Deformation of Hyperbolic 3-Cone-Structures: Study of the non-Collapsing case

Alexandre Paiva Barreto

Dedicated to my wife Cynthia.

ABSTRACT. This work is devoted to the study of deformations of hyperbolic cone structures under the assumption that the length of the singularity remains uniformly bounded over the deformation. Let (M_i, p_i) be a sequence of pointed hyperbolic cone-manifolds with cone angles $\leq 2\pi$ and topological type (M, Σ), where M is a closed, orientable and irreducible 3-manifold and Σ an embedded link in M. Assuming that the length of the singularity remains uniformly bounded, we prove that either the sequence M_i collapses and M is Seifert fibered or a Sol manifold, or the sequence M_i does not collapse and, in this case, a subsequence of (M_i, p_i) converges to a complete three dimensional Alexandrov space endowed with a hyperbolic metric of finite volume on the complement of a finite union of quasi-geodesics. We apply this result to a question proposed by Thurston and to provide universal constants for hyperbolic cone structures when Σ is a small link in M.

1. Introduction

This text focusses on deformations of hyperbolic cone structures on a closed, orientable and irreducible 3-manifold M which are singular along a fixed embedded link $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$. Unlike complete hyperbolic structures, which are rigid by Mostow's Theorem, the hyperbolic cone structures can be deformed (see [**HK2**]). The difficulty in understanding these deformations lies in the possibility that the structure degenerates. In other words, the Hausdorff-Gromov limit (see section 2 for the definition) of the deformation is only an Alexandrov space which may have dimension strictly smaller than 3, although its curvature remains bounded from below by -1 (see [**Koj**]).

The works of Kojima, Hodgson-Kerckhoff and Fuji (see [**Koj**], [**HK**] and [**Fuj**]) show that the degeneration of the hyperbolic cone structures occurs if and only if the singular link of these structures intersects itself during the deformation. When the cone angles vary between 0 and π, the Dirichlet polyhedron of the hyperbolic cone structures is convex and we can use this fact to avoid self intersections of

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the singular link over deformations (see [**Koj**]). In this article we will not use this restrictive assumption and allow the cone angles vary until 2π.

We are interested in studying the following question that was proposed by W.Thurston in 80's:

QUESTION 1. *Let M be a closed and orientable hyperbolic 3-manifold and suppose there exists a simple closed geodesic Σ in M. Can the hyperbolic structure of M be deformed to the complete hyperbolic structure on $M-\Sigma$ through a path M_α of hyperbolic cone structures with topological type (M,Σ) and parametrized by the cone angles $\alpha \in [0,2\pi]$?*

If the deformation proposed by Thurston in question 1 exists, it is a consequence of Thurston's hyperbolic Dehn surgery Theorem that the length of the singular link must converge to zero. In particular, we have that its length remains uniformly bounded over the deformation. This conclusion give us a necessary condition for the existence of Thurston's desired deformation. For this reason, we will focus only on deformations of hyperbolic cone structures with this additional hypothesis on the singularity's length. We remark that this assumption is automatically verified when the holonomy representations of the hyperbolic cone structures are convergent.

We started studying this question in [**Bar2**]. In that paper we obtained the following result (see section 3 for the definition of collapse):

THEOREM 1. *Let M be a closed, orientable and irreducible 3-manifold and let $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ be an embedded link in M. Suppose there exists a sequence M_i of hyperbolic cone-manifolds with topological type (M,Σ) and having cone angles $\alpha_{ij} \in (0,2\pi]$ along Σ_j for $i \in \mathbb{N}$. Denote by $\mathcal{L}_{M_i}(\Sigma_j)$ the length of the connected component Σ_j of Σ in the hyperbolic cone-manifold M_i. If*

$$\sup\left\{\mathcal{L}_{M_i}(\Sigma_j)\ ;\ i \in \mathbb{N} \text{ and } j \in \{1,\ldots,l\}\right\} < \infty \tag{1.1}$$

and the sequence M_i collapses, then M is Seifert fibered or a Sol manifold.

As a consequence of this theorem, we obtained the following result yielding some information on Thurston's question 1 :

COROLLARY 1. *Let M be a closed and orientable hyperbolic 3-manifold and suppose there exists a finite union of disjoint simple closed geodesics Σ in M. Let M_α be a (angle decreasing) deformation of this structure along a continuous path of hyperbolic cone-structures with topological type (M,Σ) and having cone angles $\alpha \in (L,2\pi] \subset [0,2\pi]$ (the same for all components of Σ). If*

$$\sup\left\{\mathcal{L}_{M_\alpha}(\Sigma_j)\ ;\ \alpha \in (L,2\pi] \text{ and } j \in \{1,\ldots,l\}\right\} < \infty, \tag{1.2}$$

then every convergent sequence M_{α_i}, with α_i converging to L, does not collapse.

In this article, we will focus on non-collapsing deformations of hyperbolic cone structures. The principal result of this paper is the following one:

THEOREM 2. *Let M be a closed, orientable and irreducible 3-manifold and let $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ be an embedded link in M. Suppose there exists a sequence M_i of hyperbolic cone-manifolds with topological type (M,Σ) and having cone angles $\alpha_{ij} \in (0,2\pi]$ along Σ_j for $i \in \mathbb{N}$. Denote by $\mathcal{L}_{M_i}(\Sigma_j)$ the length of the connected component Σ_j of Σ in the hyperbolic cone-manifold M_i. If*

$$\sup\left\{\mathcal{L}_{M_i}(\Sigma_j)\ ;\ i \in \mathbb{N} \text{ and } j \in \{1,\ldots,l\}\right\} < \infty,$$

then one of the following statements holds:

i. the sequence M_i collapses and M is Seifert fibered or a Sol manifold,

ii. the sequence M_i does not collapse and there exists a sequence of points $p_{i_k} \in M - \Sigma$ such that the sequence (M_{i_k}, p_{i_k}) converges to a three-dimensional pointed Alexandrov space (Z, z_0). The Alexandrov space Z is endowed with a (non-complete) hyperbolic metric of finite volume on the complement of a finite union Σ_Z of quasi-geodesics. Moreover, Z is homeomorphic to M (in particular, Z is compact) if there exists $\varepsilon \in (0, 2\pi)$ such that the cone angles α_{ij} belong to $(\varepsilon, 2\pi]$. Moreover, the following three statements are equivalent:

(a) *Z is compact*

(b) $\inf \left\{ \textit{cone angle}_{M_{i_k}} (\Sigma_j) \ ; \ k \in \mathbb{N} \textit{ and } \Sigma_j \subset \Sigma \right\} > 0$

(c) $\inf \left\{ \mathcal{L}_{M_{i_k}} (\Sigma_j) \ ; \ k \in N \right\} > 0$, *for each component Σ_j of Σ.*

REMARK 1. *A by-product of the above theorem is that the length of a connected component Σ_j of Σ shrinks down to zero if and only if the same arises for its cone angles α_{ij} (when i goes to infinity). If the cone angles are supposed to be the same on all of the connected components of Σ, it follows from this fact (see Corollary 4) that the sequence of cone angles converges to zero if and only if the following three statements hold:*

i. $\sup \{ \mathcal{L}_{M_i} (\Sigma) \ ; \ i \in \mathbb{N} \} < \infty$

ii. $\lim\limits_{i \to \infty} \textit{diam} (M_i) = \infty$

iii. the sequence M_i does not collapse.

