Limit cycles in discontinuous classical Liénard equations

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Abstract

We study the number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed by nonlinear functions inside the class of all classical polynomial Liénard differential equations allowing discontinuities.

In particular our results show that for any $n \geq 1$ there are differential equations of the form $\ddot{x} + f(x)\dot{x} + x + sgn(\dot{x})g(x) = 0$, with f and g polynomials of degree n and 1 respectively, having $[n/2] + 1$ limit cycles, where $[\cdot]$ denotes the integer part function.

Keywords: Limit cycles, Liénard systems, Averaging theory.

1. Introduction

One of the main problems in the qualitative theory of real planar differential equations is the determination of limit cycles. The non-existence, existence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc (see for instance the books [3, 13]).

The second part of the sixteen Hilbert's problem [6] is related with the least upper bound on the number of limit cycles of polynomial vector fields having

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a fixed degree. This problem together with the Riemann conjecture are the unique two problems of the list of Hilbert which has not been solved.

The generalized polynomial Liénard differential equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

introduced in [7], is a simplified version of Hilbert's problem in which a subclass of polynomial vector fields is considered. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.

We remark that in 1977 Lins, de Melo and Pugh [8] studied the classical polynomial Liénard differential equations (1) obtained when $g(x) = x$ and stated the following conjecture: *if $f(x)$ has degree $n \geq 1$ and $g(x) = x$, then* (1) *has at most* $[n/2]$ *limit cycles.* They also proved the conjecture for $n = 1, 2$. For $n = 3$ this conjecture has been proved recently by Chengzhi Li and Llibre in [5]. Recently De Maesschalck and Dumortier proved in [11] that the classical Liénard equation of degree $n \geq 5$ can have $[(n-1)/2] + 2$ limit cycles, where $[\cdot]$ denotes the integer part function. For $n \geq 6$ Dumortier, Panazzolo and Roussarie proved this conjecture is not true in [4]. The conjecture for $n = 4$ is still open.

In [9], the authors apply the averaging theory of first, second and third order to the class of generalized polynomial Liénard differential equations. Their main result shows that for any $n, m \geq 1$ there are differential equations of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with f and g polynomials of degree n and m respectively, having at least $[(n+m-1)/2]$ limit cycles.

A large number of problems from mechanics and electrical engineering, theory of automatic control, economy, impact systems among others cannot be described with smooth dynamical systems (see for instance the book [2] and the references quoted therein). This is one of the reasons that the study of non-smooth dynamical systems has attracted many mathematicians. And of course in these problems the detection of limit cycles is of fundamental importance.

Thus we have been motivated by the Liénard equations and by importance of the non-smooth systems to study the limit cycles of the discontinuous classical polynomial Liénard differential equations

$$\ddot{x} + f(x)\dot{x} + x + sgn(\dot{x})g(x) = 0, \tag{2}$$

with f and g polynomials of degree n and 1, respectively.

We study the number of limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed inside the following class of discontinuous classical polynomial Liénard differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(f(x)y + sgn(y)(k_1 x + k_2)) \end{aligned} \tag{3}$$

where f is a polynomial of degree $n \in \mathbb{N}$ and k_1, $k_2 \in \mathbb{R}$.

In order to prove our main result we first study the piecewise linear classical polynomial Liénard differential systems:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(f(x)y + \varphi_w(y)(k_1 x + k_2)), \end{aligned} \tag{4}$$

where $\varphi_w : \mathbb{R} \to \mathbb{R}$ is the piecewise linear function

$$\varphi_w(y) = \begin{cases} -1, & \text{if } y < -w, \\ \dfrac{y}{w}, & \text{if } -w < y < w, \\ 1, & \text{if } y > w. \end{cases} \tag{5}$$

Observe that taking $w \to 0$ in (4) we obtain discontinuous classical polynomial Liénard differential systems (3).

The classical results for studying the periodic orbits of differential systems require that the systems involved are of class at least $\mathcal{C}^2$. In 2004, Buica and Llibre [1] extended the averaging theory for studying periodic orbits to continuous differential systems using mainly the Brouwer degree theory.

