DENJOY-CARLEMAN CLASSES: BOUNDARY VALUES, APPROXIMATE SOLUTIONS AND APPLICATIONS

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Abstract. This paper investigates and extends results proved in [PV] to solutions of systems of complex vector fields. A key ingredient is the existence of approximate solutions for such systems. The wedge-of-the-edge theory in such structures and microlocal analysis of solutions of non-linear system of vector fields are presented as applications.

1. Introduction

In this paper we prove the existence, in the sense of the ultradistributions, of boundary values of continuous functions that are solutions of an M-locally integrable structure —a new concept that we introduce. The main ingredient is the existence of approximate solutions in the Denjoy-Carleman classes, proved in Section 2.

It is well known that tempered growth of holomorphic functions completely characterizes the existence of boundary values, in the sense of distributions [H, Theorems 3.1.11 and 3.1.14] in one variable. In several complex variables, one considers the more general situation of holomorphic functions defined on wedges and studies their boundary values at the edges —in this situation, results are well know [BER, Ch. VII]. Holomorphic functions are solutions of a (system of) complex vector field(s) and it is natural to study vector fields L for which the solutions of the homogeneous (system of) equation(s) $Lf = 0$ show similar behavior. Recently, S. Berhanu and J. Hounie, gave necessary and sufficient conditions for homogeneous solutions, $Lf = 0$ (L an analytic vector field), to possess boundary values in the sense of distributions based on the tempered growth of $f(x,t)$, when $t \to 0$. Later, they also considered solutions for systems of (smooth) complex vector fields that are also locally integrable, see [BH1, BH2, BH3, BH4] for more on this subject.

On the other hand, if one considers faster growth for f, Komatsu [K], gave a necessary and sufficient condition for an ultradistribution, defined in $\mathbb{R}^n$, to be

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the boundary value, in the sense of ultradistributions, of a "sum" of holomorphic functions defined in wedges.

After that, H.-J. Petzsche and D. Vogt, [PV], characterized ultradifferentiable functions by their almost analytic extensions. They used these extensions in order to show how growth properties of a holomorphic function $f(x+iy)$ determine the classes of ultradistributions which contain the boundary value of f. Their results are simpler and more general than Komatsu's results, [K].

We shall generalize these results to solutions of systems of C^M (see Section 2) complex vector fields.

For the existence of approximate solutions, suppose that we are given a complex bundle $\mathcal{V}$ that is involutive and locally generated by C^M complex vector fields L_j, $1 \le j \le n$, i.e.,

$$\mathcal{V} = \operatorname{span}_{\mathbb{C}}\{L_j : 1 \le j \le n\},$$

defined in a neighborhood of a point, say the origin for simplicity, in $\mathbb{R}^{m+n} = \mathbb{R}^m_x \times \mathbb{R}^n_t$ (we refer to this structure as M–involutive structure). One can assume that L_j is given by

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_j^k\,(x,t)\,\frac{\partial}{\partial x_k},$$

where the coefficients $a_j^k\,(x,t)$ are in the class C^M. Let $f = f\,(x)$ be a function in the class C^M. Can we (explicitly) construct a C^M function $u = u\,(x,t)$ in a neighborhood of $\mathbb{R}^m$ in $\mathbb{R}^{m+n}$ which is an *approximate solution* of $\mathcal{V}u = 0$ (in the sense of (2.2)) and such that $u\,(x,0) = f\,(x)$?

Recently we showed how to construct C^M-approximate solutions for a single complex vector field with C^M coefficients, see [AH3]. For similar results for the Gevrey class (a particular case), see [AH2, BP1, BP2].

The paper is organized as follows: in Section 2, we prove the existence of approximate solutions. Section 3 is devoted to prove results about existence of boundary values for solutions in different situations. The edge-of-the-wedge theory for structures considered here is given in Section 4. Microlocal results, regarding solutions of non-linear systems, similar to those appearing in [As, BP2, B, HT] are given in Section 5. Finally, an appendix on the standard conditions that we assume on the class of ultradifferentiable functions is included.

2. Existence of approximate solutions in Denjoy-Carleman classes

Given a sequence $M = (M_j)$ of nonnegative numbers and an open set $U \subset \mathbb{R}^m$, we define the non-quasi analytic Denjoy-Carleman class $C^M(U)$ as follows:

Definition 2.1. *$C^M(U)$ is the space of all C^∞ functions $f = f(x)$ defined in U with the property that for any compact subset $K \subset\subset U$, there exists a constant $C > 0$ such that*

$$|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1} M_{|\alpha|}, \quad \forall x \in K, \quad \forall \alpha \in \mathbb{Z}_+^m.$$

*$C^M(U)$ is called the space of C^M-**ultradifferentiable functions** in U.*

The choice $M_j = (j!)^s$ gives the standard Gevrey class $G^s(U)$. Under suitable assumptions on the sequence $M = (M_j)$ one obtains for $C^M(U)$ results similar to those valid for $G^s(U)$.

In the appendix we recall some results and definitions on Denjoy-Carleman classes and some conditions on the sequence M that will be used throughout this paper.

From now on the sequence M will be assumed to satisfy conditions (A.1), (A.2), (A.3), and (A.8).

We are now ready to state and prove our theorem regarding the existence of C^M approximate first integrals in C^M involutive structures of arbitrary rank, generalizing previous results in [AH2, AH3, BP1, BP2]. Although our theorem can be stated in a more general setup (see [AH3, BP2]), we restrict ourselves to structures $\mathcal{V}$ which are locally generated by

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_j^k(x,t)\frac{\partial}{\partial x_k}, \quad j = 1, 2, \ldots, n. \tag{2.1}$$

Theorem 2.1. *Let $\mathcal{V} = \{L_j\}_{1\leq j\leq n}$ be an involutive system of C^M complex vector fields defined in a neighborhood $\Omega = U \times V \subset \mathbb{R}_x^m \times \mathbb{R}_t^n$ of the origin. Let $f(x) \in C^M(U)$. There exists a function $Z(x,t) \in C^M(\Omega)$ which is an approximate solution of $\mathcal{V}Z = 0$ in the sense*

$$|L_j Z(x,t)| \leq C^{p+1}\frac{|t|^p M_p}{p!}, \quad \forall p \in \mathbb{N}, \quad \forall j = 1, \ldots, n. \tag{2.2}$$

and such that $Z(x,0) = f(x)$.

PROOF: Observe that the involution condition implies that the vector fields $L_1, \dots, L_n$ commute pairwise, that is, for all $i, j = 1, \dots, n$,

$$\sum_{k=1}^{m} \partial_{t_j} a_i^k \partial_{x_k} + \sum_{k,l=1}^{m} a_j^k (\partial_{x_k} a_i^l) \partial_{x_l} = \sum_{k=1}^{m} \partial_{t_i} a_j^k \partial_{x_l} + \sum_{k,l=1}^{m} a_i^l (\partial_{x_l} a_j^k) \partial_{x_k}. \tag{2.3}$$

The conditions that Z has to satisfy determine the Taylor coefficients of the formal power series

$$Z(x,t) = \sum_{\beta} u_\beta(x) t^\beta$$

where the equal sign means that the formal series of Z is equal to the right hand side and

$$u_\beta(x) = \frac{\partial_t^\beta Z}{\beta!}(x,0).$$

Set

$$u_0(x) = f(x).$$

For each β since $L_j Z(x,t) = O\left(t^{|\beta|+1}\right)$, $j = 1, 2, \dots, n$, we must have

$$\partial_t^\beta \left(L_j Z\right)(x,0) = 0, \quad j = 1, 2, \dots, n. \tag{2.4}$$

This leads to the following formula valid for all $\beta \neq 0$

$$u_\beta(x) = -\frac{1}{A_\beta} \sum_{\substack{l:\beta_l \neq 0 \\ \nu+\mu=\beta-e_l}} \frac{1}{\beta_l} \frac{1}{\nu!} \left(\sum_{k=1}^{m} (\partial_t^\nu a_l^k)(x,0)(\partial_{x_k} u_\mu)(x) \right) \tag{2.5}$$

where

$$A_\beta = \#\{l : \beta_l \neq 0\}, \qquad \beta = (\beta_1, \beta_2, \dots, \beta_n).$$

