ON THE BOREL PROPERTY FOR SOLUTIONS TO SYSTEMS OF COMPLEX VECTOR FIELDS

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Abstract. In this work we study the Borel property for smooth solutions to systems of complex vector fields associated to locally integrable structures. Inspired by the recent article [LdS], in which the Borel property was studied for generic submanifolds of the complex space, we prove similar results in this more general set up. In particular we obtain, for the case of corank one structures, a necessary and sufficient condition for the validity of the Borel property.

Introduction

The classical Borel Theorem states that any formal power series $\mathfrak{s} \in \mathbb{C}[[X_1, \dots, X_N]]$ in N indeterminates is the (formal) Taylor series at the origin of a smooth function f on $\mathbb{R}^N$, that is, $\mathfrak{s} = \sum_\alpha (\partial^\alpha f)(0) X^\alpha/\alpha!$. A natural and relevant question is the extension of such a statement for solutions to a system of (partial) differential equations: given a formal power series $\mathfrak{s}$, which is a formal solution to the system, can we find a smooth solution to the system whose (formal) Taylor series at a given point equals $\mathfrak{s}$?

Such a question was addressed to in [LdS] when the system is defined by the CR vector fields defined on a generic submanifold $\mathcal{M}$ of the complex space. The authors obtained a sufficient condition for the validity of the Borel property for CR functions at $p \in \mathcal{M}$ stated in terms of the existence of a "peak" function for $\mathcal{M}$ near p, that is, a smooth CR function defined near and vanishing to finite order at p whose image avoids a half-line issuing from the origin (cf. Example 5 below).

This result was the inspiration for the present work. Our goal here is to investigate to what extent a similar result remains valid for solutions to systems of the kind

$$(\dagger) \qquad \mathrm{Y}_j u = 0, \ j = 1, \dots, n,$$

where $\mathrm{Y}_1, \dots, \mathrm{Y}_n$ are smooth, complex vector fields defined on a smooth manifold Ω. We assume that $\mathrm{Y}_1, \dots, \mathrm{Y}_n$ are linearly independent and that (†) admits the maximum number of independent integrals. The span of $\{\mathrm{Y}_1, \dots, \mathrm{Y}_n\}$ defines an integrable subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ of rank n, and our results are described invariantly in terms of $\mathcal{V}$.

We first prove, by functional analytic methods, that a necessary condition for the validity of the Borel property for the solutions for $\mathcal{V}$ at a point p (Theorem 2) is that the space of solutions that vanish to infinite order at p must be non trivial. The remaining part of the article is devoted to the proof of a sufficient condition (Theorem 3), based on the

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existence of a first integral W for $\mathcal{V}$ such that (i) W vanishes at p, its differential belongs to the characteristic set of $\mathcal{V}$ and its image avoids a half-line issuing from the origin, and (ii) W dominates, in a precise sense, a set of complementary first integrals for $\mathcal{V}$. This is what we call Property ($\mathfrak{B}$) in Section 2, where we also discuss some of its aspects. The particular case of structures of hypersurface type provides a link between Property ($\mathfrak{B}$) and the sufficient condition in [LdS]. Furthermore, when $\mathcal{V}$ is a corank one structure, that is, when the orthogonal bundle $\mathcal{V}^{\perp}$ is a complex line bundle, Property ($\mathfrak{B}$) is equivalent to the validity of the Borel property at p for the solutions for $\mathcal{V}$ (Corollary 1).

For the proof of our main theorem (Section 5) we recall, in Section 4, basic facts about asymptotic expansions of holomorphic functions defined in sectors of the complex plane. Of foremost importance is the classical Borel-Ritt theorem (Proposition 5(3)) and, even more, its multidimensional version which we were able to prove (Proposition 7). Finally, in Section 6 we also show how to obtain a particular case of the Borel property for structures of hypersurface type by applying a version of the Borel–Ritt theorem with parameters (Proposition 6) that is proved in [W].

1. The Borel property

A. Let Ω be a smooth (paracompact) manifold. Fix $p \in \Omega$ and let C^∞_p denote the ring of germs of smooth functions at p. For each $k \geq 0$ let $\mathfrak{m}^k_p$ denote the ideal of C^∞_p formed by all $f \in C^\infty_p$ for which there is a constant $C > 0$ such that $|f(q)| \leq Cd(q,p)^{k+1}$ for q in a neighborhood of p.[1] It is clear that $\mathfrak{m}^{k+1}_p \subset \mathfrak{m}^k_p$ for every $k \geq 0$. For $k \geq 0$ we define the ring of k-jets at p as being the quotient ring $\mathfrak{J}^k_p = C^\infty_p/\mathfrak{m}^k_p$. Let $\gamma^k_p : C^\infty_p \to \mathfrak{J}^k_p$ denote the quotient homomorphism and let $\iota^k_p : \mathfrak{J}^k_p \to \mathfrak{J}^{k-1}_p$ be the natural homomorphism induced by the inclusion $\mathfrak{m}^k_p \subset \mathfrak{m}^{k-1}_p$. We have $\iota^{k+1}_p \circ \gamma^{k+1}_p = \gamma^k_p$ for every $k \geq 0$. The Borel theorem can be stated as follows:

Theorem 1. *Let $f_k \in \mathfrak{J}^k_p$, $k \geq 0$, be such that $\iota^{k+1}_p(f_{k+1}) = f_k$ for every $k \geq 0$. Then there is $f \in C^\infty_p$ such that $\gamma^k_p(f) = f_k$ for every $k \geq 0$.*

B. Now we assume that Ω is endowed with a locally integrable structure $\mathcal{V}$. Thus $\mathcal{V}$ is a vector subbundle of $\mathbb{C}T\Omega$ of rank n whose orthogonal bundle $\mathcal{V}^{\perp} \subset \mathbb{C}T^*\Omega$ is locally spanned by the differentials of $m = N - n$ smooth functions. Here $\dim \Omega = N$.

If $p \in \Omega$ we set

$$\mathfrak{S}_p \doteq \{f \in C^\infty_p : \mathrm{L}f = 0, \ \forall \text{ section L of } \mathcal{V} \text{ near } p\}.$$

It is clear that $\mathfrak{S}_p$ is a ring and that $\mathfrak{m}^k_p \cap \mathfrak{S}_p$ is an ideal of $\mathfrak{S}_p$. We can then form the quotient ring $\mathfrak{J}(\mathcal{V})^k_p \doteq \mathfrak{S}_p/(\mathfrak{m}^k_p \cap \mathfrak{S}_p)$, which is called the *ring of k-jets of solutions at p*. Notice that $\mathfrak{J}(\mathcal{V})^k_p$ is a subring of $\mathfrak{J}^k_p$ and it is clear that γ^k_p maps $\mathfrak{S}_p$ into $\mathfrak{J}(\mathcal{V})^k_p$ and that ι^k_p maps $\mathfrak{J}(V)^k_p$ into $\mathfrak{J}(\mathcal{V})^{k-1}_p$.

[1] Here d is any distance function defined near p by using local coordinates. It is easily seen that the definition of the ideals $\mathfrak{m}^k_p$ is invariant.

