PATTERNS ON SURFACES OF REVOLUTION IN A DIFFUSION PROBLEM WITH VARIABLE DIFFUSIVITY

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Abstract. In this paper we address the question of existence of non-constant stable stationary solutions to the the diffusion equation $u_t = \mathrm{div}(a\nabla u) + f(u)$ on a surface of revolution whose border is supplied with zero Neumann boundary condition. Sufficient conditions on the geometry of the surface and on the diffusivity function a are given for the existence of a function f such the problem possesses such solutions.

1. Introduction

The main concern in this paper is to find sufficient conditions for existence of nonconstant stable stationary solutions (herein referred to as *patterns*, for short) to the diffusion problem

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \mathrm{div}_g[a(x)\nabla_g u] + f(u), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{D} \\[2ex] \dfrac{\partial u}{\partial \nu} &= 0, \qquad\qquad\qquad\qquad\qquad x \in \partial\mathcal{D} \end{cases} \tag{1.1}$$

where $a \in C^2(\mathcal{D})$ is positive in $\overline{\mathcal{D}}$, $\mathcal{D} \subset \mathbb{R}^3$ is a surface of revolution with border $\partial\mathcal{D}$, metric g and ν the outer co-normal vector to $\partial\mathcal{D}$. This work should be seen as an attempt to understand the role played by the diffusivity function a, the geometry of the surface $\mathcal{D}$ and the reaction term f (a source or sink, depending on its sign) in existence of patterns to (1.1).

Typically this kind of problem appears as a mathematical model in many distinct areas and, roughly speaking, a solution models the time evolution of the concentration of a diffusing substance in a heterogeneous medium whose diffusivity is given by a positive function a, under the effect of the term f. Usually the diffusivity is a property of the material which the surface is made of.

The question of existence and nonexistence of patterns for scalar diffusion equations of the type considered here seems to have been first addressed in [7, 3] in bounded domains of $\mathbb{R}^n$ when diffusivity is constant; indeed it is proved that no pattern exists if the domain is convex regardless of the function f and in [7] a non-convex domain and a source term f for which there exists a nonconstant stable stationary solution are provided.

Still in the case when a is a constant function, non-existence of patterns to the diffusion equation appearing in (1.1) on a Riemannian manifold without boundary

2000 *Mathematics Subject Classification.* Primary: 35K57, 35B36, 35R01; Secondary: 35B25, 35B35, 34K20, 58J32.

Key words and phrases. Patterns, diffusivity, diffusion, surface of revolution, stability.

with non-negative Ricci curvature was proved in [10], thus generalizing a similar result for surface of revolution found in [14].

For bounded domains in $\mathbb{R}^N$ the question of how the diffusivity function can give rise to patterns, or not, has been considered by some authors.

For the diffusion equation considered in one-dimensional domains, i.e. when $\mathcal{M}$ is an interval, subjected to zero Neumann boundary condition, a sufficient condition for nonexistence of patterns was found to be $a'' < 0$ in [4] and $(\sqrt{a})'' < 0$ in [16]. Such result in domains with dimension $N \geq 2$ still remains an open problem.

When the domain is an interval, existence of a diffusivity function a which gives rise to patterns to (1.1) was addressed in [5, 6]. These results were generalized to two-dimensional domains in [11], using a $\Gamma-$convergence approach, and for any dimension in [9], employing a variational approach to dynamical systems.

Let us now briefly state our main result. To this end consider a smooth curve C in $\mathbb{R}^3$ parametrized by $(\psi(s), 0, \chi(s))$, $s \in [l_1, l_2]$ and the borderless surface of revolution $\mathcal{M}$ generated by C. We consider $\mathcal{D} \subset \mathcal{M}$ a surface of revolution with boundary. Moreover, we suppose that the diffusivity function does not depend on the angular variable θ, so that, abusing notation, we set $a(x(s,\theta)) = a(s)$, $s \in [l_1, l_2]$. We prove that if $a \in C^2([l_1, l_2])$, $a > 0$ in $[l_1, l_2]$ and

$$\left(\frac{(a\psi)'}{\psi}\right)'(s_0) > 0 \tag{1.2}$$

for some $s_0 \in (0, l) \subset [l_1, l_2]$ and if $a(\cdot)$ satisfies

$$a'(s) \geq 0 \text{ in } (0, s_0) \text{ and } a'(s) \leq 0 \text{ in } (s_0, l), \tag{1.3}$$

then there exists $f \in C^1(\mathbb{R})$ such that problem (1.1) admits a nonconstant stable stationary solution.

