Klein-Gordon type wave models with non-effective time-dependent potential

Dedicated to Sergei Rogosin[1] on his 60th birthday

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Abstract

The goal of this note is to derive some energy estimates for solutions to the Cauchy problem for Klein-Gordon type models

$$u_{tt} - \Delta u + m(t)^2 u = 0$$

with a so-called non-effective time-dependent potential $m(t)^2 u$. In particular, we are interested in precise estimates for the potential energy. Some scattering result completes our considerations.

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1 Introduction

The Cauchy problem for the wave equation of Klein-Gordon type

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.1}$$

where $m(t)^2 u$ is a time-dependent potential, encompass two important models: the classical wave model ($m(t)^2 \equiv 0$) and the classical Klein-Gordon model ($m(t)^2 \equiv m_0^2$). Their respective energies

$$E_W(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\big)dx \tag{1.2}$$

and

$$E_{KG}(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + m_0^2|u(t,x)|^2\big)dx \tag{1.3}$$

are conserved.

Remark 1.1. The difference between the relations (1.2) and (1.3) is that to prove energy conservation we can include in $E_{KG}(u)(t)$ not only the elastic or kinetic energy as in $E_W(u)(t)$ but also the potential energy.

There appear already some questions after these few observations:

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1. How do we define the energy for the solutions to (1.1) with a general time-dependent coefficient $m(t)^2$?

2. What kind of a-priori estimates do we expect for a suitable energy for solutions to (1.1)?

To find answers to these questions is not a trivial thing as the following example from [2] shows:

Example 1.1. Let us consider with $\mu^2 > 0$ the scale invariant model

$$u_{tt} - \Delta u + \frac{\mu^2}{(1+t)^2} u = 0, \;\; u(0,x) = u_0(x), \;\;\; u_t(0,x) = u_1(x).$$

We introduce the energy $E^{(\mu)}(u) = E^{(\mu)}(u)(t)$ in the form

$$E^{(\mu)}(u)(t) := \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla_x u(t,\cdot)\|^2_{L^2} + p_\mu(t)^2 \|u(t,\cdot)\|^2_{L^2}\Big), \tag{1.4}$$

where

$$p_\mu(t) = \begin{cases} (1+t)^{-\frac{1}{2}}, & \mu^2 > \frac{1}{4}, \\ (1+t)^{-\frac{1}{2}}\big(1+\ln(1+t)\big)^{-1}, & \mu^2 = \frac{1}{4}, \\ (1+t)^{-\frac{1}{2}-\frac{1}{2}\sqrt{1-4\mu^2}}, & \mu^2 \in \big(0, \frac{1}{4}\big). \end{cases} \tag{1.5}$$

Then the *generalized energy conservation*

$$p_\mu(t)^2 E^{(\mu)}(u)(0) \lesssim E^{(\mu)}(u)(t) \lesssim E^{(\mu)}(u)(0) \tag{1.6}$$

holds.

Remark 1.2. The estimate (1.6) excludes a blow-up behavior of the energy $E^{(\mu)}(u)(t)$ for $t \to \infty$. Moreover, it yields a lower bound of the decay behavior for this energy. We see that the potential energy can be estimated in the following way:

$$\|u(t,\cdot)\|^2_{L^2} \lesssim p_\mu(t)^{-2} E^{(\mu)}(u)(0).$$

If $\mu \to +0$, then $p_\mu(t)^{-2}$ tends to $(1+t)^2$, an asymptotic profile which is known for the potential energy of solutions to the Cauchy problem for the free wave equation. If $\mu \to \infty$, then $p_\mu(t)^{-2} = 1+t$, so the potential energy has a smaller growth for $t \to \infty$.

The Remark 1.2 motivates the above questions for coefficients $m(t)^2$ which are above or below those from Example 1.1. The PhD thesis [1] are devoted to this issue. The author studies decreasing coefficients $m = m(t)$ which are "above" the coefficients from Example 1.1, that is, which satisfy among other things $\lim_{t\to\infty} tm(t) = \infty$. In this case models (1.1) are called models of Klein-Gordon type. Generalized energy conservation and $L^p - L^q$ decay estimates on the conjugate line are proved. Moreover, some scattering result to free waves (cf. with Section 4) is presented. What remains open in the thesis is to explain qualitative properties of solutions to (1.1) with a potential which does not allow on the one hand scattering to free waves, and which is "below" the critical case from Example 1.1. Typical examples are decreasing $m(t)$ satisfying $m \notin L^1(\mathbb{R}^+)$ and $\lim_{t\to\infty} tm(t) = 0$. The present note concerns with this topic. In Section 2.1 we will explain our main results. Some examples are given in Section 2.2. The proofs are given in Section 3. First we describe the philosophy of our approach in Section 3.1. In Sections 3.2 to 3.4 we give the proofs to Theorems 2.1 to 2.3. Section 4 is devoted to scattering results. Some remarks complete the note (see Section 5).

2 Main results and examples

2.1 Main results

Let us consider the Cauchy problem of Klein-Gordon type (1.1) under the following assumptions:

Hypothesis 2.1. Let $m(t) \in C(\mathbb{R}^+)$ satisfy

$$|m(t)| \lesssim^{3} \frac{1}{1+t}. \tag{2.1}$$

Hypothesis 2.2. There exists a positive increasing function $\psi \in C^2(\mathbb{R}^+)$ with $\psi(0) = 1$ such that

$$\limsup_{t\to\infty} 2(1+t)\frac{\psi'(t)}{\psi(t)} < 1, \quad \Big|\frac{\psi''(t)}{\psi(t)}\Big| \lesssim \frac{1}{(1+t)^2}. \tag{2.2}$$

Besides (2.2) the following relation between $m(t)$ and $\psi(t)$ must be satisfied:

$$\int_0^\infty (1+\tau)\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| d\tau \lesssim 1. \tag{2.3}$$

Moreover, we define the energy

$$E(u)(t) = \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla u(t,\cdot)\|^2_{L^2} + p(t)^2\|u(t,\cdot)\|^2_{L^2}\Big), \quad \text{where } p(t) = (1+t)^{-1}\psi(t). \tag{2.4}$$

Theorem 2.1. *Under Hypotheses* 2.1 *and* 2.2 *the solution of the Cauchy problem* (1.1) *satisfies the energy estimate*

$$E(u)(t) \lesssim E(u)(0). \tag{2.5}$$

Here we assume additionally, that the data (u_0, u_1) *belong to the energy space* $H^1 \times L^2$.

Remark 2.1. Applying Theorem 2.1 gives the following estimate for the potential energy:

$$\|u(t,\cdot)\|^2_{L^2} \lesssim (1+t)^2\psi(t)^{-2}E(u)(0).$$

Remark 2.2. After differentiation of the Klein-Gordon energy $E_{KG}(u)(t)$ with respect to t we get

$$E'_{KG}(u)(t) = m(t)m'(t)\int_{\mathbb{R}^n} |u(t,x)|^2 dx.$$

If $m(t)$ is a positive decreasing function, then $E_{KG}(u)(t) \le E_{KG}(u)(0)$. In particular

$$\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla u(t,\cdot)\|^2_{L^2} \lesssim E(u)(0).$$

What remains to prove in this case is the desired estimate to $\|u(t,\cdot)\|^2_{L^2}$ in Theorem 2.1. However, one needs to be more careful if m has some oscillations.