Definition 1. We say that $\mathcal{V}$ satisfies the Borel property at p if given $f_k \in \mathfrak{J}(\mathcal{V})_p^k$, $k \geq 0$, satisfying $\iota_p^{k+1}(f_{k+1}) = f_k$ for every $k \geq 0$ there is $f \in \mathfrak{S}_p$ such that $\gamma_p^k(f) = f_k$ for every $k \geq 0$.

C. Let Ω and $\mathcal{V}$ be as above. Following [T2, I.5] and [BCH, I.10], each point $p \in \Omega$ is the center of a coordinate system $(x_1, \dots, x_m,\, t_1, \dots, t_n)$, which can be assumed defined in a product $U = B \times \Theta$, where B (respectively Θ) is an open ball centered at the origin in $\mathbb{R}_x^m$ (respectively $\mathbb{R}_t^n$), over which there are defined smooth functions $\Phi_1(x,t), \dots, \Phi_m(x,t)$ satisfying

$$\Phi_k(0,0) = 0, \quad \mathrm{d}_x\Phi_k(0,0) = 0, \quad k = 1, \dots, m,$$

such that the differentials of the functions

$$Z_k(x,t) = x_k + i\Phi_k(x,t), \quad k = 1, \dots, m$$

span $\mathcal{V}^\perp$ over U.

Moreover we can also assume $Z_x(x,t)$ is invertible on U and thus there are well defined vector fields

$$\mathrm{M}_k = \sum_{k'=1}^{m} \mu_{kk'}(x,t) \frac{\partial}{\partial x_{k'}}, \quad k = 1, \dots, m,$$

characterized by the rule

$$\mathrm{M}_k Z_{k'} = \delta_{k,k'}\,, \quad k, k' = 1, \dots, m.$$

If we further set

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m-1} \frac{\partial \Phi_k}{\partial t_j}(x,t)\mathrm{M}_k, \quad j = 1, \dots, n,$$

then $\mathrm{L}_1, \dots, \mathrm{L}_n$ span $\mathcal{V}$ over U.

The properties below are of foremost importance:

- $\mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m$ are linearly independent and pairwise commuting;
- $\{\mathrm{d}t_1, \dots, \mathrm{d}t_n, \mathrm{d}Z_1, \dots, \mathrm{d}Z_m\}$ is the dual basis to $\{\mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m\}$.

D. From these properties if follows that the Taylor series of $u \in \mathfrak{S}_0$ at the origin has the expression

$$Q_u = \sum_{\alpha \in \mathbb{Z}_+^m} \frac{(\mathrm{M}^\alpha u)(0,0)}{\alpha!} Z^\alpha \in \mathbb{C}[[Z_1, \dots, Z_m]]\,.$$

Let

$$\mathfrak{n}_k \doteq \left\{ \sum_{\alpha \in \mathbb{Z}_+^m} b_\alpha Z^\alpha \in \mathbb{C}[[Z_1, \dots, Z_m]] :\, b_\alpha = 0 \text{ if } |\alpha| \leq k \right\};$$

$\mathfrak{n}_k$ is an ideal of $\mathbb{C}[[Z_1, \dots, Z_m]]$ and the ring homomorphism $\mathfrak{S}_0 \ni u \mapsto Q_u \in \mathbb{C}[[Z_1, \dots, Z_m]]$ induces a ring isomorphism

$$\theta_k : \mathfrak{J}(\mathcal{V})_0^k \longrightarrow \mathbb{C}[[Z_1, \dots, Z_m]]/\mathfrak{n}_k\,.$$

Moreover, for each $k \geq 1$, we have commutative diagrams

$$\begin{array}{ccc} \mathfrak{J}(\mathcal{V})_0^k & \longrightarrow & \mathbb{C}[[Z_1,\dots,Z_m]]/\mathfrak{n}_k \\ \iota_0^k \downarrow & & \downarrow \\ \mathfrak{J}(\mathcal{V})_0^{k-1} & \longrightarrow & \mathbb{C}[[Z_1,\dots,Z_m]]/\mathfrak{n}_{k-1}\,, \end{array}$$

in which the second vertical arrow is the ring homomorphism induced by the inclusion $\mathfrak{n}_k \subset \mathfrak{n}_{k-1}$.

After these considerations we can state:

Proposition 1. *Under the preceding set-up, a system $f_k \in \mathfrak{J}(\mathcal{V})_0^k$, $k \geq 0$, satisfying $\iota_0^{k+1}(f_{k+1}) = f_k$ for every $k \geq 0$, can be identified to a formal power series $\sum_\alpha c_\alpha Z^\alpha \in \mathbb{C}[[Z_1,\dots,Z_m]]$ in such a way that $f \in \mathfrak{S}_0$ satisfies $\gamma_0^k(f) = f_k$ for every $k \geq 0$ if and only if $c_\alpha = (\mathrm{M}^\alpha f)(0,0)/\alpha!$ for every $\alpha \in \mathbb{Z}_+^m$.*

Example 1. Suppose that $\Phi_k = 0$ for $k = 1,\dots,m$. Then $\mathfrak{S}_0$ is the space of (germs of) smooth functions $f(x)$ which only depend on x and $\mathrm{M}_k = \partial/\partial x_k$, $k = 1,\dots,m$. Now it follows from Theorem 1 that given $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}_+^m$, there exists $f \in \mathfrak{S}_0$ such that $(\partial^\alpha f)(0) = c_\alpha$. Thus, the locally integrable structure $\mathcal{V}$ spanned by the vector fields $\mathrm{L}_j = \partial/\partial t_j$, $j = 1,\dots,n$ satisfies the Borel property at the origin.

Example 2. Suppose that Φ_k is of Gevrey class of order $s \geq 1$ and that every $u \in \mathfrak{S}_0$ is also of Gevrey class of order s (this occurs when, for instance, the system $\mathrm{L}_1,\dots,\mathrm{L}_n$ is Gevrey-hypoelliptic of order s). Under this condition, if $f \in \mathfrak{S}_0$ then for some $C > 0$ we have $|(\mathrm{M}^\alpha f)(0,0)| \leq C^{|\alpha|+1}\alpha!^s$ and thus Proposition 1 implies that the Borel property at the origin is not verified in this case.

In order to present the next example we recall the following definition ([T2], III.5]).

Definition 2. Let Ω be a smooth manifold over which a locally integrable structure is defined. We say that $\mathcal{V}$ is *hypocomplex at a point* $p \in \Omega$ if there are an open neighborhood U of p in Ω and smooth functions $Z_j : U \to \mathbb{C}$, $Z_j(p) = 0$, $j = 1,\dots,m$, whose differentials span $\mathcal{V}^\perp$ over U, and such that the following is true: given any solution u, defined near p there is a holomorphic function H, defined in an open neighborhood of $0 \in \mathbb{C}^m$, such that $u = H \circ Z$ near p.

Example 3. By the same argument of Example 2, if $\mathcal{V}$ is hypocomplex at the origin then $\mathcal{V}$ does not satisfy the Borel property at the origin.