In general, the limiting singular locus Σ_Z need not be a disjoint union of quasi-geodesics since the singular link could intersect itself as cone angles are changed. It seems possible that the components of Σ_Z are continuous geodesics and that the limit is a hyperbolic cone manifold in a more general sense allowing singularities along a graph instead of a link. The main problem in understanding the limiting singular locus lies in the possibility that the singularity intersects itself infinitely many times at the limit. More precisely, Σ_Z may be a graph with infinite degree vertices. A better comprehension of the limiting singular locus is an interesting problem for further investigation.

As an application of Theorem 2, we obtain the following result related to the Thurston's question 1.

COROLLARY 2. *Let M be a closed and orientable hyperbolic 3-manifold and suppose there exists a finite union of disjoint simple closed geodesics Σ in M. Let M_α be a deformation of this structure along a continuous path of hyperbolic cone structures with topological type (M, Σ) and having cone angles $\alpha \in (\theta, 2\pi] \subset [0, 2\pi]$ (the same for all components of Σ). Then the following statements are equivalent*

i. $\theta = 0$ and the path M_α extends continuously to $[0, 2\pi]$, where M_0 denotes $M - \Sigma$ with the complete hyperbolic metric

ii. $\lim\limits_{\alpha \to \theta} \mathcal{L}_{M_\alpha} (\Sigma) = \lim\limits_{\alpha \to \theta} \sum\limits_{i=1}^{l} \mathcal{L}_{M_\alpha} (\Sigma_j) = 0$

iii. There exists a sequence $\alpha_i \in (\theta, 2\pi]$ converging to θ satisfying

$$\sup \{ \mathcal{L}_{M_\alpha} (\Sigma_j) \ ; \ \alpha \in (\theta, 2\pi] \ \textit{ and } j \in \{1, \ldots, l\} \} < \infty$$

and such that the sequence diam (M_{α_i}) *goes to infinity with* i.

Remark 2. *Note that Corollary 2 provides a necessary and sufficient condition for the existence of the deformation proposed by Thurston. Using the notations in the statement of Thurston's question 1, we have that*

$$\theta = 0 \Longleftrightarrow \lim_{\alpha\to\theta} \mathcal{L}_{M_\alpha}(\Sigma) = 0.$$

Supposing in addition that M is not Seifert fibered and that Σ is a small link in M, we have also the following theorem (see Corollaries 5 and 6) providing universal constants for hyperbolic cone structures with topological type (M, Σ).

Theorem 3. *Let* M *be a closed, orientable, irreducible and non Seifert fibered 3-manifold and let* Σ *be a small link in* M. *There exists a constant* $V = V(M, \Sigma) > 0$ *and a constant* $K = K(M, \varepsilon) > 0$*, for each* $\varepsilon \in (0, 2\pi)$*, such that:*

- *i.* $Vol(\mathcal{M}) > V$*, for every hyperbolic cone-manifold* $\mathcal{M}$ *with topological type* (M, Σ),
- *ii.* $diam(\mathcal{M}) < K$*, for every hyperbolic cone-manifold* $\mathcal{M}$ *with topological type* (M, Σ) *and having cone angles in the interval* $(\varepsilon, 2\pi]$.

2. Metric Geometry

In this section, we recall some definitions about Alexandrov spaces and Hausdorff-Gromov convergence. We refer to [**BBI**], [**BGP**], [**Gro**] and [**PP**] for details.

Given a metric space Z, the metric on Z will always be denoted by $d_Z(\cdot, \cdot)$. The open ball of radius $r > 0$ about a subset A of Z will be denoted by

$$B_Z(A, r) = \bigcup_{a\in A} \{z \in Z \,;\, d_Z(z, a) < r\}.$$

A metric space Z is called a *length space* (and its metric is called *intrinsic*) when the distance between every pair of points in Z is given by the infimum of the lengths of all rectifiable curves connecting them. When a minimizing geodesic between every pair of points exists, we say that Z is *complete*.

For all $k \in \mathbb{R}$, denote by $\mathbb{M}^2_k$ the complete and simply connected two dimensional Riemannian manifold of constant sectional curvature equal to k.

Let $\triangle(x, y, z) \subset Z$ be a geodesic triangle in Z with vertices $x, y, z \in Z$. The angle of $\triangle(x, y, z)$ at vertex x, for example, will be denoted by $\measuredangle_\triangle(x)$. A *comparison triangle* for $\triangle(x, y, z) \subset Z$ in $\mathbb{M}^2_k$ is a geodesic triangle $\overline{\triangle}_k(\overline{x}, \overline{y}, \overline{z}) \subset \mathbb{M}^2_k$ verifying

$$d_{\mathbb{M}^2_k}(\overline{x}, \overline{y}) = d_Z(x, y)\ ,\ d_{\mathbb{M}^2_k}(\overline{y}, \overline{z}) = d_Z(y, z) \text{ and } d_{\mathbb{M}^2_k}(\overline{z}, \overline{x}) = d_Z(z, x).$$

Definition 1. *A length space* Z *is called an Alexandrov space of curvature not smaller than* $k \in \mathbb{R}$ *if every point of* Z *has a neighborhood* U *such that, the angles of every triangle* $\triangle(x, y, z) \subset U$ *are well defined and satisfy the inequalities*

$$\measuredangle_\triangle(x) \geq \measuredangle_{\overline{\triangle}_k}(\overline{x})\ ,\ \measuredangle_\triangle(y) \geq \measuredangle_{\overline{\triangle}_k}(\overline{y}) \text{ and } \measuredangle_\triangle(z) \geq \measuredangle_{\overline{\triangle}_k}(\overline{z}).$$

for every comparison triangle $\overline{\triangle}_k(\overline{x}, \overline{y}, \overline{z}) \subset \mathbb{M}^2_k$ *of* $\triangle$.

Suppose from now on that Z is a n dimensional Alexandrov space of curvature not smaller than $k \in \mathbb{R}$ and fix a point $O \in \mathbb{M}_k^2$. Let us recall the definition of quasi-geodesics on an Alexandrov space (see [**PP**]):

Let $\gamma : [a,b] \to Z$ be 1-Lipschitz curve and let $z \in Z$ be a point verifying

$$0 < d_Z(z, \gamma(t)) < \frac{\pi}{\sqrt{k}} \tag{2.1}$$

for all $t \in [a,b]$. We say that a curve $\widetilde{\gamma} : [a,b] \to \mathbb{M}_k^2$ is a *development* of γ with respect to $z \in Z$ when

$$d_Z(z, \gamma(t)) = d_{\mathbb{M}_k^2}(O, \widetilde{\gamma}(t)),$$

for all $t \in [a,b]$.

Definition 2. *A 1-Lipschitz curve $\gamma : [a,b] \to Z$ is a quasi-geodesic of Z if it is parametrized by arc length and, for every point $z \in Z$ verifying (2.1) and every development $\widetilde{\gamma} : [a,b] \to \mathbb{M}_k^2$ of γ with respect to $z \in Z$, the curvilinear triangle bounded by the segments $O\widetilde{\gamma}(t \pm \delta)$ and the arc $\widetilde{\gamma}_{|[t-\delta,t+\delta]}$, where $t \in (a,b)$ and $\delta > 0$ sufficiently small, is convex.*

Given three points $x, y, z \in Z$, let $\overline{\triangle}_k(\overline{x}, \overline{y}, \overline{z})$ be a triangle in $\mathbb{M}_k^2$ verifying

$$d_{\mathbb{M}_k^2}(\overline{x}, \overline{y}) = d_Z(x,y) \ , \ d_{\mathbb{M}_k^2}(\overline{y}, \overline{z}) = d_Z(y,z) \ \text{ and } \ d_{\mathbb{M}_k^2}(\overline{z}, \overline{x}) = d_Z(z,x).$$

We denote by $\measuredangle_k(x\ ;\ y,z)$ the angle of $\overline{\triangle}_k(\overline{x}, \overline{y}, \overline{z})$ at $\overline{x}$. Note that this definition does not depend on the choice of the triangle $\overline{\triangle}_k(\overline{x}, \overline{y}, \overline{z})$.