[3] Let $f, g : [0,\infty) \to (0,\infty)$ be two strictly positive functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all $t \ge 0$. If the inequality is one-sided, namely, if $f(t) \le Cg(t)$ (resp. $f(t) \ge Cg(t)$) for all $t \ge 0$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

Now, let us consider the Cauchy problem (1.1) with a coefficient $m = m(t)$ having the special structure

$$m(t) = \frac{\mu}{(1+t)g(t)} \quad \text{with a positive constant } \mu. \tag{2.6}$$

We suppose the following conditions:

Hypothesis 2.3. Let $g \in C^1(\mathbb{R}^+)$ be a positive, increasing function with $g(0) = 1$ satisfying

$$g'(t) \lesssim \frac{g(t)}{1+t}. \tag{2.7}$$

Remark 2.3. From (2.7) functions m which are given by (2.6) satisfy $|m'(t)| \lesssim \frac{m(t)}{1+t}$.

Hypothesis 2.4. There exists an integer $N \geq 0$ such that

$$\int_0^\infty \frac{1}{(1+\tau)g(\tau)^{2(N+1)}} d\tau \lesssim 1. \tag{2.8}$$

Remark 2.4. From (2.8) we get $\lim_{t\to\infty} g(t) = \infty$. This implies $tm(t) \to 0$ as $t \to \infty$. So, under Hypothesis 2.4 we are really consider a class of masses $m(t)^2 u$ in (1.1) "below" the scale invariant case.

For the model (2.6) we can explicitly give the function ψ in Hypotheses 2.2. Under Hypothesis 2.4 it turns out that in the case $(1+t)m(t)^2 \in L^1$ we can take $\psi \equiv 1$. Otherwise, we choose

$$\psi(t) = \exp\Big(\sum_{k=1}^{N} \gamma_k \mu^{2k} \int_0^t \frac{1}{(1+\tau)g(\tau)^{2k}} d\tau \Big), \ \ \gamma_k = \sum_{\ell=1}^{k-1} \gamma_\ell \gamma_{k-\ell}, \ \gamma_1 = 1. \tag{2.9}$$

Remark 2.5. The sequence $\{\gamma_k\}_k$ in (2.9) is well-known as Segner's recurrence formula given by Segner in 1758. It gives the solution to Euler's polygon division problem. The solution is described by the Catalan numbers which are given by the explicit formula [7]

$$\gamma_k = \frac{(2k-2)!}{k!(k-1)!}. \tag{2.10}$$

By using (2.10) one can explicitly compute the convergence ratio R of the power series $\sum_{k=1}^\infty \gamma_k \mu^{2k}$ by

$$R = \lim_{k\to\infty} \Big| \frac{\gamma_k}{\gamma_{k+1}} \Big| = \lim_{k\to\infty} \frac{k^2}{2k(2k-1)} = \frac{1}{4},$$

and the series converges uniformly for $\mu^2 < \frac{1}{4}$.

In the definition of function ψ given by (2.9) we can take N as the smallest integer satisfying Hypothesis 2.4.

Theorem 2.2. *Under Hypotheses* 2.3 *and* 2.4 *the solution of the Cauchy problem* (1.1) *with* $m(t)$ *given by* (2.6) *satisfies the energy estimate*

$$E(u)(t) \lesssim E(u)(0), \tag{2.11}$$

where the energy $E(u)(t)$ *is defined by (*2.4*), with* $\psi(t)$ *given by* (2.9)*. The data* (u_0, u_1) *are from the energy space* $H^1 \times L^2$*.*

If we can not find any N satisfying Hypothesis 2.4, then we replace N by infinity in (2.9). In addition, in the case of $g_\infty = \lim_{t\to\infty} g(t) < \infty$ we introduce the following condition:

Hypothesis 2.5. With γ_k from (2.9) we assume

$$\sum_{k=1}^{\infty} \gamma_k \frac{\mu^{2k}}{g_\infty^{2k}} < \frac{1}{2} \text{ for all } \mu^2 < \frac{g_\infty^2}{4}. \tag{2.12}$$

Remark 2.6. For given $t_0 > 0$ and $\mu^2 < \frac{g(t_0)^2}{4}$ the series

$$\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t)^{2k}} \tag{2.13}$$

converges uniformly for all $t \geq t_0$. Indeed, by using that $1 \leq g(t)$ and by taking into account the benefit of $g(t)$ is an increasing function, then for all $t \geq t_0$ we have

$$\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t)^{2k}} \leq \sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}.$$

By using that $\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}$ converges for $\mu^2 < \frac{g(t_0)^2}{4}$ we can apply the Weierstrass M-test to conclude the uniformly convergence of the series in (2.13) for all $t \geq t_0$. Moreover, the power series $\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k-1}}{g(t_0)^{2k}}$ has the same radius of convergence $\mu^2 < \frac{g(t_0)^2}{4}$, because it is the derivative on μ of the series $\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}$. This implies, together with (2.7), that for $\mu^2 < \frac{g(t_0)^2}{4}$, $\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k} g'(t)}{g(t)^{2k+1}}$ converges uniformly for all $t \geq t_0$.

Remark 2.6 allows us to choose a function $\psi_{t_0} \in C^2[t_0, \infty)$ with a large t_0 if necessary, which is defined by

$$\psi_{t_0}(t) = \exp\Big(\sum_{k=1}^{\infty} \gamma_k \mu^{2k} \int_{t_0}^{t} \frac{1}{(1+\tau) g(\tau)^{2k}} d\tau \Big). \tag{2.14}$$

Theorem 2.3. *Under Hypotheses* 2.3 *and* 2.5 *the solution of the Cauchy problem* (1.1) *satisfies*

$$E(u)(t) \lesssim E(u)(0), \tag{2.15}$$

where the energy $E(u)(t)$ is defined by (2.4) with $\psi(t) = \psi_{t_0}(t)$ from (2.14). *The data* (u_0, u_1) *are from the energy space $H^1 \times L^2$.*

2.2 Examples

Example 2.1. If $g(t)$ in (2.6) is given by $g(t)^2 = \ln(e+t) \cdots \ln^{[m]}(e^{[m]} + t)$ with $e^{[k+1]} = e^{e^{[k]}}$ and $\ln^{[k+1]}(t) = \ln(\ln^{[k]}(t))$, then we have (2.8) for $N = 1$, i.e., the conclusion of Theorem 2.2 holds with $\psi(t)$ given by (2.9). We have that $\psi(t) \approx (\ln^{[m]}(e^{[m]} + t))^{\mu^2}$.

Example 2.2. Let $g(t)^2 = (\ln(e+t))^\gamma$ for some $0 < \gamma < 1$. In order to have (2.8) one should take N such that $(N+1)\gamma > 1$. Then the conclusion of Theorem 2.2 holds with $\psi(t)$ given by (2.9).