Our proof stems from [1] which by its turn was inspired in [16]. In [16] the problem is considered in a interval and it is proved that if $a''(s_0) \geq 0$ for some s_0 in this interval then there exists f such that the corresponding problem possesses patterns. In [1] problem (1.1) is considered with constant diffusivity and the sufficient condition found for existence of a function f such that (1.1) possesses patterns is $(\psi'/\psi)'(s_0) > 0$ for some $s_0 \in (0, l)$. Note that our result generalizes [16] and [1] and although the mathematical procedure used in the proof is basically the same, the problem of diffusion on surfaces made of inhomogeneous material - and therefore with variable diffusivity - is a basic one in Mathematical Physics and therefore worthy been studied.

Still regarding problem (1.1) it was proved in [8] that if

$$-\left(\frac{\psi'}{\psi}\right)'(s) \geq \frac{(a'\psi)'}{2(a\psi)}(s)$$

for all $s \in (0, l)$ then (1.1) has no pattern. As expected, our conditions (see (1.2) and (1.3)) for existence of patterns implies that

$$-\left(\frac{\psi'}{\psi}\right)'(s_0) < \frac{(a'\psi)'}{2(a\psi)}(s_0).$$

The expression $-\left(\frac{\psi'}{\psi}\right)'(s)$ has a interesting geometrical meaning. Note that the Gaussian curvature of $\mathcal{M}$ is given by

$$K(s) = \frac{-\psi''(s)}{\psi(s)} \quad (s \in (0,l))$$

whereas

$$K_g(s) = \frac{\pm\psi'}{\psi} \quad (s \in (0,l))$$

represents the geodesic curvature of the parallel circles $s =$ constant on $\mathcal{M}$. Here the sign depends on the orientation of the parametrization. Therefore

$$K(s) + [K_g(s)]^2 = -\left(\frac{\psi'}{\psi}\right)'(s).$$

2. Preliminaries

We begin with some definitions and known results from Differential Geometry which will be used in the following sections.

2.1. **Surface of revolution.** Consider $\mathcal{M} = (\mathcal{M}, g)$ a $n-$dimensional Riemannian manifold with a metric in local coordinates $x = (x^1, x^2, \cdots, x^n)$ given by (using Einstein summation convention)

$$ds^2 = g_{ij}dx^i dx^j, \quad (g^{ij}) = (g_{ij}^{-1}), \quad |g| = \det(g_{ij}).$$

Given a smooth vector field X on $\mathcal{M}$, the divergence operator of X is defined as

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}X^i)$$

and the Riemannian gradient, denoted by ∇_g, of a sufficiently smooth real function ϕ defined on $\mathcal{M}$, as the vector field

$$(\nabla_g \phi)^i = g^{ij}\partial_j \phi.$$

We will see how the operator $\mathrm{div}_g(a(x)\nabla_g u)$ can be expressed for the particular case where $\mathcal{M}$ is a surface of revolution. Let C be the curve of $\mathbb{R}^3$ parametrized by

$$\begin{cases} x_1 = \psi(s) \\ x_2 = 0 \qquad (s \in I := [l_1, l_2]) \\ x_3 = \chi(s) \end{cases}$$

where $\psi, \chi \in C^2(I)$, $\psi > 0$ in (l_1, l_2) and $(\psi')^2 + (\chi')^2 = 1$ in I. Moreover,

$$\psi(l_1) = \psi(l_2) = 0, \tag{2.4}$$

and

$$\psi'(l_1) = -\psi'(l_2) = 1. \tag{2.5}$$

Let $\mathcal{M}$ be the surface of revolution parametrized by

$$\begin{cases} x_1 = \psi(s)\cos(\theta) \\ x_2 = \psi(s)\sin(\theta) \qquad (s,\theta) \in [l_1, l_2] \times [0, 2\pi). \\ x_3 = \chi(s) \end{cases} \tag{2.6}$$

Setting $x^1 = s, x^2 = \theta$ then a surface of revolution in $\mathbb{R}^3$ with the above parametrization is a 2-dimensional Riemannian manifold with metric

$$g = ds^2 + \psi^2(s)d\theta^2.$$

By (2.4) and (2.5) $\mathcal{M}$ has no boundary and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [2], for instance). The area element on $\mathcal{M}$ is $d\sigma = \psi d\theta ds$ and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2} \partial_\theta u \right).$$

Now, consider $\mathcal{D} \subset \mathcal{M}$ a surface of revolution with boundary, i.e., $\mathcal{D}$ is delimited by two circles C_0 and C_l, $l_1 < 0 < l < l_2$, parametrized in the local coordinates (s, θ) as follows:

$$C_0 : \begin{cases} s(t) = 0 \\ \theta(t) = t \end{cases} \quad \text{and} \quad C_l : \begin{cases} s(t) = l \\ \theta(t) = t \end{cases}$$

with $t \in [0, 2\pi)$.