Example 4. Consider the CR structure in $\mathbb{R}^3$, where the coordinates are written as (x_1, x_2, t), defined by the first integrals $Z_1(x,t) = x_1 + it$, $W(x,t) = x_2$. The bundle $\mathcal{V}$ in this case is spanned by the vector field $\mathrm{L} = \partial/\partial t - i\partial/\partial x_1$. If u is a smooth solution to $\mathrm{L}u = 0$ then $u(x,t) = h(x_1 + it, x_2)$, where $h(z, x_2)$ is holomorphic in z. In particular, for some constant $C > 0$, $|(\partial_{x_1}^k h)(0,0)| \leq C^{k+1}k!$, which shows that $\mathcal{V}$ does not satisfy the Borel property at the origin. Notice that $\mathcal{V}$ is not hypocomplex at the origin.

Example 5. ([LdS]) Let $\mathcal{M}$ be a generic submanifold of $\mathbb{C}^N$ and suppose that, for some $p \in \mathcal{M}$ there is a germ of CR function ψ at p such that, for some neighborhood U of p in $\mathcal{M}$ and for some $\mu > 0$ we have

$$\psi(p) = 0, \quad \arg \psi(q) \neq \pi, \quad |\psi(q)| \geq |q - p|^{\mu}, \ q \in U, \, q \neq p.$$

Then the CR structure on $\mathcal{M}$ induced by $\mathbb{C}^N$ satisfies the Borel property at p. In particular this is verified when $\mathcal{M}$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^N$.

2. A necessary condition for the validity of the Borel property

In this section we prove:

Theorem 2. *If $\mathcal{V}$ satisfies the Borel property at p then there are non trivial germs $u \in \mathfrak{S}_p$ which vanish to infinite order at p.*

Proof. We shall work in the set up described in Section 1(C). In particular we shall apply the characterization of the Borel property given by Proposition 1.

If V is an open subset of U denote by $\mathfrak{S}(V)$ the space of all smooth solutions for $\mathcal{V}$ on V. The *Borel map* $B_V : \mathfrak{S}(V) \to \mathbb{C}[[Z_1, \dots, Z_m]]$ on V is defined as

$$B_V(u) = \sum_{\alpha} (\mathrm{M}^{\alpha} u)(0, 0) Z^{\alpha} / \alpha!, \qquad u \in \mathfrak{S}(V).$$

Notice that both spaces $\mathfrak{S}(V)$ and $\mathbb{C}[[Z_1, \dots, Z_m]]$ are Fréchet spaces and that B_V is a continuous linear map.

Lemma 1. *Assume that the Borel property for $\mathcal{V}$ holds at the origin. Then there is an open neighborhood V of the origin such that $B_V : \mathfrak{S}(V) \to \mathbb{C}[[Z_1, \dots, Z_m]]$ is surjective.*

Proof. Let $\{V_n\}_{n \in \mathbb{N}}$ be a fundamental system of open neighborhoods of the origin and for each $n \in \mathbb{N}$ let $E_n \doteq B_{V_n}(\mathfrak{S}(V_n)) \subset \mathbb{C}[[Z_1, \dots, Z_m]]$. By hypothesis we have $\cup_n E_n = \mathbb{C}[[Z_1, \dots, Z_m]]$ and then, by Baire's theorem, there is $n_0 \in \mathbb{N}$ such that the subspace E_{n_0} is of second category in $\mathbb{C}[[Z_1, \dots, Z_m]]$. It follows from the open mapping theorem that $B_{V_{n_0}}$ is surjective. ∎

End of proof of Theorem 2. By Lemma 1 we can take an open neighborhood V of the origin such that B_V is surjective and hence by the homomorphism theorem for Fréchet spaces[2] we conclude that

$$(\star) \qquad (\ker B_V)^{\mathrm{o}} = {}^t B_V(\mathbb{C}[[Z_1, \dots, Z_m]]^*).$$

We now compute the transpose of B_V. Firstly notice that we can identify the dual of $\mathbb{C}[[Z_1 \dots, \mathrm{Z}_m]]$ with the space of polynomials in $Z_1, \dots, Z_m$, denoted by $\mathbb{C}[Z_1, \dots, Z_m]$. If we take $P(Z) = \sum_{|\alpha| \leq M} a_{\alpha} Z^{\alpha} \in \mathbb{C}[Z_1, \dots, Z_m]$ and $\mathfrak{s}(Z) = \sum_{\alpha} b_{\alpha} Z^{\alpha} \in \mathbb{C}[[Z_1, \dots, Z_m]]$ the duality just mentioned is given by $\langle P(Z), \mathfrak{s}(Z) \rangle = \sum_{|\alpha| \leq M} a_{\alpha} b_{\alpha}$. Under this duality

[2] If $T : E \to F$ is a continuous linear map between Fréchet spaces then T has closed image if and only if ${}^tT : F^* \to E^*$ has weakly closed image. Since the weak closure of the image of ${}^tT$ equals $(\ker T)^{\mathrm{o}}$, it follows that T has closed image if and only if $(\ker T)^{\mathrm{o}} = {}^tT(F^*)$ [T1, Theorem 37.2 and Proposition 35.4].

the transpose of the Borel map ${}^tB_V : \mathbb{C}[Z_1, \dots, Z_m] \to \mathfrak{S}(V)^*$ can then be computed as follows: if $P(Z) = \sum_{|\alpha| \leq M} a_\alpha Z^\alpha \in \mathbb{C}[Z_1, \dots, Z_m]$ and $u \in \mathfrak{S}(V)$ then

$$ {}^tB_V(P(Z))(u) = \langle P(Z), B_V u \rangle = \sum_{|\alpha| \leq M} \frac{a_\alpha \, (\mathrm{M}^\alpha u)(0,0)}{\alpha!} . $$

Assume finally that $\ker B_V = 0$, that is, that the only smooth solution for $\mathcal{V}$ in V that vanishes to infinite order at the origin is the zero one. Then ($\star$) would imply that ${}^tB_V$ is surjective and, consequently, for *any* $v \in \mathcal{E}'(V)$ there would exist $M \in \mathbb{Z}_+$ such that $v(Z(x,t)^\beta) = 0$ if $|\beta| > M$. This is certainly not possible. ∎

3. Property ($\mathfrak{B}$)

A. Let $\mathcal{V}$ be a locally integrable structure as before.

The characteristic set of $\mathcal{V}$ is, by definition, the set $\mathrm{T}^{\mathrm{o}} \doteq (\mathcal{V}^\perp \cap \mathrm{T}^*\Omega)$.

Definition 3. *We shall say that $\mathcal{V}$ satisfies Property ($\mathfrak{B}$) at $p \in \Omega$ if*

1. *There exists a solution W for the structure $\mathcal{V}$, defined near p, $W(p) = 0$, such that $\mathrm{d}W|_p \in \mathrm{T}^{\mathrm{o}}_p \setminus 0$ and $\arg W \neq -\pi/2$ near p;*
2. *There are smooth solutions $W_1, \dots, W_{m-1}$ for $\mathcal{V}$, defined in a neighborhood of p, $W_j(p) = 0$, such that $\mathrm{d}W_1, \dots, \mathrm{d}W_{m-1}, \mathrm{d}W$ are linearly independent, and positive constants μ and C such that $(|W_1| + \ldots + |W_{m-1}|)^\mu \leq C|W|$ near p.*

We emphazyse one of the main differences between (1) in property ($\mathfrak{B}$) and the condition imposed in [LdS]: although we allow the possibility for W to vanish outside $\{p\}$ it will be crucial for us to know, in particular, that $\mathrm{d}W|_p \neq 0$.

