BOUNDARY BEHAVIOR OF GENERALIZED ANALYTIC FUNCTIONS

S. BERHANU AND J. HOUNIE

ABSTRACT. We study the boundary properties of the solutions of the elliptic equation $a(z)\partial_{\bar{z}}u+b(z)\partial_z u+A(z)u+B(z)\bar{u}=0$ under the assumption that a and b are Hölder continuous and A and B are in L^p for some $p>2$. These properties include the H^p property, the F. and M. Riesz property and the Rudin-Carleson property.

INTRODUCTION

Let $A(x,y)$ and $B(x,y)$ be complex-valued functions defined on a domain $\Omega\subset\mathbb{C}$ of the complex plane. A function u that satisfies, in the sense of distributions, the equation

$$\overline{\partial}u+Au+B\bar{u}=0 \tag{0.1}$$

on a sub-domain $D\subset\Omega$, where $A,B\in L^p(D)$ for some $p>2$, is said to be a generalized analytic function on D. In particular, if A and B vanish identically, u is a holomorphic (or analytic) function. It is a remarkable fact that holomorphic functions and generalized analytic functions share many local qualitative properties. Indeed, many of these properties follow from the Similarity Principle for equation (0.1) which states that for any solution u of (0.1) in D, there exist a holomorphic function h on D and a Hölder continuous function g on D such that

$$u=e^g h \text{ on } D. \tag{0.2}$$

In particular, (0.2) implies that if u is not constant, its zeros are isolated and of finite order. The Radó property for equation (0.1) is another consequence of (0.2).

On the other hand, there are important boundary properties of holomorphic functions and solutions of linear and nonlinear equations that are not shared by the boundary values of solutions of (0.1) even when A and B are real analytic functions. For example, it is well known that if h is holomorphic on D with a weak boundary value bh and bD is smooth, then bh is microlocally smooth (real analytic if bD is real analytic) in a direction. We will show that there is a solution of $\overline{\partial}u=\bar{u}$ with the property that the C^∞ wave front set of the boundary value contains both directions. The delimitation of the wave front set of the boundary value for holomorphic

2010 *Mathematics Subject Classification.* Primary 30G20 35F15; Secondary 35A18 35J46 42B30.

Key words and phrases. Generalized analytic functions, weak boundary values, F. and M. Riesz property, Rudin-Carleson property, Vekua equation.

Work supported in part by NSF DMS 1001283, CNPq and FAPESP.

functions and more generally solutions of locally integrable vector fields has been used to establish the localized version of the F. and M. Riesz property for the homogeneous solutions of some classes of (not necessarily elliptic) vector fields ([BH1], [BCH]). In this paper we will employ a different method to establish the F. and M. Riesz property for the solutions of (0.1). We will be concerned with other boundary properties such as necessary and sufficient conditions for the existence of weak boundary values, convergence in L^p to the boundary value when the latter belongs to L^p, $1 \le p < \infty$, and the Rudin-Carleson property.

More generally, we will show that some of these boundary properties are valid for solutions of the equation

$$Lu = a(z)\partial_{\bar z}u + b(z)\partial_z u + A(z)u + B(z)\bar u = 0 \tag{0.3}$$

where $a(z)$ and $b(z)$ are Hölder continuous and $A(z)$ and $B(z)$ are L^p functions for some $p > 2$ and $a(z)\partial_{\bar z} + b(z)\partial_z$ is an elliptic vector field. An important example is the classical and widely studied Beltrami vector field $\partial_{\bar z} - q(z)\partial_z$, $|q(z)| \le c < 1$, that plays an important role in the theory of quasiconformal mappings ([AIM]). The solutions will be defined on a simply connected domain D with boundary of class $C^{1,\alpha}$ for some $0 < \alpha < 1$ which, by the Riemann mapping theorem, will be taken as the unit disc without loss of generality.

Equation (0.1) arises in several geometrical and physical problems. In geometry, it is intimately connected with the infinitesimal bending of surfaces in $\mathbb{R}^3$ of positive Gaussian curvature $K > 0$. The equation also arises in the study of the elasticity of thin shells. Among the numerous works on the equation we mention [K], [Kr], [P], [S], and [V]. In recent years, A. Meziani (see for example [M1] and [M2]) has shown that the more general equation $Lu + Au + B\bar u = 0$ where L is a locally solvable complex vector field is closely related with the existence of infinitesimal bendings for surfaces whose curvature $K \ge 0$.

The paper is organized as follows. In Section 1 we show that equation (0.3) can be transformed via a $C^{1,\alpha}$ diffeomorphism to the simpler equation (0.1). In Sections 2 and 3 we establish necessary and sufficient conditions for the existence of a weak boundary value for solutions of (0.1). Section 4 presents an example where the boundary value is not smooth in any direction. In Sections 5 and 6 we prove the F. and M. Riesz and the Rudin-Carleson properties for solutions of (0.1) when A and B are in L^p for some $p > 2$. Sections 7 and 8 extend these two properties to solutions of (0.3).

1. Reduction to a canonical form

We consider the equation

$$Lu = a(z)\partial_{\bar z}u + b(z)\partial_z u + A(z)u + B(z)\bar u = 0 \tag{1.1}$$

on the unit disc $\mathbb{D}$ and assume that

(i) $a(z)$, $b(z) \in C^\beta(\bar{\mathbb{D}})$ for some $0 < \beta < 1$;

(ii) The ellipticity condition

$$\inf_{z\in\bar{\mathbb{D}}} \inf_{|\zeta|=1} \left|a(z)\bar{\zeta} + b(z)\zeta\right| > 0$$

holds;

(iii) A and $B \in L^p(\mathbb{D})$ for some $p > 2$.

We wish to transform equation (1.1) into the simpler equation

$$\tilde{L}w = \partial_{\bar{z}} w + A'w + B'\bar{w} = 0 \tag{1.2}$$

on the unit disc Δ by means of a diffeomorphism of manifolds with boundary $Z : \mathbb{D} \longrightarrow \Delta$ of class $C^{1,\alpha}, 0 < \alpha < \beta$. The main point is to transform the principal part $L_1 = a(z)\partial_{\bar{z}} + b(z)\partial_z$ of L into $\bar{\partial}$.

1.1. **Local first integrals.** We start by finding local solutions $L_1 Z = 0$ of class $C^{1,\alpha}, 0 < \alpha < \beta$ such that dZ does not vanish at a given point where L_1 is the vector field

$$L_1 = a(z)\frac{\partial}{\partial \bar{z}} + b(z)\frac{\partial}{\partial z}$$

assumed to be elliptic with C^β coefficients. If we want to find Z, say, in a neighborhood of the origin with $dZ(0) \neq 0$, we may assume after a linear change of variables that $a(0) = 1$ and $b(0) = 0$. Hence, dividing by $a(z)$, there is no loss of generality in assuming that L_1 has the form

$$L_1 = \frac{\partial}{\partial \bar{z}} + b(z)\frac{\partial}{\partial z}.$$

with $b(0) = 0$. When $b(z) \equiv 0$, the function $Z(z) \equiv z$ is a first integral of $L_1 = \bar{\partial}$, so in the general case we try $Z(z) = z + f(z)$ with $f(z)$ satisfying

$$\frac{\partial f}{\partial \bar{z}} + b(z)\frac{\partial f}{\partial z} = -b \doteq \psi. \tag{1.3}$$

Let D_r be a disc around the origin of radius r, and consider the operator

$$T_r g(z) = \frac{1}{\pi}\iint_{D_r} \frac{g(z')}{z - z'}\, dx'dy'$$

and the singular integral operator

$$P_r f(z) = \frac{1}{\pi}\iint_{D_r} \frac{f(z')}{(z - z')^2}\, dx'dy'$$

both acting on $g, f \in C^\alpha(\overline{D_r})$, for a fixed $0 < \alpha < \beta$. If we denote by δ_ρ the dilation operator $\delta_\rho f(z) = f(\rho z)$, a change of variables gives

$$\delta_r P_r f(z) = P_r f(rz) = \frac{1}{\pi}\iint_{\mathbb{D}} \frac{f(rz')}{(rz - rz')^2} r^2\, dx'dy' = P_1[\delta_r f](z), \quad f \in C^\alpha(\overline{D_r}),$$

that may be written as

$$P_r = \delta_r^{-1} P_1\, \delta_r. \tag{1.4}$$

The norm in $C^\alpha(\overline{D_r})$ is given by

$$\|g\|_r = |g|_r + \|g\|_{L^\infty}, \quad |g|_r \doteq \sup_{z\neq z'} \frac{|g(z)-g(z')|}{|z-z'|^r}.$$

A simple computation using (1.4) shows that the operator norm of P_r on $C^\alpha(\overline{D_r})$ is equal to the operator norm of P_1 on $C^\alpha(\overline{\mathbb{D}})$ which is finite (see, e.g., [V, pp. 57-59]). Hence, P_r is continuous on $C^\alpha(\overline{D_r})$ with operator norm independent of $r > 0$.

To solve (1.3) in a sufficiently small disc D_r, we try $f = T_r w$ with w such that

$$w + b(z)P_r(w) = \psi. \tag{1.5}$$

Let us write $K_r g(z) = b(z)P_r g(z)$. To solve $(I + K_r)w = \psi$ we will show that the Neumann series

$$w \doteq \psi - K_r\psi + K_r^2\psi - K_r^3\psi + \cdots \tag{1.6}$$

converges in $C^\alpha(\overline{D_r})$ provided $r > 0$ is small enough.

Since $b(0) = 0$ and $b(z)$ is of class C^β, it follows that for any fixed $\alpha \in (0, \beta)$, the norm of the multiplication operator $C^\alpha(\overline{D_r}) \ni \phi \mapsto b\phi \in C^\alpha(\overline{D_r})$ tends to zero as $r \searrow 0$. Hence, given $\varepsilon > 0$, we can find a sufficiently small r such that the operator norm of $K_r : C^\alpha(\overline{D_r}) \to C^\alpha(\overline{D_r})$ is less than ε. This implies that the Neumann series in (1.6) converges for such r and since this leads to a solution w whose C^α norm is as small as we wish, we get f as desired. Note that $f = T_r w \in C^{1,\alpha}(\overline{D_r})$. In particular, since

$$\frac{\partial f}{\partial \bar z} = w \quad \text{and} \quad \frac{\partial f}{\partial z} = P_r w,$$

we can choose r so that $\nabla f(0)$ is as small as we wish showing that $dZ(0) \neq 0$. The ellipticity of the equation $L_1 Z = 0$ implies now that $Z_x(0)$ and $Z_y(0)$ are $\mathbb{R}$-linearly independent so Z is a local diffeomorphism of class $C^{1,\alpha}$.