We will prove formula (2.5) by induction. For $\beta = e_j$, $1 \leq j \leq n$ we have, in view of (2.4),

$$u_\beta(x) = u_{e_j}(x) = \frac{\partial_{t_j} Z}{1!}(x,0) = -\sum_{k=1}^{m} a_j^k(x,0) \frac{\partial Z}{\partial x_k}(x,0).$$

Suppose now that formula (2.5) is valid for all multiindices α with $|\alpha| \leq N$, $N > 1$. If β is a multiindex with $|\beta| = N+1$, we can write $\beta = \alpha + e_j$ for some

$1 \leq j \leq n$ and $|\alpha| = N$, we have

$$\begin{aligned} u_\beta(x) &= \frac{\partial_t^\beta Z}{\beta!}(x,0) = \frac{1}{\beta_j}\frac{1}{(\beta-e_j)!}(\partial_t^\alpha \partial_{t_j} Z)(x,0) \\ &= \frac{1}{\beta_j}\frac{1}{\alpha!}\,\partial_t^\alpha\left(L_j Z - \sum_{k=1}^m a_j^k \partial_{x_k} Z\right)(x,0) \\ &= \frac{1}{\beta_j}\frac{1}{\alpha!}\,\partial_t^\alpha\left(-\sum_{k=1}^m a_j^k \partial_{x_k} Z\right)(x,0) \\ &= -\frac{1}{\beta_j}\frac{1}{\alpha!}\sum_{k=1}^m \sum_{\nu+\mu=\alpha} \frac{\alpha!}{\nu!\mu!}(\partial_t^\nu a_j^k)(x,0)(\partial_{x_k}\partial_t^\mu Z)(x,0) \\ &= -\frac{1}{\beta_j}\sum_{k=1}^m \sum_{\nu+\mu=\alpha} \frac{1}{\nu!}(\partial_t^\nu a_j^k)(x,0)(\partial_{x_k} u_\mu)(x). \end{aligned}$$

Note that in the second equality we could have written $\beta = \alpha + e_j$ in exactly A_β different ways. The formula follows by taking the mean among all different ways.

Given $K \subset\subset U$, we claim that there exist constants $B, D > 0$ such that

$$|\partial_x^\alpha u_\beta(x)| \leq \frac{B^{|\beta|}}{\beta!} D^{|\alpha|+1} M_{|\alpha|+|\beta|}, \quad \forall x \in K, \quad \alpha, \beta \in \mathbb{Z}_+^m. \tag{2.6}$$

In fact, since $f \in C^M(U)$ and $a_j^k \in C^M(\Omega)$, there exists a constant $A > 1$ such that for all $n \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^m$, $x \in K$, and $k = 1, .., m$, we have

$$|\partial_x^\alpha f(x)| \leq A^{|\alpha|+1} M_{|\alpha|} \quad \text{and} \quad \left|\partial_x^\alpha \partial_t^\nu a_j^k(x,0)\right| \leq A^{|\alpha|+|\nu|+1} M_{|\alpha|+|\nu|}. \tag{2.7}$$

We now choose $L, G > 1$ such that Lemma 4.2 in [BP1] holds (where the constant $C > 1$ is the same as in condition (A.4) and we define $B = mAL$ and $D = AG$).

We will prove (2.6) using induction on $|\beta|$. The case $|\beta| = 0$ is trivial. Suppose now that (2.6) holds for any multiindex with length $\leq n-1$, $n \geq 1$. If $|\beta| = n$, we have

$$|\partial_x^\alpha u_\beta(x)| \leq \frac{1}{\beta_j}\sum_{k=1}^m \sum_{\mu+\nu=\beta-e_j} \frac{1}{\nu!}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}|\partial_x^{\gamma+e_k} u_\mu(x)|\,|\partial_x^{\alpha-\gamma}\partial_t^\nu a_j^k(x,0)| \tag{2.8}$$

where $\{e_k\}_{k=1}^m$ is the standard basis of $\mathbb{R}^m$. It follows from (2.7), [BP1, Lemma 4.2], and our induction hypothesis that

$$\begin{aligned}
&\left|\partial_x^\alpha u_\beta\left(x\right)\right| \leq \\
\leq\ & \frac{C}{\beta_j}\sum_{k=1}^{m}\sum_{\mu+\nu=\beta-e_j}\frac{1}{\nu!}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{B^{|\mu|}}{\mu!}D^{|\gamma|+2}M_{|\mu|+|\gamma|+1}A^{|\alpha|-|\gamma|+|\nu|+1}M_{|\alpha|-|\gamma|+|\nu|} \\
\leq\ & \frac{C}{\beta!}M_{|\alpha|+|\beta|}\sum_{k=1}^{m}\sum_{\mu+\nu=\beta-e_j}\sum_{\gamma\leq\alpha}B^{|\mu|}D^{|\gamma|+2}A^{|\alpha|-|\gamma|+|\nu|+1} \\
\leq\ & \frac{Cm}{\beta!}B^{|\beta|}D^{|\alpha|+1}M_{|\alpha|+|\beta|}\sum_{\mu+\nu=\beta-e_j}\sum_{\gamma\leq\alpha}\frac{1}{B^{|\nu|+1}}D^{|\gamma|-|\alpha|+1}A^{|\alpha|-|\gamma|+|\nu|+1} \\
\leq\ & \frac{Cm}{\beta!}B^{|\beta|}D^{|\alpha|+1}M_{|\alpha|+|\beta|}\sum_{\mu+\nu=\beta-e_j}\sum_{\gamma\leq\alpha}\left(\frac{1}{mAL}\right)^{|\nu|+1}(AG)^{|\gamma|-|\alpha|+1}A^{|\alpha|-|\gamma|+|\nu|+1} \\
\leq\ & \frac{Cm}{\beta!}B^{|\beta|}D^{|\alpha|+1}M_{|\alpha|+|\beta|}\ \frac{A}{mL}\sum_{q=0}^{\infty}\left(\frac{1}{L}\right)^{q}\sum_{\gamma\leq\alpha}G^{|\gamma|-|\alpha|+1} \\
=\ & \frac{B^{|\beta|}}{\beta!}D^{|\alpha|+1}M_{|\alpha|+|\beta|}\ \frac{AC}{L-1}\sum_{\gamma\leq\alpha}G^{|\gamma|-|\alpha|+1} \\
\leq\ & \frac{B^{|\beta|}}{\beta!}D^{|\alpha|+1}M_{|\alpha|+|\beta|},
\end{aligned}$$

where we have used the estimate

$$\begin{aligned}
\binom{\alpha}{\gamma}\frac{M_{|\gamma|+|\mu|+1}M_{|\alpha|-|\gamma|+|\nu|}}{\mu!\nu!} &\leq \binom{|\alpha|}{|\gamma|}\binom{|\beta|}{|\mu|+1}\frac{M_{|\gamma|+|\mu|+1}M_{|\alpha|-|\gamma|+|\nu|}}{(\beta-e_j)!} \\
&\leq \binom{|\alpha|+|\beta|}{|\gamma|+|\mu|+1}\frac{M_{|\gamma|+|\mu|+1}M_{|\alpha|-|\gamma|+|\nu|}}{(\beta-e_j)!} \\
&\leq \frac{M_{|\alpha|+|\beta|}}{(\beta-e_j)!}.
\end{aligned}$$

that follows from property (A.6). This completes the proof of the claim.

It follows from [AH3, Lemma 15] that, shrinking U, there exists $Z \in C^M(\Omega)$ such that $\partial_t^\beta Z(x,0) = \beta! u_\beta(x)\ \ \forall x \in U$, and so $u_\beta(x) = \frac{1}{\beta!}\partial_t^\beta Z(x,0)\ \ \forall x \in U$. In particular, we have $Z(x,0) = u_0(x) = f(x)$. It is now easy to see that Z is our desired $M-$approximate solution of L.