Let ν be the outer normal vector of $\partial\mathcal{D} = C_0 \cup C_l$ lying in the tangent space $T_p(\mathcal{M})$ for any $p \in \partial\mathcal{D}$. We shall assume that $\partial\mathcal{D}$ is orientable for the outer normal to be well-defined and continuous.

The derivative of u in the direction of ν at $\partial\mathcal{D}$ is given by

$$\frac{\partial u}{\partial \nu} = \langle \nabla_g u, \nu \rangle,$$

where $\nu = \nu_1 \frac{\partial}{\partial s} + \nu_2 \frac{\partial}{\partial \theta}$ and $\left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial \theta} \right\}$ is the basis of $T_p(\mathcal{M})$.

Moreover it is supposed that

$$\chi'(s) \geq 0, \quad s \in (0, \delta) \cup (l - \delta, l) \tag{2.7}$$

for some $\delta > 0$. Thus there holds $\nu = \frac{\partial}{\partial s}$ on C_l and $\nu = -\frac{\partial}{\partial s}$ on C_0.

Although the diffusivity function a may depend on (s, θ) throughout this work we suppose, for the sake of simplicity, that it depends just on the variable s. Thus abusing notation, for simplicity sake, we set

$$a(x(s, \theta)) = a(s), \text{ for } x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)) \in \mathcal{D} \tag{2.8}$$

and therefore

$$\operatorname{div}_g(a(x)\nabla_g u) = a u_{ss} + \frac{(\psi a)_s}{\psi} u_s + \frac{a}{\psi^2} u_{\theta\theta}. \tag{2.9}$$

Hence throughout the text, problem (1.1) on $\mathcal{D}$ reduces to

$$\left. \begin{array}{l} u_t = a u_{ss} + \dfrac{(\psi a)_s}{\psi} u_s + \dfrac{a}{\psi^2} u_{\theta\theta} + f(u), \quad (s, \theta) \in (0, l) \times [0, 2\pi) \\ u'(0) = u'(l) = 0. \end{array} \right\} \tag{2.10}$$

2.2. **Stability analysis.** By a stationary solution of problem (1.1) we mean a solution to the problem

$$\begin{cases} \operatorname{div}_g(a(x)\nabla_g u) + f(u) = 0, \quad x \in \mathcal{D} \\ \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \partial\mathcal{D} \end{cases} \tag{2.11}$$

or equivalently, in our setting, a solution to (2.10) which does not depend on time. A stationary solution U of (2.10) is called *stable* (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|u(\cdot, t) - U\|_\infty < \epsilon$ for all $t > 0$, whenever $\|u_0 - U\|_\infty < \delta$, where $\|\cdot\|_\infty$ stands for the norm of the space $L^\infty(\mathcal{D})$. If there exists $\delta_1 > 0$ such that $\|u_0 - U\|_\infty < \delta_1$ implies that $\|u(\cdot, t) - U\|_\infty \to 0$, as $t \to \infty$, then U is called *asymptotically stable*. We say that U is *unstable* if it is not stable.

Regarding the eigenvalue problem for the linearized problem

$$\begin{cases} \operatorname{div}_g(a\nabla_g\phi) + f'(U)\phi + \lambda\phi = 0 \text{ in } \mathcal{D} \\ \dfrac{\partial\phi}{\partial\nu} = 0 \text{ in } \partial\mathcal{D} \end{cases} \tag{2.12}$$

the first eigenvalue λ_1 is given by

$$\lambda_1 = \min\left\{R_U(\phi) : \phi \in H^1(\mathcal{D}), |\phi|_{L^2(\mathcal{D})} = 1\right\} \tag{2.13}$$

where

$$R_U(\phi) = \int_{\mathcal{D}} \{a\,|\nabla_g\phi|^2 - f'(U)\phi^2\} d\sigma.$$

It is well known that if $\lambda_1 > 0$ then U is asymptotically stable and if $\lambda_1 < 0$ then U is unstable. If $\lambda_1 = 0$ then stability or instability can occur.

3. Existence of Patterns

Next theorem is the main result of this work.

Theorem 3.1. *If for some $s_0 \in (0, l)$ it holds that*

$$\left(\frac{(a\psi)'}{\psi}\right)'(s_0) > 0 \tag{3.14}$$

and if a satisfies

$$a'(s) \geq 0 \text{ in } (0, s_0) \text{ and } a'(s) \leq 0 \text{ in } (s_0, l), \tag{3.15}$$

then there exists $f \in C^1(\mathbb{R})$ such that problem (1.1) admits a nonconstant asymptotically stable solution.