1.2. **Global first integrals.** We have shown that any point $z_0 \in \mathbb{D}$ is contained in a neighborhood $U = U(z_0) \subset \mathbb{D}$ on which a local first integral $Z : U \to \mathbb{C}$ of class $C^{1,\alpha}$ is defined. Since Z transforms L_1 into a multiple of the Cauchy-Riemann vector field, any continuous homogeneous solution u of the equation $L_1 u = 0$ defined on a neighborhood of z_0 is, on a neighborhood V of z_0, of the form $u = U \circ Z$, with U holomorphic on $Z(V)$. It is easy to see that if $du(z_0) \neq 0$ then $U'(Z(z_0)) \neq 0$ so U is a local biholomorphism. This endows $\mathbb{D}$ with a natural structure of a Riemann surface such that the local holomorphic functions u are the solutions of the equation $L_1 u = 0$. Since $\mathbb{D}$ is simply connected and not compact, by the uniformization theorem, it must be conformal to X where X is either the complex plane $\mathbb{C}$ or the half plane $\{\operatorname{Re} z > 0\}$ equipped with the standard holomorphic structure in each case. However, note that L_1 can be extended to a slightly larger disc D' preserving the ellipticity and the regularity properties of its coefficients, so reasoning with D' as we did with $\mathbb{D}$, we would find a conformal map $F : D' \to X$ which, in particular, satisfies the equation $L_1 F = 0$. Hence, F is a diffeomorphism of class $C^{1,\alpha}$ that maps $\mathbb{D}$ to the region of $\mathbb{C}$ bounded by the Jordan curve $t \mapsto F(e^{it})$ of class $C^{1,\alpha}$ and also maps $\overline{\mathbb{D}}$ to $F(\overline{\mathbb{D}}) = \overline{F(\mathbb{D})}$ as a diffeomorphism of manifolds with boundary.

By the Riemann mapping theorem (boundary version) there is a diffeomorphism of class $C^{1,\alpha}$ of manifolds with boundary that maps $\overline{F(\mathbb{D})}$ onto the closed unit disc $\overline{\Delta}$ (we will denote by Δ this new copy of the unit disc) and is holomorphic on $F(\mathbb{D})$. If we set $Z = G \circ F$, we see that $Z : \mathbb{D} \cup \partial\mathbb{D} \longrightarrow \Delta \cup \partial\Delta$ has the following properties:

- $Z : \mathbb{D} \longrightarrow \Delta$ is a diffeomorphism of class $C^{1,\alpha}$ that satisfies the equation $L_1 Z = 0$.
- $Z|_{\partial\mathbb{D}} : \partial\mathbb{D} \longrightarrow \partial\Delta$ is a diffeomorphism of class $C^{1,\alpha}$, in particular, a set $E \subset \partial\mathbb{D}$ is in the σ-algebra $\mathcal{M}(\partial\mathbb{D})$ of Lebesgue measurable subsets of $\partial\mathbb{D}$ if and only if $Z(E) \in \mathcal{M}(\partial\Delta)$ and furthermore E has Lebesgue measure $|E| = 0$ if and only if $|Z(E)| = 0$.
- If a continuous function $u \in C^0(\Delta)$ satisfies (1.1) then $w \doteq u \circ Z^{-1}$ satisfies

$$\partial_{\bar{z}} w + A' w + B' \bar{w} = 0$$

where

$$A'(z) = \frac{A(Z^{-1}(z))}{d(z)}, \quad B'(z) = \frac{B(Z^{-1}(z))}{d(z)}, \quad d(z) = L_1(\bar{Z}).$$

Note that $d(z)$ is of class C^α and never vanishes on $\overline{\Delta}$. The functions A', $B' \in L^p(\Delta)$ because A, $B \in L^p(\mathbb{D})$.

Suppose now that the coefficients $a(z)$, $b(z) \in C^\infty(\overline{\mathbb{D}})$. In this case, by standard elliptic regularity the local first integrals are smooth and so is the global first integral F which maps the boundary of $\mathbb{D}$ diffeomorphically onto a smooth Jordan curve. It follows that the first integral $Z = G \circ F$ constructed as before is a C^∞ diffeomorphism of manifolds with boundary that maps $\overline{\mathbb{D}}$ onto $\overline{\Delta}$. We summarize the results of this section as follows:

Theorem 1.1. *Let $a(z)$, $b(z) \in \mathcal{C}^\beta(\overline{\mathbb{D}})$, $A(z)$, $B(z) \in L^p(\mathbb{D})$ for some $0 < \beta < 1$, $p > 2$. For any $0 < \alpha < \beta$, there exists a diffeomorphism of class $C^{1,\alpha}$ mapping $\overline{\mathbb{D}}$ onto $\overline{\Delta}$ and functions $A'(z)$, $B'(z) \in L^p(\Delta)$ such that a function $u \in C(\overline{\mathbb{D}})$ satisfies equation* (1.1) *if and only if $u \circ Z^{-1}$ satisfies equation* (1.2). *If $a(z)$ and $b(z)$ are smooth the same result holds with a smooth diffeomorphism Z. If, in addition, $A(z)$ and $B(z)$ are smooth, so are $A'(z)$ and $B'(z)$*

Due to this theorem, many qualitative properties proved for solutions of equation (1.1) will automatically hold for the solutions of (1.2).

2. On the existence of weak boundary values

We will be first concerned with the weak boundary values when the coefficients are smooth. Let $a(z)$, $b(z)$, $A(z)$ and $B(z)$ be smooth functions defined on $\overline{\mathbb{D}}$. Let $z_0 \in$

$\partial\mathbb{D}$ and consider a complex-valued function $u(x,y)$ defined on $Q=\mathbb{D}\cap\{|z-z_0|<r\}$, $r>0$, that satisfies equation

$$a(z)\,\overline{\partial}u+b(z)\,\partial u+A(z)\,u+B(z)\,\bar{u}=0 \quad \text{on } Q, \tag{2.1}$$

where the vector field $L=a\,\overline{\partial}+b\,\partial$ is assumed to be elliptic. Let $z_0=e^{i\theta_0}$ and write $I=\{e^{i\theta}:\ |\theta-\theta_0|<r\}$. We say that u has a weak trace (or a weak boundary value) at $r=1$ on I if the limit

$$\langle bu,\phi\rangle \doteq \lim_{r\nearrow 1}\int u(re^{i\theta})\,\phi(\theta)\,d\theta, \quad \phi(\theta)\in C_c^\infty(I),$$

exists for every $\phi\in C_c^\infty(I)$.

Theorem 2.1. *Let u satisfy equation (2.1) and assume that the coefficients $a(z)$, $b(z)$, $A(z)$ and $B(z)$ are smooth and $L_1=a(z)\overline{\partial}_z+b(z)\partial_z$ is elliptic. Then the following conditions are equivalent:*

(1) *The solution u has a weak trace at $r=1$ on I.*
(2) *For every compact subinterval $J\subset\subset I$, there is a positive integer k and a constant $C>0$ such that for each $re^{i\theta}\in Q$ with $\theta\in J$ the estimate*

$$|u(re^{i\theta})|\le C(1-r)^{-k} \quad \textit{holds.} \tag{$\star$}$$

(3) *For every compact subinterval $J\subset\subset I$, there is a positive integer k and a constant $C>0$ such that for $r<1$ and close enough to 1 the estimate*

$$\int_J |u(re^{i\theta})|\,d\theta\le C(1-r)^{-k} \quad \textit{holds.} \tag{$\star\star$}$$

Note that this is a theorem of local nature that roughly states that a solution has a weak trace on some neighborhood of a boundary point if and only if it has at most a tempered growth when approaching the boundary on some neighborhood of that boundary point z_0. Let F be a diffeomorphism that maps the closure of $K=\mathbb{D}\cap\{|z-z_0|<r\}$ onto the closure of the rectangle $Q=(-a_1,a_1)\times(0,b)\subset\mathbb{R}^2\simeq\mathbb{C}$ in such a way that it sends $\{z\in K:|z|=1\}$ onto $\{(x,y)\in Q:y=0\}$ and the circular arcs $\{z\in K:|z|=\rho\}$ onto the horizontal segments in Q. In the new coordinates, we may assume that for some $p,A,B\in C^\infty(Q)$,

$$\frac{\partial u}{\partial y}+p(x,y)\frac{\partial u}{\partial x}+Au+B\bar{u}=0 \text{ on } Q \tag{2.2}$$

The tempered growth conditions ($\star$) and ($\star\star$) become, for $J=[-a,a]$, $a<a_1$,

$$|u(x,y)|\le\frac{C}{y^k}, \quad x\in J,\ y\in(0,b) \quad\text{and}\quad \int_{-a}^{a}|u(x,y)|\,dx\le\frac{C}{y^k}, \quad y\in(0,b).$$

It is obvious that (2) $\Longrightarrow$ (3). We will now show that (3) implies (1). By elliptic regularity, $u\in C^\infty(Q)$. Assume that u satisfies the estimate

$$\int_{-a}^{a}|u(x,y)|\,dx\le\frac{C}{y^k}, \quad y\in(0,b),$$

for some $C > 0$ and $k \in \mathbb{Z}_+$. We want to show that u has a weak trace at $y = 0$, that is, the limit

$$\langle bu, \phi \rangle \doteq \lim_{\varepsilon \searrow 0} \int u(x, \varepsilon)\, \phi(x)\, dx, \quad \phi(x) \in C_c^\infty(-a, a),$$

exists. Let $\psi(x) \in C^\infty(-a, a)$. For $0 < y < b$, define

$$T(y) = \int u(x, y)\, \psi(x)\, dx.$$

Using (2.2), we have:

$$\frac{\partial u}{\partial y} = -p(x, y)\frac{\partial u}{\partial x} - A(x, y)u - B(x, y)\overline{u} \tag{2.3}$$

and we can write after an integration by parts

$$T'(y) = \int \Big(u(x, y)(p(x, y)\psi' + p_x(x, y)\psi - A(x, y)\psi) - B(x, y)\overline{u(x, y)}\psi \Big)\, dx$$

This implies that

$$|T'(y)| \le C_1 \frac{||\psi||_{C^1}}{y^k} \quad \text{for some } C_1 > 0,\ \ 0 < y < b.$$

We can iterate this method, using (2.3) at each stage to conclude that for some constants $C_j > 0$,

$$|T^{(j)}(y)| \le C_j \frac{||\psi||_{C^j}}{y^k} \quad \text{for } 0 < y < b,\ j = 1, 2, \dots.$$

Hence $T(y)$ extends as a smooth function to $y = 0$. In particular, the weak boundary value bu of u exists and bu is a distribution of order $k + 1$ on $(-a, a)$.

