilon_j \in \{0, 1\}\Big\}$$

and subject to the relations

$$u^{n_1+1} = 0 \text{ and } v_j^2 = 0 \text{ for each } 2 \leq j \leq r.$$

Recall that $u \in H^1(P_{\overline{n}})$ and $v_j \in H^{n_j}(P_{\overline{n}})$. By Leray-Hirsch Theorem, there exist elements $a \in H^1(\overline{E})$ and $b_j \in H^{n_j}(\overline{E})$ such that the restriction to a typical fiber

$$H^*(\overline{E}) \to H^*(P_{\overline{n}})$$

maps $a \mapsto u$ and $b_j \mapsto v_j$ for each $2 \leq j \leq r$. Note that $H^*(\overline{E})$ is a $H^*(B)$-module with a basis

$$\Big\{a^i b_2^{\epsilon_2} \ldots b_r^{\epsilon_r} \mid 0 \leq i \leq n_1 \text{ and } \epsilon_j \in \{0, 1\}\Big\}.$$

For simplicity, we set

$$\epsilon = (\epsilon_2, \ldots \epsilon_r) \text{ and } b^\epsilon = b_2^{\epsilon_2} \ldots b_r^{\epsilon_r}.$$

Then a basis for $H^*(\overline{E})$ as a $H^*(B)$-module is

$$\Big\{a^i b^\epsilon \mid 0 \leq i \leq n_1 \text{ and } \epsilon \in \{0, 1\}^{r-1}\Big\}. \tag{1}$$

In view of Theorem 3.1, we have the following two cases.

4.2. **n_1 is odd or $n_1 < n_2$.** Consider the elements

$$a^{n_1+1} \in H^{n_1+1}(\overline{E}) \quad \text{and}$$

$$b_j^2 \in H^{2n_j}(\overline{E}) \qquad \text{for } 2 \leq j \leq r.$$

These elements can be expressed uniquely in terms of the basis (1). Therefore, there exist unique elements

$$w_{i,\epsilon}^0 \in H^{d^0(i,\epsilon)}(B) \quad \text{for } 0 \leq i \leq n_1 \text{ and } \epsilon \in \{0,1\}^{r-1} \text{ and}$$

$$w_{i,\epsilon}^j \in H^{d^j(i,\epsilon)}(B) \quad \text{for } 0 \leq i \leq n_1,\ \epsilon \in \{0,1\}^{r-1} \text{ and } 2 \leq j \leq r,$$

such that

$$a^{n_1+1} = \textstyle\sum_{i,\epsilon} w_{i,\epsilon}^0 a^i b^\epsilon \quad \text{and}$$

$$b_j^2 = \textstyle\sum_{i,\epsilon} w_{i,\epsilon}^j a^i b^\epsilon \qquad \text{for } 2 \leq j \leq r.$$

Note that

$$\deg(w_{i,\epsilon}^0) = d^0(i,\epsilon) = (n_1+1) - \deg(a^i b^\epsilon) \quad \text{and}$$

$$\deg(w_{i,\epsilon}^j) = d^j(i,\epsilon) = 2n_j - \deg(a^i b^\epsilon) \qquad \text{for } 2 \leq j \leq r.$$

Let x be an indeterminate of degree 1. For each $2 \leq j \leq r$, let y_j be an indeterminate of degree n_j. For each $\epsilon = (\epsilon_2, \ldots, \epsilon_r) \in \{0,1\}^{r-1}$, set

$$y^\epsilon = y_2^{\epsilon_2} \ldots y_r^{\epsilon_r}.$$

Then the characteristic polynomials in the indeterminates $\{x, y_2, \ldots, y_r\}$ associated to the fiber bundle $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r} \hookrightarrow E \to B$ are defined by

$$p_0(x, y_2, \ldots, y_r) = x^{n_1+1} + \textstyle\sum_{i,\epsilon} w_{i,\epsilon}^0 x^i y^\epsilon,$$

$$p_j(x, y_2, \ldots, y_r) = y_j^2 + \textstyle\sum_{i,\epsilon} w_{i,\epsilon}^j x^i y^\epsilon \qquad \text{for } 2 \leq j \leq r.$$