Consider $z \in Z$ and $\lambda \in (0, \pi)$. The point z is said to be λ-*strained* if there exists a set $\{(a_i, b_i) \in Z \times Z\ ;\ i \in \{1, \ldots, n\}\}$, called a λ-strainer at z, such that $\measuredangle_k(z\ ;\ a_i, b_i) > \pi - \lambda$ and

$$\max\left\{\left|\measuredangle_k(z\ ;\ a_i, a_j) - \frac{\pi}{2}\right|,\ \left|\measuredangle_k(z\ ;\ b_i, b_j) - \frac{\pi}{2}\right|, \left|\measuredangle_k(z\ ;\ a_i, b_j) - \frac{\pi}{2}\right|\right\} < \lambda$$

for all $i \neq j \in \{1, \ldots, n\}$. The set $R_\lambda(Z)$ of λ-strained points of Z is called the *set of λ-regular points of Z*. It is a remarkable fact that $R_\lambda(Z)$ is an open and dense subset of Z.

Recall now, the notion of (pointed) Hausdorff-Gromov convergence (see [**BBI**]):

Definition 3. *Let (Z_i, z_i) be a sequence of (pointed) metric spaces. We say that the sequence (Z_i, z_i) converges in the (pointed) Hausdorff-Gromov sense to a (pointed) metric space (Z, z_0), if the following holds: For every $r > \varepsilon > 0$, there exist $i_0 \in \mathbb{N}$ and a sequence of (may be non continuous) maps $f_i : B_{Z_i}(z_i, r) \to Z$ ($i > i_0$) such that*

i. $f_i(z_i) = z_0$,
ii. $\sup\{d_{Z'}(f_i(z_1), f_i(z_2)) - d_Z(z_1, z_2)\ ;\ z_1, z_2 \in Z\} < \varepsilon$,
iii. $B_Z(z_0, r - \varepsilon) \subset B_Z(f_i(B_{Z_i}(z_i, r)), \varepsilon)$,
iv. $f_i(B_{Z_i}(z_i, r)) \subset B_Z(z_0, r + \varepsilon)$.

Let us point out that, throughout the rest of the paper, the term "converges" is going to stand for "converges in the (pointed) Hausdorff-Gromov sense".

Let (Z_i, z_i) be a convergent sequence of Alexandrov spaces with the same lower curvature bound $k \in \mathbb{R}$ and the same dimension $n \in \mathbb{N}$. The limit Alexandrov space must have the same lower curvature bound k, but can have dimension less than or equal to n (see [**BBI**, Corollary 10.8.25]). When the limit Alexandrov space has dimension n, Perelman's stability Theorem (see [**Kap**]) assures that it is homeomorphic to Z_i, for sufficiently large indexes.

It is a fundamental fact that the class of Alexandrov spaces of curvature not smaller than $k \in \mathbb{R}$ is precompact with respect to the Hausdorff-Gromov convergence (see [**Gro**, Proposition 5.2] and [**BBI**, Corollary 10.8.25]). More precisely, every sequence of pointed Alexandrov spaces of curvature not smaller than $k \in \mathbb{R}$ admits a convergent subsequence to an Alexandrov space with the same lower bound for the curvature.

Another important fact concerning Alexandrov spaces (see [**PP**]) is that the Hausdorff-Gromov limit of quasi-geodesics is a quasi-geodesic. More precisely, if $\gamma_i : [a,b] \to Z_i$ is a convergent sequence of quasi-geodesics, then the limit curve is a quasi-geodesic on the limit space.

3. Sequences of Hyperbolic Cone-Manifolds

Let M be a closed, orientable and irreducible differential manifold of dimension 3 and let $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ be an embedded link in M. A *hyperbolic cone-structure* with topological type (M, Σ) is a complete intrinsic metric on M such that every non-singular point (i.e. every point in $M - \Sigma$) has a neighborhood isometric to an open set of $\mathbb{H}^3$, the hyperbolic space of dimension 3, and that every singular point (i.e. every point in Σ) has a neighborhood isometric to an open neighborhood of a singular point of $\mathbb{H}^3(\alpha)$, the space obtained by identifying the sides of a wedge of angle $\alpha \in (0, 2\pi]$ in $\mathbb{H}^3$ by a rotation about the axis of the wedge. The angles α are called *cone angles* and they may vary from one connected component of Σ to the other. We emphasize that we only allow cone angles $\leq 2\pi$ in this paper. By convention, the complete hyperbolic structure M_0 on $M - \Sigma$ (see [**Koj2**]) is considered as a hyperbolic cone-structure with topological type (M, Σ) and cone angles equal to zero.

We point out that every hyperbolic cone-manifold is an Alexandrov space of curvature not smaller than -1. Furthermore, every geodesic on it is a quasi-geodesic.

A natural way to study degenerating deformations of hyperbolic cone-structures on (M, Σ) is to consider sequences of hyperbolic cone-structures converging (in the pointed Hausdorff-Gromov sense) to the limit Alexandrov space. To study these kind of sequences, we need the important notion of collapse which illustrates the intuitive fact that the volume of the sequence may or may not go to zero.

Definition 4. *We say that a sequence M_i of hyperbolic cone-manifolds with topological type (M, Σ) collapses if, for every sequence of points $p_i \in M - \Sigma$, the sequence $r_{inj}^{M_i - \Sigma}(p_i)$ consisting of their Riemannian injectivity radii in $M_i - \Sigma$ converges to zero. Otherwise, we say that the sequence M_i does not collapse.*

When a convergent sequence of hyperbolic cone-manifolds collapses, most of the geometric information can be lost. This happens because the dimension of the limit Alexandrov space is strictly smaller than 3 (see [**Bar2**]). On the non-collapsing case, however, the limit Alexandrov space must have dimension 3 and, in this case, many kinds of geometric information are preserved and can be used to study the deformation.

Given a sequence M_i of hyperbolic cone-manifolds with topological type (M, Σ), fix indices $i \in \mathbb{N}$ and $j \in \{1, \ldots, l\}$. For sufficiently small radius $R > 0$, the metric neighborhood

$$B_{M_i}(\Sigma_j, R) = \{x \in M_i \; ; \; d_{M_i}(x, \Sigma_j) < R\}$$

of Σ is a solid torus embedded in M_i. The supremum of the radius $R > 0$ satisfying the above property will be called *normal injectivity radius of* Σ_j *in* M_i and it is going to be denoted by $R_i(\Sigma_j)$. Analogously we can define $R_i(\Sigma)$, the *normal injectivity radius of* Σ. It is a remarkable fact (see [**Fuj**] and [**HK**]) that the existence of a uniform lower bound for $R_i(\Sigma)$ ensures the existence of a sequence of points $p_{i_k} \in M$ such that the sequence (M_{i_k}, p_{i_k}) converges to a pointed hyperbolic cone-manifold (M_∞, p_∞) with topological type (M, Σ). Moreover, M_∞ must be compact provided that the cone angles of M_{i_k} are uniformly bounded from below.