The remaining implication (1) $\implies$ (2) will be proved in the next section.

3. Necessary conditions for the existence of a trace

Let $A(x, y)$, $B(x, y)$ be two smooth functions defined on $\mathbb{R}^2$. Suppose we have a complex-valued function $u(x, y)$ defined on $Q = I \times (0, 2)$, $I = (-a, a)$, $a > 0$, that satisfies the Vekua equation

$$\overline{\partial} u + Au + B\bar{u} = 0 \text{ on } Q, \quad \overline{\partial} \doteq \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{3.1}$$

Since we have omitted the factor $1/2$ from the standard definition of $\overline{\partial}$ and ∂ we have $\partial\overline{\partial} = \overline{\partial}\partial = \Delta$, $\Delta = \partial_x^2 + \partial_y^2$. By elliptic regularity, $u \in C^\infty(Q)$. Assume that u has a weak trace at $y = 0$, which means that the limit

$$\langle bu, \phi \rangle \doteq \lim_{\varepsilon \searrow 0} \int u(x, \varepsilon)\, \phi(x)\, dx, \quad \phi(x) \in C_c^\infty(-a, a),$$

exists. By an application of Baire's category theorem it follows that the limit above defines a distribution, i.e., $bu \in \mathcal{D}'(-a, a)$. We want to prove that, if $Q' = J \times (0, 1)$ with $J \subset\subset I$, u satisfies an estimate of the form

$$|u(x, y)| \le \frac{C}{y^k}, \quad (x, y) \in Q',$$

for some $C > 0$ and $k \in \mathbb{Z}_+$. Notice that $\bar{u}$ satisfies the conjugate equation $\partial\bar{u} + \bar{A}\bar{u} + \bar{B}u = 0$, so the $\mathbb{C}^2$-valued function $U = (u, \bar{u})$ satisfies the first-order system

$$DU + EU = 0, \quad D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad E = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} \tag{3.2}$$

for some smooth complex-valued functions a, b, c, d that have simple expressions in terms of A, B and their conjugates. In order to prove that u has a tempered growth as $y \searrow 0$, it will be enough to prove that

$$|U(x,y)| \leq \frac{C}{y^k}, \quad (x,y) \in Q',$$

for a generic solution $U = (u_1, u_2)$ of (3.2) such that

$$\langle bU, \phi \rangle \doteq \lim_{\varepsilon \searrow 0} \int U(x,\varepsilon)\, \phi(x)\, dx, \quad \phi(x) \in C_c^\infty(-a,a), \tag{3.3}$$

exists. In doing so, we will allow arbitrary smooth complex-valued functions a, b, c, d.

The principal symbol of $D + E$ is $\left(\begin{smallmatrix} i\xi-\eta & 0 \\ 0 & i\xi+\eta \end{smallmatrix}\right)$ whose determinant is the elliptic symbol $-(\xi^2 + \eta^2)$. Hence we may find a 2×2 matrix P whose entries are pseudo-differential operators of order -1, with symbols in the class $S^{-1}_{1,0}$ such that

$$f = P(D+E)f + Rf, \quad f \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2), \tag{3.4}$$

where R is a regularizing operator. Since P has order -1, its distribution kernel $k(X, X')$, $X = (x,y)$, $X' = (x', y') \in \mathbb{R}^2$, which is smooth off the diagonal of $\mathbb{R}^2 \times \mathbb{R}^2$, satisfies the estimates (see e.g., [Ta])

$$|\partial_x^\alpha \partial_y^\beta k(X, X')| \leq \frac{c_{\alpha\beta}}{|X - X'|^{1+|\alpha|+|\beta|}}, \quad \alpha, \beta \in \mathbb{Z}_+, \quad X \neq X'. \tag{3.5}$$

Let V be an open interval such that $\overline{J} \subset V \subset \overline{V} \subset I$ and choose $\psi(x) \in C_c^\infty(I)$ satisfying $\psi(x) \equiv 1$, $x \in V$. Next we pick two functions $\phi_1(y), \phi_2(y) \in C^\infty(\mathbb{R})$ such that $\phi_1(y) \equiv 0$ for $y \leq 1/4$, $\phi_1(y) \equiv 1$ for $y \geq 1/2$, so $\phi_1'(y)$ is supported in $[1/4, 1/2]$, and $\phi_2(y) \equiv 1$ for $y \leq 1$, $\phi_2(y) \equiv 0$ for $y \geq 3/2$. Let $y_0 \in (0,1)$ and set $\phi(y) = \phi_1(y/y_0)\phi_2(y)$.

We will apply (3.4) to the test function $f(x,y) = \psi(x)\phi(y)U(x,y)$. By Leibniz's rule and equation (3.2) we have

$$(D+E)f = \rho(x,y)U(x,y)$$

with $\rho(x,y)$ supported in the support of $\nabla(\psi(x)\phi(y))$. Note that for $X_0 \doteq (x_0, y_0) \in J \times (0,1)$,

$$U(x_0, y_0) = f(x_0, y_0) \text{ and } |X_0 - X| \geq y_0/2 \text{ for } X = (x,y) \in \operatorname{supp} \rho,$$

if y_0 is small enough. In particular, $(x_0, y_0) \notin \operatorname{supp} \rho(x,y)$. We have

$$U(x_0, y_0) = f(x_0, y_0) = P(\rho U)(x_0, y_0) + Rf(x_0, y_0). \tag{3.6}$$

We will first focus our attention on the growth as $y_0 \searrow 0$ of the first term on the right hand side of the last equation. We may write

$$P(\rho U)(x_0, y_0) = \int k(x_0, y_0; x, y)]\, (\rho U)(x, y)\, dxdy$$

where the integral is well defined because $(x_0, y_0) \notin \operatorname{supp} \rho(x, y)$. Let $\rho_\varepsilon(x, y)$ be a test function supported in an ε-neighborhood of $\operatorname{supp} \rho(x, y)$ such that $\rho_\varepsilon(x, y) \equiv 1$ on the support of $\rho(x, y)$ so $\rho_\varepsilon \rho = \rho$. We may assume that $|\partial^\alpha_{x,y} \rho_\varepsilon| \leq C_\alpha \varepsilon^{-|\alpha|}$, $\alpha \in \mathbb{Z}^2_+$.

Denote by $\| \, \|_s$ the norm in the Sobolev space $H^s(\mathbb{R})$ in the variable x; it follows from the uniform boundedness principle that for some $s \in \mathbb{R}$, $\|\psi(x) U(x, y)\|_s \leq C$, $0 < y < 1$. Let us denote by $J_s = (1 - \partial_x^2)^{-s/2}$ the pseudo-differential operator of order s defined on $\mathcal{S}(\mathbb{R})$ with symbol $(1 + \xi^2)^{-s/2}$. Choose an even integer $2n > s$ and note that

$$\sup_{0<y<1} \|J_{2n}\big(\psi(\cdot) U(\cdot, y)\big)\|_0 \leq C.$$

We may write

$$P(\rho U)(x_0, y_0) = \int (1 - \partial_x^2)^n \big(\rho_\varepsilon k(x_0, y_0; x, y)\big)\, J_{2n}(\rho U)(x, y)\, dxdy$$

and majorize the right hand side by

$$\sup_y \|(1 - \partial_x^2)^n (\rho_\varepsilon k(x_0, y_0; x, y))\|_0 \leq C \sup_{x,y} |(1 - \partial_x^2)^n (\rho_\varepsilon k(x_0, y_0; x, y))|.$$

Taking $\varepsilon = y_0/4$, so $|X - X_0| \geq y_0/4$ for $X \in \operatorname{supp} \rho_\varepsilon$, we get

$$|P(\rho U)(x_0, y_0)| \leq \frac{C}{y_0^{2n+1}}, \quad (x_0, y_0) \in J \times (0, 1)$$

in view of (3.5).

The analysis of the term $Rf(x_0, y_0)$ is similar and somewhat simplified by the fact that the kernel $r(x, y; x', y')$ of the remainder operator R is a smooth function defined on $\mathbb{R}^2$. We may choose a function $\beta(x) \in C_c^\infty(\mathbb{R})$ identically one on the support of $\psi(x)$ and write

$$Rf(x_0, y_0) = \int (1 - \partial_x^2)^n \big(\beta(x)\, r(x_0, y_0; x, y)\big)\, \phi(y) J_{2n}[\psi(\cdot) U(\cdot, y)](x)\, dxdy$$

to get

$$|Rf(x_0, y_0)| \leq C, \quad (x_0, y_0) \in J \times (0, 1).$$

Keeping in mind these estimates, (3.6) yields

$$|U(x_0, y_0)| \leq \frac{C}{y_0^{2n+1}}, \quad (x_0, y_0) \in J \times (0, 1).$$

We have proved

Proposition 3.1. *Let U be a solution of* (3.2) *on $Q = I \times (0, 2)$ and assume that* (3.3) *holds. For any interval $J \subset \overline{J} \subset I$ there exist a nonnegative integer k and a constant $C > 0$ such that*

$$|U(x, y)| \leq \frac{C}{y^k}, \quad (x, y) \in Q' = J \times (0, 1). \tag{3.7}$$

Thus, the proof of Theorem 2.1 is now complete.

In Section 4 we will need a result on regularity up to the boundary for solutions with smooth traces. This is also a consequence of (3.4) and is stated in the following proposition.

Proposition 3.2. *Let U be a solution of*

$$DU + EU = \Phi, \quad on\ Q_+ = \mathfrak{I} \times (0, 2) \tag{3.8}$$

with D and E as in (3.2) *and $\Phi \in C^\infty(\mathfrak{I} \times (-2, 2))$. Assume that the trace*

$$\langle bU, \phi\rangle \doteq \lim_{\varepsilon \searrow 0} \int U(x, \varepsilon)\, \phi(x)\, dx, \quad \phi(x) \in C_c^\infty(-a, a).$$

exists and $bU \in C^\infty(\mathfrak{I})$. Then U can be smoothly extended to $\mathfrak{I} \times [0, 2)$ by setting $U(x, 0) = bU(x)$.