We shall devote the remaining of this section to discuss several aspects of Property ($\mathfrak{B}$). Our first observation concerns real-analytic structures.

Proposition 2. *Assume that $\mathcal{V}$ is a real-analytic structure and that there is a real-analytic solution W near p satisfying property (1). Then $\mathcal{V}$ satisfies Property ($\mathfrak{B}$) near p if the following property holds:*

- *There are real-analytic solutions $W_1, \dots, W_{m-1}$ for $\mathcal{V}$, defined in a neighborhood of p, such that $W_j(p) = 0$, $\mathrm{d}W_1, \dots, \mathrm{d}W_{m-1}, \mathrm{d}W$ are linearly independent, and $\{q : W(q) = 0\} \subset \{q : W_j(q) = 0,\ j = 1, \dots, m-1\}$.*

Proof. It is a consequence of the Lojasiewicz inequality. See ([BM]).

Example 6. In the analytic category we conclude that Property ($\mathfrak{B}$) is valid at p if $N \geq 3$ and if there exists a real-analytic solution W for the structure $\mathcal{V}$, defined near $p \in \Omega$, $W(p) = 0$, such that $\mathrm{d}W|_p \in \mathrm{T}^{\mathrm{o}}_p \setminus 0$ and $W^{-1}\{0\} = \{p\}$. Indeed it suffices to show that either $\arg W \neq -\pi/2$ or $\arg W \neq \pi/2$ near the origin. Select local coordinates $(y_1, \dots, y_N)$ centered at p such that $\Re W = y_1$. Hence $W(y) = y_1 + i\Psi(y)$ and then $\Psi(0, y_2, \dots, y_N) = 0$ if and only if $y_2 = 0, \dots, y_N = 0$. Since $N \geq 3$ it follows that Ψ has constant sign near the origin in $\mathbb{R}^{N-1}$, and our claim follows. ∎

B. Tube structures. Let us consider a tube structure $\mathcal{V}$ on $\mathbb{R}^N$ defined by the first integrals

$$Z_j(x,t) = x_j + i\Phi_j(t)\,, \quad j = 1, \dots, m.$$

Here $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$ and the real-valued, smooth functions Φ_j are defined in an open neighborhood of the origin in $\mathbb{R}^n$ and satisfy $\Phi_j(0) = 0$.

Proposition 3. *Suppose that $\Phi_m(t) \geq 0$ and that there is a constant $C > 0$ such that*

$$\Phi_1(t)^2 + \ldots + \Phi_{m-1}(t)^2 \leq C\Phi_m(t)\,.$$

Then Property ($\mathfrak{B}$) holds at the origin.

Proof. Define

$$W(x,t) = Z_m(x,t) + i\kappa \sum_{j=1}^{m-1} Z_j(x,t)^2\,,$$

with $\kappa > 0$ to be chosen. Notice that $\mathrm{d}Z_1, \dots, \mathrm{d}Z_{m-1}, \mathrm{d}W$ also span $\mathcal{V}^\perp$. Moreover

$$W(x,t) = x_m - 2\kappa \sum_{j=1}^{m-1} x_j \Phi_j(t) + i \left[\Phi_m(t) + \kappa \sum_{j=1}^{m-1} x_j^2 - \kappa \sum_{j=1}^{m-1} \Phi_j(t)^2 \right] .$$

We take $\kappa = (2C)^{-1}$ and redefine $x_m - 2\kappa \sum_{j=1}^{m-1} x_j \Phi_j(t)$ as a new variable x_m. Then

$$W(x,t) = x_m + i\Psi(x,t)\,,$$

where $\Psi(x,t) \geq \Phi_m(t)/2 \geq 0$. Furthermore

$$\begin{aligned} |W(x,t)| &\geq \frac{1}{2} \left[|x_m| + \Phi_m(t) + \kappa \sum_{j=1}^{m-1} x_j^2 - \kappa \sum_{j=1}^{m-1} \Phi_j(t)^2 \right] \\ &\geq \frac{1}{2} \left[|x_m| + \kappa \sum_{j=1}^{m-1} x_j^2 + \kappa \sum_{j=1}^{m-1} \Phi_j(t)^2 \right] \\ &\geq \frac{\kappa}{2} \left[\sum_{j=1}^{m-1} x_j^2 + \sum_{j=1}^{m-1} \Phi_j(t)^2 \right] \\ &= \frac{\kappa}{2} \sum_{j=1}^{m-1} |Z_j(x,t)|^2 . \qquad \blacksquare \end{aligned}$$

C. Structures of hypersurface type. Suppose that $\dim \mathrm{T}_0^\circ = 1$ and furthermore that there is a solution W for $\mathcal{V}$, defined in a neighborhood of the origin, such that $W(0) = 0$, $\mathrm{d}(\Re W)|_0$ spans T_0^0 and $\mathrm{d}(\Im W)|_0 = 0$. According to ([T2], pp. 368), these hypotheses imply the following: there are coordinates near the origin $(x_1, \dots, x_m, t_1, \dots, t_n)$, such that $\mathcal{V}^\perp$ is spanned by the differential of the functions

$$z_j = x_j + it_j,\ j = 1, \dots, m-1,\ \ W(z, x_m, t) = x_m + i\Phi(x,t),$$

where Φ is a smooth function defined near the origin and satisfying $\Phi(0,0)=0$, $\mathrm{d}\Phi|_{(0,0)}=0$. Here $m-1\leq n$ ($m-1=n$ means "CR of hypersurface type"). Hence, if in addition to the hypothesis $\arg W\neq -\pi/2$ we have, near the origin,

$$(|z_1|+\ldots+|z_r|)^\mu\leq C|W|$$

for suitable constants $\mu>0$, $C>0$ then Property ($\mathfrak{B}$) is satisfied.