Let us also emphasize that the sequence $Vol(M_i)$ consisting of the Riemannian volumes of the hyperbolic manifolds $M_i - \Sigma$ is always uniformly bounded. More precisely, we have (see [**Dun**] and [**Fra**])

$$Vol(M_i) < Vol(M_0), \qquad (3.1)$$

where M_0 denotes the complete hyperbolic manifold that is homeomorphic to $M - \Sigma$.

The purpose of this section is to prove Theorem 2. It is divided into two parts. The first part contains some preliminary results whereas the remaining part deals with the proof of Theorem 2.

Let us point out that, throughout the rest of the paper, the term "component" is going to stand for "connected component"

3.1. Preliminary results. Let us recall some definitions and elementary results which will be important for the proof of Theorem 2. We will begin with the classification of two dimensional embedded tori in $M - \Sigma$ (see [**Bar2**]).

LEMMA 1. *Suppose that $M - \Sigma$ is hyperbolic and let T be a two dimensional torus embedded in $M - \Sigma$. Then T separates M. Moreover, one and only one of the following statements holds:*

i. T is parallel to a component of Σ (hence it bounds a solid torus in M),
ii. T is not parallel to a component of Σ and it bounds a solid torus in $M - \Sigma$,
iii. T is not parallel to a component of Σ and it is contained in a ball B of $M - \Sigma$. Furthermore, T bounds a region in B which is homeomorphic to the exterior of a knot in S^3.

Now let us recall the geometric classification of the thin part of a hyperbolic manifold.

DEFINITION 5. *Consider $\delta > 0$ and let M be a hyperbolic manifold of dimension 3 (without boundary and perhaps noncomplete). Define $M_{thin}(\delta)$, the δ-thin part of M, by*

$$M_{thin}(\delta) = \left\{q \in M \; ; \; r^M_{inj}(q) < \delta \;\; \text{and} \;\; \exp_q \;\text{is defined on}\; B_{T_qM}(0, 3\delta)\right\}.$$

The following result concerning the thin part of hyperbolic manifolds will be needed later.

PROPOSITION 1. *Let M be a hyperbolic manifold of dimension 3 (without boundary and perhaps noncomplete) of finite volume. If $\delta > 0$ is small enough, then each component of $M_{thin}(\delta)$ contains a maximal region which is isometric to one of the following models:*

i. the quotient of a metric neighborhood of a geodesic γ in $\mathbb{H}^3$ by a loxodromic element of $PSL_2(\mathbb{C})$ leaving γ invariant and whose translation length is not bigger than δ,

ii. a parabolic cusp of rank 2.

In addition, when $vol(M) < \infty$, it follows that M has finitely many ends.

This proposition is a consequence of the existence of a Margulis foliation for the thin part of a hyperbolic manifold. A proof for this proposition is given in [**BLP**, Theorem 5.3 and Corollary 5.5] where the authors study the thin part of hyperbolic cone-manifolds with topological type (M, Σ) and whose cone angles are not bigger than π. Note that the condition imposed on the cone angles is used only in the description of the singular components of the thin part. We summarize below their proof for the first part of the above proposition which, indeed, dispenses the angle condition.

Consider a hyperbolic manifold M and denote by $\pi : \widetilde{M} \to M$ the universal cover of M. Let $\delta > 0$ be the constant given by the Margulis lemma (see [**KM**, KM], [**BGS**] and [**BLP**]). Then for every component $\mathcal{P}$ of $M_{thin}(\delta)$, the stabilizer of a component of $\pi^{-1}(\mathcal{P}) \subset \widetilde{M}$ is an elementary subgroup of $PSL_2(\mathbb{C})$ generated by a loxodromic element or by at most two parabolic elements. Associated to this group we have a canonical foliation of $\mathbb{H}^3$. The pull-back of this foliation by a developing map gives a foliation on $\pi^{-1}(\mathcal{P})$ which is equivariant by the action of $\pi_1 M$. The quotient of this foliation is the Margulis foliation on $\mathcal{P}$.

To finish the proof, it is sufficient to show that the leaves of this foliation are two-dimensional tori.

First, we remark that the leaves are complete. This is a consequence of the fact that injectivity radius is constant on them (see [**Bar1**]). When the stabilizer of a component of $\pi^{-1}(\mathcal{P})$ is generated by a loxodromic element, the conclusion follows immediately. In the second case, we need to use the fact that the leaves are flat (they were obtained from horospheres) and the Gauss-Bonnet Theorem. The hypothesis that the volume of the manifold is finite excludes undesirable euclidean surfaces other than torus.

3.2. Proof of the Theorem 2. The purpose of this section is to study a non-collapsing sequence M_i. Without loss of generality, this hypothesis implies the existence of a sequence $p_i \in M - \Sigma$ satisfying

$$r_0 = \inf \left\{ r_{inj}^{M_i}(p_i) \ ; \ i \in \mathbb{N} \right\} > 0 \ ,$$

and such that the sequence (M_i, p_i) converges to a pointed Alexandrov space (Z, z_0). By definition of the pointed Hausdorff-Gromov convergence, the ball $B_Z(z_0, r_0)$ is isometric to a ball of radius r_0 in $\mathbb{H}^3$ and this implies that Z has dimension equal to 3.

We are interested in the case where the length of the singularity remains uniformly bounded, i.e. where

$$\sup \left\{ \mathcal{L}_{M_i}(\Sigma_j) \ ; \ i \in \mathbb{N} \, , \, j \in \{1, \ldots, l\} \right\} < \infty \ .$$

Fix $j \in \{1, \ldots, l\}$. We can suppose (passing to a subsequence if necessary) that

$$\sup \left\{ d_{M_i}(p_i, \Sigma_j) \ ; \ i \in \mathbb{N} \right\} < \infty \qquad \text{or} \qquad \lim_{i \in \mathbb{N}} d_{M_i}(p_i, \Sigma_j) = \infty.$$

In the first case, we can use again the precompactness to suppose that the component $\Sigma_j \subset M_i$, viewed as a sequence of Alexandrov spaces of dimension 1, converges to a closed curve Σ_j^Z in Z. Since Z has dimension 3 and it is the limit of a sequence of Alexandrov spaces with same dimension 3 and same lower curvature bound -1, we can conclude that Σ_j^Z is a quasi-geodesic in Z (see [**PP**]).

Summarizing, each component Σ_j of Σ satisfies one, and only one, of the following statements:

(1) $\sup\{d_{M_i}(p_i, \Sigma_j) \ ; \ i \in \mathbb{N}\} < \infty$ and Σ_j converges to a quasi-geodesic $\Sigma_j^Z \subset Z$,
(2) $\lim_{i\in\mathbb{N}} d_{M_i}(p_i, \Sigma_j) = \infty$.

This dichotomy allows us to write $\Sigma = \Sigma_0 \sqcup \Sigma_\infty$, where Σ_0 contains the components Σ_j of Σ which satisfy Item (1) and Σ_∞ those that satisfy Item (2).

The following lemma shows that the hypothesis of non-collapsing imposes restrictions on the length and on the cone angles of the singular components of Σ contained in Σ_0.

LEMMA 2. *Suppose that the sequence M_i does not collapse and let $p_i \in M - \Sigma$ be a sequence of points such that $r_0 = \inf\left\{r_{inj}^{M_i}(p_i) \ ; \ i \in \mathbb{N}\right\} > 0$. If*

$$L = \sup\{\mathcal{L}_{M_i}(\Sigma_j) \ ; \ i \in \mathbb{N} \ , \ j \in \{1, \dots, l\}\} < \infty \ ,$$

then the following inequalities holds:

i. $\inf\{\mathcal{L}_{M_i}(\Sigma_j) \ ; \ i \in \mathbb{N} \ , \ \Sigma_j \subset \Sigma_0\} > 0$,
ii. $\inf\{\alpha_{ij} \ ; \ i \in \mathbb{N} \ , \ \Sigma_j \subset \Sigma_0\} > 0$,
iii. $\sup\{R_i(\Sigma_j) \ ; \ i \in \mathbb{N}$ *and* $\Sigma_j \subset \Sigma_0\} < \infty$.