It remains to show estimate (2.2). To do so, we will actually prove a more general statement. We claim that for $K \subset\subset \Omega$, there is a constant $E > 0$ independent of α and p such that, for all $(x,t) \in K$, we have

$$\left|\partial_x^\alpha L_j Z(x,t)\right| \leq C^{|\alpha|+p+1}\frac{|t|^p M_{|\alpha|+p}}{p!}, \quad \forall p \in \mathbb{N}, \quad \forall j = 1, \ldots, n. \tag{2.9}$$

In fact, from the way we defined Z it follows from Taylor's theorem that for some $t_0 \neq 0$, $t_0 = t_0\,(x, t, N)$ that

$$|\partial_x^\alpha L_j Z(x,t)| = \left|\partial_t^\beta \partial_x^\alpha L_j Z(x,t_0)\right| \frac{|t|^{|\beta|}}{\beta!}.$$

Using that $L_j Z \in C^M$ we obtain (2.9). □

3. Existence of traces in the sense of ultradifferentiable functions

In 2003, S. Berhanu and J. Hounie proved the existence of boundary values for $f \in C^1(U_+)$, $U_+ = X \times (0,1)$, $X \subset \mathbb{R}^m$, under the following hypotheses:

(1) $Lf \in L^1(U_+)$;

(2) for any compact set $K \subset X$ there exist an $N = N(K)$ and $C = C(K) > 0$ such that

$$\int_K |f(x,t)|dx < \frac{C}{t^N}. \tag{3.1}$$

where L is a smooth vector field in $\mathbb{R}^m \times [0,1]$. The inequality (3.1) is known as tempered growth. Later, assuming local integrability of the vector fields, they weakened the hypotheses to the following: $f \in C^0(U_+)$, $X = (-A, A) \subset \mathbb{R}$ and

(1) $Lf \in L^1(U_+)$;

(2) There exist an N such that

$$\int_0^B \int_{-A}^A |f(x,t)||\varphi(x,t) - \varphi(x,0)|^N dxdt < \infty. \tag{3.2}$$

The condition (3.2) is called tempered growth with respect to the first integral $Z(x,t) = x + i\varphi(x,t)$. They also showed that there are solutions that do not satisfy (3.1) but do satisfy (3.2). Quite recently, J. Hounie and E. R. da Silva, [HdaS], weakened condition (3.2) and proved similar results for systems of vector fields.

Now, consider an $M-$involutive structure $(\mathcal{W}, \mathcal{V})$, where $\dim_{\mathbb{R}} \mathcal{W} = m + n$ and $\mathcal{V} \subset \mathbb{C}T\mathcal{W}$ a subbundle of rank n. The involutive structure $(\mathcal{W}, \mathcal{V})$ is called locally integrable if the orthogonal of $\mathcal{V}$ in $\mathbb{C}T^*\mathcal{W}$ is locally generated by exact forms.

In this section we will use the existence of C^M-approximate solutions to generalize the results proved in [AB, BH1, BH2, BH3, K, PV].

In some cases, we will need tempered growth with respect to the first integrals, but in other situations, we will ask for a more restricted tempered growth. Whether one can improve our theorem or not is still an open question.

3.1. **The locally integrable case.** In this subsection the involutive structure will be assumed to be locally integrable, i.e., we shall reason under the following setup: Let $x = (x_1, ..., x_m) \in \mathbb{R}^m$, $t = (t_1, ..., t_n) \in \mathbb{R}^n$, and suppose that $U \subset \mathbb{R}^{m+n}$ is an open set, $0 \in U$, and $\Phi(x,t) : U \to \mathbb{R}^m$ is a C^M function satisfying

$$\Phi(0,0) = 0 \quad \text{and} \quad \Phi_x(0,0) = 0. \tag{3.3}$$

For simplicity, suppose that $U = B_r(0) \times B_\delta(0) \subset \mathbb{R}^m \times \mathbb{R}^n$. Let

$$\begin{aligned} Z(x,t) &= x + i\Phi(x,t) \\ &= (x_1 + i\Phi_1(x,t), ..., x_m + i\Phi_m(x,t)) \\ &= (Z_1(x,t), ..., Z_m(x,t)). \end{aligned} \tag{3.4}$$

For $1 \le k \le m$, let M_k be C^M vector fields in x-space satisfying

$$M_k Z_l = \delta_{kl} \quad \text{for } 1 \le k, l \le m,$$

and consider the $M-$locally integrable structure $\mathcal{V} = \{L_1, ..., L_n\}$ generated by the vector fields

$$L_j = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} \frac{\partial Z_k}{\partial t_j}(x,t) M_k. \tag{3.5}$$

Note that $L_j Z_k = 0$ for all $1 \le j \le n$, $1 \le k \le m$. In the following theorem, we will give sufficient conditions which guarantee the existence of the boundary value, bf, of a continuous solution f of $\mathcal{V}$.

Let $M(t)$ be the associated function with respect to the sequence M (see Definition A.2), its Young conjugate $w^* : [0,\infty) \to [0,\infty]$ is defined by $w^*(r) = \sup_{t \ge 0}\{M(t) - rt\}$. Let

$$M^*(s) = -\log \inf_{p \in \mathbb{N}} \left\{ \frac{s^p M_p}{p!} \right\}. \tag{3.6}$$

It is well known that w^* and M^* are comparable in the sense that for every $H > 1$ there exists a positive constant C such that

$$M^*(Hs) - C \le w^*(s) \le M^*(s), \quad \text{for all } s > 0, \tag{3.7}$$

see Lemma 5.6 in [PV].

Remark 1. *Using the function M^* one can rewrite estimate (2.2) as*

$$|L_j Z(x,t)| \le C e^{-M^*(s|t|)}, \quad \forall j = 1, \dots, n.$$

Let $\mathcal{W} = B_r(0) \times \Gamma_\delta \subset \mathbb{R}^m_x \times \mathbb{R}^n_t$, where $\Gamma_\delta \doteq \Gamma \cap B_\delta(0)$ and Γ is an open acute convex cone in $\mathbb{R}^n_t$.

Theorem 3.1. *Suppose that* $f \in C^0(\mathcal{W})$, *satisfies:*

(1) $L_j f \in L^\infty(\mathcal{W})$, $1 \le j \le n$;

(2) *For all* $\lambda > 0$,

$$|f(x,t)| e^{-\lambda M^*(|t|/\lambda)} < \infty.$$

Then

$$bf \doteq \lim_{\Gamma \ni t \to 0} f(\cdot + it)$$

exists in $\mathcal{D}'_M(B_r(0))$.

Proof: Fix $\varphi \in \mathcal{D}^M(B_r(0))$. We want to show that

$$\lim_{\Gamma \ni t \to 0} \langle f(\cdot + it), \varphi \rangle$$

exists. For this, let $\Psi(x,t) \in C^M(B_r(0) \times (-1,1)^n)$ be the function given by Theorem 2.1 and such that $\Psi(x,0) = \varphi(x)$.

Note that for any C^1 function $g(x,t)$ defined near the origin in $\mathbb{R}^m \times \mathbb{R}^n$,

$$\mathrm{d}g(x,t) = \sum_{j=1}^n L_j g(x,t)\,\mathrm{d}t_j + \sum_{k=1}^m M_k g(x,t)\,\mathrm{d}Z_k(x,t).$$

Consider the m-form

$$\omega(x,t) = g(x,t)\,\mathrm{d}Z(x,t).$$

Then

$$\mathrm{d}\omega = \mathrm{d}\,(g dZ) = \mathrm{d}g \wedge \mathrm{d}Z = \sum_{j=1}^n L_j g\,\mathrm{d}t_j \wedge \mathrm{d}Z.$$

If

$$g(x,t) = f(x,t)\Psi(x,t),$$

we have

$$\mathrm{d}\omega = \sum_{j=1}^n f(x,t) L_j \Psi(x,t)\,\mathrm{d}t_j \wedge \mathrm{d}Z + \sum_{j=1}^n L_j f(x,t)\Psi(x,t)\,\mathrm{d}t_j \wedge \mathrm{d}Z.$$

Fix $T \in \Gamma_\delta$, $|T| < 1$ and let $\delta' = \delta - |T|$. For $s \in \Gamma_{\delta'}$, define

$$\gamma_s(\tau) = (1-\tau)\,s + \tau T.$$

We now avail ourselves of Stokes Theorem:

$$\int_{B_r(0)} \int_{\gamma_s} \mathrm{d}\omega(x,t) = \int_{B_r(0)} \omega(x,T) - \int_{B_r(0)} \omega(x,s).$$