Example 2.3. Let us consider the Cauchy problem (1.1) with $m(t) = \frac{\mu}{1+t}$ and $\mu \neq 0$, i.e., we consider the scale invariant case from [2]. Here $g(t) \equiv 1$, hence, there does not exist a positive integer N such that (2.8) holds. In order to apply Theorem 2.3 one has to verify Hypothesis 2.5. Let us take the function ψ from Theorem 2.3 as

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_0^t \frac{\gamma_k \mu^{2k}}{(1+\tau)}d\tau\Big) = (1+t)^{\sigma}$$

with $\sigma = \sum_{k=1}^{\infty}\gamma_k\mu^{2k}$. By using the infinite Cauchy product and from the definition of γ_k we get

$$\sigma^2 = \Big(\sum_{k=1}^{\infty}\gamma_k\mu^{2k}\Big)^2 = \sum_{n=2}^{\infty}\gamma_n\mu^{2n} = \sigma - \mu^2.$$

Therefore, $\sigma_{\pm} = \frac{1\pm\sqrt{1-4\mu^2}}{2}$ and $\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2 = 0$. If we take $\sigma_- = \frac{1-\sqrt{1-4\mu^2}}{2}$, then $2\sigma_- < 1$ and (2.12) holds. In this way, for $\mu^2 \in \big(0, \frac{1}{4}\big)$, we derived the decay estimate which is proposed by (1.5) and (1.6).

Example 2.4. If $g(t) = \ln(\ln(e^e + t))$, then we can take for $t \geq t_0$, $t_0 \gg 1$, the function

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_{t_0}^t \frac{\gamma_k \mu^{2k}}{(1+\tau)g(\tau)^{2k}}d\tau\Big)$$

which is well defined for $\mu^2 < \frac{g(t_0)^2}{4}$. It is clear that the condition (2.12) holds and the statement of Theorem 2.3 is applicable.

3 Proof of the main results

3.1 Philosophy of our approach

We have a good knowledge [5] about asymptotic properties of solutions to wave equations with a time-dependent dissipation. In order to get some feeling for the behavior of solutions to (1.1) we can transform the time-dependent potential to a time-dependent damping and a new potential. If we consider the change of variables given by $u(t,x) = \psi(t)v(t,x)$, then the Cauchy problem (1.1) takes the form

$$v_{tt} - \triangle v + 2\frac{\psi'(t)}{\psi(t)}v_t + \Big(\frac{\psi''(t)}{\psi(t)} + m(t)^2\Big)v = 0,\ v(0,x) = \frac{u_0(x)}{\psi(0)},\ v_t(0,x) = v_1(x) \qquad (3.1)$$

with $v_1(x) = \frac{u_1(x) - \frac{\psi'(0)}{\psi(0)}u_0(x)}{\psi(0)}$. Therefore, if we take ψ such that $\psi''(t) + m(t)^2\psi(t) = 0$, then we can apply directly the result of [5]. The main difficulty is that, in general, it is not easy to obtain an explicit representation of ψ in terms of $m(t)^2$. Fortunately, it will be sufficient to find an asymptotic solution, that is, we will look for a solution ψ such that (2.3) is satisfied. This approach was motivated by some ideas of [3], where the authors studied the long time behavior of the wave-type energy for a class of Cauchy problems of the form

(3.1) by given sufficient conditions in order to exclude contributions to the energy coming from the time-dependent potential.
Our main contributions are to give explicit representations to such asymptotic solutions ψ, at least for the model (2.6). Depending on the function g itself in Theorems 2.2 and 2.3 the function ψ is given by (2.9) and (2.14), respectively.

3.2 Proof of Theorem 2.1

Proof. The proof is divided into several steps.

3.2.1 Preliminaries

We perform the partial Fourier transformation of (1.1) with respect to x and obtain

$$\widehat{u}_{tt}+|\xi|^2\widehat{u}+m(t)^2\widehat{u}=0,\ \ \widehat{u}(0,\xi)=\widehat{u_0}(\xi),\ \ \widehat{u}_t(0,\xi)=\widehat{u_1}(\xi). \tag{3.2}$$

We divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into the *pseudo-differential zone* $Z_{pd}(N)$ and into the *hyperbolic zone* $Z_{hyp}(N)$ which are defined by

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n:(1+t)|\xi|\le N\},\\ Z_{hyp}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n:(1+t)|\xi|\ge N\}. \end{aligned}$$

The *separating curve* is given by

$$\theta:(0,N]\to[0,\infty),\quad (1+\theta_{|\xi|})|\xi|=N.$$

We put also $\theta_0=\infty$, and $\theta_{|\xi|}=0$ for any $|\xi|\ge N$. The pair (t,ξ) from the extended phase space belongs to $Z_{pd}(N)$ (resp. to $Z_{hyp}(N)$) if and only if $t\le\theta_{|\xi|}$ (resp. $t\ge\theta_{|\xi|}$).

3.2.2 Considerations in the pseudo-differential zone

In the pseudo-differential zone $Z_{pd}(N)$ we put

$$V=\Big(\frac{\widehat{u}}{1+t},\widehat{u}_t-\frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T,\ \ V_0(\xi)=\Big(\widehat{u_0}(\xi),\widehat{u_1}(\xi)-\frac{\psi'(0)}{\psi(0)}\widehat{u_0}(\xi)\Big)^T,\ \text{ and } V=\psi(t)\widetilde{V}.$$

So we have

$$\partial_t\widetilde{V}(t,\xi)=\mathcal{A}(t,\xi)\widetilde{V}=\begin{pmatrix} -\frac{1}{1+t} & \frac{1}{1+t}\\ -(1+t)\Big(\frac{\psi''}{\psi}+m(t)^2+|\xi|^2\Big) & -2\frac{\psi'(t)}{\psi(t)}\end{pmatrix}\widetilde{V}. \tag{3.3}$$

We have to prove that the fundamental solution $E=E(t,s,\xi)$ to (3.3), that is, the solution to

$$\partial_tE=\mathcal{A}(t,\xi)E\,,\quad E(s,s,\xi)=I,$$

satisfies the estimate $|E(t,0,\xi)|\lesssim\psi(t)^{-2}$. If we put $E=(E_{ij})_{i,j=1,2}$, then we can write for $j=1,2$ the following system of coupled integral equations:

$$E_{1j}(t,0,\xi)=(1+t)^{-1}\Big(\delta_{1j}+\int_0^tE_{2j}(\tau,0,\xi)d\tau\Big), \tag{3.4}$$

$$E_{2j}(t,0,\xi)=\psi(t)^{-2}\Big(\delta_{2j}-\int_0^t(1+\tau)\psi(\tau)^2\Big(\frac{\psi''}{\psi}(\tau)+m(\tau)^2+|\xi|^2\Big)E_{1j}(\tau,0,\xi)d\tau\Big). \tag{3.5}$$

By replacing (3.5) into (3.4) and after integration by parts we get

$$E_{1j}(t,0,\xi) = (1+t)^{-1}\Big(\delta_{1j} + \delta_{2j}\int_0^t \psi(\tau)^{-2}d\tau\Big)$$
$$-(1+t)^{-1}\int_0^t (1+\tau)\psi(\tau)^2\Big(\frac{\psi''}{\psi}(\tau) + m(\tau)^2 + |\xi|^2\Big)E_{1j}(\tau,0,\xi)\int_\tau^t \psi(s)^{-2}dsd\tau. \quad (3.6)$$

By using that ψ is increasing and by the first estimate in (2.2) (see Proposition 7 of [6]) we have

$$\int_0^t \psi(s)^{-2}ds \sim \frac{t}{\psi(t)^2}, \quad (3.7)$$

and $\frac{t}{\psi(t)^2}$ is increasing for large t. Introducing

$$h_j(t,\xi) := |E_{1j}(t,0,\xi)|\psi(t)^2$$

and by using $\psi(t)^2 \le 1+t$ for large t we conclude from (3.6) and (3.7) that

$$h_j(t,\xi) \le C + C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| + |\xi|^2\Big)h_j(\tau,\xi)d\tau.$$