PROOF. Consider $\mathcal{R} > \sup\{d_{M_i}(p_i, \Sigma_j) \ ; \ i \in \mathbb{N} \ , \ \Sigma_j \subset \Sigma_0\} + r_0$. Note that, by construction, $\mathcal{R} < \infty$ and $B_{M_i}(p_i, r_0) \subset B_{M_i}(\Sigma_j, \mathcal{R})$, for all $i \in \mathbb{N}$ and all components Σ_j of Σ_0.

Fix $i \in \mathbb{N}$ and fix a component Σ_j of Σ_0. Let $\mathcal{A}$ be a region of $\mathbb{H}^3(\alpha_{ij})$ which is bounded by two planes orthogonal to the singular geodesic σ of $\mathbb{H}^3(\alpha_{ij})$ and having distance $\mathcal{L}_{M_i}(\Sigma_j)$ between them. Using a developing map for $M_i - \Sigma$ and the minimizing geodesics leaving Σ_j orthogonally, the manifold M_i can be developed in a compact domain $K \subset \mathcal{A}$ such that $Vol(K) = Vol(M_i)$.

Since $B_{M_i}(p_i, r_0) \subset B_{M_i}(\Sigma_j, \mathcal{R})$, the development of $B_{M_i}(p_i, r_0)$ in K is contained in $B_{\mathbb{H}^3(\alpha_{ij})}(\sigma, \mathcal{R}) \cap \mathcal{A}$. If V_0 represents the volume of a ball of radius r_0 in $\mathbb{H}^3$, we have

$$V_0 = Vol(B_{M_i}(p_i, r_0)) \leq Vol\left(B_{\mathbb{H}^3(\alpha_{ij})}(\sigma, \mathcal{R}) \cap \mathcal{A}\right) = \frac{\alpha_{ij}}{2}\mathcal{L}_{M_i}(\Sigma_j)\sinh^2(\mathcal{R})$$

and therefore

$$\mathcal{L}_{M_i}(\Sigma_j) \geq \frac{V_0}{\pi.\sinh^2(\mathcal{R})} > 0 \qquad \text{and} \qquad \alpha_{ij} \geq \frac{2V_0}{L.\sinh^2(\mathcal{R})} > 0 \ .$$

Finally, item (iii) follows from the fact that the sequence $Vol(M_i)$ is uniformly bounded from above (see 3.1). □

With the preceding notations, set

$$\Sigma_Z = \bigcup_{\Sigma_j \subset \Sigma_0} \Sigma_j^Z \subset Z \ .$$

We present now the main result for the non-collapsing:

THEOREM 4 (non-collapsing). *Suppose that there exists a sequence $p_i \in M - \Sigma$ satisfying*

$$r_0 = \inf \left\{ r_{inj}^{M_i}(p_i) \ ; \ i \in \mathbb{N} \right\} > 0$$

and such that the sequence (M_i, p_i) converges to a pointed Alexandrov space (Z, z_0) of dimension 3. If

$$\sup \{ \mathcal{L}_{M_i}(\Sigma_j) \ ; \ i \in \mathbb{N} \ , j \in \{1, \ldots, l\} \} < \infty \ ,$$

then the following assertions hold:

i. *$Z - \Sigma_Z$ is a hyperbolic 3-manifold of finite volume whose convex and unbounded ends are finite in number and are parabolic cusps of rank 2,*

ii. *Z is compact (and therefore homeomorphic to M) if and only if $\Sigma_\infty = \emptyset$,*

iii. *if Z is not compact, there is a bijection between the connected components of Σ_∞ and the complete ends of $Z - \Sigma_Z$. In fact, each unbounded end C_j of $Z - \Sigma_Z$ is the Hausdorff-Gromov limit of metric neighborhoods (homeomorphic to solid tori) $B_{M_i}(\Sigma_j, r_i)$ of a component Σ_j of Σ_∞, where $r_i > 0$ is an increasing sequence going off to infinity. In addition, the cone angles α_{ij} and the lengths of these components converge to 0.*

PROOF OF ITEM (I). According to [**Fuj**, Lemma 2], every point of $Z - \Sigma_Z$ is the limit of a sequence of points of $M_i - \Sigma$ whose injectivity radius is uniformly bounded from below. This implies that $Z - \Sigma_Z$ is a (without boundary and noncomplete) hyperbolic manifold. Note that the unbounded ends of Z are those of $Z - \Sigma_Z$. In view of Proposition (1), to prove item (i) it is sufficient to show the following:

Claim: $Vol\,(Z - \Sigma_Z) < \infty$.

Proof of Claim : Suppose for contradiction the statement is false. Let K_∞ be a compact set of $Z - \Sigma_Z$ whose Riemannian volume is strictly greater than $Vol(M_0)$, where M_0 is $M - \Sigma$ with its complete hyperbolic metric. Since the convergence is bilipschitz on compact subsets (see [**CHK**, Theorem 6.20]), there exists an index $i_0 \in \mathbb{N}$ and a compact subset K_{i_0} of $M_{i_0} - \Sigma$ (near K_∞) such that

$$Vol\,(M_0) < Vol_{M_{i_0}}(K_{i_0}) \leq Vol\,(M_{i_0}) \, .$$

This is however impossible since $Vol\,(M_{i_0}) < Vol\,(M_0)$ (see (3.1)). ◇

□

PROOF OF ITEMS (II) AND (III). If Z is compact then $\Sigma_\infty = \emptyset$. Suppose now that Z is not compact. By Lemma (2) we can choose $R > 0$ such that

$$B_{M_i}(\Sigma_j, R_i(\Sigma_j)) \subset B_{M_i}(p_i, R/2)$$

for all connected component Σ_j of Σ_0 and all $i \in N$. Let K be a compact subset of Z which contains the ball $B_Z(z_0, R)$ (and hence Σ_Z) in its interior and satisfies

$$\mathcal{Z} = Z - int\,(K) = C_1 \sqcup \ldots \sqcup C_m \ ,$$

where each $C_k \approx T^2 \times [0, \infty)$ is a cuspidal end of Z.

Consider a sequence $C_{1i} = T^2 \times [0, t_i]$ of compact subsets of C_1, where $t_i > 0$ is an unbounded and strictly increasing sequence.

Let $\varepsilon_i > 0$ be a sequence converging to zero. Without loss of generality, there exists (according to [**CHK**, Theorem 6.20]) a sequence of $(1+\varepsilon_i)$-bilipschitz embeddings $f_{1i} : C_{1i} \to M_i - \Sigma$ onto their images. Therefore, the sequence $B_{1i} = f_{1i}(C_{11})$ converges in the bilipschitz sense to the compact set C_{11}.

Consider now a sequence of holonomy representations $\zeta_{1i} : \mathbb{Z}\times\mathbb{Z} \to PSL_2(\mathbb{C})$ for the hyperbolic structures on the interior sets B_{1i}. According to [**CHK**, Theorem 6.22], we can assume that

$$\zeta_{1i} \circ (f_{1i})_* \longrightarrow \varphi_1 \ , \tag{3.2}$$

where $\varphi_1 : \mathbb{Z}\times\mathbb{Z} \to PSL_2(\mathbb{C})$ is a holonomy representation of the hyperbolic structure in the interior of C_1 and where $(f_{1i})_* : \mathbb{Z}\times\mathbb{Z} \to \pi_1(M-\Sigma)$ is the canonical homomorphism induced by the map f_{1i}.