Writing things out explicitly, we get

$$\begin{aligned}\int_{B_r(0)} f(x,s)\Psi(x,s)\,\mathrm{d}Z(x,s) &= \int_{B_r(0)} f(x,T)\Psi(x,T)\,\mathrm{d}Z(x,T)\\ &\quad -\sum_{j=1}^{n}\int_{B_r(0)}\int_{\gamma_s} L_j f(x,t)\Psi(x,t)\,\mathrm{d}t_j\wedge \mathrm{d}Z(x,t)\\ &\quad -\sum_{j=1}^{n}\int_{B_r(0)}\int_{\gamma_s} f(x,t)L_j\Psi(x,t)\,\mathrm{d}t_j\wedge \mathrm{d}Z(x,t) \qquad (3.8)\end{aligned}$$

The first integral on the RHS clearly exists. The second integral on the RHS exists, independently of s, by assumption (1) of the theorem. Now, since

$$|L_j\Psi(x,t)| \le C^{N+1}\frac{M_N}{N!}|t|^N, \quad \forall N\in\mathbb{N},\ 1\le j\le n,$$

say $C>1$, we have

$$|L_j\Psi(x,t)| \le C\inf_N C^N\frac{M_N}{N!}|t|^N = Ce^{-M^*(C|t|)} \le Ce^{-\frac{1}{C}M^*(C|t|)}, \quad 1\le j\le n.$$

By assumption (2) of the theorem, we get that the third integral on the RHS exists, independently of s, and hence

$$\lim_{\Gamma_\delta\ni s\to 0}\int_{B_r(0)} f(x,s)\Psi(x,s)\,\mathrm{d}Z(x,s)$$

exists as well. □

3.2. **The involutive case.** In [AB] the authors showed the existence of boundary values, in the smooth case, even if the structure is not locally integrable. One can show that the same result is true in the C^M case.

Here the $M-$involutive structure $(M,\mathcal{V})$ is locally generated by (2.1) in a neighborhood U of the origin in $\mathbb{R}^m_x\times\mathbb{R}^n_t$. For simplicity, say $U=B_r(0)\times B_\delta(0)$ and let $\mathcal{W}=B_r(0)\times\Gamma_\delta$ be a wedge with edge $\mathbb{R}^m_x$, where $\Gamma_\delta\subset\mathbb{R}^n_t$ is a truncated open convex cone. The following theorem generalizes Theorem 1.1 in [BH2].

Theorem 3.2. *Let $\mathcal{W}=B_r(0)\times\Gamma_\delta$ be as above and suppose that $f(x,t)\in C(\mathcal{W})$ satisfies:*

(i) *$L_jf\in L^1(\mathcal{W})$, $j=1,\dots,n$; and*
(ii) *For all $\lambda>0$, $|f(x,t)|\,e^{-\lambda M^*(|t|/\lambda)}<C_\lambda<\infty$.*

Then $bf=\lim_{\Gamma_\delta\ni t\to 0} f(\cdot,t)$ exists in $\mathcal{D}'_M(B_r(0))$.

Proof: Let $Z_1,\dots,Z_m: U\to\mathbb{C}$ be a complete set of $M-$approximate first integrals for $\mathcal{V}$ near the origin in U given by Theorem 2.1, with $Z_k(x,0)=x_k$, $1\le k\le m$.

Define

$$b_{jk}(x,t) = L_j Z_k(x,t). \tag{3.9}$$

Write

$$\begin{aligned} Z(x,t) &= (Z_1(x,t), ..., Z_m(x,t)); \text{ and} \\ Z_k(x,t) &= \Psi_{1k}(x,t) + i\Psi_{2k}(x,t). \end{aligned}$$

For $j = 1, ..., m$, let

$$M_j = \sum_{k=1}^{m} c_{jk}(x,t) \frac{\partial}{\partial x_k}$$

be vector fields in x-space satisfying

$$M_j Z_k = \delta_{jk}, \qquad [M_j, M_k] = 0. \tag{3.10}$$

Note that for each j, k,

$$[M_j, L_k] = \sum_{l=1}^{m} d_{jkl}(x,t) M_l, \tag{3.11}$$

where each $d_{jkl}(x,t)$ also satisfies (2.2). Indeed, the latter can be seen by expressing $[M_j, L_k]$ in terms of the basis $\{L_1, \ldots, L_n, M_1, \ldots, M_m\}$ and applying both sides to the $n+m$ functions $\{t_1, \ldots, t_n, Z_1, \ldots, Z_m\}$.

In particular, (3.9) and (3.10) implies

$$M_k b_{jk} = [M_k, L_j]\, Z_k = d_{kjk}(x,t). \tag{3.12}$$

Using (3.9), we obtain

$$L_j \Psi_{2k} = i L_j \Psi_{1k} - i b_{jk} \quad \text{and} \quad M_j \Psi_{2k} = i M_j \Psi_{1k} - i\delta_{jk}. \tag{3.13}$$

Now, if $g(x,t)$ is any C^1 function defined in U, observe that the differential

$$\mathrm{d}g = \sum_{k=1}^{m} M_k(g)\, \mathrm{d}Z_k + \sum_{j=1}^{n} L_j(g)\, \mathrm{d}t_j - \sum_{j=1}^{n}\sum_{k=1}^{m} M_k(g)\, b_{jk} \mathrm{d}t_j. \tag{3.14}$$

Hence, if we consider the m-form $\omega = g\, \mathrm{d}Z$, we get

$$\mathrm{d}\omega = \mathrm{d}g \wedge \mathrm{d}Z = \sum_{j=1}^{n} L_j(g)\, \mathrm{d}t_j \wedge \mathrm{d}Z - \sum_{j=1}^{n}\sum_{k=1}^{m} M_k(g)\, b_{jk} \mathrm{d}t_j \wedge \mathrm{d}Z. \tag{3.15}$$

We will first show that hypothesis (ii) in the theorem implies that $\forall \varphi \in C_0^M(B_r(0))$

$$\left| \int_{\Gamma_\delta} \langle M_k f(\cdot,t), b_{jk}(\cdot,t)\varphi \rangle \mathrm{d}t \right| \quad \leq \quad C_2, \tag{3.16}$$

where $C_2 > 0$ is a constant that depends only on $\sum_{|\alpha|\leq 1} \|D^\alpha \varphi(x)\|_{L^\infty}$.

To do this, fix $\varphi \in C_0^M(B_r(0))$, we have

$$\begin{aligned}\langle M_k f(\cdot,t), b_{jk}(\cdot,t)\varphi\rangle &= -\sum_{l=1}^{m}\langle f(\cdot,t), \frac{\partial}{\partial x_l}((c_{kl}b_{jk})(\cdot,t)\varphi)\rangle \\ &= -\sum_{l=1}^{m}\int_{B_r(0)} f(x,t)\frac{\partial}{\partial x_l}((c_{kl}b_{jk})(\cdot,t)\varphi)(x)\,\mathrm{d}x \\ &= \int_{B_r(0)} f(x,t)\varphi(x)M_k b_{jk}(x,t)\mathrm{d}x \\ &\quad -\int_{B_r(0)} f(x,t)b_{jk}(x,t)M_k\varphi(x)\mathrm{d}x \\ &\quad -\int_{B_r(0)} f(x,t)b_{jk}(x,t)\varphi(x)\sum_{l=1}^{m}\frac{\partial c_{kl}}{\partial x_l}(x,t)\mathrm{d}x.\end{aligned}$$

Since

$$|\varphi M_k b_{jk}| + |b_{jk}M_k\varphi| + \left|b_{jk}\varphi\sum_{l=1}^{m}\frac{\partial c_{kl}}{\partial x_l}\right| \le C^{N+1}\frac{M_N}{N!}|t|^N,\ \ N\in\mathbb{N}$$

where $C>0$ is a constant that depends only on $\sum_{|\alpha|\le 1}\|D^\alpha\varphi(x)\|_{L^\infty}$, one obtains that

$$|\langle M_k f(\cdot,t), b_{jk}(\cdot,t)\varphi\rangle| \le C_1\int_{B_r(0)}|f(x,t)|e^{-M^*(C|t|)}\,\mathrm{d}x$$

proving (3.16). We now proceed as in the proof of Theorem 3.1. This shows that $bf = \lim_{\Gamma_\delta\ni t\to 0} f(\cdot,t)$ exists in $\mathcal{D}'_M(B_r(0))$. $\square$