Proof: We will extend $U(x, y)$ to $Q = \mathfrak{I} \times (-2, 2)$ by solving an approximate backwards Cauchy problem for equation (3.8) (on the subject of approximate Cauchy problems we refer the reader to [Tr] and [H]). More precisely, given any interval $J \subset\subset \mathfrak{I}$, we may find a smooth function $U^\#(x, y)$ on $J \times (-1, 0]$ such that

- $U^\#(x, 0) = bU(x)$, $x \in J$,
- $DU^\# + EU^\# - \Phi \doteq h^\#(x, y)$ vanishes to infinite order at $y = 0$.

The existence of bU allows us to consider U as an element in $C^0([0, 2); \mathcal{D}'(\mathfrak{I}))$. We now define $\tilde{U} \in C^0\big((-2, 2); \mathcal{D}'(J)\big)$ and $\tilde{h} \in C^\infty\big((-2, 2); \mathcal{D}'(J)\big)$ by

$$\tilde{U}(x, y) = \begin{cases} U(x, y) & \text{on } J \times [0, 2), \\ U^\#(x, y) & \text{on } J \times (-2, 0] \end{cases}, \quad \tilde{h}(x, y) = \begin{cases} 0 & \text{on } J \times [0, 2), \\ h^\#(x, y) & \text{on } J \times (-2, 0] \end{cases}.$$

It is not hard to check that $\tilde{U}$ satisfies $D\tilde{U} + E\tilde{U} = \Phi + \tilde{h}$ on Q in the sense of distributions, due to the fact that the equation is satisfied for $y \neq 0$ and $\tilde{U} \in C^0((-2, 2); \mathcal{D}'(J))$. We must show that

$$\langle \tilde{U}, D^t\psi\rangle = -\langle E\tilde{U} - \Phi - \tilde{h}, \psi\rangle, \quad \psi(x, y) \in C_c^\infty\big((-2, 2) \times J\big), \tag{3.9}$$

and it is enough to check that for $\psi(x, y)$ of the special form $\psi(x, y) = \psi_1(x)\psi_2(y)$. Let $\rho(y) \in C_c^\infty(-1, 1)$ satisfy $\rho(y) = 1$ for $|y| \leq 1/2$ and set $\psi_\varepsilon(x, y) = \psi_1(x)\psi_2(y)(1 - \rho(y/\varepsilon)$. Then (3.9) holds for $\psi_\varepsilon(x, t)$ and integrating with respect to x first, we are led to an identity

$$\int \mu(y)\big((1 - \rho(y/\varepsilon))\psi_2'(y) - \varepsilon^{-1}\rho'(y/\varepsilon)\psi_2(y)\big)\, dy = \int \nu(y)(1 - \rho(y/\varepsilon))\psi_2(y)\, dy$$

where $\mu(y)$ and $\nu(y)$ are continuous functions. Letting $\varepsilon \searrow 0$ we get

$$\int \mu(y)\psi_2'(y)\,dy = \int \nu(y)\,\psi_2(y)\,dy$$

which implies (3.9) for $\psi(x,y) = \psi_1(x)\psi_2(y)$.

Given $p = (x_0, 0)$, $x_0 \in J$, pick a test function $\psi(x,y) \in C_c^\infty(J \times (-2,2))$ equal to one in a neighborhood of p. By Leibniz's rule

$$(D+E)\psi\tilde{U} = \psi(D+E)\tilde{U} + g = \psi(\tilde{h} + \Phi) + g$$

with g vanishing in a neighborhood of p. Applying (3.4) with $f = \psi\tilde{U}$, we see that $\psi\tilde{U} = P(\psi\tilde{h} + g) + Rf$ is smooth on the neighborhood of p where g vanishes because P preserves singular supports and we conclude that $\tilde{U}$ is smooth at p. Since p is an arbitrary point in $\mathfrak{I} \times 0$, it follows that U is smooth on $\mathfrak{I} \times [0,2)$. □

4. On the wave front set of the trace of a solution

It is well known that if h is a holomorphic function on a rectangle $Q = (-a,a) \times (0,b)$ with a weak trace bh at $y = 0$, then bh is microlocally analytic and hence also microlocally smooth at the covector $(0,-1)$. This boundary regularity is also shared by traces of solutions of some classes of not necessarily elliptic vector fields such as locally integrable vector fields (see [BH1], Lemma 3.1) and nonlinear first order pdes ([Be], [LMX]). This fact is relevant in the study of the F. and M. Riesz property due to the well known fact that if the wave front set of a Radon measure does not contain a line, it must be absolutely continuous with respect to Lebesgue measure ([Br], [U]).

We will show here that, in general, generalized analytic functions do not share with holomorphic functions this type of microlocal boundary behavior. Consider the equation

$$\bar{\partial} u = \bar{u}, \quad \overline{\partial} = \partial_x + i\partial_y. \tag{4.1}$$

Suppose the trace bu of every solution u of (4.1) in Q is microlocally smooth at $(0,-1)$ (smoothness at $(0,1)$ will also lead to a contradiction). Notice that u_y is also a solution of (4.1) and since u_x and $\bar{u}$ have a trace at $y = 0$ whenever u does, (4.1) implies that u_y has a trace $b(u_y)$. By assumption, the traces of u_x and u_y are smooth at $(0,-1)$, and so using (4.1), we see that $b\bar{u}$ is also smooth at $(0,-1)$. But $b\bar{u}$ is also smooth at $(0,1)$ since bu is smooth at $(0,-1)$. It follows that bu is C^∞ near the origin. By Proposition 3.2, we conclude that u is smooth up to $y = 0$. We will show next that this is not true for all u. Let $f \in C^{2,\alpha}(\overline{\mathbb{D}})$ (for some $0 < \alpha < 1$) be real-valued, where $\mathbb{D}$ is the unit disc and assume that the restriction of f is not in $C^3(\partial\mathbb{D})$. It is well known that the Dirichlet problem

$$\begin{cases} \Delta h - h = \Delta f - f & \text{if } |z| < 1; \\ h(z) = 0 & \text{if } z \in \partial\mathbb{D}. \end{cases}$$

has a solution $h \in C^{2,\alpha}(\overline{\mathbb{D}})$. If $v = f - h$, and we define $v_1 = \bar{v}_z - v$, and $v_2 = v + \bar{v}_z$, then both iv_1 and v_2 satisfy equation (4.1), and hence their traces are smooth on $\partial\mathbb{D}$. But then $v_2 - v_1$ and hence v are smooth up to the boundary. Since $h = 0$ on $\partial\mathbb{D}$, this leads to the contradiction that the trace of f is smooth.

We have shown that there exists a solution v of (4.1) that is not microlocally smooth in the direction $(0, -1)$ at the origin and a similar reason shows that there is a solution w of (4.1) that is not microlocally smooth in the direction $(0, 1)$ at the origin. Thus, $v + w$ is a solution of (4.1) that has full wave front set at the origin. Nevertheless, we will show that generalized analytic functions enjoy the F. and M. Riesz property.

5. On boundary L^p convergence and the F. and M. Riesz property

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$, and u satisfy the equation

$$\bar{\partial} u + Au + \bar{B}u = 0 \tag{5.1}$$

in $\mathbb{D}$ where $A(x, y)$ and $B(x, y)$ are in L^p on $\mathbb{C}$, $p > 2$. Suppose u has a weak boundary value $bu \in L^p(\partial D)$ for some $1 \le p < \infty$, that is, for each $\psi \in C^\infty(\partial D)$,

$$\langle bu, \psi \rangle \doteq \lim_{r \to 1} \int_0^{2\pi} u(re^{i\theta}) \psi(e^{i\theta})\, d\theta \quad \text{exists}$$

and $|\langle bu, \psi \rangle| \le C \|\psi\|_{L^q}$, $1/p + 1/q = 1$. We will show that

$$u(re^{i\theta}) \to bu(e^{i\theta}) \quad \text{in } L^p \text{ as } r \to 1.$$

We will use the representation formula (see (10.6) in Chapter III of [V])

$$\frac{1}{2\pi i}\left(\int_{|\zeta|=1} \Omega_1(z, \zeta) u(\zeta)\, d\zeta - \int_{|\zeta|=1} \Omega_2(z, \zeta) \bar{u}(\zeta)\, d\bar{\zeta} \right) = \begin{cases} u(z), & \text{if } |z| < 1; \\ 0, & \text{if } |z| > 1. \end{cases}$$

where the $\Omega_j(z, t)$ are C^∞ functions when $z \ne t$ and satisfy

$$\Omega_1(z, t) = \frac{1}{t - z} + O(|z - t|^{\frac{-2}{q}}),\ \Omega_2(z, t) = O(|z - t|^{\frac{-2}{q}}) \tag{5.2}$$

with $q > 1$ arbitrary. The preceding formula can be written as

(5.3)
$$\frac{1}{2\pi i}\left(\int_{|\zeta|=1} (\Omega_1(z, \zeta) - \Omega_1(1/z, \zeta)) u(\zeta)\, d\zeta - \int_{|\zeta|=1} (\Omega_2(z, \zeta) - \Omega_2(1/z, \zeta)) \overline{u(\zeta)}\, d\bar{\zeta} \right)$$

for $|z| < 1$. Using (5.2), we have

$$\Omega_1(z, \zeta) - \Omega_1(1/z, \zeta) = \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{1}{z}} + O(|\zeta - z|^{\frac{-2}{q}}) + O(|\zeta - 1/z|^{-\frac{2}{q}})$$

and

$$\Omega_2(z,\zeta) - \Omega_2(1/z,\zeta) = O(|\zeta - z|^{\frac{-2}{q}}) + O(|\zeta - 1/z|^{-\frac{2}{q}}).$$

It follows that for any $|z| < 1$,

$$u(z) = \int_0^{2\pi} P(z,e^{it})u(e^{it})\,dt + \int_0^{2\pi} E_1(z,e^{it})u(e^{it})\,dt + \int_0^{2\pi} E_2(z,e^{it})\overline{u(e^{it})}\,dt \tag{5.4}$$

where $P(z,e^{it})$ is the Poisson kernel for the unit disc, and hence by choosing $q > 1$, we conclude that there is a constant $C > 0$ independent of u such that

$$\text{for any } 0 < r < 1, \quad \int_0^{2\pi} |u(re^{i\theta})|^p\,d\theta \leq C \int_0^{2\pi} |u(e^{i\theta})|^p\,d\theta. \tag{5.5}$$

By the Similarity Principle ([V]), there exist a holomorphic function h on $\mathbb{D}$ and $g \in C^\alpha(\overline{\mathbb{D}})$ $(0 < \alpha < 1$ arbitrary) such that

$$u(z) = e^{g(z)}h(z) \quad z \in \mathbb{D}.$$

Since the function $g(z)$ is bounded on $\overline{\mathbb{D}}$, it follows that the holomorphic function $h \in H^p(\mathbb{D})$, and hence,

$$u(re^{i\theta}) \to bu(e^{i\theta}) \quad \text{in } L^p.$$

The argument also shows that the nontangential maximal function $u^*(\theta) \in L^p$, that

$$u(re^{i\theta}) \to bu(e^{i\theta}) \quad \text{nontangentially for a.e. } \theta \in [0,2\pi]$$

and that $u \in L^\infty(\mathbb{D})$ if $bu \in L^\infty(\partial\mathbb{D})$.

