On positive solutions for classes of positone/semipositone systems with multiparameters

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Abstract

We will deal with existence and nonexistence of solution for a system involving p,q-Laplacian and nonlinearity with multiparameter. We will employ the method of lower and upper solutions for prove the existence of solution.

keyword: Existence of positive solution, nonexistence of nontrivial solution, positone/semipositone systems

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1 Introduction

We will deal with existence of solution for the positone/semipositone system involving p,q-Laplacian and nonlinearity with multiparameter

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) + \mu g_1(x,u,v) & \text{in} \quad \Omega, \\ -\Delta_q v = \lambda f_2(x,u,v) + \mu g_2(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary C^2 and $f_i, g_i : \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$, $i = 1,2$, are Carathéodory functions, g_i, $i = 1,2$, are bounded on bounded sets. Moreover, there exists $h_i : \mathbb{R} \to \mathbb{R}$ continuous and nondecreasing such that $h_i(0) = 0$, $0 \leq h_i(s) \leq C(1 + |s|^{r-1})$, for all $s \in \mathbb{R}$, $r = \min\{p,q\}$, $C > 0$, $i = 1,2$, and the maps

$$\begin{array}{l} s \to f_1(x,s,t) + h_1(s) \text{ and } t \to f_2(x,s,t) + h_1(t), \\ s \to g_1(x,s,t) + h_2(s) \text{ and } t \to g_2(x,s,t) + h_2(t), \end{array} \tag{2}$$

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are nondecreasing for almost everywhere $x \in \Omega$. Also, we will prove the nonexistence of nontrivial solution for system (1) in the positone case.

In the scalar case, Castro, Hassanpour, and Shivaji in [4], using the lower and upper solutions method, focused their attention on a class of problems, so called semipositione problems, of the form

$$-\Delta u = \lambda f(u) \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a positive parameter, and $f : [0,\infty) \to \mathbb{R}$ is a monotone and continuous function satisfying the conditions $f(0) < 0$, $\lim\limits_{s\to\infty} f(s) = +\infty$, and also the sublinear condition at infinity, namely, $\lim_{s\to\infty} f(s)/s = 0$. Recently, in 2008, Perera and Shivaji [11] proved the existence of solution for problem

$$\begin{cases} -\Delta_p u &= \lambda f(x,u) + \mu g(x,u) \quad \text{in} \quad \Omega, \\ \quad u &= 0 \qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary C^2 and $f, g : \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$, are Carathéodory functions, g is bounded on bounded sets and $|f(x,t)| \geq a_0$ for all $t \geq t_0$, where a_0, t_0 are positive constants. Moreover, the existence of solution is assured for $\lambda \geq \lambda_0$ and small $0 < |\mu| \leq \mu_0$, for some $\lambda_0 > 0$ and $\mu_0 = \mu(\lambda_0) > 0$.

Many authors has been studied the existence of positive solutions for elliptic systems, due to great number of applications, for instance, reaction-diffusion problems, in fluid mechanics, in newtonian fluids, glaciology, population dynamics, etc, see [3, 8] and references therein.

Hai and Shivaji [9] applied the lower and upper solutions method for obtain the existence of solution for semipositone systems of the type

$$\begin{cases} -\Delta_p u &= \lambda f_1(v) \quad \text{in} \quad \Omega, \\ -\Delta_p v &= \lambda f_2(u) \quad \text{in} \quad \Omega, \\ u = v &= 0 \qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter, and $f_1, f_2 : [0,\infty) \to \mathbb{R}$ are monotone and continuous functions satisfying conditions $f_i(0) < 0$, $\lim\limits_{s\to+\infty} f_i(s) = +\infty$, $i = 1, 2$, and

$$\lim_{s\to+\infty} \frac{f_1(M(f_2(s))^{1/(p-1)})}{s^{p-1}} = 0, \quad \text{for all } M > 0. \tag{4}$$

While, Chhetri, Hai, and Shivaji in [6] proved an existence result for system (3) with the condition

$$\lim_{s\to+\infty} \frac{\max\{f_1(s), f_2(s)\}}{s^{p-1}} = 0, \tag{5}$$

instead of condition (4) above mentioned.

In 2007, Ali and Shivaji [1] proved of positive solution for system

$$\begin{cases} -\Delta_p u &= \lambda_1 f_1(v) + \mu_1 g_1(u) \quad \text{in} \quad \Omega, \\ -\Delta_q v &= \lambda_2 f_2(u) + \mu_2 g_2(v) \quad \text{in} \quad \Omega, \\ u = v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

when Ω is a smooth bounded domain in $\mathbb{R}^N$, λ_i, μ_i, $i = 1, 2$, are nonnegative parameters with $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large and

$$\lim_{x\to+\infty} \frac{f_1(M[f_2(x)]^{1/q-1})}{x^{p-1}} = 0,$$

for all $M > 0$, $\lim_{x\to+\infty} \frac{g_1(x)}{x^{p-1}} = 0$, and $\lim_{x\to+\infty} \frac{g_2(x)}{x^{q-1}} = 0$.

