# Solutions for a Kirchhoff equation with weight and nonlinearity with subcritical and critical Caffarelli-Kohn-Nirenberg growth

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**Abstract**

In this paper, we study the multiplicity of nontrivial solutions to a class of nonlinear boundary value problems of Kirchhoff type. We prove existence results when the problem has nonlinearities with subcritical and with critical Caffarelli-Kohn-Nirenberg exponent.

**keyword:** Variational methods, Critical exponents, Singular perturbations, Kirchhoff equation, Nonlinear elliptic equations.

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## 1 Introduction

We are concerned with existence of infinitely many solutions to the class of nonlinear boundary value problems of Kirchhoff type

$$\left\{ \begin{array}{rcll} L(u) & = & \lambda |x|^{-\delta} f(x,u) + |x|^{-\beta}|u|^{q-2}u & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where

$$L(u) := -\left[ M \left( \int_\Omega |x|^{-ap} |\nabla u|^p \; dx \right) \right] div \left( |x|^{-ap} |\nabla u|^{p-2} \nabla u \right),$$

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and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq 3$, $1 < p < N$, $a < \frac{N-p}{p}$, $p \leq q \leq p^*$, $\beta \leq (a+1)q + N(1-\frac{q}{p})$, $p^* = \frac{Np}{N-dp}$ the critical Caffarelli-Kohn-Nirenberg exponent, where $d = 1 + a - b$ with $a \leq b < a + 1$, and $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ is a continuous function.

Problem (1) with $a = b = \delta = \beta = 0$ and $p = 2$, that is,

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

is called nonlocal because of the presence of the term $M\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)$ which implies that the equation (2) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which makes the study of such a class of problem, particularly interesting. Besides, this class of problems has physical motivation. Indeed, the operator $M(\int_\Omega |\nabla u|^2 dx)\Delta u$ appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x,u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{3}$$

Such a hyperbolic equation is a general version of Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

presented by Kirchhoff [25]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (4) have the following meanings: $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density, and $P_0$ is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time which introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical d'Alembert equation. Later on, Woinowsky-Krieger (Nash - Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends. See, for example, [4], [5] and the references therein. The reader may consult [1], [2], [10], [11], [19], [29], and the references therein, for more information on nonlocal problems.

For enunciate the main result, we need to give some hypotheses on the functions $M$ and $f$. The hypotheses on the continuous function $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ are the following:

There exists $m_0 > 0$ such that

$(M_1)$ $$M(t) \geq m_0, \forall\, t \geq 0.$$

$(M_2)$ The function $M$ is increasing.

The hypotheses on the Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following:

$$(f_1) \qquad f(x,-t) = -f(x,t), \forall\,(x,t) \in \Omega \times \mathbb{R}.$$

There exists $r \in [1, p^*)$ and $C_1, C_2$ positive constants with $C_1 < C_2$, such that

$$(f_2) \qquad C_1|t|^{r-1} \leq f(x,t) \leq C_2|t|^{r-1}, \forall\,(x,t) \in \Omega \times (\mathbb{R}^+ \cup \{0\}).$$

Also, the function $f$ satisfies the well known Ambrosetti-Rabinowitz superlinear condition, that is,

$$(f_3) \qquad 0 < \xi \int_0^t f(x,s)ds \leq tf(x,t), \forall (x,t) \in \Omega \times \mathbb{R}^+, \text{ for some } \xi \in (p, p^*).$$

Moreover, we suppose $\delta \leq (a+1)r + N(1 - \frac{r}{p})$.

The main results of this paper are listed below.

The first result gives us infinitely many solutions for the problem (1) for the subcritical case.

**Theorem 1.1** *Assume* $(M_1)$, $(f_1)$, $(f_2)$, $p \leq q < p^*$, $\beta < (a+1)q + N(1 - \frac{q}{p})$, *and* $1 \leq r < p$. *Then, there exists* $\lambda_0 > 0$ *such that problem (1) has infinitely many solutions for each* $\lambda \in (0, \lambda_0)$.

In the last two results, we obtain infinitely many solutions for the problem (1) for the critical case.

**Theorem 1.2** *Assume* $(M_1)$, $(M_2)$, $(f_1)$, $(f_2)$, $q = p^*$, $\beta = bp^*$, *and* $1 \leq r < p$. *Then, there exists* $\lambda^* > 0$ *such that problem (1) has infinitely many solutions for each* $\lambda \in (0, \lambda^*)$.

**Theorem 1.3** *Assume* $(M_1)$, $(M_2)$, $(f_1)$, $(f_2)$, $(f_3)$, $q = p^*$, $\beta = bp^*$, *and* $p < r < p^*$. *Then, there exists* $\lambda^{**} > 0$ *such that problem (1) has a nontrivial solution for each* $\lambda \in (\lambda^{**}, +\infty)$.

In the last years the study on nonlocal problems of the type (2) grew exponentially. That was, probably, by the difficulties existing in this class of problems and that do not appear in the study of local problems, as well as due to their significance in applications. Without hope of being thorough, we mention some articles with multiplicity results and that are related with our main result. We will restrict our comments to the works that have emerged in the last four years.

The problem (2) was studied in [19]. The version with p-Laplacian operator was studied in [14]. In both cases, the authors showed a multiplicity result using genus theory. In [17], [22], [23], [24], [28], and [30] the authors showed a multiplicity result for the problem (2) using the Fountain theorem and the Symmetric Mountain Pass theorem. The case with nonlinearity discontinuous was studied in [15]. The authors showed a existence of two solutions via Mountain Pass Theorem and Ekeland's Variational Principle.

In [31] the authors used Fountain theorem and showed a multiplicity result of solutions for a problem involving a nonlocal operator and nonlinearity of Caffarelli-Kohn-Nirenberg type and subcritical growth.

In this article we study a different class of nonlocal operators that was considered in [31]. Moreover, our class of nonlocal operators include, but are not restricted to the type (4). Besides, our results are true for the subcritical and critical case. For this, the arguments found in [31] could not be repeated. It was necessary make a truncation on function $M$ in the critical case.

Our proof of the main theorems are inspired by the work [6], where the authors showed a multiplicity result for a problem involving $p-$Laplacian operator. But, since we are working with nonlocal operator, some refined estimates were necessary, for instance, the proof of Lemma 4.1 that does not appeared in [6].

Before concluding this introduction, it is very important to say that recently in the literature, we find many papers where the authors study the existence and multiplicity of solution to problems involving the nonlinearities of Caffarelli-Kohn-Nirenberg type, see, for example, [12], [21], [32], [35], and references therein.

This paper is organized as follows. In Section 2 we provide some preliminary results on the Krasnoselskii genus and the variational framework. Section 3 is devoted to the Palais-Smale condition for the Euler-Lagrange functional associated to problem (1) and to prove Theorem 1.1. In Section 4 by using the concentration-compactness principle we prove the Palais-Smale condition, some auxiliary results and Theorems 1.2 and 1.3.