Applying Gronwall's type inequality we conclude

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| + |\xi|^2\Big)d\tau\Big).$$

In $Z_{pd}(N)$ we have $(1+t)|\xi| \le C$. So, from the last estimate we get

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big|\Big)d\tau\Big).$$

Finally, by using (2.3) we get $|E_{1j}(t,0,\xi)| \lesssim \psi(t)^{-2}$. From the boundedness of $|E_{1j}(t,0,\xi)|\psi(t)^2$, using again (2.3), we can estimate $|E_{2,j}(t,0,\xi)| \lesssim \psi(t)^{-2}$. Therefore, we proved that $|E(t,0,\xi)| \lesssim \psi(t)^{-2}$, that is,

$$|V(t,\xi)| \le C\psi(t)^{-1}|V_0(\xi)|. \quad (3.8)$$

3.2.3 Considerations in the hyperbolic zone

In the hyperbolic zone $Z_{hyp}(N)$ we put

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta_{|\xi|},\xi), \widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|},\xi)\Big)^T,$$

and $U = \psi(t)\widetilde{U}$, so that

$$\partial_t\widetilde{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i|\xi|\widetilde{U} + \begin{pmatrix} 0 & 0 \\ 0 & -2\frac{\psi'(t)}{\psi(t)} \end{pmatrix}\widetilde{U} + \begin{pmatrix} 0 & 0 \\ -\frac{\psi''}{\psi}(t) - m(t)^2 & 0 \end{pmatrix}(i|\xi|)^{-1}\widetilde{U} \quad (3.9)$$

for $t \geq \theta_{|\xi|}$ with initial datum $\widetilde{U}(\theta_{|\xi|}, \xi) = \psi(\theta_{|\xi|})^{-1} U_0(\xi)$. Let P be the diagonalizer of the principal part (with respect to powers of $|\xi|$) of (3.9) given by

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If we put $V(t,\xi) := P^{-1}\widetilde{U}(t,\xi)$, then we get

$$\partial_t V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| V + B_0(t,\xi)\, V + B_1(t)(i|\xi|)^{-1} V, \tag{3.10}$$

where

$$B_0(t,\xi) = -\frac{\psi'(t)}{\psi(t)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B_1(t) = -\frac{1}{2} \left(\frac{\psi''}{\psi}(t) + m(t)^2 \right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now we define the *second diagonalizer* that depends on the anti-diagonal entries of $B_0(t)$:

$$K(t,\xi) := \begin{pmatrix} 1 & \frac{h(t)}{2i|\xi|} \\ -\frac{h(t)}{2i|\xi|} & 1 \end{pmatrix}, \qquad h(t) = \frac{\psi'(t)}{\psi(t)}. \tag{3.11}$$

Thanks to (2.2) we have

$$\frac{|h(t)|}{|\xi|} \leq \frac{C}{(1+t)|\xi|} \leq \frac{C}{N}$$

for $t \geq \theta_{|\xi|}$, hence, $|\det K| \geq 1 - C^2/N^2$. Therefore, $K(t,\xi)$ and $K^{-1}(t,\xi)$ are bounded for a sufficiently large N. We replace $V(t,\xi) =: K(t,\xi)W(t,\xi)$. We get

$$\partial_t W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| W - \frac{\psi'(t)}{\psi(t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W + J(t,\xi) W, \tag{3.12}$$

where $J(t,\xi) = K^{-1}(t,\xi)R(t,\xi)$ with $D_0(t,\xi) = \operatorname{diag}(-i|\xi|, i|\xi|)$, $H(t,\xi) = K(t,\xi) - I$ and

$$\begin{aligned} R &= D_0 K + B_0 K - \partial_t K - K D_0 - K \operatorname{diag} B_0 + (i|\xi|)^{-1} B_1 K \\ &= B_0 + D_0 H - H D_0 - \operatorname{diag} B_0 - H \operatorname{diag} B_0 - \partial_t H + B_0 H + (i|\xi|)^{-1} B_1 K. \end{aligned}$$

By construction the sum of the first four terms of $R(t,\xi)$ vanishes. Thanks to condition (2.2) and Hypothesis 2.1 the matrix $R(t,\xi)$, and therefore also $J(t,\xi)$, satisfies the following estimate in $Z_{hyp}(N)$:

$$\|J(t,\xi)\| \leq \frac{C}{|\xi|(1+t)^2}. \tag{3.13}$$

After substituting $W(t,\xi) =: \frac{\psi(\theta_{|\xi|})}{\psi(t)} D(t,\xi) Z(t,\xi)$, where

$$D(t,\xi) = \operatorname{diag}\Big(\exp(-i|\xi|(t-\theta_{|\xi|})), \exp(i|\xi|(t-\theta_{|\xi|})) \Big),$$

we obtain the following Cauchy problem in $Z_{hyp}(N)$:

$$\begin{cases} \partial_t Z = \widetilde{J}(t,\xi)\, Z, & t \geq \theta_{|\xi|}, \\ Z(\theta_{|\xi|}, \xi) = K^{-1}(\theta_{|\xi|}, \xi) P^{-1} \widetilde{U}(\theta_{|\xi|}, \xi), \end{cases} \tag{3.14}$$

where the matrix $\widetilde{J}(t,\xi) = D^{-1}(t,\xi)J(t,\xi)D(t,\xi)$ satisfies again (3.13). For any $s,t \geq \theta_{|\xi|}$ we have

$$\int_s^t \|\widetilde{J}(\tau,\xi)\|\, d\tau \leq C\int_{\theta_{|\xi|}}^\infty \frac{1}{|\xi|(1+\tau)^2}\, d\tau \leq \frac{C'}{|\xi|(1+\theta_{|\xi|})} = \frac{C'}{N},$$

hence, $|Z(t,\xi)| \leq C|Z(\theta_{|\xi|},\xi)|$ and, by using Liouville's formula, $|Z(t,\xi)| \geq C'|Z(\theta_{|\xi|},\xi)|$. Indeed, let $E(t,\xi)$ be the fundamental solution of (3.14), then $Z(t,\xi) = E(t,\xi)Z(\theta_{|\xi|},\xi)$. By Liouville's formula, $\det E(t,\xi) = \exp(\int_0^t \operatorname{tr} \widetilde{J}(s,\xi)ds) \approx 1$. Therefore,

$$|Z(\theta_{|\xi|},\xi)| = |E^{-1}(t,\xi)Z(t,\xi)| \leq C|Z(t,\xi)|.$$

Summarizing we have proved in $Z_{hyp}(N)$

$$C_1\,\frac{\psi(\theta_{|\xi|})^2}{\psi(t)^2}\,|\widetilde{U}(\theta_{|\xi|},\xi)|^2 \leq |\widetilde{U}(t,\xi)|^2 \leq C_2\,\frac{\psi(\theta_{|\xi|})^2}{\psi(t)^2}\,|\widetilde{U}(\theta_{|\xi|},\xi)|^2. \tag{3.15}$$