Recently Llibre, Novaes and Teixeira [10], using the theory of regularization, developed the averaging theory of first order for studying periodic orbits to discontinuous piecewise differential systems with two systems. The displacement function that we construct here is the same as in [10], but adapted to the family of differential equations we consider.

More precisely our main result are the following.

Theorem 1. *For every $n \geq 1$ and $|\varepsilon|$ sufficiently small the maximum number of limit cycles of piecewise classical polynomial Liénard differential systems* (4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is $[n/2]+1$. Moreover there are systems* (4) *having exactly $[n/2] + 1$ limit cycles.*

In order to guarantee that the limit cycles dont vanishes then $w \to 0$, that is, to obtain the number of limit cycles of discontinuous classical polynomial Liénard differential systems (3) we have the following result.

Corollary 2. *For every $n \geq 1$ and $|\varepsilon|$ sufficiently small the maximum number of limit cycles of discontinuous classical polynomial Liénard differential systems* (3) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is $[n/2] + 1$. Moreover there are systems* (3) *having exactly $[n/2] + 1$ limit cycles.*

Comparing the mentioned result from [9], that smooth generalized Lienard polynomial differential systems have at least $[(n + m - 1)/2]$ limit cycles with Corollary 2, we can say that the non-smooth classical Lienard polynomial differential systems can have at least one more limit cycle than the smooth ones.

The proof of Theorem 1 is based on the first-order averaging method. In section 2 we will present this method in the form obtained in [1] where differentiability of the vector field is not needed. Theorem 1 and Corollary 2 are proved in sections 3 and 4 respectively.

2. The first-order averaging theory

The first-order averaging theory developed in [1] is presented in this section. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{6}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $F_{10} : D \to \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

and we assume that the following hypotheses (i) and (ii) hold.

(i) F_1 and R are locally Lipschitz with respect to x;

(ii) $F_{10}(0) = 0$ and there exists a neighborhood V of 0 such that $F_{10}(z) \neq 0$ for all $z \in \bar{V} \backslash \{0\}$ and $d_B(F_{10}, V, 0) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\psi(\cdot, \varepsilon)$ of system (6) such that $\psi(0,\varepsilon) \to 0$ as $\varepsilon \to 0$.

The expression $d_B(F_{10}, V, 0) \neq 0$ means that the Brouwer degree of the function $F_{10} : V \to \mathbb{R}^n$ at the fixed point 0 is not zero.

3. Proof of Theorem 1

We shall need the first-order averaging theory to prove Theorem 1.

In order to apply the first-order averaging method we write system (4) in polar coordinates (r,θ) where $x = r\cos\theta$, $y = r\sin\theta, r > 0$. In this way system (4) is written in the standard form for applying the averaging theory. If we write $f(x) = \sum_{i=0}^{n} a_i x^i$ then system (4) becomes

$$\begin{aligned}
\dot{r} &= \varepsilon \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^i\theta \sin^2\theta + \varphi_w(r\sin\theta)(k_1 r\cos\theta + k_2)\sin\theta \right), \\
\dot{\theta} &= -1 + \frac{\varepsilon}{r} \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^{i+1}\theta \sin\theta + \varphi_w(r\sin\theta)(k_1 r\cos\theta + k_2)\cos\theta \right).
\end{aligned} \tag{7}$$

Now taking θ as the new independent variable system (7) becomes

$$\frac{dr}{d\theta} = \varepsilon \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^i \theta \sin^2 \theta + \varphi_w(r \sin\theta)(k_1 r \cos\theta \sin\theta + k_2 \sin\theta) \right) + O(\varepsilon^2),$$

where

$$\varphi_w(r\sin\theta) = \begin{cases} -1, & \text{if } \sin\theta < -\frac{w}{r}, \\ \dfrac{r\sin\theta}{w}, & \text{if } -\frac{w}{r} < \sin\theta < \frac{w}{r}, \\ 1, & \text{if } \sin\theta > \frac{w}{r}, \end{cases} \tag{8}$$

and

$$F_{10}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^i \theta \sin^2 \theta + \varphi_w(r \sin\theta)(k_1 r \cos\theta \sin\theta + k_2 \sin\theta) \right) d\theta.$$