## 2 Preliminary results and variational framework

We start by considering some basic notions on the Krasnoselskii genus that we will use in the proof of our main results.

Let $E$ be a real Banach space. We denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E\setminus\{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

**Definition 2.1** *Let $A \in \mathfrak{A}$. The Krasnoselskii genus of $A$, $\gamma(A)$, is defined as being the least positive integer $k$ such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$, for all $x \in A$. If $k$ does not exists we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

In the sequel we enunciate some results on the Krasnoselskii genus that may be found in the references [3], [9], [16], and [26].

**Proposition 2.2** *Suppose that* $E = \mathbb{R}^N$ *and* $\partial\Omega$ *the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^N$ *with* $0 \in \Omega$*. Then* $\gamma(\partial\Omega) = N$*.*

**Corollary 2.3** *If* $\partial\Omega = S^{N-1}$ *is the unit sphere in* $\mathbb{R}^N$*, then* $\gamma(S^{N-1}) = N$*.*

We now establish a result due to Clark [13].

**Proposition 2.4** *Consider* $\Phi \in C^1(X, \mathbb{R})$ *a functional satisfying the Palais-Smale condition and suppose that*
*i)* $\Phi$ *is bounded from below and even;*
*ii) there is a compact set* $K \in \mathfrak{A}$ *such that* $\gamma(K) = k$ *and* $\sup_{x \in K} \Phi(x) < \Phi(0)$*.*
*Then* $\Phi$ *possesses at least* $k$ *distinct pairs of critical points and their corresponding critical values are less than* $\Phi(0)$*.*

We point out that this result is a consequence of a basic multiplicity theorem involving an invariant functional under the action of a compact topological group.

**Proposition 2.5** *If* $K \in \mathfrak{A}$*,* $0 \notin K$*, and* $\gamma(K) \geq 2$*, then* $K$ *has infinitely many points.*

Now, we will introduce the basic variational framework. Consider $\Omega \subset \mathbb{R}^N$ a bounded smooth domain with $0 \in \Omega$, $N \geq 3$, $1 < p < N$, $a < (N-p)/p$, $a \leq b < a+1$, and $p^* = Np/(N-dp)$, where $d = 1 + a - b$. From [8, 33] we have

$$\left( \int_\Omega |x|^{-\alpha} |u|^r dx \right)^{\frac{p}{r}} \leq C \int_\Omega |x|^{-ap} |\nabla u|^p dx, \ \ \forall\, u \in C_0^\infty(\Omega), \tag{5}$$

where $1 \leq r \leq Np/(N-p)$, $\alpha \leq (a+1)r + N(1 - \frac{r}{p})$, and $\mathcal{D}_a^{1,p}$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u|| = \left( \int_\Omega |x|^{-ap} |\nabla u|^p dx \right)^{1/p};$$

that is, we have the continuous embedding of $\mathcal{D}_a^{1,p}$ in $L^r(\Omega, |x|^{-\alpha})$, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted $L^r(\Omega)$ space with the norm

$$||u||_{r,\alpha} = \left( \int_\Omega |x|^{-\alpha} |u|^r dx \right)^{1/r}.$$

Moreover, this embedding is compact if $\Omega$ is a bounded smooth domain, $1 \leq r < Np/(N-p)$, and $\alpha < (a+1)r + N(1 - \frac{r}{p})$. The best constant of the weighted Caffarelli-Kohn-Nirenberg type (see [8]) inequality will be denoted by $C^*_{a,p}$, which is characterized by

$$C^*_{a,p} = \inf_{u\in\mathcal{D}^{1,p}_a\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\left(\int_\Omega |x|^{-bp^*}|u|^{p^*}dx\right)^{\frac{p}{p^*}}}.$$

We will look for solutions of problem (1) by finding critical points of the Euler-Lagrange functional $I:\mathcal{D}^{1,p}_a\to\mathbb{R}$ given by

$$I(u) = \frac{1}{p}\widehat{M}(||u||^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\ dx - \frac{1}{q}\int_\Omega |x|^{-\beta}|u|^q\ dx,$$

where $\widehat{M}(t) := \int_0^t M(s)ds$ and $F(x,t) = \int_0^t f(x,s)ds$. Note that $I\in C^1$ and

$$\begin{aligned} I'(u)(\phi) &= M(||u||^p)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\phi\ dx \\ &\quad -\lambda\int_\Omega |x|^{-\delta}f(x,u)\phi\ dx - \int_\Omega |x|^{-\beta}|u|^{q-2}u\phi\ dx, \end{aligned}$$

for all $\phi\in\mathcal{D}^{1,p}_a$.

The Theorems 1.1 and 1.2 will be proved by using Proposition 2.4. From $(f_1)$ and $(f_2)$ we have that $I$ is even and $I(0)=0$, however we have a common difficulty to assure that $I$ is bounded from below in $\mathcal{D}^{1,p}_a$, so we will use a modified functional to obtain the critical points of $I$. In the following, we will construct the auxiliary functional.

We obtain by $(M_1)$, $(f_1)$, $(f_2)$, and Caffarelli-Kohn-Nirenberg inequality (5) that

$$I(u)\ \geq\ \frac{m_0}{p}\|u\|^p - \lambda\tilde{C}_2\|u\|^r - \frac{1}{q}\tilde{C}\|u\|^q\ =\ g_\lambda(||u||^p),$$

where $g_\lambda:[0,+\infty)\to\mathbb{R}$ is given by

$$g_\lambda(t) = \frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} - \frac{1}{q}\tilde{C}t^{q/p}.$$

Note that $r<p\leq q\leq p^*$ implies $\lim\limits_{t\to+\infty} g_\lambda(t) = -\infty$. Hence $I$ is not bounded from below in $\mathcal{D}^{1,p}_a$ and we can not apply the Proposition 2.4. But, there exists $\lambda_0>0$ such that, for each $\lambda\in(0,\lambda_0)$, the function $g_\lambda(t)$ achieves a positive maximum and there exist $t_1,t_2\in(0,+\infty)$ with $t_1<t_2$ and $g_\lambda(t_1)=g_\lambda(t_2)=0$. Now, consider $\phi\in C^1_0([0,+\infty))$ with $0\leq\phi\leq 1$, $\phi(t)=1$ for all $t\in[0,t_1]$, $\phi(t)=0$ for all $t\in[t_2,+\infty)$, and $\phi'(t)\leq 0$, for all $t\in[0,+\infty)$. Define the function $\bar{g}_\lambda:[0,+\infty)\to\mathbb{R}$ given by

$$\bar{g}_\lambda(t) = \frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} - \frac{\tilde{C}}{q}\phi(t)t^{q/p}.$$

Note that $\bar{g}_\lambda(0)=0$, $\bar{g}_\lambda(t)\geq 0$ for all $t\geq t_1$, and

$$\lim_{t\to+\infty}\bar{g}_\lambda(t) = \lim_{t\to+\infty}\frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} = +\infty, \tag{6}$$

because $\frac{r}{p} < 1$ and $\phi(t) = 0$ for all $t \in [t_2, +\infty)$.