The evaluation map gives a homomorphism of $H^*(B)$-algebras

$$H^*(B)[x, y_2, \ldots, y_r] \to H^*(\overline{E})$$

given by

$$(x, y_2, \ldots, y_r) \mapsto (a, b_2, \ldots, b_r).$$

Since the kernel is the ideal $\langle p_0, p_2, \ldots, p_r\rangle$, we have the following isomorphism of $H^*(B)$-algebras

$$\frac{H^*(B)[x, y_2, \ldots, y_r]}{\langle p_0, p_2, \ldots, p_r\rangle} \cong H^*(\overline{E}). \tag{2}$$

4.3. **n_1 is even and $n_1 = n_2$.** Let $k = \max\{j \mid 2 \leq j \leq r \text{ and } n_1 = n_j\}$. In this case, along with the elements

$$a^{n_1+1} \in H^{n_1+1}(\overline{E}) \quad \text{and}$$

$$b_j^2 \in H^{2n_j}(\overline{E}) \qquad \text{for } k+1 \leq j \leq r,$$

we consider the elements

$$b_j^2 + a^{n_1} b_j \in H^{2n_j}(\overline{E}) \quad \text{for } 2 \leq j \leq k.$$

Just as in the previous case, there exist unique elements

$$W^j_{i,\epsilon} \in H^{D^j(i,\epsilon)}(B) \quad \text{for } 0 \leq i \leq n_1,\ \epsilon \in \{0,1\}^{r-1} \text{ and } 2 \leq j \leq k,$$

such that

$$b_j^2 + a^{n_1} b_j = \textstyle\sum_{i,\epsilon} W^j_{i,\epsilon} a^i b^\epsilon \quad \text{for } 2 \leq j \leq k,$$

where

$$\deg(W^j_{i,\epsilon}) = D^j(i,\epsilon) = 2n_j - \deg(a^i b^\epsilon) \text{ for } 2 \leq j \leq k.$$

Thus, along with

$$p_0(x, y_2, \ldots, y_r) = x^{n_1+1} + \textstyle\sum_{i,\epsilon} w^0_{i,\epsilon} x^i y^\epsilon \quad \text{and}$$

$$p_j(x, y_2, \ldots, y_r) = y_j^2 + \textstyle\sum_{i,\epsilon} w^j_{i,\epsilon} x^i y^\epsilon \qquad \text{for } k+1 \leq j \leq r,$$

in this case, we have the following additional characteristic polynomials

$$P_j(x, y_2, \ldots, y_r) = y_j^2 + x^{n_1} y_j + \sum_{i,\epsilon} W^j_{i,\epsilon} x^i y^\epsilon \text{ for } 2 \leq j \leq k.$$

Then the evaluation map gives the following isomorphism of $H^*(B)$-algebras

$$\frac{H^*(B)[x, y_2, \ldots, y_r]}{\langle p_0, p_{k+1}, p_{k+2}, \ldots, p_r, P_2, \ldots, P_k\rangle} \cong H^*(\overline{E}). \tag{3}$$

4.4. Characteristic polynomial for $E' \to B$ with $\mathbb{Z}/2$-action.

Next we define the characteristic polynomial associated to the vector bundle $\mathbb{R}^m \hookrightarrow E' \to B$ equipped with a fiber preserving $\mathbb{Z}/2$-action on E' which is free outside the zero section. The construction is originally due to Dold [5] and Nakaoka [21]. Let

$$\mathbb{S}^{m-1} \hookrightarrow SE' \to B$$

be the associated sphere bundle. The free $\mathbb{Z}/2$-action on SE' gives the projective space bundle

$$\mathbb{R}P^{m-1} \hookrightarrow \overline{SE'} \to B,$$

and the principal $\mathbb{Z}/2$-bundle $SE' \to \overline{SE'}$. It is easy to see that