Our first result deal with the existence of solution for the system (1) with p,q-Laplacian operators and nonautonomous nonlinearity with multiparameter.

Notice that, we make no suppositions about the signs of $g_1(x, 0, 0)$ and $g_2(x, 0, 0)$, and hence can occur the positone case, $\lambda f_i(x, 0, 0) + \mu g_i(x, 0, 0) \geq 0$, $i = 1, 2$, the semipositone case, $\lambda f_i(x, 0, 0) + \mu g_i(x, 0, 0) < 0$, $i = 1, 2$, the case $\lambda f_1(x, 0, 0) + \mu g_1(x, 0, 0) \geq 0$ and $\lambda f_2(x, 0, 0) + \mu g_2(x, 0, 0) < 0$, or the case $\lambda f_1(x, 0, 0) + \mu g_1(x, 0, 0) < 0$ and $\lambda f_2(x, 0, 0) + \mu g_2(x, 0, 0) \geq 0$; for almost everywhere $x \in \Omega$.

Theorem 1.1 *Consider the system (1) and suppose (2) and that there exist* $a_0, \gamma, \delta > 0$ *and* $\alpha, \beta \geq 0$ *such that* $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, $(p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, *and*

$$|f_1(x, s, t)| \leq a_0|s|^\alpha|t|^\gamma \text{ and } |f_2(x, s, t)| \leq a_0|s|^\delta|t|^\beta, \tag{7}$$

for all $s, t \in (0, +\infty)$ *and* $x \in \Omega$. *In addition, suppose there exist* $a_1 > 0$, $a_2 > 0$, *and* $R > 0$ *such that*

$$f_i(x, s, t) \geq a_1, \quad i = 1, 2, \text{ for all } s > R \text{ and } t > R, \tag{8}$$

and

$$f_i(x, s, t) \geq -a_2, \quad i = 1, 2, \text{ for all } s, t \in (0, +\infty), \tag{9}$$

uniformly in $x \in \Omega$. *Then, there exists* $\lambda_0 > 0$ *such that for each* $\lambda > \lambda_0$, *there exists* $\mu_0 = \mu_0(\lambda) > 0$ *for which system* (1) *has a solution* $(u, v) \in C^{1,\rho_1}(\Omega) \times C^{1,\rho_2}(\Omega)$ *for some* $\rho_1, \rho_2 > 0$, *where each component is positive, whenever* $|\mu| \leq \mu_0$.

Let $\lambda_p > 0$ and $\lambda_q > 0$ be the first eigenvalue of $p-$Laplacian and $q-$Laplacian, respectively, where $\phi_p \in C^{1,\alpha_p}(\Omega)$ and $\phi_q \in C^{1,\alpha_q}(\Omega)$ are the respective positive eigenfunctions (see [7]).

Chen [5] proved the nonexistence of nontrivial solution for system

$$\begin{cases} -\Delta_p u &= \lambda u^\alpha v^\gamma, \quad \text{in} \quad \Omega, \\ -\Delta_q v &= \lambda u^\delta v^\beta, \quad \text{in} \quad \Omega, \\ u = v &= 0 \quad \text{on} \quad \Omega, \end{cases}$$

when Ω is a smooth bounded domain in $\mathbb{R}^N$, $p\gamma = q(p-1-\alpha)$, $(p-1-\alpha)(q-1-\beta)-\gamma\delta = 0$, and $0 < \lambda < \lambda_0$ where $\lambda_0 = \min\{\lambda_p, \lambda_q\}$ (see also [10]). We note that due to Young's inequality we have

$$u^{\alpha+1}v^{\gamma} \le \frac{1+\alpha}{p}u^p + \frac{p-1-\alpha}{p}v^q \text{ and } u^{\alpha+1}v^{\gamma} \le \frac{q-1-\beta}{q}u^p + \frac{\beta+1}{q}v^q.$$

Now, we will enunciated the nonexistence theorem for the system (1) that improves the result proved by Chen in [5].