We start with a lemma concerning stationary solutions of (1.1). This result was observed in [14] for $a \equiv 1$ and for convenience of the reader we will prove it in our case.

Lemma 3.2. *Every stationary solution u of problem (1.1) on $\mathcal{D}$, which depends on the angular variable θ, is unstable.*

Proof. By (2.9) u satisfies the equation

$$au_{ss} + \frac{(\psi a)_s}{\psi}u_s + \frac{a}{\psi^2}u_{\theta\theta} + f(u) = 0.$$

As the function a does not depend on θ we have that u_θ is an eigenfunction of (2.12) with corresponding eigenvalue $\lambda = 0$. Since u_θ must change sign it cannot be the eigenfunction corresponding to the lowest eigenvalue. Hence $\lambda_1 < 0$. □

Lemma 3.3. *Let v be a solution of problem (1.1). Let there exist $w \in C^2(\mathcal{D}) \cap C^1(\overline{\mathcal{D}})$ such that $w \geq 0$, w not identically zero on $\overline{\mathcal{D}}$ and*

$$\left.\begin{array}{ll} div_g(a\nabla_g w) + f'(v)w \leq 0, & \mathcal{D} \\ \dfrac{\partial w}{\partial\nu} > 0, & \partial\mathcal{D}. \end{array}\right\} \tag{3.16}$$

Then v is asymptotically stable.

Proof. Let λ_1 be the smallest eigenvalue of the linearized problem

$$\left.\begin{array}{ll} \mathrm{div}_g(a\nabla_g\phi)+f'(v)\phi+\lambda\phi=0, & \mathcal{D} \\ \dfrac{\partial\phi}{\partial\nu}=0, & \partial\mathcal{D} \end{array}\right\} \tag{3.17}$$

and let ϕ_1 be the corresponding eigenfunction. As $\phi_1>0$ in $\overline{\mathcal{D}}$ (see [15]), we have

$$\begin{aligned} 0 &\geq \int_{\mathcal{D}} \phi_1\left[\mathrm{div}_g(a\nabla_g w)+f'(v)w\right]d\sigma \\ &= -\int_{\mathcal{D}} \nabla_g\phi_1(a\nabla_g w)d\sigma+\int_{\partial\mathcal{D}}\phi_1 a\frac{\partial w}{\partial\nu}d\gamma+\int_{\mathcal{D}}\phi_1 f'(v)w d\sigma \\ &= \int_{\mathcal{D}} w\left[\mathrm{div}_g(a\nabla_g\phi_1)+f'(v)\phi_1\right]d\sigma+\int_{\partial\mathcal{D}} a\left[\phi_1\frac{\partial w}{\partial\nu}-w\frac{\partial\phi_1}{\partial\nu}\right]d\gamma \\ &= -\lambda_1\int_{\mathcal{D}} w\phi_1 d\sigma+\int_{\partial\mathcal{D}} a\phi_1\frac{\partial w}{\partial\nu}d\gamma \\ &> -\lambda_1\int_{\mathcal{D}} w\phi_1 d\sigma. \end{aligned}$$

It follows that $\lambda_1>0$ and then v is asymptotically stable. □

The following steps are essential for the proof of Theorem 3.1. Using (3.14) and the regularity of ψ and a, we can find a neighborhood V of s_0 such that

$$\left(\frac{(a\psi)'}{\psi}\right)'(s)>0, \quad \forall\ s\in V. \tag{3.18}$$

Consider four points R_1, R_2, R_3, R_4 in V such that

$$R_1<R_2<s_0<R_3<R_4.$$

Obviously,

$$\left(\frac{(a\psi)'}{\psi}\right)'(s)>0, \quad \forall\ s\in[R_1,R_4]. \tag{3.19}$$

Let $z_1=z_1(s)$ be the solution of the Cauchy problem

$$\left.\begin{array}{l} \left[\dfrac{(a\psi z)'}{\psi}\right]'-Bz=0 \quad \text{in } [0,R_2) \\ z(0)=0, \quad z'(0)=1 \end{array}\right\} \tag{3.20}$$

where

$$B>\overline{B}:=\max_{[0,l]}\left|\left[\frac{(a\psi)'}{\psi}\right]'\right|. \tag{3.21}$$

Analogously let $z_2=z_2(s)$ be the solution of the Cauchy problem

$$\left.\begin{array}{l} \left[\dfrac{(a\psi z)'}{\psi}\right]'-Bz=0 \quad \text{in } (R_3,l] \\ z(l)=0, \quad z'(l)=-1. \end{array}\right\} \tag{3.22}$$

We shall write $z_i(s)=z_i(s,B)$ $(i=1,2)$ to indicate the dependence of the solution on the parameter B.