3.3. **The real analytic case.** We will start with an example. Let $R=(-1,1)\times(-1,1)$. Set

$$Z(x,y) = x + ie^{-1/y}, y>0 \quad\text{and}\quad L = \frac{\partial}{\partial y} - \frac{Z_y(x,y)}{Z_x(x,y)}\frac{\partial}{\partial x}.$$

Z is a first integral of L. If $f(x,y)=e^{i/Z(x,y)}$, then f is a solution of L in $(-1,1)\times(0,1)$. We have, for $x=0$,

$$|f(0,y)| = \left|e^{\frac{i}{Z(0,y)}}\right| = e^{e^{1/y}}.$$

Theorem 3.1 does not apply in this situation. Since $|Z(x,y)-Z(x,0)| = e^{-1/y}$, the results proved in [BH1, BH2, BH3, BH4, HdaS] cannot by applied either.

Now, our $M-$involutive structure $(M,\mathcal{V})$ will be real analytic, i.e, the subbundle $\mathcal{V}$ is locally generated by real analytic vector fields. In particular, $(M,\mathcal{V})$, is locally integrable and one can assume that the structure is locally generated by (3.5), where Z is given by (3.4), Φ is real analytic, satisfies (3.3) and $\Phi(x,0)=0$.

Theorem 3.3. *Suppose that the structure is real analytic, i.e., the vector fields in (3.5) are real analytic. Suppose $f\in C(\mathcal{W})$, satisfies:*

(1) *$L_j f \in L^\infty(\mathcal{W})$, $1 \le j \le n$;*

(2) *For all $\lambda > 0$, $|f(x,t)|e^{-\lambda M^*(|Z(x,t)-Z(x,0)|/\lambda)} < C_\lambda < \infty$.*

Then

$$bf \doteq \lim_{\Gamma \ni t \to 0} f(\cdot + it)$$

exists in $\mathcal{D}'_M(B_r(0))$.

Remark 2. *Let $M_p = (p!)^2$ be the Gevrey sequence of order 2. It is not difficult to verify that the associated function, in this case, is given by*

$$M^*(t) = t^{-1}.$$

For the example given in the beginning of this subsection, we have

$$|f(x,y)|e^{-M^*(|Z(x,y)-Z(x,0)|)} \le e^{e^{1/y}} e^{-M^*(e^{-1/y})} = 1.$$

Therefore, Theorem 3.3 applies in this situation and $bf \in \mathcal{D}'_M(-1,1)$, i.e., bf is a Gevrey ultradistribution of order 2.

For the proof of Theorem 3.3 we will need the following definition.

Definition 3.1. *Let $U \subset \mathbb{R}^m$ be an open set and $f = f(x) \in C^M(U)$. We say that a function $\tilde{f} \in C^M(U \times (-1,1)^m)$ is an M-**almost analytic extension** of f if the following is true:*

i) *$\tilde{f}(x,0) = f(x)$ for all $x \in U$; and*

ii) *For every $z = x + iy \in U \times (-1,1)^m$ and for all $N = 1, 2, \ldots$, there exists a constant $C > 0$ independent of N such that*

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}_j}(z) \right| \le C^{N+1} \frac{M_N}{N!} |y|^N. \tag{3.17}$$

We recall that every C^M function has an $M-$almost analytic extension, see Lemma 17 in [AH3].

Proof:(of Theorem 3.3). Fix $\varphi \in \mathcal{D}^M(B_r(0))$. We want to show that

$$\lim_{\Gamma \ni t \to 0} \langle f(\cdot + it), \varphi \rangle$$

exists. For this, let $\tilde{f}(x,y) \in C^M(B_r(0) \times (-1,1)^n)$ be an M-almost analytic extension of f. Define $\Psi(x,t) = \tilde{f}(x, \Phi(x,t))$ (where Φ is given by (3.4), is real analytic, satisfies (3.3) and $\Phi(x,0) = 0$). We have

a) $\Psi(x,0) = f(x)$;

b) Ψ satisfies (2.2) with t replaced by $Z(x,t)-Z(x,0)=i\Phi(x,t)$.

In fact:

$$\begin{aligned}|L_j\Psi(x,t)| &= \sum_{l=1}^{m}\frac{\partial\tilde f}{\partial x_l}(x,\Phi(x,t))L_j(x_l)+\frac{\partial\tilde f}{\partial y_l}(x,\Phi(x,t))L_j(\Phi_l(x,t))\\ &= \sum_{l=1}^{m}\frac{\partial\tilde f}{\partial x_l}(x,\Phi(x,t))L_j(x_l)+i\frac{\partial\tilde f}{\partial y_l}(x,\Phi(x,t))L_j(x_l)\\ &= 2\sum_{l=1}^{m}\frac{\partial\tilde f}{\partial \bar z_l}(x,\Phi(x,t))L_j(x_l).\end{aligned}$$

The second equality follows since $L_jZ_l(x,t)=0$. This shows that

$$|L_j\Psi(x,t)|\le C^{N+1}\frac{M_N}{N!}\left|Z(x,t)-Z(x,0)\right|^N,\quad N\in\mathbb{N},\ 1\le j\le n,$$

and the proof now follows as before. $\square$

3.4. **The CR case.** For the rest of this section, let $(M,\mathcal{V})$ be $\mathbb{R}^{m+n}=\mathbb{R}^m_x\times\mathbb{R}^n_t$ with a smooth CR structure $\mathcal{V}$ near the origin; i.e., $\mathcal{V}\cap\overline{\mathcal{V}}=\{0\}$ in a neighborhood $U=B_r(0)\times B_\delta(0)$ of the origin in $\mathbb{R}^m_x\times\mathbb{R}^n_t$. Suppose that $\mathcal{V}$ is generated in U by the complex vector fields $\{L_1,\dots,L_n\}$, where

$$L_j=\frac{\partial}{\partial t_j}+\sum_{k=1}^{m}a_{jk}(x,t)\frac{\partial}{\partial x_k}.$$

Let $Z_1,\dots,Z_m:U\to\mathbb{C}$ be a complete set of smooth approximate first integrals for $\mathcal{V}$ in U.

Theorem 3.4. *Let $\mathcal{W}=B_r(0)\times\Gamma_\delta$ be a wedge with edge $B_r(0)$, where $\Gamma\subset\mathbb{R}^n_t$ is an open cone with vertex at the origin, and suppose that $f(x,t)\in C(\mathcal{W})$ satisfies:*

(i) $\displaystyle\int_{B_r(0)}|L_jf(x,t)|\,dx<\infty$, $1\le j\le n$;

(ii) *For all* $\lambda>0$, $|f(x,t)|\,e^{-\lambda M(|Z(x,t)-Z(x,0)|/\lambda)}<\infty$.

Then $bf=\lim_{\Gamma_\delta\ni t\to 0}f(\cdot,t)$ *exists in* $\mathcal{D}'(B_r(0))$.

Proof: Follows from Theorem 3.2 and from the fact that, for each $t\in\Gamma_\delta$, one can find $\tilde\Gamma\subset\subset\Gamma$ and $\tilde r<r$ such that, if $\widetilde{\mathcal{W}}=B_{\tilde r}(0)\times\tilde\Gamma$, then

$$|t|\le\text{const.}\,|Z(x,t)-Z(x,0)|\,.$$

The latter follows from Corollary 4.1 in [AB]. $\square$

4. Edge of the Wedge Theorem

Consider an $M-$involutive structure $(M, \mathcal{V})$. A distribution f on M is called a solution if $Lf = 0$ for all smooth sections L of $\mathcal{V}$.

From now on, we will follow closely Section 5 in [AH3]. The next result extends Theorem 24 in the referred paper to C^M structures of any rank.