The auxiliary Euler-Lagrange functional that we will use is $J : \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p}\widehat{M}(||u||^p) - \lambda \int_\Omega |x|^{-\delta} F(x,u)\ dx - \frac{\phi(||u||^p)}{q} \int_\Omega |x|^{-\beta}|u|^q\ dx,$$

where $\widehat{M}(t) := \int_0^t M(s)ds$ and $\lambda \in (0, \lambda_0)$. Since $J(u) \geq \bar{g}_\lambda(||u||^p)$ and (6) hold we get that $J$ is coercive in $\mathcal{D}_a^{1,p}$, which implies $J$ bounded from below in $\mathcal{D}_a^{1,p}$. Thus the functional $J$ is appropriate to prove the Theorem 1.1.

The next Lemma was proved by Ghoussoub and Yuan in [20, Lemma 4.1]

**Lemma 2.6 ($S_+$ condition)** *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $(u_n) \subset \mathcal{D}_a^{1,p}$, and $u \in \mathcal{D}_a^{1,p}$ are such that $u_n \rightharpoonup u$, as $n \to \infty$, and*

$$\limsup_{n\to\infty} \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx \leq 0,$$

*then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.*

# 3 Subcritical case: Theorem 1.1

We will prove in the next lemma that a critical point of $J$ with energy less than 0 is a critical point of $I$. We also note that the critical level from Proposition 2.4 are less than $I(0) = J(0) = 0$.

**Lemma 3.1** *If $u_0$ is critical point of $J$ with $J(u_0) < 0$ then $u_0$ is a critical point of $I$.*

**Proof.** Let $u_0$ be a critical point of the functional $J$ with $J(u_0) < 0$. Since that $J$ is continuous, there exists $R_0 > 0$ such that $J(u) < 0$, for all $u \in B(u_0, R_0) \subset \mathcal{D}_a^{1,p}$, which implies $||u||^p < t_1$ because $J(u) \geq \bar{g}_\lambda(||u||^p)$ and $\bar{g}_\lambda(t) \geq 0$ if $t \geq t_1$. Therefore, for all $u \in B(u_0, R_0)$, we have $\phi(||u||^p) = 1$, and consequently $J(u) = I(u)$ for all $u \in B(u_0, R_0)$, in particular, it follows that $u_0$ is a critical point of $I$. ■

**Lemma 3.2** *Assume $(M_1)$, $(f_1)$, $(f_2)$, $q < p^*$, and $\beta < (a+1)q + N(1 - \frac{q}{p})$. Then $J$ satisfies the Palais-Smale condition.*

**Proof.** Let $(u_n) \subset \mathcal{D}_a^{1,p}$ be a Palais-Smale sequence at level $c$, that is, $J(u_n) \to c$ and $J'(u_n) \to 0$ (in the dual of $\mathcal{D}_a^{1,p}$), as $n \to +\infty$. Since that $J$ is coercive, we have that $(u_n) \subset \mathcal{D}_a^{1,p}$ is bounded. Then, passing to a subsequence, if necessary, we have $u \in \mathcal{D}_a^{1,p}$ such that

$$\begin{gathered} u_n \rightharpoonup u \text{ in } \mathcal{D}_a^{1,p},\ u_n \to u \text{ in } L^s(\Omega, |x|^{-\sigma}), \\ u_n(x) \to u(x) \text{ a.e. in } \Omega, \text{ and } \|u_n\| \to t_0 \geq 0, \end{gathered} \tag{7}$$

as $n \to +\infty$, where $1 \leq s < p^*$ and $\sigma < (a+1)s + N(1 - \frac{s}{p})$. Hence $J'(u_n)(u_n - u) = o_n(1)$, that is,

$$\begin{aligned} M(||u_n||^p)\int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}}dx - \lambda\int_\Omega \frac{f(x,u_n)(u_n - u)}{|x|^\delta}dx \\ -\frac{p}{q}\phi'(||u_n||^p)\int_\Omega \frac{|u_n|^q}{|x|^\beta}dx \int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}}dx \\ -\phi(||u_n||^p)\int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta}dx \;=\; o_n(1). \end{aligned} \tag{8}$$

From Hölder inequality, (7), and as $\phi$ is continuous, we obtain

$$\int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta}dx = o_n(1) \text{ and } \phi(||u_n||^p)\int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta}dx = o_n(1).$$

By using $(f_2)$, Hölder inequality, and (7), we get

$$\left|\int_\Omega |x|^{-\delta} f(x,u_n)(u_n - u)dx\right| \leq C_2\int_\Omega |x|^{-\delta}|u_n|^{r-1}|u_n - u|dx = o_n(1).$$

Therefore, substituting in (8), we have

$$\left[M(||u_n||^p) - \frac{p}{q}\phi'(||u_n||^p)\int_\Omega \frac{|u_n|^q}{|x|^\delta}dx\right]\int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}}dx = o_n(1).$$

Since $M$ and $\phi'$ are continuous functions, $M(t) \geq m_0$, and $\phi'(t) \leq 0$, for all $t \in [0, \infty)$, there exists $C > 0$, such that

$$m_0 \leq M(||u_n||^p) - \frac{p}{q}\phi'(||u_n||^p)\int_\Omega |x|^{-\beta}|u_n|^q dx \leq C.$$

Thus,

$$\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\ dx = o_n(1).$$

Then by using Lemma 2.6 the proof is finished. ■

#### 3.0.1 Proof of Theorem 1.1

We have that $\mathcal{D}_a^{1,p}$ is a reflexive and separable Banach space, then, for any $k \in \mathbb{N}$, there is a $k$-dimensional linear subspace $\mathcal{X}_k$ of $\mathcal{D}_a^{1,p}$ with $\mathcal{X}_k \subset C_0^{+\infty}(\Omega)$. Therefore, all norms on $\mathcal{X}_k$ are equivalent. Hence, there exists a positive constant $C(k)$ which depends on $k$, such that $rC(k)\|u\|^r \leq C_1\|u\|^r_{L^r(\Omega,|x|^{-\delta})}$, for all $u \in \mathcal{X}_k$. Hence, if $u \in \mathcal{X}_k$, we get by $(f_2)$ that

$$\int_\Omega |x|^{-\delta}F(x,u)dx \geq \frac{C_1}{r}\int_\Omega |x|^{-\delta}|u|^r dx \geq C(k)\|u\|^r.$$

Thus, for all $u \in \mathcal{X}_k$ with $\|u\| \leq 1$, from continuity of the function $M$, we conclude that, there exists $C > 0$ such that

$$J(u) \leq C\|u\|^p - \lambda C(k)\|u\|^r.$$

Take $R = \min\{1, \left(\frac{\lambda C(k)}{C}\right)^{\frac{1}{p-r}}\}$ and consider $\mathcal{S} = \{u \in \mathcal{X}_k : \|u\| = s\}$ with $0 < s < R$. Since $1 \leq r < p$, for all $u \in \mathcal{S}$, we get

$$J(u) \leq s^r \left[ C s^{p-r} - \lambda C(k) \right] < 0 = J(0), \tag{9}$$

which implies $\sup_{\mathcal{S}} J(u) < 0 = J(0)$.