Corollary 5.1. *Suppose u is a solution of* (5.1) *with a weak trace that is continuous. Then u is continuous on $\overline{\mathbb{D}}$.*

Indeed, the preceding arguments show that $bu(\zeta) = e^{g(\zeta)}bh(\zeta)$ on $\partial\mathbb{D}$, and so the corollary follows from the continuity of h on $\overline{\mathbb{D}}$.

Corollary 5.2. *Suppose u is a solution in $\mathbb{D}$ with a weak trace bu which is in L^p for some $p \geq 1$. If $bu(\zeta) = 0$ on a subset $E \subseteq \partial\mathbb{D}$ of positive measure, then $u \equiv 0$ on $\mathbb{D}$.*

Again since $bu(\zeta) = e^{g(\zeta)}bh(\zeta)$ on ∂D, the corollary follows from the corresponding property for functions in $H^p(\mathbb{D})$ (the Riesz uniqueness theorem).

Corollary 5.3. *(The F. and M. Riesz Property). Suppose u is a solution in $\mathbb{D}$ with a weak trace which is a measure μ. Then $\mu \in L^1(\partial\mathbb{D})$.*

To see this, observe first that (5.4) this time takes the form

$$u(z) = \int_0^{2\pi} P(z,e^{it})\,d\mu(t) + \int_0^{2\pi} E_1(z,e^{it})\,d\mu(t) + \int_0^{2\pi} E_2(z,e^{it})\overline{d\mu(t)}\,dt$$

which implies the uniform boundedness of the L^1 norms

$$\int_0^{2\pi} |u(re^{i\theta})|\,d\theta, \quad 0 < r < 1.$$

It follows that for some constant $C > 0$,

$$\int_0^{2\pi} |h(re^{i\theta})|\, d\theta \le C \quad 0 < r < 1$$

and hence by the F. and M. Riesz Theorem, there is an L^1 function bh such that

$$h(re^{i\theta}) \to bh(e^{i\theta}) \quad \text{in } L^1.$$

This in turn implies that

$$u(re^{i\theta}) \to bu(e^{i\theta}) \quad \text{in } L^1$$

and so $\mu = bu \in L^1$.

We consider next the local versions of the preceding results.

Corollary 5.4. *Let A and B in (5.1) be smooth. Suppose u is a solution in $\mathbb{D}$ with a weak trace bu on a smooth piece $\Sigma \subseteq \mathbb{D}$. If $bu \in L^p(\Sigma)$, $p \ge 1$, then for any smooth $\Sigma' \subseteq\subseteq \Sigma$, the norms*

$$\int_{\Sigma'} |u(re^{i\theta})|^p\, d\theta$$

are uniformly bounded as $r \to 1$. In particular, if bu is a measure on Σ, then it is absolutely continuous with respect to Lebesgue measure.

Proof: By Proposition 3.1, u has a tempered growth near Σ. Therefore, given Σ' compact and smooth in Σ, we can find a smoothly bounded simply connected domain $G \subseteq \mathbb{D}$ with $\Sigma' \subseteq \partial G$ such that for some C and k,

$$|u(z)| \le C\text{dist}(z, \partial G)^{-k} \;\; \forall z \in G.$$

By (2) $\Longrightarrow$ (1) in Theorem 2.1, it follows that u has a weak trace which we still denote by bu on ∂G. After using a conformal map (which transforms equation (5.1) into an equation of the same type), we may assume that u is a solution in $\mathbb{D}$, it has a weak trace bu on $\partial\mathbb{D}$, and $bu \in L^p(\Sigma)$. Let $\Sigma' \subseteq\subseteq \Sigma$ be a smooth piece and choose $\psi \in C_0^\infty(\Sigma)$ real-valued, $\psi \equiv 1$ on Σ'. Write $u(z) = u_1(z) + u_2(z)$ where for $|z| < 1$,

$$u_1(z) = \int_0^{2\pi} P(e^{it}, z)\psi(e^{it})u(e^{it})\, dt + \int_0^{2\pi} E(e^{it}, z)\psi(e^{it})(u(e^{it}) + \overline{u(e^{it})})\, dt$$

and

$$u_2(z) = \int_0^{2\pi} P(e^{it}, z)(1-\psi(e^{it}))u(e^{it})\, dt + \int_0^{2\pi} E(e^{it}, z)(1-\psi(e^{it}))(u(e^{it}) + \overline{u(e^{it})})\, dt.$$

Observe that u_2 is smooth for z near Σ' and it is easy to see that the integrals

$$\int_{\Sigma'} |u_2(re^{i\theta})|^p\, d\theta$$

are uniformly bounded. It follows that the integrals

$$\int_{\Sigma'} |u(re^{i\theta})|^p\, d\theta$$

are also uniformly bounded. If bu is a measure on Σ, we can use the boundedness of the L^1 norms

$$\int_{\Sigma'} |u(re^{i\theta})|\, d\theta$$

and the Similarity Principle as before to conclude that bu is absolutely continuous with respect to Lebesgue measure on Σ. □

6. The Rudin-Carleson Property

In this section we will prove the following generalization of the Rudin-Carleson Theorem.

Theorem 6.1. *Let Δ be the unit disc. Consider a closed subset E of the unit circle $\partial\Delta = \{z : |z| = 1\}$ of Lebesgue measure $|E| = 0$ and a continuous function ϕ on E. Then there is a continuous function $u \in C(\overline{\Delta})$ which satisfies*

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}} + Au + B\bar{u} = 0 \ &\text{on } \Delta, \\ u = \phi \ &\text{on } E, \\ \sup_{z\in\partial\Delta} |u(z)| = \sup_{z\in E} |\phi(z)|&. \end{aligned} \tag{6.1}$$

To prove this theorem, we will use a theorem due to E. Bishop [B]. We will use the following strengthened version (Theorem 12.5 in [G]):

Theorem 6.2. *Let $C(X)$ be the uniformly normed Banach space of all continuous complex-valued functions on a compact Hausdorff space X. Let H be a closed subspace of $C(X)$. Let $H^\perp$ consist of all complex measures μ on X such that $\int h\, d\mu = 0$ for all h in H. Let S be a closed subset of X with the property that $\mu(T) = 0$ for every Borel subset T of S and every μ in $H^\perp$. Let ϕ be a continuous complex-valued function on S and G a positive function on X such that $|\phi(x)| \le G(x)$ for all x in S. Then there exists Φ in H with $|\Phi(x)| \le G(x)$ for all x in X and $\Phi(x) = \phi(x)$ for all x in S.*

We will now prove Theorem 6.1.

Proof: Let the set $E \subset \partial\Delta$ and $\phi \in C(E)$ be as in the statement of Theorem 6.1. We will apply Bishop's theorem choosing $X = \partial\Delta$ with the standard topology, $S = E$, $G(z) = \text{constant} = \sup_{z\in E} |\phi(z)|$ and H equal to the space of functions in $C(\Delta)$ that are boundary values of functions $u \in C(\overline{\Delta})$ that are solutions of

$$\frac{\partial u}{\partial \bar{z}} + Au + B\bar{u} = 0 \text{ in } \Delta. \tag{6.2}$$

To check that the hypotheses of Theorem 6.2 are satisfied it will be enough to show that

(i) H is a closed subspace of $C(\partial\Delta)$;

(ii) For every $\mu \in H^\perp$ and every closed subset $T \subset \partial\Delta$, $|T| = 0 \implies \mu(T) = 0$.

Once this is shown, Theorem 6.1 will follow by a direct application of Theorem 6.2. By a variant of the maximum principle for Vekua's equation ((6.18) in Chapter III of [V]), there is a constant $M > 0$ depending only on the coefficients A and B such that for any solution u that is continuous on $\overline{\Delta}$,

$$|u(z)| \leq M \sup\{|u(w)| : \, |w| = 1\}.$$

It follows right away that (i) holds.