Theorem 4.1. *Let $(M, \mathcal{V})$ be a C^M-structure, $\dim_{\mathbb{R}} M = m + n$, rank of $\mathcal{V} = n$, $X \subset M$ a C^M-maximally real submanifold, and $\mathcal{W}$ a wedge in M with edge X. Suppose that $u \in \mathcal{D}'_M(X)$, is the boundary value of an M-approximate solution $f \in \mathcal{D}'_M(\mathcal{W})$. Then*

$$WF_M(u) \subset \left(\Gamma^T(\mathcal{W})\right)^0.$$

Where $\left(\Gamma^T(\mathcal{W})\right)^0$ denotes the polar of $\Gamma^T(\mathcal{W})$ in the cotangent space T^*X).

Proof: Since $\mathcal{W}$ is a wedge in M with edge X, in a neighborhood Ω of a point $p \in X$, there are coordinates $(x, t) = (x_1, \ldots, x_m, t_1, \ldots, t_n)$ vanishing at p so that in Ω

$$X = \{(x, 0) : |x| < r\} = B_r(0),$$

$$\mathcal{W} = X \times \Gamma \text{ for some open convex cone } \Gamma \subset \mathbb{R}^n_t.$$

Since X is maximally real,

$$\mathbb{C}TM = \mathbb{C}TX \oplus \mathcal{V}$$

and so for each $j = 1, \ldots, n$, there exists a smooth section L_j of $\mathcal{V}$ (near 0) and smooth functions $a_{jk}(x, t)$, $1 \leq j \leq n$, $1 \leq k \leq m$ such that

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk}(x, t) \frac{\partial}{\partial x_k} \quad (1 \leq j \leq n).$$

Observe that the L_j's are linearly independent over $\mathbb{C}$, and so

$$\mathcal{V} = \text{span}_{\mathbb{C}}\{L_j : 1 \leq j \leq n\}.$$

Let

$$\{Z_1(x, t), ..., Z_m(x, t)\}$$

be a complete set of approximate first integrals. We can write

$$Z(x, t) = (Z_1, \ldots, Z_m)(x, t) = x + A(x, t)t.$$

We have (see, for instance [AH2]):

$$\mathcal{V}_0^X = \{L \in \mathcal{V}_0 : \Re L \in T_0X\} = \text{span}_{\mathbb{R}}\{iL_j|_0 : 1 \leq j \leq n\}.$$

Hence,

$$\begin{aligned} \left(\Gamma_0^T(\mathcal{W})\right)^0 &= \left\{\xi \in T_0^*X\backslash\{0\} \simeq \mathbb{R}^m\backslash\{0\} : \xi \cdot v \geq 0 \text{ for all } v \in \Gamma_0^T(\mathcal{W})\right\} \\ &= \{\xi \in \mathbb{R}^m\backslash\{0\} : \xi \cdot \Im A(0,0)b \geq 0 \text{ for all } b \in \Gamma\}. \end{aligned} \tag{4.1}$$

Therefore, since $\left(\Gamma_0^T(\mathcal{W})\right)^0$ is closed in $\mathbb{R}^m\backslash\{0\}$, we obtain

$$\xi^0 \notin \left(\Gamma_0^T(\mathcal{W})\right)^0 \Leftrightarrow \exists \text{ an open convex cone } \widetilde{\Gamma} \subset\subset \Gamma : \xi^0 \cdot \Im A(0,0)\widetilde{\Gamma} < 0. \tag{4.2}$$

Using Stoke's theorem and FBI transform estimates, like in [AH3, Pag. 2284] (see also [AH2]), one can show that there are constants $A_1,\ A_2,\ A_3 > 0$ such that

$$|\mathcal{F}_{\eta u}(y,\xi)| \leq A_1 e^{-M(A_2|\xi|)}$$

for all $(y,\xi) \in V \times \mathcal{C}$ with $|\xi| \geq A_3$. Hence,

$$\left(0,\xi^0\right) \notin WF_M(u),$$

as desired. □

Corollary 4.1 (Edge-of-the-Wedge Theorem)**.** *Let $\mathcal{W}^+$ and $\mathcal{W}^-$ be wedges in Ω with edge X whose directions are opposite: $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$. If $u \in \mathcal{D}'_M(X)$ is the boundary value of an approximate solution $f^+ \in \mathcal{D}'_M(\mathcal{W}^+)$ of $\mathcal{V}$ on $\mathcal{W}^+$ and also the boundary value of an approximate solution $f^- \in \mathcal{D}'_M(\mathcal{W}^-)$ of $\mathcal{V}$ on $\mathcal{W}^-$, then*

$$WF_M(u)|_p \subset i_X^*(T^0).$$

PROOF: Follows from Theorem 4.1. □

Corollary 4.2. *If $(M,\mathcal{V})$ is an elliptic C^M-structure and we have the same hypothesis as in the previous corollary, then u is C^M in X.*

4.1. **The converse.** We will now prove a converse of Theorem 4.1. Before we do this, we recall the Paley-Wiener theorem for non-quasi analytic ultradistributions (see, for instance, [Ta, Theorem 4.1]).

Theorem 4.2. *Let M satisfy (A.1), (A.5), (A.8). Assume further that C^M does not define the analytic class. For a compact convex set $K \subset \mathbb{R}^n$, the following are equivalent:*

(i) *$\hat{f}$ is the Fourier-Laplace transform of $f \in \mathcal{E}'_M(K)$;*

(ii) *For all $L > 0$ there exists $C > 0$ such that*

$$|\hat{f}(\xi)| \leq Ce^{M(L|\xi|)}.$$

Remark 3. *Condition* (A.4) *implies that there exists $K \geq 1$ such that* $\liminf_{t\to\infty} \frac{M(Kt)}{M(t)} > 1$*, see* [PV, Lemma 5.3]*. Let $L_1 > 1$ and $L_2 > 0$ such that $M(Kt) > L_1 M(t)$ for all $t > L_2$. It is possible to show that (see* [PV, Page 34]*) exists $C > 0$ such that*

$$M(\lambda t) \leq 2\lambda L_2 M(t) + C, \qquad \forall \lambda \geq 1. \tag{4.3}$$

In the next theorem we will use the same notation as in Theorem 4.1. It is a generalization, even for Gevrey classes, of Theorem 6.1 in [AH2].

Theorem 4.3. *Let $(M, \mathcal{V})$ be a C^M-structure, $\dim_{\mathbb{R}} M = m + n$, rank of $\mathcal{V} = n$, $X \subset M$ a C^M-maximally real submanifold, and $\mathcal{W}$ a wedge in M with edge X. Suppose $u \in \mathcal{E}'_M(X)$ is such that*

$$WF_M(u) \subset \left(\Gamma^T(\mathcal{W})\right)^0.$$

Then in a slightly smaller wedge $\mathcal{W}' \subset\subset \mathcal{W}$ with edge X, there exists an approximate solution $f \in \mathcal{D}'_M(\mathcal{W}')$ of $\mathcal{V}f = 0$ such that

$$u = bf \qquad \text{on } X.$$

Proof: We take off from (4.1). For some open convex cone $\Gamma' \subset\subset \Gamma$, one can write

$$\mathcal{W}' = B_r(0) \times \Gamma'.$$

Using (4.1) and the fact that $\Gamma' \subset\subset \Gamma$, one can find an open convex cone $\mathcal{C} \subset \mathbb{R}^m \backslash \{0\}$ containing $\left(\Gamma^T_0(\mathcal{W})\right)^0$ and a constant $0 < c \leq 1$ such that

$$\xi \cdot \Im A(0,0)t \geq c\,|\xi|\,|t| \qquad \text{for all } (\xi, t) \in \mathcal{C} \times \Gamma'. \tag{4.4}$$

For $(x, t) \in \mathcal{W}'$ and $\xi \in \mathcal{C}$, define

$$\begin{aligned} Q(x,t,\xi) &= i\xi \cdot Z(x,t) \\ &= i\xi \cdot (x + \Re A(x,t)t) - \xi \cdot \Im A(x,t)t. \end{aligned}$$