We denote $F_{10}(r) = \dfrac{1}{2\pi}(F_{10a}(r)+F_{10b}(r))$, where $F_{10a}(r) = \displaystyle\int_0^{2\pi} \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^i \theta \sin^2 \theta \right) d\theta$ and $F_{10b}(r) = \displaystyle\int_0^{2\pi} \left(\varphi_w(r\sin\theta)(k_1 r \cos\theta \sin\theta + k_2 \sin\theta) \right) d\theta$.

In order to calculate the exact expression of F_{10a} we use the following formulas

$$\begin{aligned} \int_0^{2\pi} \cos^{2k+1}\theta \sin^2\theta d\theta \quad &= \quad 0, \qquad k = 0, 1, ... \\ \int_0^{2\pi} \cos^{2k}\theta \sin^2\theta d\theta = \pi\alpha_k \quad &\neq \quad 0, \qquad k = 0, 1, ... \\ \int_0^{2\pi} \cos^{k}\theta \sin\theta d\theta \quad &= \quad 0, \qquad k = 0, 1, ... \end{aligned}$$

Hence

$$F_{10a}(r) = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \pi a_i \alpha_i r^{i+1}. \tag{9}$$

In order to calculate the expression of F_{10b} we define for each $r_1 > 0$ the function

$$
\begin{aligned}
I_1(r_1, w) = &\int_0^{2\pi} \left(\varphi_w(r_1 \sin\theta)(k_1 r \cos\theta \sin\theta + k_2 \sin\theta)\right) d\theta \\
= &\begin{cases} \pi k_2 \dfrac{r_1}{w} & 0 < r_1 \leq w, \\ \\ 2k_2 \left(\dfrac{r_1}{w} \operatorname{arccsc}\left(\dfrac{r_1}{w}\right) + \dfrac{\sqrt{r_1^2 - w^2}}{r_1} \right) & r_1 \geq w \end{cases}
\end{aligned} \tag{10}
$$

Thus the averaged function F_{10} is given by

$$
F_{10}(r_1) = \sum_{\substack{i=0 \\ i \text{ even}}}^{n} \pi a_i \alpha_i r_1^{i+1} + I_1(r_1, w)
$$

We have to find the zeroes of equation $F_{10}(r_1) = 0$. We shall divide our study in two cases.

At really, we are interested just in the zeroes for $r_1 > w$, but we shall also consider the case $0 < r_1 \leq w$ for completion.

If we write

$$\begin{aligned}
F_{10}^{I}(r_1) &= \frac{1}{2\pi}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n} \pi a_i \alpha_i r_1^{i+1} + \pi k_2 \frac{r_1}{w}\right)\\
&= \frac{r_1}{2}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n} a_i \alpha_i r_1^{i} + \frac{k_2}{w}\right)\\
&= \frac{r_1}{2}\left(\sum_{\substack{i=2\\ i \text{ even}}}^{n} a_i \alpha_i r_1^{i} + \left(a_0\alpha_0 + \frac{k_2}{w}\right)\right)
\end{aligned}$$

and

$$F_{10}^{II}(r_1) = \frac{1}{2\pi}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n} \pi a_i \alpha_i r_1^{i+1} + 2k_2\left(\frac{r_1}{w}\operatorname{arccsc}\left(\frac{r_1}{w}\right) + \frac{\sqrt{r_1^2 - w^2}}{r_1}\right)\right),$$

then

$$F_{10}(r_1) = \begin{cases} F_{10}^{I}(r_1) & , \quad r_1 < w, \\ F_{10}^{II}(r_1) & , \quad r_1 \geq w. \end{cases}$$

From now we take w small so that there are no zeros of F_{10} in the interval $(0, w)$ – as F_{10}^{I} is a polynomial and $r_1 = 0$ is a root, we can assure that there is such interval.