This example provides a connection between (2) in Property ($\mathfrak{B}$) and the assumption made in [LdS] of finite order vanishing for the peak function. ∎

D. Nondegenerate structures. For such classes of structures we have the following result:

Proposition 4. *Let $\mathcal{V}$ be a locally integrable structure defined on an open neighborhood Ω of the origin in $\mathbb{R}^{m+n}$ of rank n and assume that for some $\sigma\in \mathrm{T}^\circ_0$ the Levi form of $\mathcal{V}$ at σ is definite positive. Then Property ($\mathfrak{B}$) is satisfied.*

Proof. According to ([BCT], Lemma 5.4, p. 387), we can choose coordinates $(x_1,\ldots,x_m,$ $t_1,\ldots,t_n)$ near and vanishing at the origin and $r\in\{1,\ldots,m-1\}$ such that $\mathcal{V}^\perp$ is spanned by the differentials of the functions

$$z_j=x_j+it_j,\ j=1,\ldots,r,\ \ W_j(x,t)=x_j+i\Phi_j(x,t),\ \ j=r+1,\ldots,m,$$

where Φ_j are smooth, real-valued, $\Phi_j=0$, $\mathrm{d}\Phi_j=0$ at the origin and moreover

$$\Phi_m(x,t)=|t|^2+\mathrm{O}(|x|^3+|t|^3)\,.$$

Let

$$W(x,t)=W_m(x,t)+\frac{i}{2}\left\{\sum_{j=1}^r z_j^2+\sum_{j=r+1}^m W_j(x,t)^2\right\}.$$

Notice that we have

$$\Im W(x,t)=|t|^2+\frac{1}{2}\left\{|x|^2-\sum_{j=1}^r t_j^2\right\}+\mathrm{O}(|x|^3+|t|^3)$$

since $\Phi_j(x,t)=\mathrm{O}(|x|^2+|t|^2)$, $j=r+1,\ldots,m$. Hence, in some neighborhood of the origin we have

$$\Im W(x,t)\geq\frac{1}{4}\left\{|x|^2+|t|^2\right\},$$

which implies $\arg W\in[0,\pi]$ and also $|W(x,t)|\geq(|x|^2+|t|^2)/4$. Condition ($\mathfrak{B}$) follows immediately. ∎

4. Asymptotic expansions

We shall make use of the notion of truncated sectors in the complex plane, that is, open sets of the form $S=\{w\in\mathbb{C}:\alpha<\arg w<\beta,\ 0<|w|<R\}$ where $\beta-\alpha\leq 2\pi, R>0$. If S^* is another truncated sector, the symbol $S^*\Subset S$ means that $S^*=\{w\in\mathbb{C}:\alpha^*<\arg w<\beta^*,\ 0<|w|<R^*\}$, where $\alpha<\alpha^*<\beta^*<\beta$ and $0<R^*<R$.

Let S be a truncated sector in the complex plane and let $f=f(w)$ be a holomorphic function on S, that is $f\in\mathcal{O}(S)$. We say that a formal power series $\sum_{j\geq 0}a_jw^j\in\mathbb{C}[[w]]$ is an *asymptotic expansion* for f (and we write $f(w) \sim \sum_{j=0}^{\infty} a_j w^j$) if for all $n \geq 0$ and all $S^* \subset\subset S$, there exists $C = C_n \geq 0$ such that

$$\left| f(w) - \sum_{j=0}^{n} a_j w^j \right| \leq C|w|^{n+1}, \quad \forall\, w \in S^*.$$

In the next statement we summarize the main results on such asymptotic expansions, whose proofs can be found, for instance, in [M]:

Proposition 5. *Let S be a truncated sector in the complex plane. Then*

(1) *If $f \in \mathcal{O}(S)$ and $f(w) \sim \sum_{j=0}^{\infty} a_j w^j$ then $f' \sim \sum_{j\geq 0}(j+1)a_{j+1}w^j$;*
(2) *If $f \in \mathcal{O}(S)$ then $f(w) \sim \sum_{j\geq 0} b_j w^j$ if and only if*

$$\lim_{w\to 0, w\in S^*} f^{(j)}(w) = j!b_j$$

for every sector $S^* \subset\subset S$;
(3) *Given $\sum_{j\geq 0} a_j w^j \in \mathbb{C}[[w]]$ there is $f \in \mathcal{O}(S)$ such that $f(w) \sim \sum_{j=0}^{\infty} a_j w^j$.*

Part (3) is known as the Borel–Ritt theorem, which also has the following version with holomorphic parameters (cf. also [W, p.44]):

Proposition 6. *Let Ω be an open set in $\mathbb{C}^p$ and $\sum_{j\geq 0} a_j(z)w^j \in \mathcal{O}(\Omega)[[w]]$. If S is a truncated sector in $\mathbb{C}$ then there is a holomorphic function $H(z,w) \in \mathcal{O}(\Omega \times S)$ such that*

$$H(z,w) \sim \sum_{j\geq 0} a_j(z)w^j$$

uniformly, in the sense that given $S^* \subset\subset S$, $K \subset \Omega$ compact and $n \geq 0$, there exists $C = C_n \geq$ such that

$$\left| H(z,w) - \sum_{j=0}^{n} a_j(z)w^j \right| \leq C|w|^{n+1},$$

for all $w \in S^*$ and $z \in K$. We also have

$$\partial_z^\alpha H(z,w) \sim \sum_{j\geq 0} (\partial^\alpha a_j)(z)w^j\,.$$

We now present an extension of Proposition 6 which will be crucial for the proof of our main result. We shall use the notation

$$S_{1/2} = \{w \in \mathbb{C} : |w| < 1,\ \Im w + |\Re w|/2 > 0\}.$$

Proposition 7. *Let $\{\lambda_{\gamma,k}\}$ be a sequence of complex numbers, $\gamma \in \mathbb{Z}_+^p$, $k \in \mathbb{Z}_+$. Given $\mu > 0$ there exists $H \in \mathcal{O}(\mathbb{C}^p \times S_{1/2})$ such that, for any $\rho > 0$,*

$$\lim_{\substack{(z,w)\to(0,0)\\ |z|^\mu \leq \rho|w|}} (\partial_z^\gamma \partial_w^k H)(z,w) = \lambda_{\gamma,k}.$$

Proof. Let $0 < \beta < 1/3$. If $-\pi/2 < \arg w < 3\pi/2$ then $-\beta\pi/2 < \arg w^\beta < 3\beta\pi/2$. Since $3\beta\pi/2 < \pi/2$ and $-\beta\pi/2 > -\pi/2$ it follows that there is $c_\beta > 0$ such that $\cos[\arg(w^\beta)] \geq c_\beta$ for all such w. It then follows that $\Re(w^{-\beta}) \geq c_\beta |w|^{-\beta}$ for $-\pi/2 < \arg w < 3\pi/2$ and consequently the function $\Psi_\theta(w) = \exp\{-\theta w^{-\beta}\}$, where $\theta > 0$, is holomorphic in

$$S = \{w \in \mathbb{C} : \, -\pi/2 < \arg w < 3\pi/2\}$$

and it satisfies $\Psi_\theta \sim 0$.

By using Proposition 5 - part (3), for each $\alpha \in \mathbb{Z}_+^p$ we select $g_\alpha \in \mathcal{O}(S)$ such that

$$g_\alpha(w) \sim \sum_{k=0}^{\infty} \frac{\lambda_{\alpha,k}}{k!} w^k \, .$$

Set $h_\alpha(w) \doteq (1 - \Psi_{\theta_\alpha}(w))^{|\alpha|} g_\alpha(w) \sim g_\alpha(w)$, where $\theta_\alpha > 0$ are to be determined and denote by $\| \cdot \|$ the L^∞-norm in $S_{1/2}$.