3.2.4 Verification

We conclude the proof of (2.5) under the use of (2.4). We claim that

$$|\xi|^2|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 \lesssim (1+|\xi|^2)|\widehat{u_0}(\xi)|^2 + |\widehat{u_1}(\xi)|^2 \tag{3.16}$$

and

$$|\widehat{u}(t,\xi)|^2 \lesssim \frac{(1+t)^2}{\psi(t)^2}\Big(|\widehat{u_0}(\xi)|^2 + \frac{|\widehat{u_1}(\xi)|^2}{1+|\xi|^2}\Big) \tag{3.17}$$

uniformly with respect to $\xi \in \mathbb{R}^n$. By integrating these inequalities with respect to ξ and by Plancherel's Theorem we have our desired estimate (2.5).
Let us first prove (3.16). By using Cauchy-Schwarz inequality, the first estimate in (2.2) and the considerations in the pseudo-differential zone we conclude for all $t \leq \theta_{|\xi|}$ the estimates

$$\begin{aligned}
|V(t,\xi)|^2 &\geq \frac{1}{(1+t)^2}|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 + \left|\frac{\psi'(t)}{\psi(t)}\right|^2|\widehat{u}(t,\xi)|^2 - 2|\widehat{u}_t(t,\xi)|\left|\frac{\psi'(t)}{\psi(t)}\widehat{u}(t,\xi)\right| \\
&\geq \frac{1}{(1+t)^2}|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 - \left|\frac{\psi'(t)}{\psi(t)}\right|^2|\widehat{u}(t,\xi)|^2 \\
&\geq \frac{3}{4(1+t)^2}|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 \geq \frac{3}{4N^2}|\xi|^2|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2.
\end{aligned}$$

Therefore, by using (3.8) we have for all $t \leq \theta_{|\xi|}$

$$|\xi|^2|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 \lesssim |V(t,\xi)|^2 \lesssim \frac{1}{\psi(t)^2}|V_0(\xi)|^2.$$

For $t \geq \theta_{|\xi|}$ we have to glue the estimate (3.8) with (3.15). By using again Cauchy-Schwarz inequality and the first estimate in (2.2) we have

$$\Big(1 - \frac{1}{4N^2}\Big)|\xi|^2|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 \lesssim |U(t,\xi)|^2 \quad \text{for all } t \geq \theta_{|\xi|}.$$

By using (3.15) we get $|U(t,\xi)|^2 \lesssim |U_0(\xi)|^2$. Moreover, since $\theta_{|\xi|} = 0$ for any $|\xi| \geq N$, then

$$|U_0(\xi)|^2 \lesssim (1+|\xi|^2)|\widehat{u_0}(\xi)|^2 + |\widehat{u_1}(\xi)|^2 \quad \text{for all } |\xi| \geq N.$$

By applying (3.8) we conclude

$$|U_0(\xi)|^2 \lesssim \frac{1}{\psi(\theta_{|\xi|})^2}\Big(|\widehat{u_0}(\xi)|^2 + |\widehat{u_1}(\xi)|^2\Big) \quad \text{for all } |\xi| \leq N.$$

Therefore, (3.16) follows by taking N sufficiently large and by using that ψ is increasing. Now let us prove (3.17). For $t \leq \theta_{|\xi|}$ we have from (3.8) the estimate

$$|\widehat{u}(t,\xi)|^2 \lesssim \frac{(1+t)^2}{\psi(t)^2}|V_0(\xi)|^2.$$

In order to estimate $|\widehat{u}(t,\xi)|^2$ in the hyperbolic zone we split our considerations for $|\xi| \leq N$ and $|\xi| \geq N$. By definition, $\theta_{|\xi|} = 0$ for all $|\xi| \geq N$, and from (3.15) we have

$$|\widehat{u}(t,\xi)|^2 \lesssim \frac{|U(t,\xi)|^2}{|\xi|^2} = \frac{\psi(t)^2|\widetilde{U}(t,\xi)|^2}{|\xi|^2} \lesssim \frac{|\widetilde{U}(0,\xi)|^2}{|\xi|^2} \lesssim |\widehat{u_0}(\xi)|^2 + \frac{|\widehat{u_1}(\xi)|^2}{|\xi|^2} \quad \text{for all } |\xi| \geq N.$$

Since $\frac{t}{\psi(t)^2}$ is increasing for large t the same is true for $\frac{t}{\psi(t)}$ and (3.17) holds. On the other hand, for $|\xi| \leq N$, from (3.15) and (3.8) we conclude

$$\begin{aligned}
|\widehat{u}(t,\xi)|^2 &\lesssim \frac{|U(t,\xi)|^2}{|\xi|^2} \lesssim \frac{\psi(\theta_{|\xi|})^2|\widetilde{U}(\theta_{|\xi|},\xi)|^2}{|\xi|^2} = \frac{|U_0(\xi)|^2}{|\xi|^2} \\
&\lesssim |\widehat{u}(\theta_{|\xi|},\xi)|^2 + \frac{1}{|\xi|^2}\Big|\widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|},\xi)\Big|^2 \\
&\underset{\text{by (3.8)}}{\lesssim} \frac{(1+\theta_{|\xi|})^2}{\psi^2(\theta_{|\xi|})}|V_0(\xi)|^2 + \frac{(1+\theta_{|\xi|})^2}{|\xi|^2(1+\theta_{|\xi|})^2\psi^2(\theta_{|\xi|})}|V_0(\xi)|^2 \underset{|\xi|(1+\theta_{|\xi|})=N}{\lesssim} \frac{(1+\theta_{|\xi|})^2}{\psi(\theta_{|\xi|})^2}|V_0(\xi)|^2,
\end{aligned}$$

and (3.17) follows again by using that $\frac{t}{\psi(t)}$ is increasing. This completes the proof of Theorem 2.1. □

3.3 Proof of Theorem 2.2

Proof. The desired statement will be a consequence of Theorem 2.1. It is clear that Hypothesis 2.3 implies that $m(t)$ from (2.6) satisfies Hypothesis 2.1. Moreover, in the case $(1+t)m(t)^2 \in L^1$ it follows from Hypothesis 2.4 that Hypothesis 2.2 holds by taking $\psi \equiv 1$. Otherwise, we have $\lim_{t\to\infty} g(t) = \infty$, which implies the first condition of (2.2). It remains to prove that the function ψ which is given by (2.9) satisfies (2.3) and the second condition in (2.2). Indeed, by using the Cauchy product, i.e.,

$$\Big(\sum_{k=0}^{n} a_k\Big)\cdot\Big(\sum_{k=0}^{n} b_k\Big) = \sum_{k=0}^{2n}\sum_{i=0}^{k} a_i b_{k-i} - \sum_{k=0}^{n-1}\Big(a_k\sum_{i=n+1}^{2n-k} b_i + b_k\sum_{i=n+1}^{2n-k} a_i\Big)$$

with $a_0 = b_0 = 0$ it follows by using the definition of the constants γ_k that

$$\begin{aligned}\frac{\psi''(t)}{\psi(t)} &= -\sum_{k=1}^{N} \frac{\gamma_k \mu^{2k}}{(1+t)^2 g(t)^{2k}} - \sum_{k=1}^{N} \frac{2k\gamma_k \mu^{2k} g'(t)}{(1+t) g(t)^{2k+1}} + \Big(\sum_{k=1}^{N} \frac{\gamma_k \mu^{2k}}{(1+t) g(t)^{2k}} \Big)^2 \\ &= -\frac{\mu^2}{(1+t)^2 g(t)^2} - \sum_{k=1}^{N} \frac{2k\gamma_k \mu^{2k} g'(t)}{(1+t) g(t)^{2k+1}} + \sum_{k=N+1}^{2N} \frac{\gamma_k \mu^{2k}}{(1+t)^2 g(t)^{2k}} \\ &\quad - \sum_{k=1}^{N-1} \Big(\frac{\gamma_k \mu^{2k}}{(1+t) g(t)^{2k}} \sum_{i=N+1}^{2N-k} \frac{\gamma_i \mu^{2i}}{(1+t) g(t)^{2i}} + \frac{\gamma_k \mu^{2k}}{(1+t) g(t)^{2k}} \sum_{i=N+1}^{2N-k} \frac{\gamma_i \mu^{2i}}{(1+t) g(t)^{2i}} \Big).\end{aligned}$$