Theorem 1.2 *Suppose that there exist $k_i > 0$, $i = 1, ..., 8$, such that*

$$\begin{aligned} |f_1(x,s,t)s| \le (k_1|s|^p + k_2|t|^q)\,,\ |f_2(x,s,t)t| \le (k_3|s|^p + k_4|t|^q)\,, \\ |g_1(x,s,t)s| \le (k_5|s|^p + k_6|t|^q)\,,\ \text{and}\ |g_2(x,s,t)t| \le (k_7|s|^p + k_8|t|^q)\,, \end{aligned} \tag{10}$$

for all $x \in \Omega$ and $s, t \in (0, +\infty)$. Then, the system (1) does not possess any nontrivial solution, for all λ, μ satisfying

$$|\lambda|(k_1+k_3) + |\mu|(k_5+k_7) < \lambda_p \text{ and } |\lambda|(k_2+k_4) + |\mu|(k_6+k_8) < \lambda_q. \tag{11}$$

Remark 1.1 *The most common functions that can be considered in Theorem* 1.1 *are as follows:*

$$f_1(x,s,t) = A(x)s^{\alpha}t^{\gamma} \text{ and } f_2(x,s,t) = B(x)s^{\delta}t^{\beta},$$

where $A(x), B(x)$ are continuous functions on Ω satisfying $\inf_{x\in\Omega} A(x) > 0$, $\sup_{x\in\Omega} A(x) < +\infty$, $\inf_{x\in\Omega} B(x) > 0$, and $\sup_{x\in\Omega} B(x) < +\infty$ for all $x \in \Omega$, $0 \le \alpha < p-1$, $0 \le \beta < q-1$, $(p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$, and $g_1(x,s,t)$ and $g_2(x,s,t)$ are any continuous functions on $\overline{\Omega} \times [0,+\infty) \times [0,+\infty)$ with $g_1(x,s,t)$ nondecreasing in variable s and $g_2(x,s,t)$ nondecreasing in variable t.

Remark 1.2 *The Theorem* 1.2 *can be applied for the functions of the kind*

$$\begin{aligned} f_1(x,s,t) = \sum_{i=1}^{m} a_i s^{\alpha_{1,i}} t^{\gamma_{1,i}},\ f_2(x,s,t) = \sum_{i=1}^{m} b_i s^{\delta_{1,i}} t^{\beta_{1,i}} \\ g_1(x,s,t) = \sum_{i=1}^{m} c_i s^{\alpha_{2,i}} t^{\gamma_{2,i}},\ \text{and}\ g_2(x,s,t) = \sum_{i=1}^{m} d_i s^{\delta_{2,i}} t^{\beta_{2,i}}, \end{aligned}$$

with $a_i, b_i, c_i, d_i \ge 0$, $p\gamma_{j,i} = q(p-1-\alpha_{j,i})$, and $(p-1-\alpha_{j,i})(q-1-\beta_{j,i}) = \gamma_{j,i}\delta_{j,i}$, for $j = 1, 2$ and $i = 1, \cdots, m$.

We will prove Theorem 1.1 in section 2 and Theorem 1.2 in section 3.

2 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using a general method of lower and upper-solution. This method, in the scalar situation, has been used by many authors, for instance, [2] and [3]. The proof for the system case can be found in [10].

2.1 Upper-solution

First of all, we will prove that system (1) possesses a upper-solution.

Consider $e_i \in C^{1,\alpha_i}(\overline{\Omega})$, with $\alpha_i > 0$, $i = 1, 2$, where (e_1, e_2) is a solution of system (1) with $f_1(x, u, v) = \frac{1}{\lambda}$, $f_2(x, u, v) = \frac{1}{\lambda}$, and $g_1(x, u, v) = g_2(x, u, v) = 0$, and each component is positive.

Claim. Since that $\delta > 0$, $\gamma > 0$, $0 \leq \alpha < p - 1$, $0 \leq \beta < q - 1$, and $(p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, there exist s_1 and s_2 such that

$$s_1 > \frac{1}{p-1}, \quad s_2 > \frac{1}{q-1}, \quad \text{and} \quad \frac{\delta}{q-1-\beta} < \frac{s_2}{s_1} < \frac{p-1-\alpha}{\gamma}. \tag{12}$$

In fact. Since that

$$0 < \frac{\delta}{q-1-\beta} < \frac{p-1-\alpha}{\gamma},$$

there exist $k > 0$ such that

$$\frac{\delta}{q-1-\beta} < k < \frac{p-1-\alpha}{\gamma}.$$

Define $\vartheta : (0, +\infty) \to \mathbb{R}$ by $\vartheta(\epsilon) = k(\frac{1}{p-1} + \epsilon)$. Evidently, we have

$$\lim_{\epsilon \to +\infty} \vartheta(\epsilon) = +\infty,$$

therefore, we get $\epsilon_0 > 0$ satisfying $\vartheta(\epsilon) > \frac{1}{q-1}$ for all $\epsilon > \epsilon_0$. Fixed $\epsilon > \epsilon_0$, we define $s_1 = \frac{1}{p-1} + \epsilon$ and $s_2 = \vartheta(\epsilon) = ks_1$. Then, $s_1 > \frac{1}{p-1}$, $s_2 > \frac{1}{q-1}$, and $\frac{s_1}{s_2} = k$, which prove the claim.