By the Radon-Nikodym theorem, any $\mu \in H^\perp$ may be written in a unique way as $\mu = g + \mu_s$, with $g \in L^1(\partial\Delta)$ and μ_s singular with respect to Lebesgue measure. To prove (ii) we will show that $H^\perp \subset L^1(\partial\Delta)$, i.e., μ has no singular part. If we could prove that $\mu \in H^\perp$ is the boundary value bh of a function h that satisfies equation (6.2), by the F. and M. Riesz property (Corollary 5.3) we would conclude that $\mu \in L^1(\partial\Delta)$. We will prove instead that this is true up to an L^1 error. Given $\mu \in H^\perp$, define $w(z)$ on the unit disc by

$$2\pi w(z) = \int_0^{2\pi} \big(\Omega_1(z, e^{i\theta}) - \Omega_1(1/\bar{z}, e^{i\theta})\big) e^{i\theta}\, d\mu + \int_0^{2\pi} \big(\Omega_2(e^{i\theta}, z) - \Omega_2(1/\bar{z}, e^{i\theta})\big) e^{-i\theta}\, d\bar{\mu}$$

For $z \in \Delta$, define the function

$$f(z) = \int_0^{2\pi} \Omega_1(1/\bar{z}, e^{i\theta}) e^{i\theta}\, d\mu.$$

Observe that the function $\zeta \longmapsto \Omega_1(\frac{1}{\bar{z}}, \zeta)$ may not be a solution of Vekua's equation. However, since

$$\Omega_1(z, t) = \frac{1}{t - z} + O(|z - t|^{\frac{-2}{q}}),$$

we know that

$$\Omega_1(w, \zeta) = -\Omega_1(\zeta, w) + O(|\zeta - w|^{\frac{-2}{q}}).$$

Moreover, by (8.12) and (8.13) in Chapter III of [V], there are two functions $X_1(z,t)$ and $X_2(z,t)$ defined for $z, t \in \mathbb{C}$, $z \neq t$ such that

$$\Omega_1(z, t) = X_1(z, t) + i\, X_2(z, t)$$

and the X_j satisfy

$$\partial_{\bar{z}} X_j(z, t) + A(z) X_j(z, t) + B(z)\overline{X_j(z, t)} = 0, \quad \text{for } z \neq t.$$

We can thus express f as

$$f(z) = -\int_0^{2\pi} \Omega_1(e^{i\theta}, 1/\bar{z}) e^{i\theta}\, d\mu + \int_0^{2\pi} K_1(e^{i\theta}, z)\, d\mu$$

where for some constant $C_1 > 0$,

$$|K_1(\zeta, z)| \leq \frac{C_1}{|\zeta - \frac{1}{\bar{z}}|^{\frac{2}{q}}}.$$

Writing Ω_1 in terms of the X_j, we have:

$$f(z) = -\int_0^{2\pi} X_1(e^{i\theta}, 1/\bar{z})e^{i\theta}\, d\mu - i\int_0^{2\pi} X_2(e^{i\theta}, 1/\bar{z})e^{i\theta}\, d\mu + \int_0^{2\pi} K_1(e^{i\theta}, z)\, d\mu. \tag{6.3}$$

By assumption on the measure μ and the fact that the $X_j(z,t)$ are solutions in the first variable for $z \neq t$, we get

$$\begin{aligned}\int_0^{2\pi} X_1(e^{i\theta}, 1/\bar{z})e^{i\theta}\, d\mu &= \frac{1}{\bar{z}}\int_0^{2\pi} X_1(e^{i\theta}, 1/\bar{z})\, d\mu + \int_0^{2\pi} X_1(e^{i\theta}, 1/\bar{z})\left(e^{i\theta} - \frac{1}{\bar{z}}\right) d\mu \\ &= \int_0^{2\pi} X_1(e^{i\theta}, 1/\bar{z})\left(e^{i\theta} - \frac{1}{\bar{z}}\right) d\mu. \end{aligned}\tag{6.4}$$

Using the equations

$$\Omega_1(z,t) = X_1(z,t) + iX_2(z,t) \quad \text{and}$$

$$\Omega_2(z,t) = X_1(z,t) - iX_2(z,t)$$

(see (8.12) and (8.13) in [V]) and the estimates on the $\Omega_j(z,t)$, we see that for some constant $C > 0$,

$$|X_j(z,t)| \leq \frac{C}{|z-t|}, \quad j = 1, 2. \tag{6.5}$$

From (6.4) and (6.5) it follows that the integral $\int_0^{2\pi} X_1(e^{i\theta}, \frac{1}{\bar{z}})e^{i\theta}\, d\mu$ is a bounded function of $z \in \Delta$. The same reasoning and conclusion applies to the integral $\int_0^{2\pi} X_2(e^{i\theta}, \frac{1}{\bar{z}})e^{i\theta}\, d\mu$. Going back to (6.3), we conclude that

$$f(z) = \int_0^{2\pi} K_2(e^{i\theta}, z)\, d\mu$$

where

$$|K_2(\zeta, z)| \leq \frac{C'}{|\zeta - \frac{1}{\bar{z}}|^{\frac{2}{q}}} \quad \text{for some } C' > 0.$$

This latter expression for $f(z)$ and the bound on the kernel Ω_2 lead to

$$w(z) = \frac{1}{2\pi}\int_0^{2\pi} \Omega_1(z, e^{i\theta})e^{i\theta}\, d\mu + \frac{1}{2\pi}\int_0^{2\pi} \Omega_2(z, e^{i\theta})e^{-i\theta}\, d\bar{\mu} + \int_0^{2\pi} K(e^{i\theta}, z)\, d\mu \tag{6.6}$$

where the kernel K satisfies

$$|K(\zeta, z)| \leq \frac{C_2}{|\zeta - \frac{1}{\bar{z}}|^{\frac{2}{q}}}$$

for some constant C_2. Define the function $h(z)$ on Δ by

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \Omega_1(z, e^{i\theta})e^{i\theta}\, d\mu + \frac{1}{2\pi}\int_0^{2\pi} \Omega_2(z, e^{i\theta})e^{-i\theta}\, d\bar{\mu}. \tag{6.7}$$

By (8.14) in Chapter III of [V], away from the diagonal, the kernels Ω_1 and Ω_2 satisfy the system of equations

$$\partial_{\bar{z}}\Omega_1(z,\zeta) + A(z)\Omega_1(z,\zeta) + B(z)\overline{\Omega_2(z,\zeta)} = 0,$$

$$\partial_{\bar{z}}\Omega_2(z,\zeta) + A(z)\Omega_2(z,\zeta) + B(z)\overline{\Omega_1(z,\zeta)} = 0$$

and this implies that h is a solution of

$$\partial_{\bar z} h + A(z)h + B(z)\bar h = 0, \quad z \in \Delta. \tag{6.8}$$

Observe that

$$w(z) = h(z) + \int_0^{2\pi} K(e^{i\theta}, z)\, d\mu, \quad z \in \Delta. \tag{6.9}$$

On the other hand, from the initial definition of $w(z)$, we have

$$w(z) = \frac{1}{2\pi}\int_0^{2\pi} (\Omega_1(z, e^{i\theta}) - \Omega_1(1/\bar z, e^{i\theta}))e^{i\theta}\, d\mu + \int_0^{2\pi} F(z, e^{i\theta}, \overline{z}) d\bar\mu$$

where

$$|F(z, \zeta, \overline{z})| \le C\left(\frac{1}{|\zeta - \frac{1}{\bar z}|^{\frac{2}{q}}} + \frac{1}{|\zeta - z|^{\frac{2}{q}}}\right).$$

As we saw before, the latter can be expressed using the Poisson kernel $P(e^{it}, z)$ for Δ:

$$w(z) = \int_0^{2\pi} P(z, e^{it})\, d\mu + \int_0^{2\pi} F_1(e^{it}, z)\, d\mu + \int_0^{2\pi} F_2(e^{it}, z)\, d\bar\mu \tag{6.10}$$

where

$$|F_1(\zeta, z)| + |F_2(\zeta, z)| \le C\left(\frac{1}{|\zeta - \frac{1}{\bar z}|^{\frac{2}{q}}} + \frac{1}{|\zeta - z|^{\frac{2}{q}}}\right).$$

Observe next that for $\zeta \in \partial\Delta$ and $z \in \Delta$,

$$\left|\zeta - \frac{1}{\bar z}\right| = \left|\frac{\zeta \bar z - 1}{\bar z}\right| = \frac{|\zeta(\bar z - \bar\zeta)|}{|\bar z|} = \frac{|\zeta - z|}{|z|}.$$

Thus when $(\zeta, z) \in \partial\Delta \times \Delta$,

$$\frac{1}{|\zeta - \frac{1}{\bar z}|} = \frac{|z|}{|\zeta - z|}.$$

It follows that if $R(e^{i\theta}, z)$ denotes any of the kernels K_1, K, F, F_1 or F_2, ν denotes one of the measures μ or $\bar\mu$ and $q > 1$ is large enough, then the function

$$\tau(z) = \int_0^{2\pi} R(e^{i\theta}, z)\, d\nu, \; z \in \Delta,$$

has a boundary value $b\tau \in L^1(\partial\Delta)$. The integrability of the boundary value of the functions $\tau(z)$ together with the expression (6.10) of $w(z)$ show that $w(z)$ has a boundary value bw which has the form

$$bw = \mu + g_1, \;\; g_1 \in L^1(\partial\Delta). \tag{6.11}$$

On the other hand, expression (6.9) for w shows that

$$bw = bh + g_2, \;\; g_2 \in L^1(\partial\Delta) \tag{6.12}$$

where bh is the weak boundary value of a solution. From (6.11) and (6.12) we conclude that

$$\mu = bh + g_2 - g_1$$

where bh is the boundary value of a solution h of Vekua's equation (6.8). By Corollary 5.3, we conclude that $\mu \in L^1(\partial\Delta)$. $\square$

7. The more general equation in the smooth case

Consider the equations

$$Lu = a(z)\partial_{\bar{z}}u + b(z)\partial_z u + Au + B\bar{u} = 0 \tag{7.1}$$

and

$$\tilde{L}w = \partial_{\bar{z}}w + Cw + D\bar{w} = 0 \tag{7.2}$$

where we assume that the coefficients are all smooth functions and $a(z)\partial_{\bar{z}} + b(z)\partial_z$ is elliptic. We wish to briefly indicate here that the results we have so far for solutions of $\tilde{L}w = 0$ also hold for solutions of $Lu = 0$. We will focus on the L^p convergence to the boundary value, the F. and M. Riesz property, and the Rudin-Carleson property. Let $\mathbb{D}$ and Δ both denote the unit disc.

7.1. **L^p convergence to the boundary value.** Suppose $Lu = 0$ in $\mathbb{D}$ and u has a weak boundary value bu in the sense that

$$\lim_{r\to 1}\int_0^{2\pi} u(re^{i\theta})\psi(e^{i\theta})\,d\theta = \langle bu, \psi(e^{i\theta})\rangle$$

exists for every smooth ψ on $\partial\mathbb{D}$. We wish to show that, if for some $1 \le p < \infty$, $bu \in L^p(\partial\mathbb{D})$ then

$$\lim_{r\to 1}\int_0^{2\pi} |u(re^{i\theta}) - bu(\theta)|^p\,d\theta = 0.$$

Let $Z : \mathbb{D} \to \Delta$ be a diffeomorphism that is smooth up to the boundary and $a(z)\partial_{\bar{z}}Z + b(z)\partial_z Z = 0$ on $\mathbb{D}$. If $w : \Delta \to \mathbb{C}$ is defined by $u(z) = w(Z(z))$, then w satisfies an equation of the form

$$\tilde{L}w = \partial_{\bar{z}}w + Cw + D\bar{w} = 0 \quad in\ \Delta$$

where C and D are smooth functions. Moreover, for $0 < r < 1$, if $C_r = \{Z(re^{i\theta}) : 0 \le \theta \le 2\pi\}$, then

$$\lim_{r\to 1}\int_{C_r} w(z)\psi(z)\,dz \doteq \langle \flat w, \psi(e^{i\theta})\rangle$$

exists for each smooth ψ on $\overline{\Delta}$. It is easy to see that there is a nonvanishing smooth function ω on $\partial\Delta$ such that

$$\langle \omega(e^{i\theta})\flat w, \psi(e^{i\theta})\rangle = \langle bu, \psi(Z(e^{i\theta}))\rangle$$

for all smooth ψ and $\flat w \in L^p(\partial\Delta)$. Furthermore, if bu is a measure so is $\flat w$.