Therefore, by using Hypotheses 2.3 and 2.4 we get the second condition of (2.2) and (2.3), respectively. □

3.4 Proof of Theorem 2.3

Proof. We can follow with a slightly modification the proof to Theorem 2.1. We split the pseudo-differential zone for $t \le t_0$ and and for $t \ge t_0$ with a large t_0. For $t \le t_0$ we are in a compact subset of this zone. Therefore, we only have to take into account the definition of ψ_{t_0} for large t. From Remark 2.6 the function ψ which is given by (2.14) is well defined. The first condition of (2.2) is immediately satisfied if $g(t)$ goes to infinity as t goes to infinity and, otherwise, it follows from (2.12). By using the Cauchy product we obtain

$$\frac{\psi''(t)}{\psi(t)} + m(t)^2 = -\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k} g'(t)}{(1+t) g(t)^{2k+1}}.$$

Therefore, $(1+t)\big|\frac{\psi''(t)}{\psi(t)} + m(t)^2\big| \in L^1$. By using Hypothesis 2.3 we get the second condition of (2.2) and the conclusion of Theorem 2.3 follows again from Theorem 2.1. □

4 A scattering result

4.1 Motivation

In this section we are interested in scattering results between the solutions of

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{4.1}$$

and

$$v_{tt} - \Delta v = 0, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x). \tag{4.2}$$

This question was studied in [1]. Let us assume the conditions

$$m \in L^1(\mathbb{R}^+), \;\; m(t)(1+t) \le C \;\text{ for }\; t \in [0,\infty). \tag{4.3}$$

Then the following result can be found in [1], Theorem 3.26:

Proposition 4.1. *Let the coefficient $m = m(t)$ satisfy (4.3). There exists a scattering operator $W_+ = W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that the Cauchy data to the problems (4.1) and (4.2) are related by $(|D|v_0, v_1)^T = W_+(D)(\langle D\rangle u_0, u_1)^T$. Then for the solutions of the problems (4.1) and (4.2) the asymptotic equivalence*

$$\Big\|(|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D\rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot)\Big)\Big\|_{L^2\times L^2} \to 0 \tag{4.4}$$

holds as t tends to infinity.

Here $|D|$, $\langle D\rangle$ and $\langle D\rangle_{m(t)}$ are pseudo-differential operators with symbols $|\xi|$, $\langle\xi\rangle := (|\xi|^2+1)^{\frac{1}{2}}$, $\langle\xi\rangle_{m(t)} := (|\xi|^2 + m(t)^2)^{\frac{1}{2}}$, respectively. We are interested in a scattering result under the assumption

$$(1+t)m(t)^2 \in L^1. \tag{4.5}$$

It is clear that (4.3) implies (4.5) but not conversely.

4.2 Scattering result and examples

Before stating the result we define for any $\epsilon > 0$ the following closed subset of $L^2 \times L^2$:

$$F_\epsilon := \big\{U_0 \in L^2 \times L^2 : \ U_0(\xi) = 0 \text{ for any } |\xi| \le \epsilon\big\}.$$

We remark that $\mathcal{L} = \cup_{\epsilon>0} F_\epsilon$ is a dense subset of $L^2 \times L^2$.

Theorem 4.2. *We assume the Hypothesis 2.1 and $(1+t)m(t)^2 \in L^1$. Then, for any initial data $(u_0, u_1) \in H^1 \times L^2$, there exists a linear, bounded operator $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that*

$$\lim_{t\to\infty} \Big\|(|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D\rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot)\Big)\Big\|_{L^2\times L^2} = 0, \tag{4.6}$$

where $u = u(t,x)$ is the solution to the Cauchy problem (4.1) and $v = v(t,x)$ is the solution to the Cauchy problem (4.2), where the initial data are related by $(|D|v_0, v_1) = W_+(D)(\langle D\rangle_N u_0, u_1)$. Moreover, on the dense subset $\mathcal{L}$ we can state the decay rate as

$$\Big\|(|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D\rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot)\Big)\Big\|_{\mathcal{L}} \lesssim \big\|(\langle D\rangle_N u_0, u_1)\big\|_{\mathcal{L}} \int_t^\infty (1+\tau)m^2(\tau)d\tau \tag{4.7}$$

as t goes to infinity.

Remark 4.1. From a scattering result to free waves for solutions to the damped wave equation (see [5])

$$w_{tt} - \triangle w + b(t)w_t = 0, \ w(0,x) = w_0(x), \ w_t(0,x) = w_1(x), \tag{4.8}$$

one can understand that $(1+t)m(t)^2 \in L^1$ is a reasonable condition to be assumed in Theorem 4.2. Indeed, let us assume that $b(t) = \frac{\mu}{(1+t)g(t)}$, where $g(t)$ satisfies Hypothesis 2.3. After performing the change of variables

$$w(t,x) = \exp\Big(-\frac{1}{2}\int_0^t b(s)ds\Big)u(t,x)$$

we get

$$u_{tt} - \triangle u + m(t)^2 u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x)$$

with

$$m(t)^2 = -\frac{1}{4}\big(b(t)^2 + 2b'(t)\big) = \frac{1}{2}\Big(\frac{\mu}{(1+t)^2 g(t)} + \frac{\mu g'(t)}{(1+t)g(t)^2} - \frac{\mu^2}{2(1+t)^2 g(t)^2}\Big).$$

Therefore, if $g(t)$ goes to infinity for t to infinity the condition $(1+t)m(t)^2 \in L^1$ is a necessary and sufficient condition to have $b \in L^1$ which guarantees scattering behavior of solutions to (4.8) to free waves (see [5]).

Remark 4.2. Due to the energy conservation for the free wave equation we conclude from Theorem 4.2 that

$$E(u)(t) = \frac{1}{2}\Big(\|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 + \frac{1}{(1+t)^2}\|u(t,\cdot)\|_{L^2}^2\Big) \to E_w(v)(0) \ as\ t \to \infty,$$

with $E_w(v)(0) = \|\nabla v_0\|_{L^2}^2 + \|v_1\|_{L^2}^2$.

Remark 4.3. If $(1+t)m(t)^2 \in L^1$, then (2.3) holds for $\psi \equiv 1$. Consequently, $p(t) = (1+t)^{-1}$ in (2.4). This already hints to a scattering behavior to free waves. This conjecture is now proved in form of Theorem 4.2.

Example 4.1. If $g(t)$ in (2.6) is given by $g(t) = \ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$, then $(1+t)m(t)^2 \in L^1$ and the conclusion of Theorem 4.2 holds.