Since $|1 - \exp\{-z\}| \leq |z|$ if $\Re z \geq 0$ we can estimate

$$|h_\alpha(w)| \leq \frac{\theta_\alpha^{|\alpha|}}{|w|^{\beta|\alpha|}} \, \|g_\alpha\| \, , \; w \in S_{1/2} \, .$$

Choose θ_α such that $\theta_\alpha^{|\alpha|} \|g_\alpha\| \leq 1$. Then for any $\varepsilon > 0$ we have $|h_\alpha(w)| \leq \varepsilon^{-\beta|\alpha|}$ if $|w| \geq \varepsilon$ and consequently

$$H(z, w) = \sum_\alpha \frac{h_\alpha(w)}{\alpha!} z^\alpha$$

defines a holomorphic function on $\mathbb{C}^p \times S_{1/2}$.

We shall now refine the choice of θ_α in order to obtain suitable estimates for the derivatives of H. For this we notice that, for any $p \geq 1$,

$$\partial_w^p \left(1 - e^{-\theta_\alpha w^{-\beta}}\right)^{|\alpha|} = \sum_{q=1}^{p} \theta_\alpha^q w^{-q\beta - p} \left[\sum_{\ell=1}^{q} C_{q,\ell}(\alpha) \left(1 - e^{-\theta_\alpha w^{-\beta}}\right)^{|\alpha| - \ell} e^{-\ell\theta_\alpha w^{-\beta}} \right]$$

and thus

$$\begin{aligned}
\left| \partial_w^p \left(1 - e^{-\theta_\alpha w^{-\beta}}\right)^{|\alpha|} \right| &\leq \sum_{q=1}^{p} \sum_{\ell=1}^{q} |C_{q,\ell}(\alpha)| \, \frac{\theta_\alpha^{|\alpha|+q-\ell}}{|w|^{q\beta + p + \beta(|\alpha| - \ell)}} e^{-\ell\theta_\alpha c_\beta |w|^{-\beta}} \\
&= \frac{\theta_\alpha^{|\alpha| - p/\beta}}{|w|^{\beta|\alpha|}} \sum_{q=1}^{p} \sum_{\ell=1}^{q} |C_{q,\ell}(\alpha)| \left(\frac{\theta_\alpha^{1/\beta}}{|w|} \right)^{\beta(q-\ell)+p} e^{-\ell\theta_\alpha c_\beta |w|^{-\beta}} \\
&\leq A(p) \frac{\theta_\alpha^{|\alpha| - p/\beta}}{|w|^{\beta|\alpha|}} \sum_{q=1}^{p} \sum_{\ell=1}^{q} |C_{q,\ell}(\alpha)| \\
&\doteq D(\alpha, p) \frac{\theta_\alpha^{|\alpha| - p/\beta}}{|w|^{\beta|\alpha|}} .
\end{aligned}$$

Here $A(p) > 0$ is such that $\rho^s \exp\{-c_\beta \rho^\beta\} \le A(p)$ for all $\rho > 0$ and $s = 1, \dots, 2p$.

By the Leibniz rule we then conclude that there exist constants $C(\alpha, p) > 0$ such that

$$|h_\alpha^{(p)}(w)| \le C(\alpha, p) \frac{\theta_\alpha^{|\alpha| - p/\beta}}{|w|^{\beta|\alpha|}}, \quad w \in S_{1/2}.$$

Finally, let $C_\star(\alpha) = \max\{C(\alpha, p) : 0 \le p < \beta|\alpha|\}$ and assume that $0 < \theta_\alpha < 1$. Then

$$|h_\alpha^{(p)}(w)| \le C_\star(\alpha) \frac{\theta_\alpha^{|\alpha| - p/\beta}}{|w|^{\beta|\alpha|}}, \quad w \in S_{1/2}, \quad |\alpha| > p/\beta$$

and hence

$$|h_\alpha^{(p)}(w)| \le C_\star(\alpha) \frac{\theta_\alpha^{|\alpha|/2}}{|w|^{\beta|\alpha|}}, \quad w \in S_{1/2}, \quad |\alpha| > 2p/\beta.$$

Choosing θ_α such that $C_\star(\alpha)\theta_\alpha^{|\alpha|/2} \le 1$ for every α we obtain

- *For every α and every $p < \beta|\alpha|/2$ we have*

$$|h_\alpha^{(p)}(w)| \le 1/|w|^{\beta|\alpha|}.$$

Finally we have, for fixed $\gamma \le \alpha$ and $k \ge 0$,

$$\frac{h_\alpha^{(k)}(w) z^{\alpha-\gamma}}{(\alpha-\gamma)!} \longrightarrow \begin{cases} \lambda_{\alpha,k} & \text{if } \alpha = \gamma \\ 0 & \text{if } \alpha \ne \gamma \end{cases}$$

when $(z, w) \to (0, 0)$. Moreover, when $|z|^\mu \le \rho|w|$, $|w| \le 1$, $k < \beta|\alpha|/2$ and $2|\gamma| \le |\alpha|$ we also have

$$\frac{|h_\alpha^{(k)}(w) z^{\alpha-\gamma}|}{(\alpha-\gamma)!} \le \frac{|z|^{|\alpha-\gamma|}}{(\alpha-\gamma)!|w|^{\beta|\alpha|}} \le \rho^{|\alpha-\gamma|/\mu} \frac{|w|^{(1/\mu-\beta)|\alpha| - |\gamma|/\mu}}{(\alpha-\gamma)!} \le \rho^{|\alpha-\gamma|/\mu}/(\alpha-\gamma)!$$

if we also require $\beta < 1/(2\mu)$, for then the exponent of $|w|$ will be > 0. The sought conclusion then follows from the dominate convergence theorem. ∎

5. Main Theorem

A. We can now state and prove the main result of this work:

Theorem 3. *Let Ω be a smooth (paracompact) manifold of dimension N which is endowed with a locally integrable structure $\mathcal{V}$ of rank n. If $\mathcal{V}$ satisfies Property ($\mathfrak{B}$) at $p \in \Omega$ then it satisfies the Borel property at p.*

Proof. Since $\mathrm{d}W|_p = \mathrm{d}(\Re W)|_p$ we can select coordinates $(y_1, \dots, y_N)$ in Ω centered at p such that $\Re W = y_N$. Since $\mathrm{d}W_1 \dots, \mathrm{d}W_{m-1}, \mathrm{d}W$ are linearly independent we can further assume, after permuting the variables $(y_1, \dots, y_{N-1})$, that $\{(\partial W_j/\partial y_k)(0)\}_{j,k=1,\dots,m-1}$ is non-singular. Let $A = (a_{jk}) \in \mathrm{GL}_{m-1}(\mathbb{C})$ be the inverse of this matrix and set $Z_j = \sum_{k=1}^{m-1} a_{jk} W_k$. Then $\{(\partial Z_j/\partial y_k)(0)\}_{j,k=1,\dots,m-1}$ is the identity matrix, which in particular implies that

$$\mathrm{d}Z_j|_0 = \mathrm{d}y_j|_0 \bmod \{\mathrm{d}y_m, \dots, \mathrm{d}y_N\}.$$