Now we study the existence of zeros for $r_1 > w$. Denote

$$I(r_1, w) = 2\left(\frac{r_1}{w}\operatorname{arccsc}\left(\frac{r_1}{w}\right) + \frac{\sqrt{r_1^2 - w^2}}{r_1}\right).$$

We have:

(i) For each w fixed $\dfrac{\partial^2 I(r_1,w)}{\partial r_1^2} = -\dfrac{4w^2}{r^3\sqrt{r^2-w^2}} < 0$ so the graph of $I(.,w)$ is convex.

(ii) For each w fixed $\dfrac{\partial I(r_1,w)}{\partial r_1} = \dfrac{2}{w}\operatorname{arccsc}\left(\dfrac{r_1}{w}\right) - \dfrac{2\sqrt{r_1^2-w^2}}{r_1^2}$ and,

$$\text{(iia)} \lim_{r_1\to w} \frac{\partial I}{\partial r_1}(r_1,w) = \frac{\pi}{w}, \text{ and}$$

$$\text{(iib)} \lim_{r_1\to\infty} \frac{\partial I}{\partial r_1}(r_1,w) = 0.$$

By (i) we have that $\dfrac{\partial I}{\partial r_1}$ is decreasing. Then by (i), (iia) and (iib) we obtain that $\dfrac{\partial I}{\partial r_1}(r_1,w) > 0$, so the graph of $I(.,w)$ is strictly increasing. Moreover as $\dfrac{\partial I}{\partial r_1}(r_1,w) < \dfrac{\pi}{w}$ it follows that the graph of $I(.,w)$ is below of the straight line $\dfrac{\pi}{w}r_1$.

(iii) $\lim_{r_1\to\infty} I(r_1,w) = 4$, $I(.,w) : (0,\infty) \to (0,4)$ and I is a C^1-diffeomorphism.

Thus the averaging function F_{10} is C^1.

Now we need solve

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{n} \pi a_i \alpha_i r_1^{i+1} + k_2 I(r_1,w) = 0. \tag{11}$$

For simplification, we denote $\alpha_i := \dfrac{\pi\alpha_i}{k_2}$.

Note that, if n is odd, then (11) writes as

$$-r_1\left(a_0\alpha_0 + a_2\alpha_2 r_1^2 + \ldots + a_{n-3}\alpha_{n-3}r_1^{n-3} + a_{n-1}\alpha_{n-1}r_1^{n-1}\right) = I(r_1,w), \tag{12}$$

while if n is even, then (11) writes as

$$-r_1\left(a_0\alpha_0 + a_2\alpha_2 r_1^2 + \ldots + a_{n-2}\alpha_{n-2}r_1^{n-2} + a_n\alpha_n r_1^{n}\right) = I(r_1,w). \tag{13}$$

The left hand sides of equations (12) and (13) are polynomials of odd degree, with zero as a root. Both systems have the same number of solutions. From now we consider n even, so we will prove the existence of $\frac{n}{2}+1$ limit cycles. The greatest integer function $[n/2]+1$ is needed just to deal with the case n odd, and the adaptation of the proof is straightforward.

Denote

$$\mathcal{P}(r_1) = -r_1\mathcal{P}_0(r_1), \tag{14}$$

where $\mathcal{P}_0(r_1) = a_0\alpha_0 + a_2\alpha_2 r_1^2 + \ldots + a_{n-2}\alpha_{n-2}r_1^{n-2} + a_n\alpha_n r_1^n$.

Note that:

i) $\lim_{r_1\to w^+} I(r_1,w) = \pi$ and $\lim_{r_1\to\infty} I(r_1,w) = 4$;

ii) $\mathcal{P}(0) = 0$ and the polynomial $\mathcal{P}$ has at most $n/2$ positive roots (the nonzero roots are symmetric);

iii) $\mathcal{P}'(r_1)$ is a polynomial of degree n, and its zeros are also symmetric, so there are at most $n/2$ positive critical points (maxima or minima).