Since $\mathcal{X}_k$ and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}$ and $S^{k-1}$ are homeomorphic, we conclude that $\gamma(\mathcal{S}) = k$. Moreover $J$ is coercive, even, and satisfies the Palais-Smale condition (see Lemma 3.2), then, follows from Proposition 2.4 that $J$ has at least $k$ pairs of different critical points. Since $k$ is arbitrary, we obtain infinitely many critical points of $J$. Then, by using (9) and Lemma 3.1, we obtain infinitely many critical points of $I$. ■

# 4 The critical case

In Theorems 1.2 and 1.3 we have $\beta = bp^*$, $q = p^*$, and by $(M_2)$ that $M(t)$ is increasing. Since we are working with critical growth and a nonlocal operator without more information about the behavior of function $M$ at infinity, we need to make a truncation on function $M$ (see Lemmas 4.2 and 4.11). In the case $p < r < p^*$, the truncation is also necessary to prove Lemma 4.10.

Since $p < p^*$, we can get $\theta \in (p, p^*)$. From $(M_2)$, there exists $t_0 > 0$ such that $m_0 \leq M(0) < M(t_0) < \frac{\theta}{p} m_0$ for the case $1 \leq r < p$ and $m_0 \leq M(0) < M(t_0) < \frac{\xi}{p} m_0$ for the case $p < r < p^*$, where $\xi$ is given by $(f_3)$. We set

$$M_0(t) := \begin{cases} M(t), & \text{if} \quad 0 \leq t \leq t_0, \\ M(t_0), & \text{if} \quad t \geq t_0. \end{cases}$$

From $(M_2)$ we get

$$m_0 \leq M_0(t) \leq \frac{\theta}{p} m_0, \ \forall t \geq 0 \tag{10}$$

and

$$m_0 \leq M_0(t) \leq \frac{\xi}{p} m_0, \ \forall t \geq 0. \tag{11}$$

The proofs of the Theorems 1.2 and 1.3 are based on a careful study of solutions of the following auxiliary problem

$$\begin{cases} L_0(u) = \lambda |x|^{-\delta} f(x,u) + |x|^{-\beta} |u|^{q-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where

$$L_0(u) :=: - \left[ M_0 \left( \int_\Omega |x|^{-ap} |\nabla u|^p \, dx \right) \right] div \left( |x|^{-ap} |\nabla u|^{p-2} \nabla u \right),$$

We will look for solutions of problem (12) by finding critical points of the Euler-Lagrange functional $I_\lambda : \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{p}\widehat{M_0}(||u||^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\ dx - \frac{1}{q}\int_\Omega |x|^{-\beta}|u|^q\ dx,$$

where $\widehat{M_0}(t) := \int_0^t M_0(s)ds$. Note that $I_\lambda$ is $C^1$ and for all $\phi \in \mathcal{D}_a^{1,p}$, we have

$$I'_\lambda(u)(\phi) = M_0(||u||^p)\int_\Omega \frac{|\nabla u|^{p-2}\nabla u \nabla \phi}{|x|^{ap}}dx - \lambda\int_\Omega \frac{f(x,u)\phi}{|x|^\delta}dx - \int_\Omega \frac{|u|^{q-2}u\phi}{|x|^\beta}dx.$$

## 4.1 Case $1 \le r < p$

**Lemma 4.1**

$$\lim_{\lambda\to 0^+} t_1(\lambda) = 0. \tag{13}$$

**Proof.** From $g_\lambda(t_1(\lambda)) = 0$ and $g'_\lambda(t_1(\lambda)) > 0$, we have

$$\frac{m_0}{p} = \lambda\tilde{C}_2(t_1(\lambda))^{\frac{r-p}{p}} + \frac{1}{p^*}\tilde{C}(t_1(\lambda))^{\frac{p^*-p}{p}} \tag{14}$$

and

$$m_0 > \lambda r\tilde{C}_2(t_1(\lambda))^{\frac{r-p}{p}} + \tilde{C}(t_1(\lambda))^{\frac{p^*-p}{p}}, \tag{15}$$

for all $\lambda \in (0,\lambda_0)$. From (14) and (15) we obtain

$$\lambda p\tilde{C}_2(t_1(\lambda))^{\frac{r-p}{p}} + \frac{p}{p^*}\tilde{C}(t_1(\lambda))^{\frac{p^*-p}{p}} > \lambda r\tilde{C}_2(t_1(\lambda))^{\frac{r-p}{p}} + \tilde{C}(t_1(\lambda))^{\frac{p^*-p}{p}},$$

which implies

$$\lambda\tilde{C}_2(p-r)(t_1(\lambda))^{\frac{r-p}{p}} > \tilde{C}\left(1-\frac{p}{p^*}\right)(t_1(\lambda))^{\frac{p^*-p}{p}}. \tag{16}$$

From (16) we conclude that

$$0 < t_1(\lambda) < \lambda^{\frac{p}{p^*-r}}\left[\frac{\tilde{C}_2(p-r)}{\tilde{C}\left(1-\dfrac{p}{p^*}\right)}\right]^{\frac{p}{p^*-r}}. \tag{17}$$

Since $p^* > r$, passing the limit as $\lambda \to 0^+$ in (17) we conclude the proof. ∎

As in the subcritical case we can construct an auxiliary functional $J_\lambda : \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\widehat{M_0}(||u||^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\ dx - \frac{\phi(||u||^p)}{q}\int_\Omega |x|^{-bp^*}|u|^{p^*}\ dx,$$

where $\widehat{M_0}(t) := \int_0^t M_0(s)ds$.

**Lemma 4.2** *Let $(u_n)$ be a bounded sequence in $\mathcal{D}_a^{1,p}$ such that*

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0 \text{ in } (\mathcal{D}_a^{1,p})^{-1}, \text{ as } n \to \infty.$$

*Suppose $(M_1)$, $(M_2)$, $(f_1)$, $(f_2)$, and*

$$c < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left[\frac{\lambda C C_2\left(\frac{1}{r}+\frac{1}{\theta}\right)}{\left(\frac{1}{\theta}-\frac{1}{p^*}\right)}\right]^{\frac{p^*}{p^*-r}} \left[\left(\frac{r}{p^*}\right)^{\frac{r}{p^*-r}} - \left(\frac{r}{p^*}\right)^{\frac{p^*}{p^*-r}}\right],$$

*where $C = \left(\int_\Omega \left(|x|^{-\delta+br}\right)^{\frac{p^*}{p^*-r}} dx\right)^{\frac{p^*-r}{p^*}}$, then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.*