$$H^*(\mathbb{R}P^{m-1}) \cong \frac{\mathbb{Z}/2[s']}{\langle s'^m \rangle}.$$

Here $s' = g^*(s)$, where $s \in H^1(B_G)$ and $g : \mathbb{R}P^{m-1} \to B_G$ is a classifying map for the principal $\mathbb{Z}/2$-bundle $\mathbb{S}^{m-1} \to \mathbb{R}P^{m-1}$. Let $h : \overline{SE'} \to B_G$ be a classifying map for the principal $\mathbb{Z}/2$-bundle $SE' \to \overline{SE'}$, and let $c = h^*(s) \in H^1(\overline{SE'})$. Then the $\mathbb{Z}/2$-module homomorphism

$$H^*(\mathbb{R}P^{m-1}) \to H^*(\overline{SE'})$$

given by $s' \mapsto c$ is a cohomology extension of the fiber. Therefore, by the Leray-Hirsch theorem, $H^*(\overline{SE'})$ is a $H^*(B)$-module with a basis

$$\{1, c, c^2, ..., c^{m-1}\}.$$

Hence we can write $c^m \in H^m(\overline{SE'})$ as

$$c^m = w_m + w_{m-1}c + \cdots + w_1 c^{m-1},$$

where $w_i \in H^i(B)$ are unique elements. Now the characteristic polynomial in the indeterminate x of degree 1, associated to $\mathbb{R}^m \hookrightarrow E' \to B$, is defined as

$$p(x) = w_m + w_{m-1}x + \cdots + w_1 x^{m-1} + x^m.$$

As earlier, the evaluation map $x \mapsto c$ gives the following isomorphism of $H^*(B)$-algebras:

$$\frac{H^*(B)[x]}{\langle p(x) \rangle} \cong H^*(\overline{SE'}).$$

5. Proofs of theorems

Let $X \hookrightarrow E \to B$ be a fiber bundle with a fiber preserving free action of a compact group G such that the quotient bundle $\overline{X} \hookrightarrow \overline{E} \to B$ admits a cohomology extension of the fiber with respect to $\mathbb{Z}/2$. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be a vector bundle with a fiber preserving G-action on E' which is free outside the zero section 0. For a fiber preserving G-equivariant map

$$f : E \to E',$$

define the zero set of f as

$$Z_f = \{x \in E \mid f(x) = 0\}.$$

Since the set Z_f is G-invariant, we denote by $\overline{Z}_f$ the quotient of Z_f by G. For brevity, let $\mathcal{X}$ denote a collection of indeterminates, and let $\mathcal{P}(\mathcal{X})$ denote a collection of polynomials on $\mathcal{X}$. In §4, we defined characteristic polynomials $\mathcal{P}(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ associated to certain fiber bundles, and showed that

$$\frac{H^*(B)[\mathcal{X}]}{\langle \mathcal{P}(\mathcal{X}) \rangle} \cong H^*(\overline{E})$$

as $H^*(B)$-algebras. Therefore, each polynomial $q(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ defines an element of $H^*(\overline{E})$, which we denote by $q(\mathcal{X})|_{\overline{E}}$. Let $q(\mathcal{X})|_{\overline{Z}_f}$ denote the image of $q(\mathcal{X})|_{\overline{E}}$ under the homomorphism

$$H^*(\overline{E}) \to H^*(\overline{Z}_f)$$

induced by the inclusion $\overline{Z}_f \hookrightarrow \overline{E}$.

Now we set $\mathcal{X} = \{x, y_2, \ldots, y_r\}$, $J = \{0, 2, \ldots, r\}$ and $K = \{2, \ldots, k\}$. With these notations, we prove the following results for product sphere bundles.