So if $\mathcal{P}'(0) > 0$ then equation (13) has at most $n/2+1$ solutions, and if $\mathcal{P}'(0) < 0$, then equation (13) has at most $n/2$ solutions (see Figure 1). We note that condition (iii) shows that there are no more than two solutions for equation (13) between two zeroes of $\mathcal{P}$.

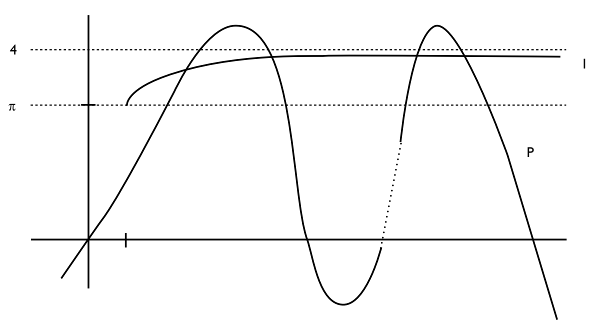

Figure 1: Representation of the case $\mathcal{P}'(0) > 0$.

Now we present, for each n, a system similar to (4) with $\frac{n}{2}+1$ limit cycles. It is suffice to show the existence of a polynomial $\mathcal{P}$, given by (14), that satisfy equation (13) for $\frac{n}{2}+1$ values of r_1.

Recall again that $\lim_{r_1\to w^+} I(r_1,w)=\pi$ and $\lim_{r_1\to\infty} I(r_1,w)=4$.

Fix n even. Consider the polynomial

$$\mathcal{P}(r_1)=\lambda r_1\prod_{\zeta=1}^{n/2}(r_1-\zeta)(r_1+\zeta). \tag{15}$$

This polynomial is in the format of (14). Now we take λ such that $\mathcal{P}(1/2)>4$ and $\mathcal{P}(1/20)<3$.

We note that $\lambda>0$ if n is a multiple of 4 and $\lambda<0$ otherwise (recall that n is even). In special, when $n=4n_1$ for some integer n_1, we can take any

$$\lambda\in\left(\frac{6}{\prod_{\xi=1}^{\frac{n}{2}}\left(\frac{1}{4}-\xi^2\right)},\frac{80}{\prod_{\xi=1}^{\frac{n}{2}}\left(\frac{1}{400}-\xi^2\right)}\right);$$

otherwise, if $n=4n_2+2$ for some n_2, take

$$\lambda\in\left(-\frac{80}{\prod_{\xi=1}^{\frac{n}{2}}\left(\frac{1}{400}-\xi^2\right)},-\frac{6}{\prod_{\xi=1}^{\frac{n}{2}}\left(\frac{1}{4}-\xi^2\right)}\right).$$

With λ choose in this way, we have $\mathcal{P}(1/2)>4$ and $\mathcal{P}(1/20)<3$. So we have a crossing point between w and $1/2$. As $\mathcal{P}(1)=0$ and $I(1,w)>\pi$, we shall have another crossing point between $1/2$ and 1.

We shall proof that with these hypotheses, the graph of $\mathcal{P}$ intercepts the graph of I in exactly $\frac{n}{2}+1$ points, for small values of $w<<\frac{1}{2}$.

Now we show that $\mathcal{P}\left(2k+\frac{1}{2}\right)>4$, for every k. This will prove that we have two crossing points in every interval $[2,3]$, $[4,5]$, …, $\left[\frac{n}{2}-2,\frac{n}{2}-1\right]$ and one crossing point after $\frac{n}{2}-1$. Together with the two intercepts in $[0,1]$, we got $\frac{n}{2}+1$ points.

As we have shown in the first part of the proof that there are no more than $\frac{n}{2}+1$ points of intersection, our result is proved.

4. Proof of Corollary 2

Let $\Sigma = \{y = 0\}$ be a section for the flow of (3) and (4). Define $P_0 : \Sigma \to \Sigma$ e $P_w : \Sigma \to \Sigma$ the first map associated to (3) and (4) respectively.