**Proof:** Since $(u_n)$ is bounded in $\mathcal{D}_a^{1,p}$, passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup u \text{ in } \mathcal{D}_a^{1,p},\ u_n \to u \text{ in } L^s(\Omega, |x|^{-\sigma}),$$
$$u_n(x) \to u(x) \text{ a.e. in } \Omega, \text{ and } \|u_n\| \to t_0 \geq 0,$$

as $n \to +\infty$, where $1 \leq s < p^*$ and $\sigma < (a+1)s + N(1 - s/p)$. Moreover, using the concentration-compactness principle due to Lions (cf. [27, 33]), we obtain at most countable index set $\Lambda$, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$, such that

$$|x|^{-ap}|\nabla u_n|^p \rightharpoonup |x|^{-ap}|\nabla u|^p + \mu \text{ and } |x|^{-bp^*}|u_n|^{p^*} \rightharpoonup |x|^{-bp^*}|u|^{p^*} + \nu, \quad (18)$$

as $n \to +\infty$, in weak$^*$-sense of measures where

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad \text{and} \quad C^*_{a,p} \nu_i^{p/p^*} \leq \mu_i, \quad (19)$$

for all $i \in \Lambda$, where $\delta_{x_i}$ is the Dirac mass at $x_i \in \Omega$.

Now let $k \in \mathbb{N}$. Without loss of generality we can suppose $B_2(0) \subset \Omega$, then for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_k)/\varrho)$ where $\psi \in C_0^\infty(\Omega, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$, and $|\nabla\psi| \leq 1$. Observe that $(\psi_\varrho u_n)$ is bounded in $\mathcal{D}_a^{1,p}$. So we have $I'_\lambda(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}dx + o_n(1)$$

$$= -M_0(||u_n||^p)\int_\Omega \frac{|\nabla u_n|^p\psi_\varrho}{|x|^{ap}}dx + \lambda\int_\Omega \frac{f(x,u_n)\psi_\varrho u_n}{|x|^\delta}dx + \int_\Omega \frac{\psi_\varrho|u_n|^{p^*}}{|x|^{bp^*}}dx.$$

It follows from (18) and $(M_1)$ that

$$M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx \leq -m_0\int_\Omega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}dx - m_0\int_\Omega \psi_\varrho d\mu$$
$$+\lambda\int_\Omega \frac{f(x,u_n)\psi_\varrho u_n}{|x|^\delta}dx + \int_\Omega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}dx + \int_\Omega \psi_\varrho d\nu + o_n(1).$$

Since $u_n \to u$ in $L^r\left(\Omega, |x|^{-\delta}\right)$, by using $(f_2)$ and the Dominated Convergence Theorem, we get

$$\lambda \int_\Omega |x|^{-\delta} f(x,u_n)\psi_\varrho u_n dx \to \lambda \int_\Omega |x|^{-\delta} f(x,u)\psi_\varrho u dx,$$

as $n \to \infty$. Thus, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\left[M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}dx\right] \ &\leq\ -m_0\int_\Omega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}dx \\ -m_0\int_\Omega \psi_\varrho d\mu + \lambda\int_\Omega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}dx &+ \int_\Omega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}dx + \int_\Omega \psi_\varrho d\nu.\end{aligned}$$

From the Dominated Convergence Theorem we obtain

$$\int_\Omega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}dx = o_\varrho(1),\ \int_\Omega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}dx = o_\varrho(1),\ \text{and} \int_\Omega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}dx = o_\varrho(1),$$

where $\lim\limits_{\varrho\to 0^+} o_\varrho(1) = 0$. So, we get

$$\begin{aligned}\lim_{\varrho\to 0^+}\left\{\limsup_{n\to\infty}\left[M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\right]\right\} \\ \leq \lim_{\varrho\to 0^+}\left[\int_\Omega \psi_\varrho d\nu - m_0\int_\Omega \psi_\varrho d\mu\right].\end{aligned} \tag{20}$$

Now, we will show that

$$\lim_{\varrho\to 0^+}\left[\limsup_{n\to\infty} M_0(||u_n||^p)\int_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho dx\right] = 0. \tag{21}$$

Indeed, by Hölder's Inequality

$$\left|\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\right| \leq \|u_n\|^{p-1}\left(\int_\Omega \frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}dx\right)^{\frac{1}{p}}.$$

Since $u_n$ is bounded in $\mathcal{D}_a^{1,p}$, $M_0$ is continuous, and $supp(\psi_\varrho) \subset B(x_k; 2\varrho)$, there exists $L > 0$ such that

$$M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx \leq L\left(\int_{B(x_k;2\varrho)} \frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}dx\right)^{\frac{1}{p}}.$$

Using the Dominated Convergence Theorem and Hölder's Inequality, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\left[M_0(||u_n||^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}dx\right] &\leq L\left(\int_{B(x_k;2\varrho)} \frac{|u|^p|\nabla\psi_\varrho|^p}{|x|^{ap}}dx\right)^{\frac{1}{p}} \\ \leq L\left(\int_{B(x_k;2\varrho)}|\nabla\psi_\varrho|^N dx\right)^{\frac{1}{N}}&\left(\int_{B(x_k;2\varrho)}\left(\frac{|u|^p}{|x|^{ap}}\right)^{\frac{N}{N-p}}dx\right)^{\frac{N-p}{Np}} \\ \leq L|B(x_k;2\varrho)|^{\frac{1}{N}}&\left(\int_\Omega \chi_{B(x_k;2\varrho)}\left(\frac{|u|^p}{|x|^{ap}}\right)^{\frac{N}{N-p}}dx\right)^{\frac{N-p}{Np}}.\end{aligned}$$

Letting $\varrho \to 0^+$ on the above expression, we obtain (21). Thus, we conclude from (20) that

$$0 \le \lim_{\rho \to 0^+} \left[ \int_\Omega \psi_\varrho d\nu - m_0 \int_\Omega \psi_\varrho d\mu \right].$$

That is,

$$\begin{aligned} 0 &\le \lim_{\rho \to 0^+} \left[ \int_{B(x_k;2\varrho)} \psi_\varrho d\nu - m_0 \int_{B(x_k;2\varrho)} \psi_\varrho d\mu \right] \\ &= \nu(\{x_k\}) - m_0 \mu(\{x_k\}) \\ &\le \sum_{i\in\Lambda} \nu_i \delta_{x_i}(\{x_k\}) - m_0 \sum_{i\in\Lambda} \mu_i \delta_{x_i}(\{x_k\}) \\ &= \nu_k - m_0 \mu_k. \end{aligned}$$

So, we have

$$m_0 \mu_k \le \nu_k.$$

It follows from (19) that

$$\nu_k \ge \left( m_0 C^*_{a,p} \right)^{\frac{p^*}{p^*-p}} \ge \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \left( m_0 C^*_{a,p} \right)^{\frac{p^*}{p^*-p}}. \tag{22}$$