Then, by using (12), we obtain $\lambda_0 > 0$ such that

$$a_\lambda := \max\{a_0 \lambda^{s_1(\alpha - p + 1) + s_2\gamma},\ a_0 \lambda^{s_1\delta + s_2(\beta - q + 1)}\} < 1, \tag{13}$$

for all $\lambda > \lambda_0$. Moreover, there exist A and B positive constants satisfying

$$A^{p-1} = \lambda A^\alpha l^\alpha B^\gamma L^\gamma \text{ and } B^{q-1} = \lambda A^\delta l^\delta B^\beta L^\beta, \tag{14}$$

where $l = ||e_1||_\infty$ and $L = ||e_2||_\infty$.

Fixed $\lambda > \lambda_0$, we define

$$(\bar{u}(x), \bar{v}(x)) := (\lambda^{s_1} A e_1(x), \lambda^{s_2} B e_2(x)).$$

Notice that $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$ and $\bar{v} \in C^{1,\alpha_2}(\overline{\Omega})$.

Let $w \in W_0^{1,p}(\Omega)$ with $w(x) \geq 0$ for a.e. (almost everywhere) $x \in \Omega$. Then, we obtain

$$\int_\Omega |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla w \, dx = \lambda^{s_1(p-1)} A^{p-1} \int_\Omega w \, dx \tag{15}$$

and, for $z \in W_0^{1,q}(\Omega)$ with $z(x) \geq 0$ for a.e. $x \in \Omega$,

$$\int_\Omega |\nabla \bar{v}|^{q-2} \nabla \bar{v} \nabla z \, dx = \lambda^{s_2(q-1)} B^{q-1} \int_\Omega z \, dx. \tag{16}$$

On the other hand, by using (7), (13), and (14), we have

$$\begin{aligned}\lambda f_1(x,\bar{u}(x),\bar{v}(x)) &\leq \lambda a_0\lambda^{s_1\alpha}A^{\alpha}l^{\alpha}\lambda^{s_2\gamma}B^{\gamma}L^{\gamma}\\ &= \lambda a_0\lambda^{s_1(\alpha-p+1)+s_2\gamma}\lambda^{s_1(p-1)}A^{\alpha}l^{\alpha}B^{\gamma}L^{\gamma}\\ &\leq a_\lambda\lambda^{s_1(p-1)}A^{p-1}\end{aligned} \tag{17}$$

and

$$\lambda f_2(x,\bar{u}(x),\bar{v}(x)) \quad\leq\quad a_\lambda\lambda^{s_2(q-1)}B^{q-1}. \tag{18}$$

But, as $a_\lambda < 1$ for $\lambda > \lambda_0$, there exists $c_\lambda > 0$ such that

$$a_\lambda\lambda^{s_1(p-1)}A^{p-1}+c_\lambda \leq \lambda^{s_1(p-1)}A^{p-1} \text{ and } a_\lambda\lambda^{s_2(q-1)}B^{q-1}+c_\lambda \leq \lambda^{s_2(q-1)}B^{q-1}. \tag{19}$$

Also, since that g_i, $i = 1,2$, are bounded on bounded sets, there exists $\mu_0 = \mu_0(\lambda) > 0$ such that

$$|\mu||g_1(x,\bar{u}(x),\bar{v}(x))| \leq c_\lambda \text{ and } |\mu||g_2(x,\bar{u}(x),\bar{v}(x))| \leq c_\lambda \tag{20}$$

for all $|\mu| < \mu_0$. Then, we get by (17), (19), and (20) that

$$\begin{aligned}\lambda f_1(x,\bar{u}(x),\bar{v}(x))+\mu g_1(x,\bar{u}(x),\bar{v}(x)) &\leq a_\lambda\lambda^{s_1(p-1)}A^{p-1}\\ &\quad+|\mu g_1(x,\bar{u}(x),\bar{v}(x))|\\ &\leq a_\lambda\lambda^{s_1(p-1)}A^{p-1}+c_\lambda\\ &\leq \lambda^{s_1(p-1)}A^{p-1},\end{aligned} \tag{21}$$

and, from (18), (19), and (20),

$$\lambda f_2(x,\bar{u}(x),\bar{v}(x))+\mu g_2(x,\bar{u}(x),\bar{v}(x)) \quad\leq\quad \lambda^{s_2(q-1)}B^{q-1}, \tag{22}$$

for all $|\mu| < \mu_0$.