Theorem 5.1. *Let $E \to B$ be a $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ bundle, where $n_1 \leq \cdots \leq n_r$. Suppose that $E \to B$ is equipped with a fiber preserving free $\mathbb{Z}/2$-action such that the induced action on each fiber is equivalent to the diagonal antipodal action, and that the quotient bundle $\overline{E} \to B$ admits a cohomology extension of the fiber with respect to $\mathbb{Z}/2$. Let $E' \to B$ be an m-dimensional vector bundle with a fiber preserving $\mathbb{Z}/2$-action which is free outside the zero section, and $f : E \to E'$ be a fiber preserving $\mathbb{Z}/2$-equivariant map. Let $q(\mathcal{X})$ be a polynomial in $H^*(B)[\mathcal{X}]$ such that $q(\mathcal{X})|_{\overline{Z}_f} = 0$. Then the following facts hold.*

1. *If n_1 is odd or $n_1 < n_2$, then there exist polynomials $\{s_j(\mathcal{X})\}_{j \in J}$ in $H^*(B)[\mathcal{X}]$ such that*
$$q(\mathcal{X})p(x) = \sum_{j \in J} s_j(\mathcal{X}) p_j(\mathcal{X}).$$
2. *If n_1 is even and $n_1 = n_2$, then there exist additional polynomials $\{S_j(\mathcal{X})\}_{j \in K}$ in $H^*(B)[\mathcal{X}]$ such that*
$$q(\mathcal{X})p(x) = \sum_{j \in J \setminus K} s_j(\mathcal{X}) p_j(\mathcal{X}) + \sum_{j \in K} S_j(\mathcal{X}) P_j(\mathcal{X}).$$

Proof. Let $q(\mathcal{X})$ in $H^*(B)[\mathcal{X}]$ be such that $q(\mathcal{X})|_{\overline{Z}_f} = 0$. By the continuity of Čech cohomology, there exists an open subset V of $\overline{E}$ such that $\overline{Z}_f \subset V$ and $q(\mathcal{X})|_V = 0$. Let

$$j_1 : \overline{E} \hookrightarrow (\overline{E}, V)$$

be the natural inclusion. Then consider the following long exact cohomology sequence for the pair $(\overline{E}, V)$

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_1^*} H^*(\overline{E}) \to H^*(V) \to H^*(\overline{E}, V) \to \cdots.$$

Since $q(\mathcal{X})|_V = 0$, there exists an element $\mu \in H^*(\overline{E}, V)$ such that $j_1^*(\mu) = q(\mathcal{X})|_{\overline{E}}$. Let

$$\overline{f} : \overline{E} - \overline{Z}_f \to \overline{E'} - 0$$

be the map induced by f on passing to quotient. Then the induced map in cohomology

$$\overline{f}^* : H^*(\overline{E'} - 0) \to H^*(\overline{E} - \overline{Z}_f)$$

is a $H^*(B)$-algebra homomorphism. Recall that $p(c) = 0$ and hence

$$p(x)|_{\overline{E}-\overline{Z}_f} = p(a) = p\big(\overline{f}^*(c)\big) = \overline{f}^*\big(p(c)\big) = 0.$$

Now consider the pair $(\overline{E}, \overline{E} - \overline{Z}_f)$. Let

$$j_2 : \overline{E} \hookrightarrow (\overline{E}, \overline{E} - \overline{Z}_f)$$

be the natural inclusion, and

$$\cdots \to H^*(\overline{E}, \overline{E} - \overline{Z}_f) \xrightarrow{j_2^*} H^*(\overline{E}) \to H^*(\overline{E} - \overline{Z}_f) \to H^*(\overline{E}, \overline{E} - \overline{Z}_f) \to \dots$$

be the long exact cohomology sequence for the pair $(\overline{E}, \overline{E} - \overline{Z}_f)$. Since $p(x)|_{\overline{E}-\overline{Z}_f} = 0$, there exists an element $\eta \in H^*(\overline{E}, \overline{E} - \overline{Z_f})$ such that $j_2^*(\eta) = p(x)|_{\overline{E}}$. Now naturality of the cup product gives

$$q(\mathcal{X})p(x)|_{\overline{E}} = j_1^*(\mu)j_2^*(\eta) = j^*(\mu\eta).$$

Since

$$\mu\eta \in H^*\big(\overline{E}, V \cup (\overline{E} - \overline{Z}_f)\big) = H^*(\overline{E}, \overline{E}) = 0,$$

we have $q(\mathcal{X})p(x)|_{\overline{E}} = 0$.