Lemma 3.4. *The solution z_1 of problem (3.20) has the following properties:*

(i) $z_1 > 0$ *in* $(0, R_2)$*;*
(ii) $z_1(\cdot, B)$ *is increasing in* $[0, R_2)$ *for any* $B > \overline{B}$*;*
(iii) $z_1(s, \cdot)$ *is increasing on* $(\overline{B}, \infty)$ *for any* $s \in (0, R_2)$*;*
(iv) $\lim_{B\to\infty} z_1(s, B) = \infty$ *for any* $s \in (0, R_2)$*.*

Similarly, for z_2 *we have that:*

(i') $z_2 > 0$ *in* (R_3, l)*;*
(ii') $z_2(\cdot, B)$ *is decreasing in* $(R_3, l]$ *for any* $B > \overline{B}$*;*
(iii') $z_2(s, \cdot)$ *is decreasing in* $(\overline{B}, \infty)$ *for any* $s \in (R_3, l)$*;*
(iv') $\lim_{B\to\infty} z_2(s, B) = \infty$ *for any* $s \in (R_3, l)$*.*

Proof. (i) Recalling that $z_1(0) = 0$ and $z_1'(0) = 1$, let us assume that z_1 vanishes in $(0, R_2)$ and let $s_1 \in (0, R_2)$ be the first root of z_1 in $(0, R_2)$. Then

$$z_1(s_1) = 0 \text{ and } z_1(s) > 0, \quad \forall\, s \in (0, s_1).$$

Then for some $s_2 \in (0, s_1)$

$$z_1(s_2) = \max_{[0,s_1]} z_1 > 0,$$

i.e., $z_1'(s_2) = 0$ and $z_1''(s_2) \le 0$.

It follows that

$$\begin{aligned}
\left[\frac{(a\psi z_1)'}{\psi}\right]'(s_2) - Bz_1(s_2) &= \left[\left(\frac{(a\psi)'}{\psi}\right)' z_1 + \frac{(a\psi)'}{\psi} z_1'\right](s_2) \\
&+ \left(a'z_1' + az_1''\right)(s_2) - Bz_1(s_2) \\
&= (az_1'')(s_2) + z_1(s_2)\left[\left(\frac{(a\psi)'}{\psi}\right)'(s_2) - B\right] \\
&\overset{(*)}{<} 0,
\end{aligned}$$

what contradicts the definition of z_1. Note that in $(*)$ the fact that $a(s) > 0$ in I and (3.21) were used.

(ii) Suppose by contradiction that $\exists\, s_1 \in (0, R_2)$ such that

$$z_1'(s) > 0, \quad \forall\, s \in (0, s_1) \text{ and } z_1'(s_1) = 0.$$

Then $z_1''(s_1) \le 0$. On the other hand,

$$z_1''(s_1) = -\left(\frac{z_1}{a}\right)(s_1)\left[\left(\frac{(a\psi)'}{\psi}\right)'(s_1) - B\right] > 0,$$

since $a(s_1) > 0$ and $z_1(s_1) > 0$ (from (i)). Therefore z_1 is increasing in $(0, R_2)$.

(iii) Let $B_1 > B_2 \ge \overline{B}$ and note that

$$\left.\begin{aligned}
&\left[\frac{(a\psi z_1(s, B_1))'}{\psi}\right]' - B_2 z_1(s, B_1) \ge 0 \quad (0, R_2) \\
&z_1(0, B_1) = 0, \quad z_1'(0, B_1) = 1.
\end{aligned}\right\} \tag{3.23}$$

The inequality in the above problem occurs by virtue of

$$\left[\frac{(a\psi z_1(s, B_1))'}{\psi}\right]' - B_2 z_1(s, B_1) = (B_1 - B_2) z_1(s, B_1) \ge 0,$$

$\forall\ s \in (0, R_2)$.

Now, as $z_1(s, B_2)$ satisfies

$$\left.\begin{array}{l}\left[\dfrac{(a\psi z_1(s,B_2))'}{\psi}\right]' - B_2 z_1(s,B_2) = 0 \quad (0,R_2) \\ z_1(0,B_2) = 0, \quad z_1'(0,B_2) = 1\end{array}\right\} \tag{3.24}$$

following the procedure used to prove Theorem 13 in Chapter 1 of [12], we can prove that

$$z_1(s, B_2) \le z_1(s, B_1),\ \forall\ s \in (0, R_2).$$

(iv) Fix any $B_1 > \overline{B}$. By integrating the equation (3.20) and remembering that z_1 is a solution to (3.24), we get for any $B \ge B_1$,

$$(a\psi z_1(\eta, B))' = \psi \int_0^\eta B z_1(t, B)dt + \psi c_1.$$

Integrating again

$$a\psi z_1(s, B) = B \int_0^s \psi(\eta) \int_0^\eta z_1(t, B)dtd\eta + c_1 \int_0^s \psi(\eta)d\eta + c_2,$$

where c_1 and c_2 are constants independent of B. As $\psi > 0$ and $a > 0$ then using (iii) we obtain

$$z_1(s, B) \ge \frac{1}{a\psi}\left[B \int_0^s \psi(\eta) \int_0^\eta z_1(t, B_1)dtd\eta + c_1 \int_0^s \psi(\eta)d\eta + c_2\right].$$