Hence, by (15) and (21), we conclude

$$\begin{aligned}\int_\Omega |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla w\,dx &\geq \lambda\int_\Omega f_1(x,\bar{u}(x),\bar{v}(x))w\,dx\\ &\quad+\mu\int_\Omega g_1(x,\bar{u}(x),\bar{v}(x))w\,dx.\end{aligned} \tag{23}$$

Analogously, from (16) and (22), we obtain

$$\begin{aligned}\int_\Omega |\nabla\bar{v}|^{q-2}\nabla\bar{v}\nabla z\,dx &\geq \lambda\int_\Omega f_2(x,\bar{u}(x),\bar{v}(x))z\,dx\\ &\quad+\mu\int_\Omega g_2(x,\bar{u}(x),\bar{v}(x))z\,dx.\end{aligned} \tag{24}$$

Thus, from (23) and (24), we see that $(\bar{u},\bar{v})$ is a upper-solution of system (1) with $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$ and $\bar{v} \in C^{1,\alpha_2}(\overline{\Omega})$.

2.2 Lower-solution

In this subsetion, we will prove that system (1) possesses a lower-solution.

Let us fix ξ and η such that

$$1 < \xi < \frac{p}{p-1} \text{ and } 1 < \eta < \frac{q}{q-1}. \tag{25}$$

From (8) and (9) we have $a_1 > 0$, $a_2 > 0$, and $R > 0$ such that

$$f_i(x,s,t) \geq a_1,\ i = 1,2, \text{ for all } s > R \text{ and } t > R, \tag{26}$$

and

$$f_i(x,s,t) \geq -a_2,\ i = 1,2, \text{ for all } s,t \in (0,+\infty), \tag{27}$$

uniformly in $x \in \Omega$.

Consider λ_p the eigenvalue associated to positive eigenfunction φ_p of the problem of eigenvalue of $p-$Laplacian operator, and λ_q the eigenvalue associated to positive eigenfunction φ_q of the problem of eigenvalue of $q-$Laplacian operator. We take a_3 and a_4 positive constants satisfying

$$a_3 > 2\frac{\lambda_p(a_2+1)\xi^{p-1}}{a_1} \text{ and } a_4 > 2\frac{\lambda_q(a_2+1)\eta^{q-1}}{a_1}, \tag{28}$$

and define

$$(\underline{u}(x), \underline{v}(x)) := (c_\lambda \varphi_p^{\xi}(x), d_\lambda \varphi_q^{\eta}(x)),$$

where

$$c_\lambda = \left(\frac{\lambda a_2 + 1}{a_3}\right)^{\frac{1}{p-1}} \text{ and } d_\lambda = \left(\frac{\lambda a_2 + 1}{a_4}\right)^{\frac{1}{q-1}}. \tag{29}$$

Thus, for $w \in W_0^{1,p}(\Omega)$ and $z \in W_0^{1,q}(\Omega)$ with $w(x) \geq 0$ and $z(x) \geq 0$ for a.e. $x \in \Omega$, we obtain

$$\begin{aligned}&\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla w\ dx \\ &= c_\lambda^{p-1}\xi^{p-1}\int_\Omega \left[\lambda_p \varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\right] w\, dx\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla z\, dx \\ &= d_\lambda^{q-1}\eta^{q-1}\int_\Omega \left[\lambda_q \varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\right] z dx.\end{aligned} \tag{31}$$

We know that $\varphi_p, \varphi_q > 0$ in Ω and $|\nabla\varphi_p|, |\nabla\varphi_q| \geq \sigma$ on $\partial\Omega$ for some $\sigma > 0$. Also, we can suppose that $||\varphi_p||_\infty = ||\varphi_q||_\infty = 1$. Furthermore, by using (25), it is easy to prove that there exists $\zeta > 0$ such that

$$\lambda_p \varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p \leq -a_3 \tag{32}$$

and

$$\lambda_q \varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q \leq -a_4, \tag{33}$$

in $\Omega_\zeta := \{x \in \Omega : dist(x, \partial\Omega) \leq \zeta\}$. But, we have by (25), (27), and (29) that

$$\begin{aligned} -c_\lambda^{p-1}\xi^{p-1}a_3 &= -(\lambda a_2 + 1)\xi^{p-1} \\ &\leq -(\lambda a_2 + 1) \\ &\leq \lambda f_1(x, \underline{u}, \underline{v}) - 1 \end{aligned} \tag{34}$$

and

$$-d_\lambda^{q-1}\eta^{q-1}a_4 \leq \lambda f_2(x, \underline{u}, \underline{v}) - 1, \tag{35}$$

for all $x \in \Omega$. Therefore, from (32), (33), (34), and (35), we get

$$c_\lambda^{p-1}\xi^{p-1}\left[\lambda_p \varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\right] \leq \lambda f_1(x, \underline{u}, \underline{v}) - 1 \tag{36}$$

and

$$d_\lambda^{q-1}\eta^{q-1}\left[\lambda_q \varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\right] \leq \lambda f_2(x, \underline{u}, \underline{v}) - 1, \tag{37}$$

in $\Omega_\zeta := \{x \in \Omega : dist(x, \partial\Omega) \leq \zeta\}$.