Now we shall prove that the above expression can not occur, and therefore the set $\Lambda$ is empty. Indeed, arguing by contradiction, let us suppose that (22) hold for some $k \in \Lambda$. Thus, once that $m_0 \le M_0(t) \le \frac{\theta}{p} m_0$, for all $t \in \mathbb{R}$, and by using $(f_1)$ and $(f_2)$, we have

$$\begin{aligned} c &= I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)(u_n) + o_n(1) \\ &\ge \left( \frac{m_0}{p} - \frac{\theta m_0}{\theta p} \right) \|u_n\|^p - \lambda \int_\Omega \frac{F(x,u_n) + \frac{1}{\theta}|x|^{-\delta} f(x,u_n) u_n}{|x|^\delta} dx \\ &\quad + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \int_\Omega \frac{|u_n|^{p^*}}{|x|^{bp^*}} dx + o_n(1) \\ &\ge -\lambda C_2 \left( \frac{1}{r} + \frac{1}{\theta} \right) \int_\Omega \frac{|u_n|^r}{|x|^\delta} dx + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \int_\Omega \frac{|u_n|^{p^*} \psi_\varrho}{|x|^{bp^*}} dx + o_n(1). \end{aligned}$$

Letting $n \to +\infty$, we get

$$\begin{aligned} c &\ge -\lambda C_2 \left( \frac{1}{r} + \frac{1}{\theta} \right) \int_\Omega \frac{|u|^r}{|x|^\delta} dx + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \int_\Omega \frac{|u|^{p^*} \psi_\varrho}{|x|^{bp^*}} dx + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \nu_k \\ &\ge -\lambda C_2 \left( \frac{1}{r} + \frac{1}{\theta} \right) \int_\Omega \frac{|u|^r}{|x|^\delta} dx + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \int_\Omega \frac{|u|^{p^*} \psi_\varrho}{|x|^{bp^*}} dx \\ &\quad + \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \left( m_0 C^*_{a,p} \right)^{\frac{p^*}{p^*-p}}. \end{aligned}$$

By Hölder's Inequality

$$\int_\Omega \frac{|u|^r}{|x|^\delta}dx \leq C\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}dx\right)^{r/p^*},$$

where $C = \left(\int_\Omega \left(|x|^{-\delta+br}\right)^{\frac{p^*}{p^*-r}}\, dx\right)^{\frac{p^*-r}{p^*}} < \infty$. So, letting $\varrho \to +\infty$, we obtain

$$\begin{aligned} c \geq & -\lambda C C_2\left(\frac{1}{r}+\frac{1}{\theta}\right)\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}dx\right)^{r/p^*} + \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}dx \\ & + \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\left(m_0 C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}. \end{aligned}$$

Define $\varphi : \mathbb{R}^+ \to \mathbb{R}$ given by $\varphi(t) = \left(\frac{1}{\theta}-\frac{1}{p^*}\right)t^{p^*} - \lambda C C_2\left(\frac{1}{r}+\frac{1}{\theta}\right)t^r$. This function attains its absolute minimum at the point

$$t_0 = \left(\frac{\lambda r C C_2\left(\frac{1}{r}+\frac{1}{\theta}\right)}{p^*\left(\frac{1}{\theta}-\frac{1}{p^*}\right)}\right)^{\frac{1}{p^*-r}} > 0.$$

Thus, we conclude that

$$\begin{aligned} c &\geq \varphi(t_0) \\ &= \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\left(m_0 C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} \\ &\quad - \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\left[\frac{\lambda C C_2\left(\frac{1}{r}+\frac{1}{\theta}\right)}{\left(\frac{1}{\theta}-\frac{1}{p^*}\right)}\right]^{\frac{p^*}{p^*-r}}\left[\left(\frac{r}{p^*}\right)^{\frac{r}{p^*-r}} - \left(\frac{r}{p^*}\right)^{\frac{p^*}{p^*-r}}\right]. \end{aligned}$$

But this is a contradiction. Thus $\Lambda$ is empty and it follows that $u_n \to u$ in $L^{p^*}\left(\Omega, |x|^{-bp^*}\right)$.

Now we will prove that $u_n \to u$ in $\mathcal{D}^{1,p}_a$.

Indeed, since $u_n \to u$ in $L^r\left(\Omega, |x|^{-\delta}\right)$ and in $L^{p^*}\left(\Omega, |x|^{-bp^*}\right)$, follows from the Dominated Convergence Theorem that

$$\lim_{n\to+\infty}\int_\Omega \frac{f(x,u_n)(u_n-u)}{|x|^\delta}dx = \lim_{n\to+\infty}\int_\Omega \frac{|u_n|^{p^*-2}u_n(u_n-u)}{|x|^{bp^*}}dx = 0.$$

Therefore, as $(u_n)$ is bounded in $\mathcal{D}^{1,p}_a$, $I'_\lambda(u_n)(u_n-u) \to 0$ in $(\mathcal{D}^{1,p}_a)^{-1}$, $\|u_n\| \to t_0$, as $n \to \infty$, and as $M$ is continuous and positive, we conclude

$$\lim_{n\to\infty}\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u)dx = 0.$$

It follows from Lemma 2.6 that $u_n \to u$ in $\mathcal{D}^{1,p}_a$. ■

**Remark 4.3** *Due to Lemma 4.1 we can consider $\lambda_0 > 0$ such that $t_1 = t_1(\lambda) \leq t_0$, for each $\lambda \in (0, \lambda_0)$. Also, we can obtain $\lambda^* \leq \lambda_0$ such that*

$$\left(\frac{1}{\theta} - \frac{1}{p^*}\right) (m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left[\frac{\lambda C C_2 \left(\frac{1}{r} + \frac{1}{\theta}\right)}{\left(\frac{1}{\theta} - \frac{1}{p^*}\right)}\right]^{\frac{p^*}{p^*-r}} \left[\left(\frac{r}{p^*}\right)^{\frac{r}{p^*-r}} - \left(\frac{r}{p^*}\right)^{\frac{p^*}{p^*-r}}\right] > 0,$$

*for each $\lambda \in (0, \lambda^*)$.*

**Lemma 4.4** *If $J_\lambda(u) < 0$, then $\|u\|^p < t_1$ and $J_\lambda(v) = I_\lambda(v)$, for all $v$ in a sufficiently small neighborhood of $u$. Moreover, $J_\lambda$ verifies a local Palais-Smale condition for $c < 0$, for all $\lambda \in (0, \lambda^*)$.*

**Proof:** Since $\bar{g}_\lambda(\|u\|^p) \leq J_\lambda(u) < 0$, arguing as in Section 3, we conclude that $\|u\|^p < t_1$, and $J_\lambda(v) = I_\lambda(v)$, for all $v \in B(u; R_0)$. Moreover, if $(u_n)$ is a sequence such that $J_\lambda(u_n) \to c < 0$ and $J'_\lambda(u_n) \to 0$ in $\mathcal{D}^{1,p}_a$, then for $n$ sufficiently large, $I_\lambda(u_n) = J_\lambda(u_n) \to c < 0$ and $I'_\lambda(u_n) = J'_\lambda(u_n) \to 0$. Since $J$ is coercive, we get $(u_n)$ bounded in $\mathcal{D}^{1,p}_a$. It follows from Remark 4.3, that for $\lambda \in (0, \lambda^*)$,

$$c < 0 < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) (m_0 C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left[\frac{\lambda C C_2 \left(\frac{1}{r} + \frac{1}{\theta}\right)}{\left(\frac{1}{\theta} - \frac{1}{p^*}\right)}\right]^{\frac{p^*}{p^*-r}} \cdot \left[\left(\frac{r}{p^*}\right)^{\frac{r}{p^*-r}} - \left(\frac{r}{p^*}\right)^{\frac{p^*}{p^*-r}}\right]$$

and from Lemma 4.2, up to a subsequence, $(u_n)$ is strongly convergent in $\mathcal{D}^{1,p}_a$. ■

Now we will construct an appropriate minimax sequence of negative critical values for the functional $J_\lambda$.