The claim follows by letting $B \to \infty$.

The proof for z_2 is analogous. □

Now, we define the following function $z : [0, l] \to \mathbb{R}$,

$$z(s) := \begin{cases} z_1(s), & \text{if } s \in [0, R_2) \\ z_3(s), & \text{if } s \in [R_2, R_3] \\ z_2(s), & \text{if } s \in (R_3, l] \end{cases} \tag{3.25}$$

where z_3 is a positive smooth function such that z is smooth at the points $s = R_2$ and $s = R_3$. Under these conditions, by Lemma 3.4 (ii) and (ii') we have that $z_3'(R_2) > 0$ and $z_3'(R_3) < 0$. Then there exist $\overline{s} \in [R_2, R_3]$ such that $z_3'(\overline{s}) = 0$, and therefore we take z_3 such that s_0 (recall that $s_0 \in [R_2, R_3]$) is the only critical point of z_3. Consequently $z_3''(s_0) < 0$.

We take z_3 in this way to ensure that the function $a'z_3'$ is positive in $[R_2, R_3]$. Thus by Lemma 3.4 and by the assumptions on $a(\cdot)$ (see (3.15)) we have

$$a'z'(s) \ge 0,\ \forall\ s \in (0, l). \tag{3.26}$$

Furthermore, the function z is smooth in $[0, l]$, $z > 0$ in $(0, l)$ and $z(0) = z(l) = 0$.

Lemma 3.5. *Let the function z be defined by (3.25). Then there exists $f \in C^1(\mathbb{R})$ such that the function*

$$Z(s) := \int_0^s z(t)dt \quad (s \in [0, l]) \tag{3.27}$$

is a stationary nonconstant solution of problem (1.1).

Proof. Since $z > 0$ in $(0, l)$ we have that $u = Z(s)$ is increasing in $(0, l)$. Hence we can define the inverse function $X(u) = Z^{-1}(u)$. Put

$$f(u) := \begin{cases} -Bu - a(0) & \text{if } u \le 0 \\ -\dfrac{\frac{d}{du}\{(a\psi)[X(u)]z[X(u)]\}}{\psi[X(u)]\frac{d}{du}[X(u)]} & \text{if } 0 < u < Z(l) \\ -Bu + BZ(l) + a(l) & \text{if } u \ge Z(l). \end{cases} \tag{3.28}$$

It is not difficult to see that f is continuous in $\mathbb{R}$ and C^1 in $\mathbb{R} - \{0, Z(l)\}$. Note that $X(\cdot)$ is smooth and satisfies $X'(u) = 1/z(X(u)) > 0$, i.e. $f(u)$ is smooth in $(0, Z(l))$.

Therefore we have to prove that f is smooth at $u = 0$ and $u = Z(l)$. The smoothness at $u = 0$ will follow if we show that

$$f(u) = -Bu - a(0),$$

$\forall\, u \in (0, Z(R_2))$.

Integrating (3.20) in $(0, s)$, for any fixed s in $(0, R_2)$, we obtain

$$\frac{(a\psi z)'}{\psi}(s) - \frac{(a\psi z)'}{\psi}(0) = B\int_0^s z(t)dt,$$

i.e.

$$\frac{(a\psi z)'}{\psi}(s) = BZ(s) + a(0). \tag{3.29}$$

By the definition of f,

$$f(Z(s)) = -\frac{(a\psi z)'}{\psi}(s) \tag{3.30}$$

for any $s \in (0, l)$. Hence by (3.29) and (3.30)

$$f(Z(s)) = -BZ(s) - a(0),$$

for any $s \in (0, R_2]$. Since Z is increasing, $f(u)$ can be written as

$$f(u) = -Bu - a(0),$$

for any $u \in (0, Z(R_2))$. Similarly it is seen that

$$f(u) = -Bu + BZ(l) + a(l)$$

for any $u \in [Z(R_3), Z(l))$. Therefore $f \in C^1(\mathbb{R})$.

