Elliptic equations involving linear and superlinear terms and critical Caffarelli-Kohn-Nirenberg exponent with sign-changing weight functions

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Abstract

In this article we establish the existence and nonexistence of a weak solution to singular elliptic equations involving linear and superlinear terms and critical Caffarelli-Kohn-Nirenberg exponent with sign-changing weight functions.

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1 Introduction

This paper deal with existence and nonexistence of a weak solution to the problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{|x|^{2a}}\right) - \mu \frac{u}{|x|^{2(a+1)}} = \lambda h(x) \frac{|u|^{q-2}u}{|x|^c} + k(x) \frac{|u|^{2_*-2}u}{|x|^{2_* b}} & \text{in } \Omega \backslash \{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain; $N \geq 3$; $0 \in \Omega$; $a < \frac{N-2}{2}$; $a \leq b < a+1$; $2 \leq q < 2_*$; $c < q(a+1) + N(1-q/2)$; $2_* := \frac{2N}{N-2+2(b-a)}$ is the critical Caffarelli-Kohn-Nirenberg exponent; $\mu < \bar{\mu}_a := \frac{(N-2(a+1))^2}{4}$; λ is a positive parameter and h, k are continuous functions which may change sign in $\bar{\Omega}$.

Similar problems to problem (1) have been studied by many authors. Bouchekif and Matallah [1] studied the same problem in the case $1 < q < 2$.

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By splitting the Nehari manifold they established the existence of two distinct nontrivial nonnegative solutions to problem (1).

In the celebrated paper of Brezis and Nirenberg [3], when $\mu = 0$, $a = b = c = 0$, $2_* = \frac{2N}{N-2}$, $h \equiv 1$, and $k \equiv 1$, it was proved that problem (1) posseses a positive solution for each $\lambda > 0$ if $2 < q < 2_*$ and for $\lambda \in (0, \lambda_0)$ if $q = 2$, for an adequate λ_0.

Janelli [10] proved that the problem (1) with $q = 2$, $2_* = \frac{2N}{N-2}$, $a = b = c = 0$, $h \equiv 1$, $k \equiv 1$, and $0 < \mu \leq \left(\frac{N-2}{2}\right)^2 - 1$ admits a positive solution for all $\lambda \in (0, \lambda_1(\mu))$; also if $\left(\frac{N-2}{2}\right)^2 - 1 < \mu < \left(\frac{N-2}{2}\right)^2$ and $\Omega = B(0; 1)$, then there exists $\lambda^* \in (0, \lambda_1(\mu))$ such that the problem (1) admits a positive solution if and only if $\lambda \in (\lambda^*, \lambda_1(\mu))$, where $\lambda_1(\mu)$ is the first eigenvalue of $-\Delta - |x|^{-2}\mu$. In [9], Han considered a problem involving the p-Laplacian operator which, for $p = 2$, is similar to problem (1) with $q = 2$, $2_* = \frac{2N}{N-2}$, $a = b = c = 0$, $h \equiv 1$, and k a nonnegative bounded function on $\bar{\Omega}$. Han proved that for $0 < \mu \leq \left(\frac{N-2}{2}\right)^2 - 1$ the problem (1) admits at least one positive solution for all $\lambda \in (0, \lambda_1(\mu))$. Xuan [15] studied the problem (1) when $q = 2$, $h \equiv 1$, and $k \equiv 1$.

For the case $2 < q < 2_*$, we refer the reader to [5], [12], and [13]. Chaudhuri and Ramaswamy [5] proved the existence of a weak solution to problem (1) when $2_* = \frac{2N}{N-2}$, $a = b = c = 0$, $k \equiv 1$, and h is positive satisfying some conditions. Also when $2_* = \frac{2N}{N-2}$, $a = b = c = 0$, $h \equiv 1$, $k \equiv 1$, and μ replaced by a continuous positive function Miyagaki [12] established the existence of a nontrivial solution to problem (1) for all $\lambda > 0$ and $N \geq 4$. Rodrigues [13] studied problem (1) when $\mu = 0$, $\lambda = 1$, $k \equiv 1$, and h is a continuous function which changes sign in $\bar{\Omega}$. Applying the Mountain Pass Theorem he obtained a nonnegative nontrivial solution to problem (1). He also considered the cases when $q = 2$ and $1 < q < 2$.

Problems involving the more general operator $L_{\mu,a}u := -\operatorname{div}(|x|^{-2a}\nabla u) - \mu|x|^{-2(a+1)}u$ have been the subject of many papers, see for example [8] for $a = 0$ and $\mu < \bar{\mu}_0$ and [7] or [15] for general case, that is $-\infty < a < \frac{N-2}{2}$ and $\mu < \bar{\mu}_a$. Looking carefully to $L_{\mu,a}$, we note that degeneracy and singularity occur in the problem (1). In these situations, we can't apply the classical methods directly due to the degenerate (or singular) character of the operator $L_{\mu,a}$.