From (4.4) and the fact that $\Im A(x,t)$ is of class C^1 near $(0,0)$, one obtains for some $K > 0$ and for all $(x,t) \in \mathcal{W}'$ and $\xi \in \mathcal{C}$:

$$\begin{aligned} \Re Q(x,t,\xi) &= -\xi \cdot \Im A(x,t)t \\ &\leq -\xi \cdot \Im A(0,0)t + M\,|\xi|\,|t|\,(|x| + |t|) \\ &\leq -c\,|\xi|\,|t| + K\,|\xi|\,|t|\,(|x| + |t|) \end{aligned}$$

Choosing $0 < r, \delta < \frac{c}{4K}$, we can insure that

$$\Re Q(x,t,\xi) \leq -\frac{c}{2}\,|\xi|\,|t| \qquad \text{for all } (x,t,\xi) \in B_r(0) \times \Gamma'_\delta \times \mathcal{C}. \tag{4.5}$$

Since $u \in \mathcal{E}'_M(X)$, there exists a constant $C > 0$ such that the Fourier transform, $\widehat{u}(\xi)$, satisfies (ii) from Theorem 4.2. This allows us to define for $(x,t) \in B_r(0) \times \Gamma'_\delta$ the continuous function

$$f_1(x,t) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}} e^{Q(x,t,\xi)}\, \widehat{u}(\xi)\, d\xi.$$

We claim that

(a) f_1 is an $M-$approximate solution of $\mathcal{V}$;
(b) For all $c > 0$, $|f_1(x,t)|e^{-cM(|t|/c)} < \infty$.

Assuming that the claims are true for the moment, we can use Theorem 3.2 to guarantee the existence of the boundary value $bf_1 = \lim_{\Gamma'_\delta \ni t \to 0} f_1(\cdot,t)$ in $\mathcal{D}'_M(B_r(0))$ and moreover,

$$bf_1(x) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}} e^{i\xi\cdot x}\, \widehat{u}(\xi)\, d\xi. \tag{4.6}$$

To show (a), we fix $t_0 \in \Gamma'_\delta$ and we consider a small open neighborhood of t_0 in Γ'_δ. In this small neighborhood, estimate (4.5) allows us to pass L_j under the integral sign

$$L_j f_1(x,t) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}} i(\xi \cdot L_j Z(x,t))\, e^{i\xi\cdot Z(x,t)}\, \widehat{u}(\xi)\, d\xi. \tag{4.7}$$

Since $Z(x,t)$ are $M-$approximate first integrals for $\mathcal{V}$, for each $l = 1, 2, \ldots,$ $j = 1, \ldots, n$ and a positive constant $C > 1$, we have

$$\begin{aligned} |L_j Z(x,t)| &\leq C^{2l+2} \frac{M_{2l+1}}{(2l+1)!}\, |t|^{2l+1} \\ &\leq C \left(\frac{M_l}{l!} C^l |t|^l \right)^2 |t|, \quad \text{for all } (x,t) \in B_r(0) \times B_\delta(0), \end{aligned} \tag{4.8}$$

where we have used that condition (A.4) implies $M_{2l} \leq H^{2l} M_l^2$ (see [PV, Lemma 5.3]), therefore

$$|L_j Z(x,t)| \leq C^{l+1} e^{2M^*(C|t|)} |t|, \qquad \text{for all } (x,t) \in B_r(0) \times B_\delta(0),$$

From (4.5), (4.7), (4.8), (4.3) and choosing $L = \dfrac{c}{16L_2C}$ we see that, for $l = 1, 2, \ldots,$ and $j = 1, \ldots, n$

$$\begin{aligned} |L_j f_1(x,t)| &\leq Ce^{-2M^*(C|t|)} \int_{\mathcal{C}} |t||\xi| e^{M(L|\xi|) - \frac{c}{2}|t||\xi|} \mathrm{d}\xi \\ &\leq Ce^{-2M^*(C|t|)} \int_{\mathcal{C}} e^{\frac{c}{8C} M(|\xi|) - \frac{c}{4C}|\xi|(C|t|)} \mathrm{d}\xi \\ &\leq Ce^{-2M^*(C|t|)} \int_{\mathcal{C}} e^{\frac{c}{4C}\{M(|\xi|) - |\xi|(C|t|)\}} e^{-\frac{c}{8C} M(|\xi|)} \mathrm{d}\xi \\ &\leq Ce^{-2M^*(C|t|)} \sup_{s \geq 0} e^{\frac{c}{4C}\{M(s) - s(C|t|)\}} \int_{\mathcal{C}} e^{-\frac{c}{4C} M(|\xi|)} \mathrm{d}\xi \\ &\leq Ce^{-2M^*(C|t|)} e^{M^*(C|t|)} \\ &= Ce^{-M^*(C|t|)} \quad \text{for all } (x,t) \in B_r(0) \times \Gamma'_\delta. \end{aligned}$$

The integral is finite since $\lim_{t\to\infty} M(t)/\log t = \infty$, see [Bj, 1, 1.8.14]. Hence, f_1 is an $M-$approximate solution of $\mathcal{V}$ and claim (a) is proved. Claim (b) follows analogously, i.e., for all $\lambda > 0$ there exists $C > 0$ such that

$$|f_1(x,t)| \leq Ce^{-\lambda M^*(|\xi|/\lambda)} \qquad \text{for all } (x,t) \in B_r(0) \times \Gamma'_\delta.$$

For $x \in B_r(0)$ define

$$v(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m \setminus \mathcal{C}} e^{i\xi\cdot x}\, \widehat{u}(\xi)\, d\xi. \tag{4.9}$$

Using the fact that $WF_M|_0(u) \subset \left(\Gamma_0^T(\mathcal{W})\right)^0$, compactness of $(\mathbb{R}^m \setminus \mathcal{C}) \cap \mathbb{S}^{m-1}$, and the characterization of the $M-$wavefront set by the rapid decay of the Fourier transform (see [K]), we get that $v \in C^M(B_r(0))$. By Theorem 2.1 one can find a C^M function $f_2 \in C^M(B_r(0) \times (-1,1)^n)$ such that f_2 is an M-approximate solution of $\mathcal{V}$ and $bf_2 = v$ on X. Thus, from (4.6) and (4.9) we get

$$u = bf_1 + bf_2 = bf,$$

where $f = f_1 + f_2$ is an approximate solution of $\mathcal{V}$ in the wedge $\mathcal{W}'$. This completes the proof. $\square$

5. On micro-local C^M regularity for solutions of non-linear PDE's

In this section we will state a result about microlocal C^M-regularity of solutions u of a system of nonlinear pdes of the form

$$F_j(x, u, u_x) = 0, \qquad 1 \leq j \leq n \tag{5.1}$$

where u is always assumed to be at least C^2, the $F_j(x, \zeta_0, \zeta)$ are complex-valued , C^M in an open subset $\mathcal{O}$ of $(x, \zeta_0, \zeta) \in \mathbb{R}^N \times \mathbb{C} \times \mathbb{C}^N$ and holomorphic in $(\zeta_0, \zeta) \in$

$\mathbb{C} \times \mathbb{C}^N$. The linear independence of the system is generalized by assuming that

$$\mathrm{d}_\zeta F_1 \wedge \cdots \wedge \mathrm{d}_\zeta F_n \neq 0 \quad \text{in} \quad \mathcal{O}. \tag{5.2}$$

If there is a point $p \in \mathcal{O}$ where $F_1(p) = \cdots = F_n(p) = 0$ the set

$$\Sigma = \{(x, \zeta_0, \zeta) \in \mathcal{O} : \ F_j(x, \zeta_0, \zeta) = 0, \ j = 1, \dots, n\} \tag{5.3}$$

is a smooth manifold of $\mathcal{O}$ whose intersection with each fiber $\mathbb{C}^{N+1}$ is a holomorphic submanifold of complex dimension $N + 1 - n$.

This nonlinear system can be seen as generalization of the linear case where one considers a pair $(\mathcal{M}, \mathcal{V})$ in which $\mathcal{M}$ is a manifold and $\mathcal{V}$ is a subbundle of the complexified tangent bundle $\mathbb{C}T\mathcal{M}$ which is involutive. We will follow the definitions and notations given in [B], see also [T, BP2].