Now we prove that $Z(\cdot)$ is a stationary solution to problem (1.1) with f defined by (3.28). For all $s \in (0, l)$, $Z(s) \in (0, Z(l))$, thus

$$f(Z(s)) = -\frac{(a\psi z)'}{\psi}(s) = -\frac{(a\psi Z')'}{\psi}(s),$$

i.e.

$$\frac{(a\psi Z')'}{\psi}(s) + f(Z(s)) = 0 \quad s \in (0, l),$$

and further

$$Z'(0) = z(0) = 0 \quad \text{and} \quad Z'(l) = z(l) = 0.$$

Moreover, $Z(\cdot)$ is nonconstant since $Z' = z > 0$ in $(0, l)$. $\square$

Proof of Theorem 3.1. Let z be the function defined by (3.25) and $m_1 > 0$ and $m_2 > 0$ constants to be chosen later. Define

$$w(s) := \begin{cases} z(s) - m_1 z(R_1)(s - R_2)^3 & \text{if } s \in [0, R_2) \\ z(s) & \text{if } s \in [R_2, R_3] \\ z(s) + m_2 z(R_4)(s - R_3)^3 & \text{if } s \in (R_3, l]. \end{cases} \tag{3.31}$$

Note that $w > 0$ in [0,l], and w depends on the parameter B, because z depends on B (see (3.20), (3.22) and (3.25)).

We have (see (2.7))

$$\frac{\partial w}{\partial \nu}(0) = -w'(0), \quad \frac{\partial w}{\partial \nu}(l) = w'(l). \tag{3.32}$$

In order to use Lemma 3.3, we prove the following claim.

Claim: Let Z be the stationary solution of (1.1) defined by (3.27). Then there exist $m_1 > 0$, $m_2 > 0$ and $B > 0$ satisfying (3.21) such that

$$\left.\begin{array}{ll} div_g(a\nabla_g w) + f'(Z)w \leq 0, & \mathcal{D} \\ \dfrac{\partial w}{\partial \nu} > 0, & \partial\mathcal{D}. \end{array}\right\} \tag{3.33}$$

The above problem can be written as

$$\left.\begin{array}{l} \dfrac{(a\psi w')'}{\psi} + f'(Z)w \leq 0, \quad (0, l) \\ w'(0) < 0, \quad w'(l) > 0. \end{array}\right\} \tag{3.34}$$

Now we partition the interval $(0, l)$ as follows:

$$(0, l) = (0, R_2) \cup [R_2, R_3] \cup (R_3, l).$$

For any $s \in (0, R_2) \cup (R_3, l)$ we have that

$$f'(Z(s)) = -B. \tag{3.35}$$

By (3.20) and (3.22),

$$\frac{(a\psi z')'}{\psi} - Bz = -\left[\frac{(a\psi)'}{\psi}\right]' z - a'z', \tag{3.36}$$

$\forall\ s \in (0, R_2) \cup (R_3, l)$.

Thus, for any $s \in (0, R_2)$,

$$\begin{aligned}
\frac{(a\psi w')'}{\psi} + f'(Z)w &= \frac{(a\psi w')'}{\psi} - Bw \\
&= \frac{(a\psi z' - 3a\psi m_1 z(R_1)(s-R_2)^2)'}{\psi} - Bz \\
&+ Bz(R_1)m_1(s-R_2)^3 \\
&= \frac{(a\psi z')'}{\psi} - Bz - 3\frac{(a\psi)'}{\psi} m_1 z(R_1)(s-R_2)^2 \\
&- 6am_1 z(R_1)(s-R_2) + Bz(R_1)m_1(s-R_2)^3 \\
&\overset{(*)}{=} -\left[\frac{(a\psi)'}{\psi}\right]' z - a'z' + m_1 z(R_1)(R_2 - s)\left[6a\right. \\
&+ \left. 3\frac{(a\psi)'}{\psi}(s-R_2) - B(s-R_2)^2\right].
\end{aligned}$$

In (*) we used (3.36). Recall that $(0, R_2) = (0, R_1) \cup [R_1, R_2)$, $a'z' \geq 0$ in $(0, l)$ (see (3.26)) and that $z = z_1$ in $(0, R_2)$. Then

- in $(0, R_1)$,

$$\begin{aligned}
-\left[\left(\frac{(a\psi)'}{\psi}\right)' z\right](s) - a'z'(s) &\leq -\left[\left(\frac{(a\psi)'}{\psi}\right)' z\right](s) \\
&\leq \left|\left(\frac{(a\psi)'}{\psi}\right)' z\right|(s) \\
&\leq \overline{B} z(R_1),
\end{aligned}$$

  with $\overline{B}$ defined by (3.21) and
- in $[R_1, R_2)$,

$$-\left[\left(\frac{(a\psi)'}{\psi}\right)' z\right](s) - a'z'(s) \leq -\hat{B}z(s) \leq -\hat{B}z(R_1)$$

  where

$$\hat{B} := \min_{[R_1, R_2]} \left[\frac{(a\psi)'}{\psi}\right]',$$

  and $\hat{B} > 0$ due to (3.19).