Consider the torus $T_{1i} = f_{1i}\left(T^2 \times \{0\}\right)$ embedded in $M-\Sigma$. Since K contains the ball $B_Z(z_0, R)$, the torus T_{1i} cannot be parallel to a component Σ_j of Σ_0. For i sufficiently large, the torus T_{1i} cannot be contained in a ball of $M-\Sigma$. To see this, consider a homotopically nontrivial loop γ_1 on $T^2\times\{0\} \subset C_{11}$. Since C_1 is a parabolic cusp, $\varphi_1(\gamma_1)$ is a nontrivial parabolic element of $PSL_2(\mathbb{C})$ and therefore the convergence (3.2) implies that $\zeta_{1i}\circ(f_{1i})_*(\gamma_1)$ is not trivial for i very large. The same then holds for the sequence $(f_{1i})_*(\gamma_1)$.

According to Lemma 1, we can suppose that the torus T_{1i} bounds a solid torus W_{1i} in M (with perhaps a singular soul). Note that

$$\lim_{i\to\infty} diam_{M_i}(W_{1i}) = \infty, \tag{3.3}$$

because $f_{1i}(C_{1i}) \subset W_{1i}$, for all $i \in \mathbb{N}$.

We can repeat the same construction for each cusp C_k of $\mathcal{Z}$ in order to obtain sequences of embedded tori $T_{ki} \subset M-\Sigma$ ($k \in \{1,\ldots,m\}$ and $i\in\mathbb{N}$), each of then bounds solid torus W_{ki} in $M-\Sigma_0$. Furthermore whose diameters become infinite with i. This yields a sequence of 3-manifolds with torus boundary

$$\mathcal{M}_i = M_i - \bigcup_{k=1}^{m} W_{ki}$$

such that M can be obtained by Dehn filling on their boundary components. By construction, the sequence $\mathcal{M}_i$ converges to the compact K and then (by Perelman's stability theorem [**Kap**]), we can assume that the manifolds $\mathcal{M}_i$ are all homeomorphic to K.

For all $i\in\mathbb{N}$ and all $k\in\{1,...,m\}$, fix a homotopically nontrivial loop μ_{ki} in $T^2\times\{0\}\subset C_k$ satisfying:

- the loop $f_{ki}\circ\mu_{ki}$ bounds a disc in W_{ki},
- if, for some index $j\in\mathbb{N}$, a loop μ_{kj} belongs to the same homotopy class of the loop μ_{ki}, then $\mu_{kj} = \mu_{ki}$.

The rest of the proof is going to be divided in two cases depending on whether or not Σ_0 is empty.

1^st case : $\Sigma_0 = \emptyset$.

Since the link Σ was supposed to be non empty, it follows that $\Sigma_\infty \neq \emptyset$. Since the distance between p_i and Σ_∞ becomes infinite, we can assume that Σ_∞ is contained in the complement of $\mathcal{M}_i$. More precisely, we can also assume (see Lemma 1) that each solid torus of $M_i - \mathcal{M}_i$ contains at most one component of Σ_∞

and, in the latter case, this component corresponds to the soul of the solid torus in question.

The singular set Σ_∞ has a finite number of components. Passing to a subsequence if necessary, we obtain an one-to-one map which associates each component Σ_j of Σ_∞ to a component C_{k_j} of $\mathcal{Z}$, that is, the component Σ_j is contained in the component $W_{k_j i}$ of $M_i - \mathcal{M}_i$, for all $i \in \mathbb{N}$.

Recall that every connected component Σ_j of Σ_∞ satisfies $\lim\limits_{i \in \mathbb{N}} d_{M_i}(p_i, \Sigma_j) = \infty$. Since the tori $T_{k_j i}$ remains at a finite distance to the points p_i and they are parallel to the components Σ_j, we must have $\lim\limits_{i \to \infty} R_i(\Sigma_j) = \infty$.

Since $\Sigma_0 = \emptyset$ and thanks to [**Fuj**, Theorem 1], we have that the cone angles of Σ converge to zero and Z has a complete hyperbolic structure whose ends are associated with components of Σ_∞. In other words, the injection defined above between the components of Σ_∞ and the components of $\mathcal{Z}$ is, indeed, a bijection.

2$^{\text{nd}}$ case : $\Sigma_0 \neq \emptyset$.

Denote by Λ the subset of $\{1, \ldots, m\}$ containing the indices that are not associated with components of Σ_∞. Denote also by Ω the subset of $\{1, \ldots, m\}$ containing the indices that are associated with components of Σ_∞ whose sequence of cone angles does not converge to zero.

LEMMA 3. *There exist $i_0 \in \mathbb{N}$ satisfying: for each $k \in \Lambda \cup \Omega$, the homotopy classes of loops μ_{ki} $(i > i_0)$ are pairwise distinct.*

Proof of Lemma 3 :

Suppose for a contradiction that the statement of the lemma does not holds. Without loss of generality, there exists $k_0 \in \Lambda \cup \Omega$ such that all loops $\mu_{k_0 i}$ $(i \in \mathbb{N})$ belongs to the same homotopy class. By construction, this implies that the loops $\mu_{k_0 i}$ $(i \in \mathbb{N})$ are the same loop, say μ.

Suppose first that $k_0 \in \Lambda$. By construction,

$$\zeta_{k_0 i} \circ (f_{k_0 i})_* (\mu) = \zeta_{k_0 i} (f_{k_0 i} \circ \mu) = 1_{PSL_2(\mathbb{C})} \,, \tag{3.4}$$

for all $i \in \mathbb{N}$. Because $\varphi_{k_0}([\mu])$ is a nontrivial parabolic element of $PSL_2(\mathbb{C})$, we have a contradiction.

Suppose now that $k_0 \in \Omega$. Then $k_0 = k_j$, for some component Σ_j of Σ_∞ whose sequence of cone angles converges to $\alpha_{\infty j} \neq 0$. Since the maps $f_{k_0 i}$ are $(1 + \varepsilon_i)$-bilipschitz embeddings (with ε_i shrinks down to zero), the loops $f_{k_0 i} \circ \mu$ must have bounded lengths.

As noted in the preceding case, the sequence $R_i(\Sigma_j)$ of the normal injectivity radii of the component Σ_j goes off to infinity. Since $\alpha_{\infty j} \neq 0$, the sequence $\mathcal{L}_{M_i}(f_{k_0 i} \circ \mu)$ formed by the lengths of the loops $f_{k_0 i} \circ \mu$ cannot be bounded. This is a contradiction with above paragraph. ◇

As a consequence of the above lemma, we will show that the set $\Lambda \cup \Omega$ is empty. To do this, the following lemma will be needed:

LEMMA 4. *Given $k \in \Lambda$, there exists $i_0 = i_0(k) \in N$ such that the solid tori W_{ki} contains a simple closed geodesic σ_{ki}, for every $i > i_0$.*

Proof of Lemma 4 : Fix $k \in \Lambda$ and let

$$\delta = \frac{\inf \left\{ r_{inj}^{Z-\Sigma z}(z) \ ; \ z \in C_{k1} \right\}}{2} > 0.$$

Since the map $f_{ki_{|C_{k1}}} : C_{k1} \to B_{ki}$ becomes closer and closer to isometries, there exists $i_1 \in \mathbb{N}$ such that

$$r_{inj}^{M_i}(q) > \delta,$$

for all $i > i_1$ and for all $q \in B_{ki}$ (in particular, for all $q \in T_{ki}$).