Note that both maps P_0 and P_w are analytic for $w > 0$, because:

i) P_0 is a composition of two analytic functions: the Poincaré maps of (3) for $y > 0$ and $y < 0$, considering the cross section $y = 0$, and

ii) P_w is a compositions of four analytic functions, the Poincaré maps of (4), with respect to the cross sections $y = \pm\omega$.

Moreover $\lim_{w\to 0} P_w = P_0$ (see [12]).

Now, from Theorem 1 we have that:

Case 1) If $r < w$, then we can discard P_w, as $r \to 0$ when $w \to 0$, and all limit cycles for $r < w$ disappears.

Case 2) If $r > w$ for each $w > 0$, P_w has at most $[n/2] + 1$ fixed point $\bar{y}^i_w \in \Sigma$, $i = 1, ..., [n/2] + 1$, with $\bar{y}^i_w \neq 0$. For each $\bar{y}^i_w$ there exists a $\bar{r}^i_{1,w} > w$ that satisfy (11). These points are the points of intersection between the graph of the function $I_1(, w)$ and the curve $h(r_1)$. We will show that these points are stable.

Now note that for each $r_1 > w$ we have

(i) $\dfrac{\partial I_1}{\partial w} = \dfrac{2}{w^2}\left(\dfrac{w\sqrt{r^2 - w^2}}{r} - r\text{arccsc}\left(\dfrac{r}{w}\right)\right)$

(ii) $\lim\limits_{r\to w} \dfrac{\partial I_1}{\partial w}(r_1, w) = -\dfrac{\pi}{w}$

(iii) $\lim\limits_{r\to \infty} \dfrac{\partial I_1}{\partial w}(r_1, w) = 0$

(iv) $\dfrac{\partial^2 I_1}{\partial r \partial w} = \dfrac{2}{w^2}\left(\dfrac{w(w^2 + r^2)}{r^2\sqrt{r^2 - w^2}} - \text{arccsc}\left(\dfrac{r}{w}\right)\right)$

(v) If $f_1(r, w) = \dfrac{w(w^2 + r^2)}{r^2\sqrt{r^2 - w^2}} - \text{arccsc}\left(\dfrac{r}{w}\right)$ then $\lim\limits_{r\to w} f_1(r, w) = +\infty$ and $\lim\limits_{r\to w} f_1(r, w) = 0$. Moreover $\dfrac{\partial f_1}{\partial r}(r, w) = -\dfrac{2w^3\sqrt{r^2 - w^2}(2r^2 - w^2)}{r(r^3 - rw^2)^2} < 0$ then we can conclude $\dfrac{\partial^2 I_1}{\partial r \partial w} = \dfrac{2}{w^2}\left(\dfrac{w(w^2 + r^2)}{r^2\sqrt{r^2 - w^2}} - \text{arccsc}\left(\dfrac{r}{w}\right)\right) > 0.$

By (i) to (v) we obtain $\dfrac{\partial I_1}{\partial w}(r,w) < 0$. So for r_1 fixed with $r_1 > w$ we have $I_1(r_1,.)$ decreasing with w. This implies that if $w_1 > w_2$ then $\bar{r}^i_{1,w_1} < \bar{r}^i_{1,w_2}$. Moreover an upper bounded of $\bar{r}^i_{1,w}$ for $i = 1, ..., [n/2] + 1$ is 4. This implies that $\lim_{w\to 0} \bar{r}^i_{1,w} = \bar{r}^i_1$ exists and $0 < w < \bar{r}^i_1 \leq 4$, for each $i = 1, ..., [n/2] + 1$.

From those arguments it follows that the fixed point $\bar{r}^i_{1,w}$, $i = 1, ..., [n/2]+1$ of the averaging equation associated to the fixed point $\bar{y}^i_w$ of P_w have a non-zero limit when $w \to 0$. This implies that $\bar{y}^i_w \to \bar{y}^i$ and $\bar{y}^i \neq 0$, for each $i = 1, ..., [n/2] + 1$. Now as $\lim\limits_{w\to 0} P_w(\bar{y}^i_w) = P_0(\bar{y}^i)$ we obtain $P_0(\bar{y}^i) = \bar{y}^i$. This concludes the proof of Corollary 2.

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