**Lemma 4.5** *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J^{-\epsilon}) \geq k,$$

*where $J^{-\epsilon} = \{u \in \mathcal{D}^{1,p}_a : J_\lambda(u) \leq -\epsilon\}$.*

**Proof:** Fix $k \in \mathbb{N}$, let $X_k$ be a $k$-dimensional subspace of $\mathcal{D}^{1,p}_a$. Thus, there exists $C(k) > 0$ such that $C(k)\|u\|^r \leq C_1 \|u\|^r_{L^r(\Omega, |x|^{-\delta})}$, for all $u \in X_k$.

Considering $\overline{\rho} > 0$ such that $0 < \|u\| = \overline{\rho}$ and $0 < \|u\|^p < t_1$, we get $J_\lambda(u) = I_\lambda(u)$. Arguing as in the proof of Theorem 1.1, we can take $R > 0$ such that

$$I_\lambda(u) < -\epsilon,$$

for all $u \in \mathcal{S} = \{u \in X_k : \|u\| = s\}$, with $0 < s < \min\{R, \overline{\rho}\}$. Hence $\mathcal{S} \subset J^{-\epsilon}$ and, since $J^{-\epsilon}$ is symmetric and closed, from Corollary 2.3,

$$\gamma(J^{-\epsilon}) \geq \gamma(\mathcal{S}) = k.$$

■

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset \mathcal{D}_a^{1,p} \backslash \{0\} : C \text{ is closed}, C = -C \text{ and } \gamma(C) \geq k\},$$

$$K_c = \{u \in \mathcal{D}_a^{1,p} : J'_\lambda(u) = 0 \text{ and } J_\lambda(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u).$$

**Lemma 4.6** *Given $k \in \mathbb{N}$, the number $c_k$ is negative.*

**Proof:** From Lemma 4.5, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J^{-\epsilon}) \geq k$. Moreover, $0 \notin J^{-\epsilon}$ and $J^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J^{-\epsilon}} J_\lambda(u) \leq -\epsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u) \leq \sup_{u \in J^{-\epsilon}} J_\lambda(u) \leq -\epsilon < 0.$$

■

The next Lemma allows us to prove the existence of critical points of $J$.

**Lemma 4.7** *If $c = c_k = c_{k+1} = ... = c_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\gamma(K_c) \geq r + 1,$$

*for all $\lambda \in (0, \lambda^*)$.*

**Proof:** Let $(u_n)$ be a sequence in $K_c$. Since Lemma 4.6 gives us $c = c_k = c_{k+1} = ... = c_{k+r} < 0$, from Lemma 4.4 we have $(u_n)$ bounded and $J_\lambda(u_n) = I_\lambda(u_n)$, for all $n \in \mathbb{N}$. Thus we can apply Lemma 4.2 and we obtain a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$, that is, $K_c$ is a compactness set. Moreover, $K_c = -K_c$. Suppose, by contratiction, that $\gamma(K_c) \leq r$, there exists a closed and symmetric set $U$ with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J^0$ because $c < 0$. By the deformation Lemma [7] we have an odd homeomorphism $\eta : \mathcal{D}_a^{1,p} \to \mathcal{D}_a^{1,p}$ such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J^{c+\delta} \subset J^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} J_\lambda(u) < c + \delta$, that is, $A \subset J^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta}. \tag{23}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (23). Hence, this Lemma is proved. ■

**Remark 4.8** *If* $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ *and since each* $c_k$ *is a critical value of* $J_\lambda$*, then we obtain infinitely many critical points of* $J_\lambda$ *and hence, the problem (12) has infinitely many solutions.*

*On the other hand, if there are two constants* $c_k = c_{k+r}$*, then* $c = c_k = c_{k+1} = ... = c_{k+r}$ *and from Lemma 4.7, there exists* $\lambda^* > 0$ *such that*

$$\gamma(K_c) \geq r + 1 \geq 2.$$

*From Proposition 2.5,* $K_c$ *has infinitely many points, that is, problem (12) has infinitely many solutions.*

#### 4.1.1 Proof of Theorem 1.2

Let $\lambda^*$ be as in the Remark 4.3 and, for $0 < \lambda < \lambda^*$, let $u_\lambda$ be the nontrivial solution of problem (12) found on the Remark 4.8. Thus $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Hence, by using Lemma 4.4 we have

$$\|u_\lambda\|^p \leq t_1 < t_0, \tag{24}$$

so we conclude that

$$M_0(\|u_\lambda\|^p) = M(\|u_\lambda\|^p)$$

and $u_\lambda$ is a solution of problem (1). ■

### 4.2 Case $p < r < p^*$

In this subsection we adapt for our study the ideas in [18]. In the sequel we prove that the functional $I_\lambda$ has the Mountain Pass Geometry.

**Lemma 4.9** *Assume that the conditions* $(M_1)$, $(f_1)$, *and* $(f_2)$ *hold. There exist positive numbers* $\rho$ *and* $\alpha$ *such that*

$$I_\lambda(u) \geq \alpha > 0, \forall u \in \mathcal{D}_a^{1,p}, \text{ with } \|u\| = \rho.$$

**Proof:** From $(M_1)$, $(f_1)$, $(f_2)$, and Caffarelli-Khon-Nirenberg inequality, we obtain

$$I_\lambda(u) \geq \frac{m_0}{p}\|u\|^p - \lambda\tilde{C}_2\|u\|^r - \frac{1}{p^*}\tilde{C}\|u\|^{p^*}.$$

Since $p < r < p^*$, the result follows by choosing $\rho > 0$ small enough. ■

**Lemma 4.10** *For all* $\lambda > 0$*, there exists* $e \in \mathcal{D}_a^{1,p}$ *with* $I_\lambda(e) < 0$ *and* $\|e\| > \rho$.