Now, if n_1 is odd or $n_1 < n_2$, then by (2), there exist polynomials $\big\{s_j(\mathcal{X})\big\}_{j\in J}$ in $H^*(B)[\mathcal{X}]$ such that

$$q(\mathcal{X})p(x) = \sum_{j\in J} s_j(\mathcal{X})p_j(\mathcal{X}).$$

Similarly, if n_1 is even and $n_1 = n_2$, then by (3), there exist polynomials $\big\{s_j(\mathcal{X})\big\}_{j\in J\setminus K}$ and $\big\{S_j(\mathcal{X})\big\}_{j\in K}$ in $H^*(B)[\mathcal{X}]$ such that

$$q(\mathcal{X})p(x) = \sum_{j\in J\setminus K} s_j(\mathcal{X})p_j(\mathcal{X}) + \sum_{j\in K} S_j(\mathcal{X})P_j(\mathcal{X}).$$

□

Next we prove a parametrized Borsuk-Ulam theorem (Theorem 1.3) for product sphere bundles.

Proof of Theorem 1.3. Let $n_1 \geq m$. First assume that n_1 is odd or $n_1 < n_2$. We claim that the $H^*(B)$-homomorphism

$$\bigoplus_{i=0}^{n_1-m} H^*(B)x^i \to H^*(\overline{Z}_f)$$

given by $x^i \to x^i|_{\overline{Z}_f}$ is injective. Let $q(x) \neq 0$ be a polynomial in $H^*(B)[\mathcal{X}]$ such that $\deg(q(x)) \leq n_1 - m$, and $q(x)|_{\overline{Z}_f} = 0$. Then by Theorem 5.1, we have

$$q(x)p(x) = \sum_{j \in J} s_j(\mathcal{X})p_j(\mathcal{X}).$$

Note that $\deg(p(x)) = m$, $\deg(p_0(\mathcal{X})) = n_1 + 1$ and $\deg(p_j(\mathcal{X})) = 2n_j$ for each $2 \leq j \leq r$. Since

$$\deg(q(x)) + m = \max_{j \in J} \Big\{ \deg(s_j(\mathcal{X})) + \deg(p_j(\mathcal{X})) \Big\},$$

we get

$$\deg(q(x)) + m \geq \deg(s_0(\mathcal{X})) + \deg(p_0(\mathcal{X})) \geq \deg(p_0(\mathcal{X})) = n_1 + 1.$$

This implies that $\deg(q(x)) \geq n_1 - m + 1$, which is a contradiction. Hence, the map

$$\bigoplus_{i=0}^{n_1 - m} H^*(B)x^i \to H^*(\overline{Z}_f)$$

is injective. This fact together with Theorem 2.1 yields

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n_1 - m).$$

The same argument works if n_1 is even and $n_1 = n_2$. This completes the proof. □

In [5], Dold considered sphere bundles associated to vector bundles. Taking $r = 1$ in the above theorem we obtain a generalization of Dold's Theorem 1.2 to arbitrary sphere bundles.

Corollary 5.2. Let $\mathbb{S}^n \hookrightarrow E \to B$ be a sphere bundle with a fiber preserving free $\mathbb{Z}/2$-action such that the induced action on each fiber is equivalent to the antipodal action. Let $\mathbb{R}^m \hookrightarrow E' \to B$ be a vector bundle with a fiber preserving $\mathbb{Z}/2$-action which is free outside the zero section, and $f : E \to E'$ be a fiber preserving $\mathbb{Z}/2$-equivariant map. If $n \geq m$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n - m).$$

Taking B to be a point yields the following result.

Corollary 5.3. Let $\mathbb{Z}/2$ act by the diagonal antipodal map on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ and antipodally on $\mathbb{R}^m$, where $n_1 \leq \cdots \leq n_r$. Let $f : \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r} \to \mathbb{R}^m$ be a $\mathbb{Z}/2$-equivariant map. If $n_1 \geq m$, then

$$cohom.dim(Z_f) \geq (n_1 - m).$$

Remark 5.4. The above corollary also follows from the classical Bourgin-Yang Theorem 1.1. Since $n_1 \leq \cdots \leq n_r$, there is a natural embedding $g : \mathbb{S}^{n_1} \to \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$ given by $g(X) = (X, X, \ldots, X)$. Further $f \circ g : \mathbb{S}^{n_1} \to \mathbb{R}^m$ is a $\mathbb{Z}/2$-equivariant map. Since $n_1 \geq m$, Theorem 1.1 gives $cohom.dim(Z_{f \circ g}) \geq (n_1 - m)$, which in turn implies that $cohom.dim(Z_f) \geq (n_1 - m)$.