In order to apply the results we proved for the solution of equation (7.2), we must show that if instead of integrating on the curves C_r, we integrate on the circles $\{re^{i\theta}\}$, the limit exists and that we get the same distribution as a boundary value.

We do this in a local situation where we assume that $\tilde{L}w = 0$ in a rectangle $Q = (-c, c) \times (0, d)$ and we have curves $C_y(x) = x + ig(x, y)$, with g smooth and real-valued on $\overline{Q}$, and $g(x, 0) = 0$. Suppose that for each $\psi(x) \in C_0^\infty(-c, c)$, the limit

$$\lim_{y\to 0} \int_{C_y} w(x, y)\psi(x)\, dx \doteq \langle \flat w, \psi\rangle$$

exists. We wish to show that for any such ψ, $\lim_{y\to 0} \int w(x, y)\psi(x)\, dx$ exists and that the limits are equal.

Observe first that the proof of the necessity for the existence of a boundary value can be used to conclude that $w(x, y)$ has a tempered growth as $y \to 0$. It follows that

$$\lim_{y\to 0} \int w(x, y)\psi(x)\, dx = \langle bw, \psi\rangle \doteq \langle f, \psi\rangle$$

exists for each ψ and we denote the boundary value by f. Consider the adjoint of $\tilde{L}$ which is given by

$$\tilde{L}^* v = \partial_{\bar{z}} v - Cv - \bar{D}\bar{v}.$$

We will use the following equation (Chapter III in [V], equation (9.3)) which is valid for the solution w in Q, and v a C^1 function in $G \subset\subset Q$ with a C^1 boundary:

$$\Re\left(\frac{1}{2i}\int_{\partial G} w(z)v(z)\, dz\right) = \Re\left(\iint_G w\tilde{L}^* v\, dxdy\right). \tag{7.3}$$

Suppose k is a positive integer such that after decreasing c and d, $|w(x, y)| \le C_1 y^{-k}$ for some $C_1 > 0$. Let $\psi \in C_0^\infty(-c, c)$. Let $v(x, y)$ be a smooth function satisfying $v(x, 0) = 0$ and $\tilde{L}^* v(x, y) = O(y^k)$, and the x-support of $v(x, y)$ contained in the support of ψ (see [Tr]). Now consider two kinds of domains in Q. Fix $0 < d' < d$. Since $g(x, 0) = 0$, there is $0 < y_1 < d'$ such that $g(x, y) < d'$ whenever $0 < y < y_1$. Define

$$G_y^1 = \{x + it : g(x, y) < t < d',\, |x| < c\}, \quad \text{and } G_y^2 = (-c, c) \times (y, d) \text{ for } 0 < y < y_1.$$

We now use equation (7.3) with $G = G_y^j, j = 1, 2$. Since the integrand $w\tilde{L}^* v$ is integrable on Q, the integrals on the right hand side converge to

$$\iint_Q w\tilde{L}^* v\, dxdy.$$

It follows that

$$\Im\{\langle \flat w, \psi\rangle\} = \Im\{\langle f, \psi\rangle\}.$$

Applying the latter to $i\psi$, we conclude that $\flat w = bw$. Since $bw \in L^p(\partial\Delta)$, we know that the nontangential maximal function $w^*(\theta) \in L^p[0, 2\pi]$ by the results in Section 5. This implies easily that for r close to 1,

$$\int_{C_r} |w(z)|^p\, |dz| \le C \int_0^{2\pi} |w^*(\theta)|^p\, d\theta < \infty.$$

For r close to 1 the curves C_r have a parametrization $[0, 2\pi] \ni \theta \mapsto \rho(r,\theta)e^{i\theta}$ and the point $\rho(r,\theta)e^{i\theta}$ belongs to the Stolz angle with vertex at θ so $|w(\rho(r,\theta)e^{i\theta})| \leq w^*(\theta)$. It follows that

$$\lim_{r\to 1}\int_0^{2\pi} |w(\rho(r,\theta)e^{i\theta}) - bw(e^{i\theta})|^p \, d\theta = 0$$

by the dominated convergence theorem. Recalling the definition of C_r and the fact that $u = w \circ Z$, this implies that

$$\lim_{r\to 1}\int_0^{2\pi} |u(re^{i\theta}) - bu(\theta)|^p \, d\theta = 0.$$

as we wished to prove.

7.2. **The F. and M. Riesz property.** If in the preceding discussion bu is a measure, bw exists and is also a measure, therefore $bw \in L^1(\partial\Delta)$ by Corollary 5.3 and this implies that $bu \in L^1(\partial\mathbb{D})$.

7.3. **The Rudin-Carleson property.** The Rudin-Carleson property for L follows directly from the same property for $\tilde{L}$ by using the mapping Z that takes null subsets of $\partial\mathbb{D}$ into null subsets of $\partial\Delta$. Then one applies Theorem 6.1 with obvious choices of E and ϕ to find the appropriate solution of $\tilde{L}w = 0$ and checks that $u = w \circ Z$ solves the problem for the original equation. Note however that to carry on this simple reasoning it is enough to assume that Z is of class C^1, and therefore it holds when $a(z)$, $b(z)$ are Hölder and $A(z)$, $B(z) \in L^p$ for some $p > 2$.

Summing up, when the coefficients are smooth, the properties that we have established for the solutions of $\tilde{L}w = 0$ are also valid for the solutions of $Lu = 0$. In the next section we discuss the case of nonsmooth coefficients.

8. On the more general equation with nonsmooth coefficients

Consider the equation

$$Lu = a(z)\partial_{\bar{z}}u + b(z)\partial_z u + Au + B\bar{u} = L_1 u + Au + B\bar{u} = 0 \tag{8.1}$$

where we assume that L_1 is elliptic, $a(z), b(z) \in C^{\beta}(D)$ for some $0 < \beta < 1$ and the coefficients $A, B \in L^p(D)$ for some $p > 2$. We wish to prove a version of the F. and M. Riesz property for L. Suppose that u satisfies the equation $Lu = 0$ on $\mathbb{D}$ and μ is a measure on $\partial\mathbb{D}$. When the principal part L_1 has rough coefficients, the natural notion one should consider when stating that μ is a weak boundary value of u will no longer be in the sense of distributions, which is appropriate for smooth coefficients. We deal instead with convergence in the space of measures considered as the dual of $C(\partial\mathbb{D})$ with the weak-$*$ topology. Thus we say that μ is the weak-$*$ boundary value of u if

$$\lim_{r\to 1}\int_0^{2\pi} u(re^{i\theta})\phi(e^{i\theta})\, d\theta = \int_0^{2\pi} \phi(e^{i\theta})\, d\mu \quad \text{for every } \phi \in C(\partial\mathbb{D}). \tag{8.2}$$

Theorem 8.1. *Let* $a(z),\, b(z) \in \mathcal{C}^{\beta}(\overline{\mathbb{D}})$*,* $A(z),\, B(z) \in L^p(\mathbb{D})$ *for some* $0 < \beta < 1$*,* $p > 2$*. Suppose* $Lu = 0$ *on* $\mathbb{D}$ *and* μ *is a measure on* $\partial\mathbb{D}$ *that is the weak-*$*$ *boundary value of* u*. Then* $\mu \in L^1(\partial\mathbb{D})$*.*

PROOF: Let $\mathbb{D}$ and Δ both denote the unit disc. By Theorem 1.1 , for any $0 < \alpha < \beta$, we can get a map $Z : \overline{D} \to \overline{\Delta}$ which is a $C^{1,\alpha}$ diffeomorphism such that $L_1 Z = a(z)\partial_{\bar z} Z + b(z)\partial_z Z = 0$ on D. Using the diffeomorphism Z, the equation $Lw = 0$ is transformed to one of the form

$$\tilde{L}w = \frac{\partial w}{\partial \bar z} + Cw + D\bar w = 0 \tag{8.3}$$

where C and D are in $L^p(\Delta)$. The restriction of Z to the boundary transforms μ into a measure ν defined on $\partial\Delta$. If we can prove that ν is the weak limit in the sense of distributions of $w = u \circ Z^{-1}$, by the results in Section 5, we would conclude that $\nu \in L^1(\partial\Delta)$ implying that $\mu \in L^1(\partial\mathbb{D})$. However, the integrals on the left hand side of (8.2) are transformed by the diffeomorphism Z into integrals where integration occurs over the C^1 curves $C_r = \{Z(re^{i\theta})\}$, $0 < r < 1$. So the question is whether the limit remains a measure if the curves C_r are replaced by concentric circles provided we deal with smooth test functions. Consider the curves $\gamma_r = \{Z^{-1}(re^{i\theta})\}$ in $\mathbb{D}$ obtained by mapping concentric circles contained in Δ by Z^{-1}. We will study the limit

$$\lim_{r\to 1} \int_{\gamma_r} u(z)\,\phi(z)\,dZ, \quad \phi \in C^1(\overline{\mathbb{D}}).$$

Since bounded sets for the weak-$*$ topology are strongly bounded, (8.2) implies that

$$\int_0^{2\pi} |u(re^{i\theta})|\,d\theta \le C_1, \quad 0 < r < 1. \tag{8.4}$$

In particular, $u \in L^1(\mathbb{D})$. Take two radii $r_2 < r_1 \in (0,1)$, both close to 1, and assume for simplicity that the disc of radius r_1 contains the the curve γ_{r_2} (this will happen if r_1 is much closer to 1 than r_2). Consider the 1-form $\omega = u(re^{i\theta})\phi(re^{i\theta})\,dZ$ for some $\phi \in C^1(\overline{\mathbb{D}})$, with exterior derivative

$$d\omega = \frac{L_1(u\phi)}{L_1\bar Z}\,d\bar Z \wedge dZ = \frac{(u - Au - B\bar u)L_1\phi}{L_1\bar Z}\,d\bar Z \wedge dZ$$

where we have used the fact that $Lu = 0$. Using Green's theorem in the region $D(r_1, r_2)$ bounded by the circle of radius r_1 and the curve γ_{r_2} we get

$$\int_{\gamma_{r_2}} u(z)\,\phi(z)\,dZ = \int_0^{2\pi} u(r_1 e^{i\theta})\phi(r_1 e^{i\theta})\,dZ - \iint_{D(r_1,r_2)} d\omega.$$