Assume that $\mathcal{M}$ is a C^M manifold and consider a $M-$involutive system Σ of rank n. Suppose u is a C^2 solution of Σ on an open set $U \subset \mathcal{M}$ which is a domain of local coordinates $x_1, \dots, x_N$. Let $\mathcal{O}$ be an open subset in $\mathbb{C}\mathscr{J}^1\mathcal{M}|_U$ and $F_1(x, \zeta_0, \zeta), \dots, F_n(x, \zeta_0, \zeta)$ satisfying the conditions (5.2) and (5.3). Consider the vectors fields on U given by

$$L_j^u = \sum_{k=1}^{N} \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x)) \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq n.$$

Since the F_j satisfy (5.2), $L_1^u, \dots, L_n^u$ are linearly independent and span a bundle V^u over U. We will refer to L_j^u as the linearized operator of $F_j(x, u(x), u_x(x)) = 0$ at u.

With $\mathcal{M}$ and Σ given as above and $\mathcal{W}$ a C^M wedge with a C^M edge E, our results can be stated as follows:

Theorem 5.1. *Let u a solution in $\mathcal{W}$ and assume that*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pE \oplus V_p^u \quad \forall p \in E.$$

If u is $C^2(\overline{\mathcal{W}})$ then $WF_M(u_0) \subset (\Gamma^T(\mathcal{W}))^0$, where $u_0 = u|_E$, and polar refers to the duality between TE and T^*E.

The theorem can be proved using the existence of M-approximate solutions together with the approach of [B, EG, HT] and a version of the implicit function theorem for C^M classes, see [K2] and we leave the details for the reader.

Appendix A. On the sequence $M = (M_j)$

Definition A.1. *Let $M = (M_j)$ be a sequence of positive real numbers satisfying the following properties:*

(***Initial Conditions***)

$$M_0 = M_1 = 1. \tag{A.1}$$

(***Strong non-quasianalyticity***) *There exists a constant $A > 1$ such that for all $p = 1, 2, ...,$ we have*

$$\sum_{j=p}^{\infty} \frac{M_j}{M_{j+1}} \leq Ap \frac{M_p}{M_{p+1}} \tag{A.2}$$

(***Strong logarithmic convexity***) *For some fixed $A > 0$ and for any r, with $0 \leq r < 1/A$, if we set $P_j = M_j / (j!)^r$, then*

$$\text{the sequence } \left(\frac{P_j}{jP_{j-1}} \right) \text{ is increasing.} \tag{A.3}$$

(***Stability under ultradifferential operators***) *There are constants $A > 1$ and $H > 1$, independent of n, such that for all $n = 1, 2, 3, ...,$ we have*

$$M_n \leq AH^n \min_{0 \leq j \leq n} M_j M_{n-j}. \tag{A.4}$$

We refer to the paper [AH3] for consequences of the conditions listed in Definition A.1. For instance, condition (A.3) implies: i) the (usual) **logarithmic convexity** condition: For all $j = 1, 2, 3, ...$

$$M_j^2 \leq M_{j-1} M_{j+1} \tag{A.5}$$

and ii) for all $0 \leq j \leq n$,

$$\binom{n}{j} M_j M_{n-j} \leq M_n. \tag{A.6}$$

Condition (A.6) insures that the class $C^M(U)$ is invariant under composition and, in particular, that for all $0 \leq j \leq n$,

$$M_j M_{n-j} \leq M_n \tag{A.7}$$

The condition (A.4) implies the (usual) ***Stability under differential operators*** condition; i.e., There are constants $A > 1$ and $H > 1$, independent of n and j, such that for all $1 \leq j \leq n$, we have

$$M_n \leq AH^{n-1} M_j M_{n-j}. \tag{A.8}$$

We will often replace AH^{n-1} with C^n.

If the sequence M satisfies conditions (A.1) and (A.3), then it satisfies the following condition: For all $n = 1, 2, 3, ...$

$$M_n \geq n! \tag{A.9}$$

Condition (A.9) insures that every analytic function belongs to the class C^M.

A.1. Associated Functions.

Definition A.2. *For each sequence (M_j) of positive numbers we define its* ***associated function*** *$M(t)$ on $(0, \infty)$ by*

$$M(t) = \sup_j \log \frac{t^j}{M_j}.$$

For the reader who is interested in learning more about associated functions and how each of the conditions which we impose on the sequence can be written in terms of the associated function, we recommend the paper by H. Komatsu [K]. In particular, it is not difficult to show that if (M_j) satisfies conditions (A.1) and (A.9), then for all $t > 0$,

$$\log t \leq M(t) \leq t. \tag{A.10}$$

A.2. The spaces $\mathcal{D}^M$ and $\mathcal{D}'_M$.

Definition A.3. *Let $U \subset \mathbb{R}^m$ be an open set. We shall denote by $\mathcal{D}^M(U)$ the vector space of all $\varphi \in C^M(U)$ with compact support in U. The space $\mathcal{D}'_M(U)$ of M-**ultradistributions** is defined to be the dual of $\mathcal{D}^M(U)$; more precisely, $\mathcal{D}'_M(U)$ is the space of all linear forms u on $\mathcal{D}^M(U)$ such that for every $K \subset\subset U$ and for all $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$|u(\varphi)| \leq C_\epsilon \sum_{\alpha \in \mathbb{Z}_+^m} \left\{ \epsilon^{|\alpha|} M_{|\alpha|}^{-1} \sup_{x \in K} |\partial^\alpha u(x)| \right\},$$

for all $\varphi \in \mathcal{D}^M(K) = C^M(U) \cap C_c^\infty(K)$.

A.3. FBI Transform and the $M-$Wavefront Set.

Following [CK], we define the FBI transform of an $M-$ultradistribution:

Definition A.4. *Let $u \in \mathcal{D}'_M(U)$, $\varphi \in \mathcal{D}^M(U)$, and $(y, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$. The* ***FBI transform*** *of φu, denoted $\mathcal{F}_{\varphi u}(y, \xi)$, is the integral (which, in reality, is a duality bracket)*

$$\mathcal{F}_{\varphi u}(y, \xi) = \int_U e^{-i\xi \cdot x - \frac{1}{2}|\xi||y-x|^2} \varphi(x) u(x) \, dx.$$

In the paper [CK], assuming that the sequence $M = (M_j)$ satisfies conditions (A.1), (A.5), (A.8), and (A.9), Chung and Kim proved the following FBI transform characterization of C^M spaces.

Proposition A.1 (Theorem 2.1 in [CK]). *Let $u \in \mathcal{D}'_M(\mathbb{R}^m)$ and $x_0 \in \mathbb{R}^m$. The following are equivalent:*

(1) *There is a neighborhood of x_0 such that $u \in C^M$; and*

(2) *There are constants $A_1,\ A_2,\ A_3 > 0$ and a neighborhood V of x_0 such that for all $\varphi \in \mathcal{D}^M(U)$, with $\varphi \equiv 1$ near x_0, we have*

$$|\mathcal{F}_{\varphi u}(y, \xi)| \leq A_1 e^{-M(A_2|\xi|)}$$

for all $y \in V$ and $|\xi| \geq A_3$.

In case $u \in \mathcal{D}'_M(U)$ is non-C^M at x_0, we can obtain additional information about the structure of the singularities at x_0 by examining the directions in which the above inequalities break down.

Definition A.5. *For fixed $x_0 \in U$ and $\xi_0 \in \mathbb{R}^m \backslash \{0\}$, we say that $u \in \mathcal{D}'_M(U)$ is $M-$**micro-regular** at (x_0, ξ_0) if there exists $\varphi \in \mathcal{D}^M(U)$, with $\varphi \equiv 1$ near x_0, a neighborhood V of x_0 in $\mathbb{R}^m$, and a conic neighborhood Γ of ξ_0 in $\mathbb{R}^m \backslash \{0\}$ such that the FBI estimate in* Proposition (A.1) holds for all $y \in V,\ \xi \in \Gamma,\ |\xi| \geq A_3$. The $M-$**wave-front set** of u, denoted $WF_M(u)$, is the complement in $U \times \mathbb{R}^m \backslash \{0\}$ of the set of all (x_0, ξ_0) where u is $M-$*micro-regular.*

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