Remark 5.5. Let us observe that results from [10] could possibly be used to extend Theorem 1.3 to products of spheres with the diagonal standard action of $\mathbb{Z}/p$ for odd primes p.

6. Bourgin-Yang Theorem for products of spheres

In this section, we prove a Bourgin-Yang type theorem [2, 27] for products of spheres equipped with the diagonal antipodal action of $\mathbb{Z}/2$. We will need the following result of Turygin [26, Lemma 1.1] (see also Munkholm [19]).

Lemma 6.1. *Let M be a topological manifold and G a finite group acting freely on M. Let $f : M \to \mathbb{R}^m$ be a continuous map and*

$$A_f = \{x \in X \mid f(x) = f(gx) \text{ for all } g \in G\}$$

be the set of coincidence points. Then $A_f \neq \emptyset$ if and only if the vector bundle

$$\xi_M : M \times_G I_{\mathbb{R}^m}(G) \to M/G$$

does not have a non-vanishing section.

Theorem 6.2. *Let $\mathbb{Z}/2$ act by the diagonal antipodal action on $M = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$, where $n_1 \leq \cdots \leq n_r$. Let $f : M \to \mathbb{R}^m$ be a continuous map and A_f be the coincidence point set. If $n_1 \geq m$, then*

$$cohom.dim(A_f) \geq \dim(M) - m.$$

Proof. Let G be a group and R be a commutative ring with unit. Let $I_R(G)$ denote the augmentation ideal of the group ring $R[G]$. In what follows, we assume $\mathbb{R}^m$ to be a ring where multiplication structure is given by multiplication of the coordinates. Now, take $G = \mathbb{Z}/2$. The mod 2 Euler class of the vector bundle

$$\eta : E_{\mathbb{Z}/2} \times_{\mathbb{Z}/2} I_{\mathbb{R}}(\mathbb{Z}/2) \to B_{\mathbb{Z}/2}$$

or its Stiefel-Whitney class is given by $e_2(\eta) = w_1(\eta) = x$, where $x \in H^1(B_{\mathbb{Z}/2})$ is the generator of $H^*(B_{\mathbb{Z}/2})$. Now, by the universal property of principal bundles, we have the following commutative diagram

$$\begin{array}{ccc} M \times_{\mathbb{Z}/2} I_{\mathbb{R}^m}(\mathbb{Z}/2) & \longrightarrow & E_{\mathbb{Z}/2} \times_{\mathbb{Z}/2} I_{\mathbb{R}^m}(\mathbb{Z}/2) \\ \Big\downarrow{\scriptstyle \xi_M} & & \Big\downarrow{\scriptstyle \xi} \\ M/\mathbb{Z}/2 & \xrightarrow{\quad \varphi \quad} & B_{\mathbb{Z}/2} \end{array}$$

such that $\xi \cong \eta \oplus \cdots \oplus \eta \cong m\eta$. Consequently, we have

$$e_2(\xi) = w_m(\xi) = w_1(\eta)^m = x^m. \tag{4}$$

Recall that, by Theorem 3.1,

$$H^*\big(M/\mathbb{Z}/2\big) \cong \frac{\mathbb{Z}/2[u]}{\langle u^{n_1+1}\rangle} \otimes \Lambda[v_2, \dots, v_r],$$

where $\deg(u) = 1$ and $\deg(v_j) = n_j$ for $2 \leq j \leq r$. Now, consider the induced homomorphism