As $r_2 \to 1$, the area of $D(r_1, r_2)$ goes to zero and since $L_1\phi$ is bounded, $L_1\bar Z \neq 0$ (because L_1 is elliptic and $dZ \neq 0$) and $u \in L^1(\mathbb{D})$, the double integral goes to zero. The pullback of dZ to a concentric circle of radius r_1 tends uniformly, as $r_1 \to 1$, to

$Z_\theta(e^{i\theta})\,d\theta$ so letting $r_2 \to 1$ we have

$$\begin{aligned}\lim_{r_2\to 1}\int_{\gamma_{r_2}} u(z)\,\phi(z)\,dZ &= \lim_{r_1\to 1}\int_0^{2\pi} u(r_1e^{i\theta})\phi(r_1e^{i\theta})Z_\theta(r_1e^{i\theta})\,d\theta \\ &= \int_0^{2\pi}\phi(e^{i\theta})Z_\theta(e^{i\theta})\,d\mu \\ &\doteq \int_0^{2\pi}\left(\phi\circ Z^{-1}\right)(e^{it})\,d\nu(t) \qquad (8.5)\end{aligned}$$

where we have used (8.4) in the second equation and written $Z_\theta(re^{i\theta}) = \partial_\theta Z(re^{i\theta})$. On the other hand, the change of variables $\zeta = Z(z)$ shows that

$$\lim_{r_2\to 1}\int_{\gamma_{r_2}} u(z)\,\phi(z)\,dZ = \lim_{r_2\to 1}\int_0^{2\pi} w(r_2e^{it})\left(\phi\circ Z^{-1}\right)(r_2e^{it})\,ir_2e^{it}\,dt. \qquad (8.6)$$

Since $\phi \in C^1(\overline{\mathbb{D}})$ is arbitrary, (8.5) and (8.6) show that ν is a measure such that

$$\lim_{r\to 1}\int_0^{2\pi} w(re^{it})\psi(e^{it})\,dt = -i\int_0^{2\pi}\psi(e^{it})\,e^{-it}\,d\nu(t), \quad \psi \in C^1(\overline{\Delta}),$$

which means that $bw = -ie^{-it}\,d\nu$. Hence, $\nu \in L^1(\partial\Delta)$ and therefore $\mu \in L^1(\partial\mathbb{D})$ as we wished to prove. $\square$

8.1. **A connection with the Rudin-Carleson property.** Bishop's theorem and the F. and M. Riesz theorem are closely related. We have already seen how using Bishop's theorem it is possible to prove the Rudin-Carleson property as a consequence of the F. and M. Riesz property. On the other hand, while the paper [BH1] established the F. and M. Riesz theorem for all smooth, locally integrable vector fields, it was shown in [BH2] and [BH3] that the Rudin-Carleson property may not be valid even for the subclass of real analytic, locally solvable vector fields. We now present an alternative proof of Theorem 8.1. The interest in the second proof lies in the fact that now the F. and M. Riesz property follows as a corollary of the Rudin-Carleson property.

Proof: We start by observing that because of (8.4), the assumption (8.2) implies an apparently stronger version of itself, namely

$$\lim_{r\to 1}\int_0^{2\pi} u(re^{i\theta})h(re^{i\theta})\,d\theta = \int_0^{2\pi} h(e^{i\theta})\,d\mu, \quad h \in C(\overline{\mathbb{D}}). \qquad (8.7)$$

We wish to show that μ is absolutely continuous with respect to $d\theta$. This will follow if we show that for any closed subset $F \subseteq \partial D$ with $|F| = 0$, $\mu(F) = 0$. Using the diffeomorphism Z, the equation $Lw = 0$ is transformed to one of the form

$$L_1 w = \frac{\partial w}{\partial \bar z} + Cw + D\bar w = 0 \qquad (8.8)$$

where C and D are in $L^p(\Delta)$. Let

$$L_1^* v = \frac{\partial v}{\partial \bar z} - Cv - \overline{D}\overline{v}.$$

If f and g are defined on $\mathbb{D}$, we define $\tilde{f}$ and $\tilde{g}$ on Δ by $f(re^{i\theta}) = \tilde{f}(Z(re^{i\theta}))$ and $g(re^{i\theta}) = \tilde{g}(Z(re^{i\theta}))$. For $0 < r < 1$, we have

$$\int_{|z|=r} f(z)g(z)\,dZ = \int_{C_r} \tilde{f}(z)\tilde{g}(z)\,dz$$

and so if $L_1\tilde{f} = 0$ and $L_1^*\tilde{g} = 0$, then as we saw in Section 7,

$$\int_{C_r} \tilde{f}(z)\tilde{g}(z)\,dz = \iint_{D_r} 2i\Im(D\overline{\tilde{f}}\tilde{g})\,d\bar{z}\wedge dz = -4\iint_{D_r} \Im(D\overline{\tilde{f}}\tilde{g})\,dx\wedge dy, \quad D_r = \{z : |z| \le r\} \tag{8.9}$$

and so it follows that

$$\Im\left(\int_0^{2\pi} f(re^{i\theta})g(re^{i\theta})Z_\theta(re^{i\theta})\,d\theta\right) = 0. \tag{8.10}$$

Let $F \subseteq \partial D$ be a closed set with $|F| = 0$. Let $g \in C(\partial D)$ satisfy $g(z) \equiv 1$ on F and $0 < |g(z)| < 1$ for $z \notin F$. For each $k = 1, 2, \dots,$ let

$$g_k(e^{i\theta}) = \frac{g(e^{i\theta})^k}{Z_\theta(e^{i\theta})}.$$

As shown in Subsection 7.1 (the Rudin-Carleson property) we can choose a corresponding sequence $G_k \in C(\overline{D})$, such that $L_1^*\widetilde{G}_k = 0$ on D, and $G_k(z) = g_k(z)$ on F. Moreover, since

$$|g_k(z)| < 2\frac{|g(z)|^k}{|Z_\theta(z)|} \quad \text{for all } z \in \partial D,$$

we may assume that $|G_k(z)| < C|g(z)|^k$ for some constant $C > 0$. Applying equation (8.10) to u and the G_k, we get:

$$\Im\left(\int_0^{2\pi} u(re^{i\theta})G_k(re^{i\theta})\,dZ\right) = 0. \tag{8.11}$$

From (8.7) and (8.11), it follows that

$$\Im\left(\int_0^{2\pi} G_k(e^{i\theta})Z_\theta(e^{i\theta})\,d\mu\right) = 0. \tag{8.12}$$

Since $G_k(z) \to 0$ for each $z \notin F$, $z \in \partial D$, and $G_k(z)Z_\theta(z) \equiv 1$ on F, (8.12) leads to $\mu(F) = 0$. It follows that L has the F. and M. Riesz property. □

References

[AIM] K. Astala, T. Iwaniec and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press (2009)

[Be] S. Berhanu, *On microlocal analyticity of solutions of first-order nonlinear PDE*, Ann. Inst. Fourier, **59**(2009),1267–1290.

[BCH] S. Berhanu, P.D. Cordaro, and J. Hounie, *An Introduction to Involutive Structures*, Cambridge University Press (2008).

[BH1] S. Berhanu and J. Hounie, *An F. and M. Riesz theorem for planar vector fields*, Math. Annalen, **320**(2001), 463–485.

[BH2] S. Berhanu and J. Hounie, *A Rudin-Carleson theorem for planar vector fields*, Mathematische Annalen, **347**(2009), 95–110.

[BH3] S. Berhanu and J. Hounie *A generalization of the Rudin-Carleson theorem*, Advances in phase space analysis of partial differential equations, Birkhauser Boston **78**(2009), 37–57.
[B] E. Bishop, *A General Rudin-Carleson Theorem*, Proc. Amer. Math. Soc., **13**(1962), 140–143.
[Br] R. G. M. Brummelhuis, *A Microlocal F. and M. Riesz Theorem with applications*, Rev. Mat. Iberoamericana, **5**(1989), 21–36.
[G] T. Gamelin, *Uniform algebras (second edition)*, Chelsea Pub. Comp., (1984)
[H] J. Hounie, *Global Cauchy problems modulo flat functions*, Advances in Math. **51** (1984), 240–252.
[K] E. Kann, *A new method for infinitesimal rigidity of surfaces*, J. Diff. Geom. **4** (1970), 5–12.
[Kr] V. V. Kravchenko, *Applied pseudoanalytic function theory*, Springer-Verlag (2009)
[LMX] N. Lerner, Y. Morimoto, and C.-J. Xu *Instability of the Cauchy-Kovalevskaya solution for a class of nonlinear systems*, American Jour. of Math. **132**(2010), 99–123.
[M1] A. Meziani, *Infinitesimal bending of homogeneous surfaces with nonnegative curvature*, Comm. Anal. Geom. **11** (2003), 697–719.
[M2] A. Meziani, *Planar complex vector fields and infinitesimal bendings of surfaces with non-negative curvature*, Contemp. Math. **400** (2006), 189–202.
[P] A. V. Pogorelov, *Bendings of surfaces and stability of the shells*, Nauka, Moscow, (1986); translation: Amer. Math. Soc., Providence, R.I., (1988).
[S] I. K. Sabitov, *Local theory of bending of surfaces*, Encyclopaedia Math. Sci. **48** , Springer-Verlag (1992).
[Ta] M. Taylor, *Pseudo-differential operators*, Princeton Univ. Press, (1970).
[Tr] F. Treves, *Approximate solutions to Cauchy problems*, J. Diff. Equations **11** (1972), 349–363.
[U] A. Uchiyama, *A constructive proof of the Fefferman-Stein decomposition of* $BMO(\mathbb{R}^n)$, Acta Math., **148**(1982), 215–241.
[V] I. N. Vekua, *Generalized Analytic Functions*, Pergamon Press (1962).

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
E-mail address: berhanu@.edu

Departamento de Matemática, UFSCAR, 13.565-905, São Carlos, Brasil
E-mail address: hounie@dm.ufscar.br