On the other hand, there exists $a_5 > 0$ such that $\varphi_p(x), \varphi_q(x) \geq a_5$ for all $x \in \Omega \setminus \Omega_\zeta$. Then, if $\lambda_0 > 0$ is provided of proof of exixtence of upper-solution, and by taking $\lambda_0 > 0$ bigger one, if necessary, we can suppose $\lambda_0 \geq \max\{1, \frac{2}{a_1}, \frac{R^{p-1}a_5^{-\xi(p-1)}a_3^{-1}}{a_2}, \frac{R^{q-1}a_5^{-\eta(q-1)}a_4^{-1}}{a_2}\} > 0$. Thus, we obtain

$$\underline{u}(x) = c_\lambda \varphi_p^\xi(x) \geq c_\lambda a_5^\xi > R \text{ and } \underline{v}(x) = d_\lambda \varphi_p^\xi(x) \geq d_\lambda a_5^\eta > R,$$

for all $x \in \Omega \setminus \Omega_\zeta$ and $\lambda > \lambda_0$. Therefore, by (26), we have

$$\lambda f_1(x, \underline{u}(x), \underline{v}(x)) - 1 \geq \lambda a_1 - 1 \text{ and } \lambda f_2(x, \underline{u}(x), \underline{v}(x)) - 1 \geq \lambda a_1 - 1 \tag{38}$$

for all $x \in \Omega \setminus \Omega_\zeta$ and $\lambda > \lambda_0$.

Claim. By (28) and $\lambda > \lambda_0 \geq \max\{1, \frac{2}{a_1}, \frac{R^{p-1}a_5^{-\xi(p-1)}a_3^{-1}}{a_2}, \frac{R^{q-1}a_5^{-\eta(q-1)}a_4^{-1}}{a_2}\}$, we have

$$a_3 > \frac{\lambda_p \xi^{p-1}(\lambda a_2 + 1)}{\lambda a_1 - 1} \text{ and } a_4 > \frac{\lambda_q \eta^{q-1}(\lambda a_2 + 1)}{\lambda a_1 - 1}. \tag{39}$$

In fact. Since that $\lambda > \frac{2}{a_1}$, we get

$$a_1 - \frac{1}{\lambda} > a_1 - \frac{a_1}{2} = \frac{a_1}{2},$$

so, as $\lambda > 1$ and by (28),

$$\begin{aligned}\frac{\lambda_p \xi^{p-1}(\lambda a_2+1)}{\lambda a_1-1} &= \frac{\lambda_p \xi^{p-1}(a_2+\frac{1}{\lambda})}{a_1-\frac{1}{\lambda}} \\ &< \frac{\lambda_p \xi^{p-1}(a_2+1)}{a_1-\frac{1}{\lambda}} \\ &< \frac{\lambda_p \xi^{p-1}(a_2+1)}{\frac{a_1}{2}} \\ &= \frac{2\lambda_p (a_2+1)\xi^{p-1}}{a_1} \\ &< a_3,\end{aligned}$$

and similarly

$$a_4 > \frac{\lambda_q \eta^{q-1}(\lambda a_2+1)}{\lambda a_1-1},$$

which prove the claim.

Then, from (30), (38), and (39), we achieve

$$\begin{aligned} c_\lambda^{p-1}\xi^{p-1} & \left[\lambda_p \varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\right](x) \\ & \leq c_\lambda^{p-1}\xi^{p-1}\lambda_p\varphi_p^{\xi(p-1)}(x) \\ & \leq \lambda_p c_\lambda^{p-1}\xi^{p-1} \\ & \leq \lambda_p \tfrac{\lambda a_2+1}{a_3}\xi^{p-1} \\ & \leq \lambda a_1 - 1 \\ & \leq \lambda f_1(x, \underline{u}(x), \underline{v}(x)) - 1 \end{aligned} \tag{40}$$

and, by (31), (38), and (39),

$$\begin{aligned} d_\lambda^{q-1}\eta^{q-1} & \left[\lambda_q \varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\right](x) \\ & \leq \lambda_q \tfrac{\lambda a_2+1}{a_4}\eta^{q-1} \\ & \leq \lambda f_2(x, \underline{u}(x), \underline{v}(x)) - 1, \end{aligned} \tag{41}$$

for all $x \in \Omega \setminus \Omega_\zeta$.