Now,

$$\begin{cases} x_j = \Re Z_j(y), \quad j = 1, \dots, m-1, \\ x_m = y_N, \\ t_k = y_{m-1+k}, \quad k = 1, \dots, n, \end{cases}$$

is a smooth diffeomorphism near the origin and then $(x_1, \dots, x_m, t_1, \dots, t_n)$ is a new coordinate system also centered at p. It follows that $Z_j(x,t) = x_j + i\Phi_j(x,t)$, $j = 1, \dots, m-1$, $W(x,t) = x_m + i\Psi(x,t)$, where Φ_j, Ψ are smooth, real-valued and vanish at the origin. Since $\mathrm{d}(\Im W)|_0 = 0$ we have $\mathrm{d}\Psi(0,0) = 0$; furthermore, for any $j = 1, \dots, m-1$, $\mathrm{d}(\Im Z_j)|_0$ only involves $\mathrm{d}x_m, \mathrm{d}t_1, \dots, \mathrm{d}t_n$ and we conclude that

$$\text{(a)} \qquad \frac{\partial \Phi_j}{\partial x_k}(0,0) = 0, \ \ j,k = 1, \dots, m-1, \ \ \mathrm{d}\Psi(0,0) = 0.$$

Next we remark that $\arg(W)(x,t) = -\pi/2$ if and only if $x_m = 0$ and $\Psi(x,t) < 0$, and then we have

$$\text{(b)} \qquad \Psi(x_1, \dots, x_{m-1}, 0, t) \geq 0\,.$$

Moreover we also have

$$\text{(c)} \qquad (|Z_1(x,t)| + \ldots + |Z_{m-1}(x,t)|)^{\mu} \leq C|W(x,t)|$$

since $(|Z_1| + \ldots + |Z_{m-1}|) \leq C_1(|W_1| + \ldots + |W_{m-1}|)$.

Now we shall consider vector fields

$$\mathrm{M}_k = \sum_{\ell=1}^{m} b_{k\ell}(x,t)\frac{\partial}{\partial x_\ell}, \ \ k = 1, \dots, m-1, \ \ \mathrm{N} = \sum_{\ell=1}^{m} c_\ell(x,t)\frac{\partial}{\partial x_\ell}$$

characterized by the relations

$$\mathrm{M}_k Z_q = \delta_{kq}, \quad k,q = 1, \dots, m-1;$$

$$\mathrm{M}_k W = 0, \quad k = 1, \dots, m-1;$$

$$\mathrm{N} Z_q = 0, \ \ q = 1, \dots, m-1;$$

$$\mathrm{N} W = 1\,.$$

Such vector fields are well defined since that for the map $(x,t) \mapsto F(x,t) \doteq (Z(x,t), W(x,t))$ we have $\det F_x(0,0) \neq 0$. Finally the vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m-1} \frac{\partial \Phi_k}{\partial t_j}(x,t)\mathrm{M}_k - \frac{\partial \Psi}{\partial t_j}(x,t)\mathrm{N}, \quad j = 1, \dots, n,$$

span $\mathcal{V}$ and satisfy $\mathrm{L}_j t_\ell = \delta_{j\ell}$, from which we derive the following property:

- $\{\mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_{m-1}, \mathrm{N}\}$ *are linearly independent and pairwise commuting.*

From the discussion presented in Section 1 it follows that $\mathcal{V}$ satisfies the Borel property at p if and only if the following holds: given $c_{\alpha,\ell} \in \mathbb{C}$, with $(\alpha, \ell) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+$, there is a solution u for $\mathcal{V}$, defined in a neighborhood of the origin, satisfying $(\mathrm{M}^\alpha \mathrm{N}^\ell u)(0,0) = c_{\alpha,\ell}$ for all α and ℓ.

Let U be a small ball centered at the the origin such that $|W(x,t)| < 1$ when $(x,t) \in U$ and also

$$|\Psi(x,t) - \Psi(x',t)| \leq \frac{1}{4}|x - x'|, \quad (x,t), (x',t) \in U.$$

Thus (b) implies

$$\begin{aligned} \Psi(x,t) &\geq \Psi(x_1,\dots,x_{m-1},0,t) - |\Psi(x,t) - \Psi(x_1,\dots,x_{m-1},0,t)| \\ &\geq -\frac{1}{4}|x_m|, \quad (x,t) \in U, \end{aligned}$$

and it follows that W maps the open set $U_0 \doteq U \setminus W^{-1}\{0\}$ into $S_{1/2}$.

Let now $c_{\alpha,\ell} \in \mathbb{C}$, with $(\alpha,\ell) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+$, be given. With the value of μ prescribed by (c) we apply Proposition 7 and select $H \in \mathcal{O}(\mathbb{C}^{m-1} \times S_{1/2})$ satisfying

$$\lim_{\substack{(z,w) \to (0,0) \\ |z|^\mu \leq C|w|}} (\partial_z^\alpha \partial_w^\ell H)(z,w) = c_{\alpha,\ell}.$$

Here $C > 0$ is the constant appearing in (b). Define

$$u(x,t) = H\left(Z_1(x,t),\dots,Z_{m-1}(x,t), W(x,t)\right), \quad (x,t) \in U_0.$$

It is clear that u is a solution in U_0 and that

$$(\mathrm{M}^\alpha \mathrm{N}^\ell u)(x,t) = (\partial_z^\alpha \partial_w^\ell H)\left(Z_1(x,t),\dots,Z_{m-1}(x,t), W(x,t)\right), \quad (x,t) \in U_0\,.$$

Consequently we obtain

$$\lim_{(x,t) \to W^{-1}\{0\}} (\mathrm{M}^\alpha \mathrm{N}^\ell u)(x,t) = c_{\alpha,\ell},$$

and hence each of the functions $\mathrm{M}^\alpha \mathrm{N}^\ell u$ on U_0 has a continuous extension to U, simply by setting it equal to $c_{\alpha,\ell}$ on $W^{-1}\{0\}$. We denote by $v_{\alpha,\ell} \in C(U)$ each of such extensions. Since $W^{-1}\{0\} \subset \{(x,t) \in U :\, x_m = 0\}$ it follows that, in the distribution sense, $\mathrm{L}_j v_{\alpha,\ell} = 0$, $j = 1,\dots,n$, $\mathrm{M}_k v_{\alpha,\ell} = v_{\alpha+e_k,\ell}$, $k = 1,\dots,m-1$, $\mathrm{N} v_{\alpha,\ell} = v_{\alpha,\ell+1}$. Consequently u is smooth in U (cf. [H, Theorem 3.1.7]) and it satisfies $(\mathrm{M}^\alpha \mathrm{N}^\ell u)|_{W^{-1}\{0\}} = c_{\alpha,\ell}$ for every $(\alpha,\ell) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+$, which concludes the proof. ∎

B. When $m = 1$ a complete answer can be given. We return to the notation established in Section 1. In particular we assume given local coordinates $(x, t_1, \dots, t_n)$ defined in an open neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^n$, such that the orthogonal of $\mathcal{V}$ is spanned by the differential of the smooth map $Z : U \to \mathbb{C}$. Here $Z(x,t) = x + i\Phi(x,t)$, $U = J \times \Theta$ and J (respectively Θ) is an open interval (resp. ball) centered at the origin in $\mathbb{R}$ (respectively $\mathbb{R}^n$), satisfying $\Phi(0,0) = \Phi_x(0,0) = 0$.