$$\varphi^* : H^*\big(B_{\mathbb{Z}/2}\big) \to H^*\big(M/\mathbb{Z}/2\big).$$

Then, from 4 and the assumption $n_1 \geq m$, it follows that

$$\varphi^*(e_2(\xi)) = e_2(\xi_M) = u^m \neq 0 \tag{5}$$

Since A_f is closed and $\mathbb{Z}/2$-invariant, the set $M \setminus A_f$ is also $\mathbb{Z}/2$-invariant. Consider the following long exact sequence associated to the pair $\big(M/\mathbb{Z}/2, (M \setminus A_f)/\mathbb{Z}/2\big)$

$$(6) \cdots \longrightarrow H^m\big(M/\mathbb{Z}/2, (M \setminus A_f)/\mathbb{Z}/2\big) \xrightarrow{\alpha} H^m(M/\mathbb{Z}/2) \xrightarrow{\beta} H^m\big((M \setminus A_f)/\mathbb{Z}/2\big) \longrightarrow \cdots$$

Let

$$f' : M \setminus A_f \to \mathbb{R}^m$$

and

$$\xi'_M : (M \setminus A_f) \times_{\mathbb{Z}/2} I_{\mathbb{R}^m}(\mathbb{Z}/2) \to (M \setminus A_f)/\mathbb{Z}/2$$

be the restrictions of f and ξ_M, respectively. Then $A_{f'} = \emptyset$ and by Lemma 6.1 the vector bundle ξ'_M has a non-vanishing section over $M \setminus A_f$. Consequently, by a well known result, $e_2(\xi'_M) = 0$. This gives $\beta(e_2(\xi_M)) = e_2(\xi'_M) = 0$. By exactness of the sequence (6), there exists a non-trivial element

$$\mu \in H^m\big(M/\mathbb{Z}/2, (M \setminus A_f)/\mathbb{Z}/2\big)$$

such that $\alpha(\mu) = e_2(\xi_M) \neq 0$. Since we are working with cohomology with coefficients in $\mathbb{Z}/2$, there exists a corresponding non-trivial element

$$\tilde{\mu} \in H_m\big(M/\mathbb{Z}/2, (M \setminus A_f)/\mathbb{Z}/2\big).$$

Now, by Alexander duality, we have $H^{\dim(M)-m}\big(A_f/\mathbb{Z}/2\big) \neq 0$. Finally, by Theorem 2.1, we obtain

$$cohom.dim(A_f) \geq \dim(M) - m.$$

This proves the theorem. □

Corollary 6.3. Let $M = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$, where $n_1 \leq \cdots \leq n_r$. Let $\mathbb{Z}/2$ act by the diagonal antipodal action on M and antipodal action on $\mathbb{R}^m$. Let $f : M \to \mathbb{R}^m$ be a $\mathbb{Z}/2$-equivariant map. If $n_1 \geq m$, then

$$cohom.dim(Z_f) \geq \dim(M) - m.$$

Proof. Since f is a $\mathbb{Z}/2$-equivariant map, we have $Z_f = A_f$. Now, by Theorem 6.2,

$$cohom.dim(Z_f) = cohom.dim(A_f) \geq \dim(M) - m.$$

□

Remark 6.4. The condition $n_1 \geq m$ cannot be dropped in Theorem 6.2. If $n_1 < m$, then the map $f : M \to \mathbb{R}^m$ given by $f(x_1, \ldots, x_r) = i(x_1)$, where $i : S^{n_1} \to \mathbb{R}^m$ is the natural inclusion, has $A_f = \emptyset$.

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Departamento de Matematica, Instituto de Ciencias Matematicas e de Computacao, Universidade de Sao Paulo, CP 668, 13560-970, Sao Carlos - SP, Brazil.
E-mail address: `deniseml@icmc.usp.br`

Departamento de Matematica, Universidade Federal de Sao Carlos, Centro de Ciencias Exatas e Tecnologia, CP 676, CEP 13565-905, Sao Carlos - SP, Brazil.
E-mail address: `pergher@dm.ufscar.br, edivaldo@dm.ufscar.br`

Indian Institute of Science Education and Research (IISER) Mohali, Sector 81, Knowledge City, S A S Nagar, Post Office Manauli, Punjab 140306, India.
E-mail address: `mahender@iisermohali.ac.in`