Thus, by combining (36), (37), (40), and (41), we obtain

$$\begin{aligned} c_\lambda^{p-1}\xi^{p-1} & \left[\lambda_p \varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\right](x) \\ & \leq \lambda f_1(x, \underline{u}(x), \underline{v}(x)) - 1 \end{aligned} \tag{42}$$

and

$$\begin{aligned} d_\lambda^{q-1}\eta^{q-1} & \left[\lambda_q \varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\right](x) \\ & \leq \lambda f_2(x, \underline{u}(x), \underline{v}(x)) - 1, \end{aligned} \tag{43}$$

for all $\lambda > \lambda_0$ and $x \in \Omega$. Moreover, if $\mu_0 = \mu_0(\lambda) > 0$ is provided of proof of existence of upper-solution; for each $\lambda > \lambda_0$, since that g_i, $i = 1, 2$, are bounded on bounded sets, replacing $\mu_0 > 0$ by another smaller, if necessary, we have

$$|\mu||g_1(x, \underline{u}(x), \underline{v}(x))| \leq 1 \text{ and } |\mu||g_2(x, \underline{u}(x), \underline{v}(x))| \leq 1 \tag{44}$$

for all $|\mu| < \mu_0$. Therefore, follows by (44) that

$$\lambda f_1(x, \underline{u}(x), \underline{v}(x)) - 1 \leq \lambda f_1(x, \underline{u}(x), \underline{v}(x)) + \mu g_1(x, \underline{u}(x), \underline{v}(x)) \tag{45}$$

and

$$\lambda f_2(x, \underline{u}(x), \underline{v}(x)) - 1 \leq \lambda f_2(x, \underline{u}(x), \underline{v}(x)) + \mu g_2(x, \underline{u}(x), \underline{v}(x)), \tag{46}$$

for all $|\mu| < \mu_0$ and $x \in \Omega$.

Hence, substituing (45) and (46) in (42) and (43), respectively, and by using (30) and (31) , we achieve

$$\begin{aligned} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla w dx &\leq \lambda \int_\Omega f_1(x, \underline{u}(x), \underline{v}(x)) w dx \\ &\quad + \mu \int_\Omega g_1(x, \underline{u}(x), \underline{v}(x)) w dx \end{aligned} \tag{47}$$

and

$$\begin{aligned} \int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla z dx &\leq \lambda \int_\Omega f_2(x, \underline{u}(x), \underline{v}(x)) z dx \\ &\quad + \mu \int_\Omega g_2(x, \underline{u}(x), \underline{v}(x)) z dx, \end{aligned} \tag{48}$$

so, we conclude that $(\underline{u}, \underline{v})$ is a lower-solution of system (1) with $\underline{u}, \underline{v} \in C^1(\Omega)$.

2.3 Proof of Theorem 1.1

We prove in the subsections 2.1 and 2.2 that there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$ there exist $\mu_0 = \mu_0(\lambda) > 0$ and $(\bar{u}, \bar{v})$, $(\underline{u}, \underline{v})$ that are upper-solution and lower-solution, respectively, of system (1), with $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$, $\overline{v} \in C^{1,\alpha_2}(\overline{\Omega})$, and $\underline{u}, \underline{v} \in C^1(\Omega)$, whenever $|\mu| < \mu_0$.

Let be $w \in W_0^{1,p}(\Omega)$ and $z \in W_0^{1,q}(\Omega)$ with $w, z \geq 0$ for a.e. in Ω. Then, from (28), (36), and (40), we have

$$\begin{aligned} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla w dx &\leq \lambda_p \tfrac{(\lambda a_2 + 1)}{a_3} \xi^{p-1} \int_\Omega w dx \\ &\leq \lambda \tfrac{a_2 + \frac{1}{\lambda}}{a_2 + 1} \tfrac{a_1}{2} \int_\Omega w dx \\ &\leq \lambda \tfrac{a_1}{2} \int_\Omega w dx, \end{aligned} \tag{49}$$

and, by (28), (37), and (41),

$$\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla z dx \quad \leq \quad \lambda \tfrac{a_1}{2}\int_\Omega z dx. \tag{50}$$

However, since that $s_1(p-1) > 1$ and $s_2(q-1) > 1$, changing $\lambda_0 > 0$ by other bigger one, if necessary, we can suppose

$$\lambda \frac{a_1}{2} \leq \min\{\lambda^{s_1(p-1)}A^{p-1},\ \lambda^{s_2(q-1)}B^{q-1}\} \tag{51}$$

for all $\lambda \geq \lambda_0$. Hence, from (15), (49), and (51), we conclude that