**Proof:** Fix $v_0 \in \mathcal{D}_a^{1,p}\backslash\{0\}$, with $v_0 \geq 0$ in $\Omega$ and $\|v_0\| = 1$. Using (11) and $(f_2)$ we get

$$I_\lambda(tv_0) \leq \frac{\xi}{p^2} m_0 t^p \|v_0\|^p - \frac{\lambda C_1}{r} t^r \int_\Omega \frac{|v_0|^r}{|x|^\delta} dx - \frac{t^{p^*}}{p^*} \int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx.$$

Since $p < r < p^*$, we have $\lim_{t \to +\infty} I_\lambda(tv_0) = -\infty$. Thus, for $\bar{t} > \rho$ large enough $I_\lambda(\bar{t}v_0) < 0$. The result follows by considering $e = \bar{t}v_0$. ■

Using a version of the Mountain Pass Theorem without the (PS) condition (see [34]), there exists a sequence $(u_n) \in \mathcal{D}_a^{1,p}$, satisfying

$$I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0 \text{ in } (\mathcal{D}_a^{1,p})^{-1},$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t))$$

and

$$\Gamma := \left\{ \gamma \in C([0,1], \mathcal{D}_a^{1,p}) : \gamma(0) = 0, \gamma(1) = e \right\}.$$

**Lemma 4.11** *If* $(M_1)$, $(M_2)$, *and* $(f_2)$ *hold, then*

$$\lim_{\lambda \to +\infty} c_\lambda = 0.$$

**Proof:** Since the functional $I_\lambda$ has the Mountain Pass Geometry, it follows that there exists $t_\lambda > 0$ verifying $I_\lambda(t_\lambda v_0) = \max_{t \geq 0} I_\lambda(t v_0)$, where $v_0$ is given by Lemma 4.10. Hence, from (11) and $(f_2)$ we get

$$0 \;=\; I'_\lambda(t_\lambda v_0)(t_\lambda v_0) \;\leq\; \frac{\xi}{p} m_0 t_\lambda^p \|v_0\|^p - \lambda C_1 t_\lambda^r \int_\Omega \frac{|v_0|^r}{|x|^\delta} dx - t_\lambda^{p^*} \int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx,$$

that is

$$\frac{\xi}{p} m_0 t_\lambda^p \;\geq\; \lambda C_1 t_\lambda^r \int_\Omega \frac{|v_0|^r}{|x|^\delta} dx + t_\lambda^{p^*} \int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx \;\geq\; t_\lambda^{p^*} \int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}} dx,$$

which implies that $(t_\lambda)$ is bounded. Thus there exists a sequence $(\lambda_n)$ and $\beta_0 \geq 0$ such that $\lambda_n \to +\infty$ and $t_{\lambda_n} \to \beta_0$, as $n \to +\infty$. Consequently, there exists $D > 0$ such that

$$\frac{\xi}{p} m_0 t_{\lambda_n}^p \leq D, \forall n \in \mathbb{N},$$

and so

$$\lambda_n C_1 t_{\lambda_n}^r \int_\Omega |x|^{-\delta} |v_0|^r dx + t_{\lambda_n}^{p^*} \int_\Omega |x|^{-bp^*} |v_0|^{p^*} dx \leq D, \forall n \in \mathbb{N}. \tag{25}$$

If $\beta_0 > 0$, we obtain

$$\lim_{n \to \infty} \left[ \lambda_n C_1 t_{\lambda_n}^r \int_\Omega |x|^{-\delta} |v_0|^r dx + t_{\lambda_n}^{p^*} \int_\Omega |x|^{-bp^*} |v_0|^{p^*} dx \right] = +\infty,$$

which is a contradiction with (25). Thus we conclude that $\beta_0 = 0$. Now, let us consider the path $\gamma_*(t) = te$, for $t \in [0,1]$, which belongs to $\Gamma$, to get the following estimate

$$0 < c_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma_*(t)) = I_\lambda(t_\lambda v_0) \leq \frac{\xi}{p^2} m_0 t_\lambda^p.$$

In this way, observing that $(c_\lambda)$ is a monotonous sequence, we conclude

$$\lim_{\lambda \to +\infty} c_\lambda = 0.$$

∎

**Remark 4.12** *Due to Lemma 4.11, there exists* $\lambda_1 > 0$ *such that* $c_\lambda < \left(\frac{1}{p}m_0 - \frac{1}{\xi}M_0(t_0)\right)t_0$*, for all* $\lambda > \lambda_1$.

**Lemma 4.13** *Suppose that* $\lambda > \lambda_1$ *and* $(M_1)$, $(M_2)$, $(f_2)$, *and* $(f_3)$ *hold. Let* $(u_n) \in \mathcal{D}_a^{1,p}$ *be a bounded sequence such that*

$$I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0 \text{ in } (\mathcal{D}_a^{1,p})^{-1}, \text{ as } n \to +\infty.$$

*Then, for all* $n \in \mathbb{N}$, *we have*

$$\|u_n\|^p \le t_0.$$

**Proof:** Suppose by contradiction that for some $n \in \mathbb{N}$ we have $\|u_n\|^p > t_0$. Thus, for each $\lambda > \lambda_1$, from the definition of $M_0(t)$, $(f_3)$, and (11), we achieve

$$\begin{aligned} c_\lambda &= I_\lambda(u_n) - \tfrac{1}{\xi}I'_\lambda(u_n)(u_n) + o_n(1) \\ &\ge \tfrac{1}{p}\widehat{M_0}(\|u_n\|^p) - \tfrac{1}{\xi}M_0(t_0)\|u_n\|^p + o_n(1) \\ &\ge \left(\tfrac{1}{p}m_0 - \tfrac{1}{\xi}M_0(t_0)\right)\|u_n\|^p + o_n(1). \end{aligned} \tag{26}$$

Since $m_0 < M(t_0) < \dfrac{\xi}{p}m_0$ we have $\frac{1}{p}m_0 - \frac{1}{\xi}M_0(t_0) > 0$. So we get $c_\lambda \ge \left(\frac{1}{p}m_0 - \frac{1}{\xi}M_0(t_0)\right)t_0 > 0$. But this contradicts the Remark 4.12. Hence we conclude that $\|u_n\|^p \le t_0$. ■

### 4.2.1 Proof of Theorem 1.3

It follows from Lemma 4.11 that there exists $\lambda^{**} \ge \lambda_1 > 0$ such that

$$c_\lambda < \left(\frac{1}{\xi} - \frac{1}{p^*}\right)\left(m_0 C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}, \tag{27}$$

for all $\lambda \ge \lambda^{**}$. Now, fix $\lambda \ge \lambda^{**}$. From Lemmas 4.9 and 4.10, there exists a bounded sequence $(u_n) \subset \mathcal{D}_a^{1,p}$ such that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$, as $n \to \infty$.

Arguing as in Lemma 4.2, from (27) we conclude that, up to a subsequence, $u_n \to u_\lambda$ in $\mathcal{D}_a^{1,p}$. Thus $u_\lambda$ is a weak solution of problem (12). Moreover, by Lemma 4.13 we conclude that $u_\lambda$ is a weak solution of problem (1) .

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