By the above remarks, in $(0, R_1)$ it holds that

$$\frac{(a\psi w')'}{\psi} + f'(Z)w \leq z(R_1)\left[\overline{B} + m_1 R_2\left(3(CR_2 + 2a(s_0)) - B(R_1 - R_2)^2\right)\right], \tag{3.37}$$

where

$$C := \max_{[0,l]} \left| \frac{(a\psi)'}{\psi} \right|.$$

Similarly, in $[R_1, R_2)$

$$\frac{(a\psi w')'}{\psi} + f'(Z)w \leq z(R_1)\left[-\hat{B} + m_1 R_2(6a(s_0) + 3CR_2)\right]. \tag{3.38}$$

Now, if

$$m_1 := \frac{\hat{B}}{3R_2(CR_2 + 2a(s_0))}$$

a simple calculation shows that the right-sides of (3.37) and (3.38) are both negative if

$$B \geq \max\left\{\overline{B}, 3\left(1 + \frac{\overline{B}}{\hat{B}}\right)\left(\frac{CR_2 + 2a(s_0)}{(R_1 - R_2)^2}\right)\right\}.$$

It follows that

$$\frac{(a\psi w')'}{\psi} + f'(Z)w < 0 \text{ in } (0, R_2]. \tag{3.39}$$

Similarly we obtain

$$\frac{(a\psi w')'}{\psi} + f'(Z)w < 0 \text{ in } (R_3, l). \tag{3.40}$$

Now consider the interval $[R_2, R_3]$. Since Z is a stationary solution to problem (1.1), in $[R_2, R_3]$ we have

$$aZ'' + \frac{(a\psi)'}{\psi} Z' + f(Z) = 0.$$

Differentiating the above equation and remembering that $Z' = z$, we have

$$az'' + \frac{(a\psi)'}{\psi} z' + f'(Z)z = -\left[\frac{(a\psi)'}{\psi}\right]' z - a'z',$$

as $w = z$ in $[R_2, R_3]$

$$\frac{(a\psi w')'}{\psi} + f'(Z)w = aw'' + \frac{(a\psi)'}{\psi} w' + f'(Z)w = -\left[\frac{(a\psi)'}{\psi}\right]' z - a'z' < 0.$$

The last inequality holds due to the fact that $z > 0$ in $[R_2, R_3]$ and using (3.19) and (3.26).

Thus we conclude the first inequality in (3.34). It remains to prove that $w'(0) < 0$ and $w'(l) > 0$. Note that by Lemma 3.4 (iv) and (iv') we can take B sufficiently large such that

$$z(R_1) = z_1(R_1, B) > \frac{1}{3m_1 R_2^2}, \text{ and } z(R_4) = z_2(R_4, B) > \frac{1}{3m_2(l - R_3)^2}$$

thus implying that

$$w'(0) = 1 - 3m_1 z(R_1) R_2^2 < 0, \text{ and } w'(l) = -1 + 3m_2 z(R_4)(l - R_3)^2 > 0.$$

Therefore our *Claim* is proved. At last, by Lemmas 3.3 and 3.5, we have that Z is a stable nonconstant stationary solution of problem (1.1) with f given by (3.28). □

We finish this work by given a simple example of a surface and a diffusivity function which satisfy the hypotheses of Theorem 3.1. Consider the domain of revolution $\mathcal{D}$ obtained from the parametrization

$$\psi(s) = \frac{s^2}{4} + \frac{1}{2}, \quad \chi(s) = \frac{s}{4}\sqrt{4-s^2} + \arcsin(\frac{s}{2}),$$

$s \in (0,1)$, which resembles a frustum of a hyperboloid. Also consider

$$a(s) = -s^2 + s + 3,$$

i.e., $a(x) = -4\sqrt{x_1^2 + x_2^2} + \sqrt{4\sqrt{x_1^2 + x_2^2} - 2} + 5$, $x \in \mathcal{D}$.

A simple computation shows that the hypotheses (1.2) and (1.3) are satisfied by taking $s_0 = \frac{1}{2}$. Hence by Theorem 3.1 we conclude that there is f such that the problem (1.1) possesses patterns.

Remark 3.6. *Theorem 3.1 should hold for the case when $\mathcal{D}$ is a surface of revolution without border. However in this case f should be required to be analytic as in* [13].

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