Corollary 1. *Assume $m = 1$. The following properties are equivalent:*

(1) *$\mathcal{V}$ is not hypocomplex at the origin;*
(2) *There is a neighborhood $V \subset U$ of the origin such that $Z(V)$ is not a neighborhood of the origin in $\mathbb{C}$;*
(3) *There is an open ball $\Theta' \subset \Theta$ centered at the origin such that the origin is not an interior point of the image of Θ' under the map $t \mapsto \Phi(0,t)$;*

(4) *There is* $u \in \mathfrak{S}_0$, $u \not\equiv 0$, *which vanishes to infinite order at the origin;*
(5) $\mathcal{V}$ *satisfies the Borel property at the origin.*

Proof. The equivalence (1)⇔(2)⇔(3)⇔(4) follows from [BT, Theorem 3.6] and [T2, Corollary III.5.3]. It is also clear that (5)⇒(1), whereas (2) implies, after changing Z to $-Z$ if necessary, that $\Phi(0,t) \geq 0$ for $t \in \Theta'$. It then follows that $\mathrm{d}Z|_{(0,0)}$ spans $\mathrm{T}^{\mathrm{o}}_{(0,0)}$ and that $\arg Z(x,t) \neq -\pi/2$ for $x \in J$, $t \in \Theta'$. Since property (2) in the definition of Property ($\mathfrak{B}$) is vacuously satisfied in this case it follows from Theorem 3 that (2)⇒(5), which concludes the proof. ∎

6. Structures of hypersurface type

As Example 4 shows, property (1) in ($\mathfrak{B}$) alone is not enough to ensure the validity of the Borel property. In this final section we turn our attention to structures of the hypersurface type (Section 3(C)) and derive a partial result which follows from property (1) in ($\mathfrak{B}$).

As in Section 3(C) we assume that $\mathcal{V}$ is a locally integrable structure defined near the origin in $\mathbb{R}^N$, whose orthogonal bundle $\mathcal{V}^{\perp}$ is spanned by the differentials of the smooth functions

$$z_j = x_j + it_j, \;\; j = 1, \dots, m-1, \;\; W(x,t) = x_m + i\Phi(x,t).$$

where Φ is a smooth function defined near the origin and satisfying $\Phi(0,0) = 0$, $\mathrm{d}\Phi|_{(0,0)} = 0$. Here $(x_1, \dots, x_m, t_1 \dots, t_n)$ are local coordinates defined near the origin.

In this situation the bundle $\mathcal{V}$ is spanned by the vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial \bar{z}_j} - i\frac{\partial \Phi}{\partial \bar{z}_j}(x,t)\,\mathrm{N}, \quad j = 1, \dots, m-1,$$

$$\mathrm{L}_k = \frac{\partial}{\partial t_k} - i\frac{\partial \Phi}{\partial t_k}(x,t)\,\mathrm{N}, \quad k = m, \dots, n,$$

where

$$\mathrm{N} = \left\{ \frac{\partial W}{\partial x_m}(x,t) \right\}^{-1} \frac{\partial}{\partial x_m}.$$

If we introduce the additional vector fields

$$\mathrm{X}_\ell = \frac{\partial}{\partial z_\ell} - i\frac{\partial \Phi}{\partial z_\ell}(x,t)\,\mathrm{N}, \quad \ell = 1, \dots, m-1,$$

it is easily seen that, under this set up, the formal Taylor expansion of a smooth solution v for $\mathcal{V}$ is given by the formal power series

$$\sum_{\alpha \in \mathbb{Z}_+^{m-1}} \sum_{k \in \mathbb{Z}_+} \frac{(\mathrm{X}^\alpha \mathrm{N}^k v)(0,0)}{\alpha! k!} z^\alpha W^k \;\in\; \mathbb{C}[[z_1, \dots, z_{m-1}, W]].$$

Proposition 8. *Assume that* $\arg W \neq -\pi/2$ *near the origin and let the sequence of complex numbers* $\{\lambda_{\alpha,k}\}$, $(\alpha,k) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+$, *satisfy the following property: there is a constant* $C > 0$ *and sequence* $C_k > 0$ *such that*

$$|\lambda_{\alpha,k}| \leq C_k C^{|\alpha|} \alpha!, \qquad (\alpha,k) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+.$$

Then there is a smooth solution u for $\mathcal{V}$, defined in a neighborhood of the origin, such that $(\mathrm{X}^\alpha \mathrm{N}^k u)(0,0) = \lambda_{\alpha,k}$, $(\alpha, k) \in \mathbb{Z}_+^{m-1} \times \mathbb{Z}_+$.

Notice that, thanks to Example 4, this is the best conclusion we can expect under the given hypothesis: we emphasize that we are requiring that $\mathcal{V}$ only satisfies the first half of Property $(\mathfrak{B})$ at the origin.

Proof. Let $\{\lambda_{\alpha,k}\}$ be as in the statement and form

$$a_k(z) \doteq \sum_{\beta \in \mathbb{Z}_+^{m-1}} \frac{\lambda_{\beta,k}}{\beta!} z^\beta , \quad k = 0, 1, \dots .$$

Each a_k is holomorphic in a fixed polydisc Δ centered at the origin of $\mathbb{C}^{m-1}$ and $(\partial_z^\alpha a_k)(0) = \lambda_{\alpha,k}$. Let $S_* = \{w \in \mathbb{C} : |w| < 2, \ -\pi/2 < \arg w < 3\pi/2\}$. By Proposition 6 there is $H(z,w) \in \mathcal{O}(\Delta \times S_*)$ such that $H(z,w) \sim \sum_{j \geq 0} a_j(z) w^j / j!$ uniformly. In particular

$$(\partial_z^\alpha H)(0,w) \sim \sum_{j \geq 0} \lambda_{\alpha,j} w^j / j!$$

and then

$$(*) \qquad \lim_{\substack{(z,\,w) \to (0,\,0) \\ (z,\,w) \in \Delta \times S_{1/2}}} (\partial_z^\alpha \partial_w^k H)(z,w) = \lambda_{\alpha,k} .$$

Next we observe that $\Phi(x_1, \dots, x_{m-1}, 0, t) \geq 0$ near the origin and thus, as in the proof of Theorem 3, we derive the existence of an open ball U centered at the origin of $\mathbb{R}^N$ such that W maps $U_0 \doteq U \setminus W^{-1}\{0\}$ into $S_{1/2}$. If we define $u(x,t) = H(z, W(x,t))$ then u is a solution in U_0 and $(\mathrm{X}^\alpha \mathrm{N}^k u)(x,t) = \partial_z^\alpha \partial_w^k H(z, W(x,t))$, $(x,t) \in U_0$. We can now proceed as in the proof of Theorem 3, taking $(*)$ into account, in order to show that u extends to U as a smooth solution satisfying the required properties. ∎

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