Example 4.2. Let us choose $m(t) = \frac{\mu}{(1+t)(\ln(e+t))^{\gamma/2}}$ in (2.6) for $\gamma > 1$. Then $(1+t)m(t)^2 \in L^1$ and the conclusion of Theorem 4.2 holds.

4.3 Proof of Theorem 4.2

Proof. With a slightly modification we can follow the proof of Theorem 3.26 of [1]. Let us define the micro-energy U by [4]

$$U = \big(h(t,\xi)\widehat{u}, D_t\widehat{u}\big)^T,\ \ h(t,\xi) = \Big(|\xi|^2 + \frac{N^2}{(1+t)^2}\Big)^{1/2}.$$

4.3.1 Considerations in the pseudo-differential zone

Here we consider the first order system

$$D_t U(t,\xi) = \mathcal{A}(t,\xi)U := \begin{pmatrix} \frac{D_t h(t,\xi)}{h(t,\xi)} & h(t,\xi) \\ \frac{m(t)^2+|\xi|^2}{h(t,\xi)} & 0 \end{pmatrix} U. \tag{4.9}$$

We can get an integral representation by using the fundamental solution $E = E(t,s,\xi)$ to (4.9), i.e., the solution to

$$D_t E = \mathcal{A}(t,\xi)E\,, \quad E(s,s,\xi) = I.$$

[4]In the definition of the micro-energy we will use $D_t\widehat{u}$, where $D_t = \frac{1}{i}\partial_t$.

If we put $E=(E_{ij})_{i,j=1,2}$, then we can write for $j=1,2$ the following integral equations:

$$E_{1j}(t,s,\xi)=h(t,\xi)\Big(\frac{\delta_{1j}}{h(s,\xi)}+\int_s^t iE_{2j}(\tau,s,\xi)d\tau\Big), \tag{4.10}$$

$$E_{2j}(t,s,\xi)=\delta_{2j}+i\int_s^t \frac{m(\tau)^2+|\xi|^2}{h(\tau,\xi)}E_{1j}(\tau,s,\xi)d\tau. \tag{4.11}$$

Under the hypothesis $(1+t)m(t)^2\in L^1$ we derive by using Gronwall's inequality the estimate $|E(t,s,\xi)|\leq C$. Indeed, by replacing (4.11) into (4.10) and after integration by parts we get

$$\begin{aligned}E_{1j}(t,s,\xi)&=h(t,\xi)\Big(\frac{\delta_{1j}}{h(s,\xi)}+i\int_s^t\Big(\delta_{2j}+i\int_s^\tau\frac{m(\sigma)^2+|\xi|^2}{h(\sigma,\xi)}E_{1j}(\sigma,s,\xi)d\sigma\Big)d\tau\Big)\\ &\underset{integration\,by\,parts}{=}h(t,\xi)\Big(\frac{\delta_{1j}}{h(s,\xi)}+i\int_s^t\delta_{2j}d\tau-\int_s^t\frac{m(\tau)^2+|\xi|^2}{h(\tau,\xi)}E_{1j}(\tau,s,\xi)(t-\tau)d\tau\Big).\end{aligned}$$

Then, after defining $w_{1j}(t,s,\xi):=\frac{E_{1j}(t,s,\xi)}{(1+t)h(t,\xi)}$ and using that $(1+t)h(t,\xi)\approx 1$ whenever $(t,\xi)\in Z_{pd}$ we get

$$\begin{aligned}|w_{1j}(t,s,\xi)|&\lesssim\frac{1}{(1+t)h(s,\xi)}+\frac{t-s}{1+t}+\int_s^t(m(\tau)^2+|\xi|^2)|w_{1j}(\tau,s,\xi)|(1+\tau)\frac{t-\tau}{1+t}d\tau\\ &\lesssim 1+\int_s^t(m(\tau)^2+|\xi|^2)|w_{1j}(\tau,s,\xi)|(1+\tau)d\tau.\end{aligned}$$

Applying Gronwall's inequality we conclude

$$|w_{1j}(t,s,\xi)|\lesssim\exp\Big(C\int_s^t(1+\tau)(m^2(\tau)+|\xi|^2)d\tau\Big).$$

Since $(1+t)|\xi|\leq N$ and $(1+t)m(t)^2\in L^1(\mathbb{R}^+)$ it follows $|w_{1j}(t,s,\xi)|\lesssim 1$. Therefore $|E_{1j}(t,s,\xi)|\leq C$. This estimate together with (4.11) gives us that $|E_{2j}(t,s,\xi)|$ is also bounded in $Z_{pd}(N)$.

4.3.2 Considerations in the hyperbolic zone

Here we define the wave type micro-energy

$$U_W=(|\xi|\widehat{u},D_t\widehat{u})^T.$$

This allows to derive from (3.2) the system

$$D_tU_W=\begin{pmatrix}0&1\\1&0\end{pmatrix}|\xi|U_W+\begin{pmatrix}0&0\\m(t)^2&0\end{pmatrix}(|\xi|)^{-1}U_W. \tag{4.12}$$

As in the proof of Theorem 2.1 let P be the constant diagonalizer of the principal part of (4.12). Defining $U_1=P^{-1}U_W$, then we get the system $(D_t-D-B_1(t,\xi))U_1=0$, where

$$D=\begin{pmatrix}-|\xi|&0\\0&|\xi|\end{pmatrix},\quad B_1(t,\xi)=\frac{m(t)^2}{2|\xi|}\begin{pmatrix}-1&-1\\1&1\end{pmatrix}.$$

Let $Q_1 = Q_1(t,s,\xi)$ be the solution of $(D_t - E_1^{-1}B_1E_1)Q_1 = 0$, $Q_1(s,s,\xi) = I$, where

$$E_1(t,s,\xi) = \begin{pmatrix} e^{-i(t-s)|\xi|} & 0 \\ 0 & e^{i(t-s)|\xi|} \end{pmatrix}.$$

Then we can estimate

$$\|B_1(t,\xi)\| \leq C(1+t)m(t)^2,$$

hence, after using $(1+t)m(t)^2 \in L^1$ brings $|Q_1(t,s,\xi)| \leq C_1$ and by Liouville's formula, $|Q_1(t,s,\xi)| \geq C_2$. Now, let us introduce

$$H(t,\xi) := \begin{pmatrix} \frac{h(t,\xi)}{|\xi|} & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that in the hyperbolic zone we have $\frac{h(t,\xi)}{|\xi|} \approx C$. Then the inverse matrix H^{-1} exists and $\|H(t,\xi)\|, \|H^{-1}(t,\xi)\| \approx C$ for all $t \geq \theta_{|\xi|}$.
Since $U(t,\xi) = HU_W = (h(t,\xi)\widehat{u}, D_t\widehat{u})^T$ we can write $U(t,\xi) = \mathcal{E}(t,\xi)U(0,\xi)$, where

$$\mathcal{E}(t,\xi) = \begin{cases} E(t,0,\xi), \ 0 \leq t \leq \theta_{|\xi|}, \\ H(t,\xi)PE_1(t,\theta_{|\xi|},\xi)Q_1(t,\theta_{|\xi|},\xi)P^{-1}H(\theta_{|\xi|},\xi)^{-1}E(\theta_{|\xi|},0,\xi), \ t \geq \theta_{|\xi|}. \end{cases}$$

We have proved that $\|\mathcal{E}(t,\xi)\| \leq C$ for all t,ξ.