Claim : *There is $i_2 \in \mathbb{N}$ such that, for all $i > i_2$, we can find a loop γ_{ki} in W_{ki} which is homotopically nontrivial in the interior $M - \Sigma$ and has length smaller than δ.*

Proof of Claim : Consider the loops consisting by two geodesic segments with same ends and equal lengths which, furthermore, are smaller than $\frac{\delta}{2}$. Note that there loops are always homotopically nontrivial, otherwise we would obtain, after development, two distinct geodesic arcs with the same ends and equal lengths in $\mathbb{H}^3$, what is not possible.

The fact that W_{ki} does not admit this type of loop in its interior is equivalent to saying that all points of W_{ki} have injectivity radius not smaller than $\frac{\delta}{2}$. This is a contradiction because the sequence $Vol(M_i)$ is uniformly bounded from above (see 3.1) and the diameter of components W_{ki} becomes infinite. $\diamond$

Consider $i_o = \max\{i_1, i_2\}$ and fix $i > i_0$. Let $\gamma_{ki} \subset W_{ki}$ be a loop as above. According to [**Koj**, Lemma 1.2.4], the loop γ_{ki} is freely homotopic (in $M - \Sigma$) to a closed geodesic $\sigma_{ki} \subset M - \Sigma$. Moreover, the length of σ_{ki} is smaller than δ because the length of loops is strictly decreasing along this homotopy. Because the points of the torus T_{ki} have injectivity radius bigger than δ, all the loops involved in this homotopy must lie entirely in the interior of W_{ki}. In particular, $\sigma_{ki} \subset W_{ki}$.

If σ_{ki} is not simple, then it gives rise to a loop γ'_{ki} consisting by two geodesic segments with same ends and equal lengths which are smaller than $\frac{\delta}{4}$. This implies that the injectivity radius of the ends of γ'_{ki} is smaller than $\frac{\delta}{4}$. We can apply the same construction for the loop γ'_{ki} in order to obtain a new closed geodesic $\sigma_{ki} \subset W_{ki}$ whose length is smaller than $\frac{\delta}{4}$. Since the injectivity radius of points of W_{ki} bounded from below by compactness, this process must end after a finite number of steps and therefore we can suppose that σ_{ki} is simple. This completes the proof of Lemma **4.** $\diamond$

The following lemma shows that Σ_∞ is not empty and the cone angles of its components goes to zero. Moreover the map between the components of Σ_∞ and the components of $\mathcal{Z}$ must be a bijection.

LEMMA 5. *The set $\Lambda \cup \Omega$ is empty.*

Proof of Lemma 5 : According to the above lemma, we can suppose there exists a simple closed geodesic σ_{ki} in the solid torus W_{ki}, for every $i \in \mathbb{N}$ and every $k \in \Lambda$. If the manifolds M_i are regarded as hyperbolic cone-manifolds with topological type (M, Σ'), where

$$\Sigma' = \Sigma \cup \bigcup_{k \in \Lambda} \sigma_{ki}$$

and the cone angles on the geodesics σ_{ki} are equal to 2π, it follows from Lemma 1 that the tori T_{ki} are parallel to the geodesics σ_{ki}. In addition, $M-\Sigma'$ admits a complete hyperbolic structure (see [**Koj2**]) that will be denoted by $\mathcal{M}_0$.

For all $i \in \mathbb{N}$ and all $k \in \Lambda$, denote the homotopy class of the loop μ_{ik} by $(p_{ki}, q_{ki}) \in \mathbb{Z}\times\mathbb{Z} \approx \pi_1 C_k$. Without loss of generality, the Thurston's hyperbolic Dehn surgery ([**CHK**, theorem 1.13]) gives a sequence of complete hyperbolic manifolds $\mathcal{M}(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im})$ diffeomorphic to $M-\Sigma$ and such that

$$V_i := Vol\left(\mathcal{M}\left(p_{1i}, q_{1i}, \ldots, p_{mi}, q_{mi}\right)\right) < Vol\left(\mathcal{M}_0\right), \tag{3.5}$$

where $(p_{ki}, q_{ki}) = \infty$, for all $i \in \mathbb{N}$ and all $k \in \{1, \ldots, m\} - \Lambda$.

Since, for each $k \in \Lambda$, the pairs $(p_{ki}, q_{ki})_{i\in\mathbb{N}}$ are pairwise distinct (the homotopy classes of μ_{ik} are pairwise distinct), a subsequence $\mathcal{M}(p_{1i_s}, q_{1i_s}, \ldots, p_{mi_s}, q_{mi_s})$ such that

$$\lim_{s\to\infty} \|(p_{ki_s}, q_{ki_s})\| = \lim_{s\to\infty} (p_{ki_s})^2 + (q_{ki_s})^2 = \infty \qquad \text{, for every } k \in \Lambda$$

always exists. Thurston's hyperbolic Dehn surgery then gives

$$\lim_{s\to\infty} V_{i_s} = Vol\left(\mathcal{M}_0\right). \tag{3.6}$$

Recall that the Riemannian volume of a complete hyperbolic manifold with finite volume is a topological invariant (Mostow's Theorem). Since the manifolds $\mathcal{M}(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im})$ are diffeomorphic, the sequence V_i must be constant. This contradicts the statements 3.5 and 3.6. Hence $M_i - \mathcal{M}_i$ cannot have nonsingular components. Therefore, $\Sigma_\infty \neq \emptyset$ and the map between the components of Σ_∞ and the components of $\mathcal{Z}$ is a bijection. □

Corollary 3. *Suppose that the sequence M_i does not collapse and verifies*

$$\sup\left\{\mathcal{L}_{M_i}(\Sigma_j)\ ;\ i \in \mathbb{N},\ j \in \{1, \ldots, l\}\right\} < \infty\ .$$

If there is $\varepsilon \in (0, 2\pi)$ such that the cone angles α_{ij} belong to $(\varepsilon, 2\pi]$, then there exists a sequence of points $p_{i_k} \in M-\Sigma$ such that the sequence (M_{i_k}, p_{i_k}) converges to a compact and 3-dimensional pointed Alexandrov space (Z, z_0) (in fact homeomorphic to M). Moreover, there exists a finite union of quasi-geodesics Σ_Z such that $Z-\Sigma_Z$ is a noncomplete hyperbolic manifold of finite volume.

Remark 3. *Suppose that Σ is not connected. If (M_i, p_i) is a sequence as in the statement of the Theorem 4, then the inequality*

$$\sup\left\{diam_{M_i}(\Sigma)\ ;\ i \in \mathbb{N}\right\} < \infty$$

is a necessary and sufficient condition to ensure that the sequence $diam(M_i)$ remains bounded.

We have also the following less immediate corollary:

Corollary 4. *Let M be a closed, orientable and irreducible 3-manifold and let Σ be an embedded link in M. Assume that there exists a sequence M_i of hyperbolic cone-manifolds with topological type (M, Σ) and having the same cone angles $\alpha_i \in (0, 2\pi]$ for all components of Σ. Then there is a pointed subsequence M_{i_k} converging to M_0 ($M-\Sigma$ with its complete hyperbolic metric) if and only if the following three conditions hold:*

i. $\sup\left\{\mathcal{L}_{M_i}(\Sigma)\ ;\ i \in \mathbb{N}\ \right\} < \infty$,
ii. $\sup\left\{diam(M_i)\ ;\ i \in \mathbb{N}\ \right\} = \infty$,

iii. the sequence M_i does not collapse.