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla w dx \quad \leq \quad \int_\Omega |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla w dx \tag{52}$$

and by (16), (50), and (51),

$$\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla z dx \quad \leq \quad \int_\Omega |\nabla \bar{v}|^{q-2}\nabla \bar{v}\nabla z dx, \tag{53}$$

so, by the weak comparison principle (see [3, Lemma 2.2]), we obtain $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ for all $x \in \Omega$. Thus, by using (2), we obtain by the standard theorem of lower and upper solution (see [10, Theorem 2.4]) a solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of system (1) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ for almost everywhere in Ω. In particular, we see that $u, v \in L^\infty(\Omega)$ and $u(x) > 0$, $v(x) > 0$ for a.e. $x \in \Omega$. Then, by Theorem 1 of Tolksdorf [12], we obtain $u \in C^{1,\rho_1}(\Omega)$ and $v \in C^{1,\rho_2}(\Omega)$ for some $\rho_1, \rho_2 > 0$, so $u(x) > 0$, $v(x) > 0$ for all $x \in \Omega$. ■

3 Proof of Theorem 1.2

We will prove this result by contradiction.

Supposing by contradiction that there exists a nontrivial solution (u, v) of system (1), for some λ, μ satisfying (11), then by variational characterization of λ_p and λ_q, we achieve

$$\begin{aligned} \lambda_p \int_\Omega |u|^p dx \quad &\leq \quad \int_\Omega |\nabla u|^p dx \\ &\leq \quad \int_\Omega [(|\lambda|k_1 + |\mu|k_5)|u|^p + (|\lambda|k_2 + |\mu|k_6)|v|^q] dx \end{aligned} \tag{54}$$

and similarly

$$\lambda_q \int_\Omega |v|^q dx \leq \int_\Omega [(|\lambda|k_3 + |\mu|k_7)|u|^p + (|\lambda|k_4 + |\mu|k_8)|v|^q]\, dx. \tag{55}$$

From (54) and (55) we get

$$\begin{aligned} 0 < \{\lambda_p - [|\lambda|(k_1 + k_3) + |\mu|(k_5 + k_7)]\} \int_\Omega |u|^p dx \\ + \{\lambda_q - [|\lambda|(k_2 + k_4) + |\mu|(k_6 + k_8)]\} \int_\Omega |v|^q dx \leq 0, \end{aligned}$$

which is a contradiction. ■

References

[1] J. Ali, R. Shivaji; Positive solutions for a class of p-Laplacian systems with multiple parameters, J. Math. Anal. Appl. 335 (2007) 1013-1019.

[2] F. Brock, L. Iturriaga, J. Sánchez, P. Ubilla, Existence of positive solutions for p-laplacian problems with weights, Comm. Pure Appl. Anal. 5 (2006) 941-952.

[3] A. Cañada, P. Drábek, J.L. Gámez, Existence of positive solutions for some problems with nonlinear diffusion, Trans. Amer. Math. Soc. 349 (1997) 4231-4249.

[4] A. Castro, M. Hassanpour, R. Shivaji, Uniqueness of non-negative solutions for a semipositone problems with concave nonlinearity, Commun. Partial Diff. Eqns. 20 (1995) 1927-1936.

[5] C. Chen, On positive weak solutions for a class of quasilinear elliptic systems, Nonlinear Anal. 62 (2005) 751-756.

[6] M. Chhetri, D.D. Hai, R. Shivaji, On positive solutions for classes of p-laplacian semipositone systems, Discr. Cont. Dyn. Syst. 9 (2003) 1063-1071.

[7] M. Cuesta, Eigenvalue problems for the p-Laplacian with indefinite weights, Electronic J. Diff. Eqns. 2001 (2001) (33) 1-9.

[8] J. Fleckinger, R. Pardo, F. Thélin, Four-parameter bifurcation for a p-Laplacian system, Eletronic J. Diff. Eqns. 2001 (2001) (06) 1-15.

[9] D.D. Hai, R. Shivaji, An existence result on positive solutions for a class of p-laplacian systems, Nonlinear Anal. TMA 56 (2004) 1007-1010.

[10] O.H. Miyagaki, R. S. Rodrigues, On positive solutions for a class of singular quasilinear elliptic systems, J. Math. Anal. Appl. 334 (2007) 818-833.

[11] K. Perera, R. Shivaji, Positive solutions of multiparameter semipositone p-Laplacian problems, J. Math. Anal. Appl. 338 (2008) 1397-1400.

[12] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Eqns. 51 (1984) 126-150.