4.3.3 Scattering operator and properties

If $m \equiv 0$, then the fundamental solution of the system (4.12) can be written as $PE_1(t,s,\xi)P^{-1}$. Then, if v solves the free wave equation (4.2), by putting $V(t,\xi) = (|\xi|\widehat{v}, D_t\widehat{v})^T$, we can write $V(t,\xi) = \tilde{E}(t,s,\xi)V(s,\xi)$, where

$$\tilde{E}(t,s,\xi) = PE_1(t,s,\xi)P^{-1}.$$

Our aim is to prove that the limit

$$W_+(\xi) := \lim_{t\to\infty} \widetilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,\xi) \tag{4.13}$$

exists in $L(L^2, L^2)$. After proving this property we are able to relate the Cauchy data by

$$V(0,\xi) = W_+(\xi)\,U(0,\xi) \quad \text{for all } \xi.$$

First we prove the existence of (4.13) for $|\xi| \geq \epsilon$. Indeed, for $t \geq \theta_{|\xi|}$ we have

$$\tilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,\xi) = P\tilde{E}_1(t,\theta_{|\xi|},\xi)Q_1(t,\theta_{|\xi|},\xi)P^{-1}H^{-1}(\theta_{|\xi|},\xi)E(\theta_{|\xi|},0,\xi)$$

with

$$\tilde{E}_1(t,\theta_{|\xi|},\xi) = E_1(0,t,\xi)P^{-1}H(t,\xi)PE_1(t,\theta_{|\xi|},\xi).$$

By using the explicit representation of $\tilde{E}_1(t,\theta_{|\xi|},\xi)$ we can prove, for all $|\xi| \geq \epsilon$, that $\lim_{t\to\infty}\tilde{E}_1(t,\theta_{|\xi|},\xi) = E_1(0,\theta_{|\xi|},\xi)$, and the existence of the limit is proved if $Q_1(t,\theta_{|\xi|},\xi)$ converges for $t \to \infty$ in L^∞. If we put

$$C_1(t,s,\xi) := E_1(s,t,\xi)B_1(t,\xi)E_1(t,s,\xi),$$

then the matrix $Q_1(t,\theta_{|\xi|},\xi)$ is given by

$$Q_1(t,\theta_{|\xi|},\xi)=I+\sum_{k=1}^{\infty}i^k\int_{\theta_{|\xi|}}^{t}C_1(t_1,\theta_{|\xi|},\xi)\int_{\theta_{|\xi|}}^{t_1}C_1(t_2,\theta_{|\xi|},\xi)\cdots\int_{\theta_{|\xi|}}^{t_{k-1}}C_1(t_k,\theta_{|\xi|},\xi)dt_k\cdots dt_1.$$

For $t,s\geq\theta_{|\xi|}$ we obtain the estimates

$$\begin{aligned}\|Q_1(t,\theta_{|\xi|},\xi)-Q_1(s,\theta_{|\xi|},\xi)\|_\infty &\leq \sum_{k=1}^{\infty}\int_s^t\|C_1(t_1,\theta_{|\xi|},\xi)\|\frac{1}{(k-1)!}\Big(\int_{\theta_{|\xi|}}^{t_1}\|C_1(t_2,\theta_{|\xi|},\xi)\|dt_2\Big)^{k-1}dt_1\\ &\leq \int_s^t\|B_1(t_1,\xi)\|\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\int_{\theta_{|\xi|}}^{t_1}\|B_1(t_2,\xi)\|dt_2\Big)^{k}dt_1\\ &= \int_s^t\|B_1(t_1,\xi)\|\exp\Big(\int_{\theta_{|\xi|}}^{t_1}\|B_1(t_2,\xi)\|dt_2\Big)dt_1\\ &\lesssim \int_s^t\|B_1(t_1,\xi\|dt_1\Big)\lesssim\int_s^t(1+t_1)m(t_1)^2dt_1.\end{aligned}$$

For the last inequality we used the representation of $B_1=B_1(t,\xi)$ and the definition of the hyperbolic zone. Since $(1+t)m(t)^2\in L^1$ it is clear that $Q_1(\infty,\theta_{|\xi|},\xi)$ exists uniformly for $|\xi|\geq\epsilon$, because $\{Q_1(t_k,\theta_{|\xi|},\xi)\}_k$ is a Cauchy sequence uniformly for $|\xi|\geq\epsilon$ in L^∞ for any sequence $\{t_k\}_k$ tending to infinity.
Then, we already proved the existence of the limit (4.13) on the dense subset $\mathcal{L}$ of $L^2\times L^2$. By using

$$\|Q_1(\infty,\theta_{|\xi|},\xi)-Q_1(t,\theta_{|\xi|},\xi)\|_\infty\lesssim\int_t^\infty(1+\tau)m^2(\tau)d\tau,$$

where $Q_1(\infty,\theta_{|\xi|},\xi)=\lim_{t\to\infty}Q_1(t,\theta_{|\xi|},\xi)$, we conclude (4.7).
According to the estimates proved in $Z_{pd}(N)$ and $Z_{hyp}(N)$, $\mathcal{E}(t,\xi)$ is uniformly bounded and the same is true for $\widetilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,\xi)$. Therefore, applying the Banach-Steinhaus' theorem we conclude that the operator $W_+(\xi)$ is well-defined for all $\xi\in\mathbb{R}^n$.
Finally, we study the difference

$$\begin{aligned}U(t,\xi)-V(t,\xi) &= \mathcal{E}(t,\xi)U(0,\xi)-\tilde{E}(t,0,\xi)V(0,\xi)\\ &= \tilde{E}(t,0,\xi)\big(\tilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,\xi)-W_+(\xi)\big)U(0,\xi).\end{aligned}$$

Under our assumption $(u_0,u_1)\in H^1\times L^2$ we conclude with a suitable positive constant C_0 the estimates $\|U(t,\cdot)\|_{L^2(\mathbb{R}^n)}\leq C_0$ and $\|V(t,\cdot)\|_{L^2(\mathbb{R}^n)}\leq C_0$ uniformly for all $t\geq0$. Then, for a given $\delta>0$ there exist positive constants ϵ and $T(\delta,\epsilon)$ such that

$$\begin{aligned}&\|U(t,\cdot)-V(t,\cdot)\|_{L^2(\mathbb{R}^n)}=\|U(t,\cdot)-V(t,\cdot)\|_{L^2(|\xi|\leq\epsilon)}+\|U(t,\cdot)-V(t,\cdot)\|_{L^2(|\xi|\geq\epsilon)}\\ &\leq\frac{\delta}{2}+\big\|\tilde{E}(t,0,\xi)\big(\tilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,\xi)-W_+(\xi)\big)U(0,\xi)\big\|_{L^2(|\xi|\geq\epsilon)}<\delta\ \text{ for all }\ t\geq T(\delta,\epsilon).\end{aligned}$$

The proof is completed. □

5 Concluding remarks

Remark 5.1. Our choice for the function ψ which is proposed in (2.9) and (2.14) is quite optimal because of Example 2.3. It shows that our choice works in a good way for the well-known scale invariant case.

Remark 5.2. The results from Theorems 2.1 to 2.3 seem to be optimal. Nevertheless a modified scattering result as proposed in [5] for damped wave models with non-effective dissipation would be of interest. Moreover, generalized energy conservation will be studied in a forthcoming project.

Remark 5.3. In a forthcoming project we are interested in $L^p - L^q$ decay estimates on the conjugate line and their applications to non-linear models.

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