PROOF. By Kojima's result (see [**Koj**]), the existence of a subsequence M_{i_k} converging to M_0 is equivalent to the convergence of the cone angles α_{i_k} to zero.

Suppose that the sequence α_i converges to zero. Without loss of generality, we can assume that $\alpha_i \in (0,\pi]$, for every $i \in \mathbb{N}$. According to [**Koj**], there exists a continuous path (parametrized by cone angles) of hyperbolic cone-structures with topological type (M,Σ) which connects the hyperbolic cone-structure of M_0 to the complete hyperbolic structure on $M-\Sigma$. Moreover, by uniqueness of the hyperbolic cone-structures with cone angles not bigger than π (see [**Koj**]), this path contains the hyperbolic cone-structures of M_i, for every $i \in \mathbb{N}$. Then for every point $p \in M$, the sequence (M_i, p) converges to $(M-\Sigma, p)$ with the complete hyperbolic structure. This implies the items (ii) and (iii). The item (i) is a consequence of Thurston's hyperbolic Dehn surgery theorem which implies that the sequence $\mathcal{L}_{M_i}(\Sigma)$ converges to zero.

Conversely, suppose now that items (i), (ii) and (iii) are true. Then there exists a sequence of points $p_{i_k} \in M-\Sigma$ satisfying

$$\inf\left\{ r^{M_i}_{inj}(p_{i_k}) \ ;\ k \in \mathbb{N} \right\} > 0$$

and such that the sequence (M_{i_k}, p_{i_k}) converges to a noncompact and 3-dimensional pointed Alexandrov space (Z, z_0). Corollary 3 then shows that the sequence α_i must converge to zero. □

4. Applications

4.1. Small links. An embedded link Σ in a 3-manifold M is called small (in M) if it has an open tubular neighborhood U such that $M-U$ does not contain an embedded essential surface whose boundary is empty or an union of meridians of Σ. An important fact due to W.Thurston and A.Hatcher (see [**HT**, Lemma 3]) is that every 3-manifold containing a small link does not admit an embedded essential surface.

Given a 3-manifold M, let Σ be an embedded link in M. Suppose there exists a sequence M_i of hyperbolic cone-manifolds with topological type (M,Σ) and consider the sequence $\mathcal{L}_{M_i}(\Sigma)$ formed by the lengths of the singular set Σ in M_i. As a consequence of the Culler-Shalen theory (see [**CS**]), the holonomy representations of M_i are convergent. Therefore, we have the following proposition:

PROPOSITION 2. *Let M_i be a sequence of hyperbolic cone-manifolds with topological type (M,Σ). If Σ is a small link in M, then*

$$\sup\left\{\mathcal{L}_{M_i}(\Sigma_j) \ ;\ i \in \mathbb{N} \textit{ and } \Sigma_j \textit{ component of } \Sigma\right\} < \infty.$$

When Σ is a small link in M, Theorem 2 yields the following corollaries:

COROLLARY 5. *Suppose that M is a closed, orientable, irreducible and non Seifert fibered 3-manifold and let Σ be an embedded small link in M. Then there exists a constant $V = V(M,\Sigma) > 0$ such that $Vol(\mathcal{M}) > V$, for every hyperbolic cone-manifold $\mathcal{M}$ with topological type (M,Σ) and having cone angles $\leq 2\pi$.*

PROOF. First note that M is not a Sol manifold. In fact every Sol manifold is foliated by essential two dimensional tori and this is not possible since Σ is small (see [**HT**, Lemma 3]).

Suppose that the lower bound V does not exist. Since Σ is small in M, the non-existence of V implies the existence of a sequence of hyperbolic cone-manifolds $\mathcal{M}_i$ with topological type (M,Σ) satisfying

- $\sup\{\mathcal{L}_{\mathcal{M}_i}(\Sigma_j)\ ;\ i\in\mathbb{N}$ and Σ_j component of $\Sigma\}<\infty$,
- the sequence $Vol(\mathcal{M}_i-\Sigma)$ formed by the Riemannian volumes of the hyperbolic manifolds $\mathcal{M}_i-\Sigma$ shrinks down to to zero (and therefore the sequence $\mathcal{M}_i$ collapses).

According to Theorem 2, M must be Seifert fibered and this contradicts our hypothesis. □

Corollary 6. *Suppose that M is a closed, orientable, irreducible and non Seifert fibered 3-manifold and let Σ be an embedded small link in M. Given $\varepsilon\in(0,2\pi)$, there is a constant $K=K(M,\varepsilon)>0$ such that $diam(\mathcal{M})<K$, for every hyperbolic cone-manifold $\mathcal{M}$ with topological type (M,Σ) and having cone angles belonging to $(\varepsilon,2\pi]$.*

Proof. As seen in the previous corollary, M is not a Sol manifold. Fix $\varepsilon\in(0,2\pi)$ and suppose that the upper bound K does not exist. Since Σ is small in M, the non-existence of K implies the existence of a sequence of hyperbolic cone-manifolds $\mathcal{M}_i$ with topological type (M,Σ), having cone angles $\alpha_{ji}\in(\varepsilon,2\pi]$ and satisfying

i. $\sup\{\mathcal{L}_{\mathcal{M}_i}(\Sigma_j)\ ;\ i\in\mathbb{N}$ and Σ_j component of $\Sigma\}<\infty$,
ii. the sequence $diam(\mathcal{M}_i)$ formed by the diameters of the hyperbolic cone-manifolds $\mathcal{M}_i$ go to infinity.

Since M is neither Seifert fibered nor a Sol manifold, it follows from item (i) and Theorem 2 that the sequence $\mathcal{M}_i$ does not collapse. Moreover, since the cone angles α_{ji} belong to $(\varepsilon,2\pi]$, it follows that the sequence $diam(\mathcal{M}_i)$ is bounded and this yields a contradiction with item (ii). □

4.2. Proof of Corollary 2. First, we would like to recall that the existence of a deformation M_α as in Corollary 2 is a consequence of the Local Deformation Theorem due to Hodgson and Kerckhoff [**HK2**].

Proof. The implication $(i\Rightarrow ii)$ is immediate (see [**Koj**]). Suppose now that the sequence $\mathcal{L}_{M_\alpha}(\Sigma)$ converges to 0 when α converges to θ. Then

$$\sup\left\{\mathcal{L}_{M_{\alpha_i}}(\Sigma_j)\ ;\ i\in\mathbb{N}\text{ and }\Sigma_j\text{ component of }\Sigma\right\}<\infty,$$

for every sequence $\alpha_i\in(\theta,2\pi]$ converging to θ. Consider such a sequence α_i. Since M is hyperbolic (and therefore is neither Seifert fibered nor a Sol manifold), it follows from Theorem 2 that the sequence M_{α_i} does not collapse. Moreover, since the sequence $\mathcal{L}_{M_{\alpha_i}}(\Sigma)$ converges to zero, we must have $\lim\limits_{i\to\infty} diam(M_{\alpha_i})=\infty$. This concludes the proof of the implication $(ii\Rightarrow iii)$.

To prove $(iii\Rightarrow i)$ take a sequence α_i satisfying item (iii). Again by Theorem 2, it follows that the sequence M_{α_i} does not collapse. Moreover, since the sequence $diam(M_{\alpha_i})$ is not bounded, we must have $\theta=0$ because all the components of Σ have the same cone-angle. Then, by Kojima's work (see [**Koj**]), it follows that M_i converges to M_0. □

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Universidade Federal de São Carlos

E-mail address: alexandre@dm.ufscar.br