The variational approach to these problems follows from Caffarelli-Kohn-Nirenberg inequality in [4]: there is a positive constant $C_{a,b}$ such that

$$\Big(\int_{\mathbb{R}^N} |x|^{-2_* b}|u|^{2_*}dx\Big)^{1/2_*} \leq C_{a,b}\Big(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx\Big)^{1/2}, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2}$$

where $-\infty < a < (N-2)/2$, $a \leq b < a+1$, and $2_* = 2N/(N-2+2(b-a))$. In (2), as $b = a+1$, then $2_* = 2$ and we have the following weighted Hardy inequality [6]:

$$\int_{\mathbb{R}^N} |x|^{-2(a+1)}u^2 dx \leq \frac{1}{\bar{\mu}_a}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

We define the weighted Sobolev space $D_a^{1,2}(\Omega)$ as the completion of the space

$C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{0,a} = \left(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\right)^{1/2}.$$

Define H_μ as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mu,a} := \left(\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx\right)^{1/2},$$

for $-\infty < \mu < \bar{\mu}_a$. Similarly we define $H_\mu(\mathbb{R}^N)$. Note that, by weighted Hardy inequality, $\|\cdot\|_{\mu,a}$ and $\|\cdot\|_{0,a}$ are equivalent.

As in Xuan [16], there exists $C > 0$ such that

$$\left(\int_\Omega |x|^{-c}|u|^p dx\right)^{1/p} \leq C\left(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\right)^{1/2}, \forall u \in D_a^{1,2}, \tag{3}$$

for $1 \leq p \leq 2N/(N-2)$, $c \leq p(a+1)+N(1-p/2)$, and $a < (N-2)/2$. Moreover, if $1 \leq p < 2N/(N-2)$, $c < p(a+1)+N(1-p/2)$, and $a < (N-2)/2$, the embedding

$$D_a^{1,2}(\Omega) \hookrightarrow L_p(\Omega, |x|^{-c})$$

is compact, where $L_p(\Omega, |x|^{-c})$ is the weighted L_p space with norm

$$\|u\|_{p,c} = \left(\int_\Omega |x|^{-c}|u|^p dx\right)^{1/p}.$$

Thus we obtain the compact embedding

$$H_\mu \hookrightarrow L_p(\Omega, |x|^{-c}),$$

where $1 \leq p < 2N/(N-2)$ and $c < p(a+1)+N(1-p/2)$.

Due to the critical exponent in problem (1), we don't have the compact embedding $H_\mu \hookrightarrow L_{2_*}(\Omega, |x|^{-2*b})$. As in Brezis and Nirenberg [3], this trouble was solved by use of the extremal functions. Xuan [15] proved that under conditions $N \geq 3$, $a < (N-2)/2$, $0 < \sqrt{\bar{\mu}_a} - \sqrt{\bar{\mu}_a - \mu} + a < (N-2)/2$, $a \leq b < a+1$, and $\mu < \bar{\mu}_a - b^2$, for $\varepsilon > 0$, the function

$$w_\varepsilon(x) = C_0 \varepsilon^{\frac{2}{2_*-2}} \left(\varepsilon^{\frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b}} |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\right)^{-\frac{2}{2_*-2}}, \tag{4}$$

with a suitable positive constant C_0, is a weak solution of

$$-\operatorname{div}\left(|x|^{-2a}\nabla u\right) - \mu|x|^{-2(a+1)}u = |x|^{-2_*b}|u|^{2_*-2}u \quad \text{in } \mathbb{R}^N\backslash\{0\}.$$

Furthermore,

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u w_\varepsilon|^2 dx - \mu\int_{\mathbb{R}^N} |x|^{-2(a+1)} w_\varepsilon^2 dx = \int_{\mathbb{R}^N} |x|^{-2_*b}|w_\varepsilon|^{2_*} dx = (A_{a,b,\mu})^{\frac{2_*}{2_*-2}},$$

where $A_{a,b,\mu}$ is the best constant,

$$A_{a,b,\mu} = \inf_{u \in H_\mu(\mathbb{R}^N)\setminus\{0\}} E_{a,b,\mu}(u) = E_{a,b,\mu}(w_\varepsilon), \tag{5}$$

with

$$E_{a,b,\mu}(u) := \frac{\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} u^2 dx}{(\int_{\mathbb{R}^N} |x|^{-2_* b}|u|^{2_*} dx)^{2/2_*}}.$$

Also, Lin [11] proved for $0 \le a < (N-2)/2$, $a \le b < a+1$, and $0 \le \mu < \bar{\mu}_a$, that the function defined for $\varepsilon > 0$ as

$$\tilde{w}_\varepsilon(x) = (2.2_*\varepsilon^2(\bar{\mu}_a - \mu))^{\frac{1}{2_*-2}} \left(\varepsilon^2 |x|^{\frac{(2_*-2)(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})}{2}} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\right)^{-\frac{2}{2_*-2}} \tag{6}$$

is a weak solution of

$$-\operatorname{div}(|x|^{-2a}\nabla u) - \mu|x|^{-2(a+1)}u = |x|^{-2_* b}|u|^{2_*-2}u \text{ in } \mathbb{R}^N\setminus\{0\},$$

and satisfies

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla \tilde{w}_\varepsilon|^2 dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} \tilde{w}_\varepsilon^2 dx = \int_{\mathbb{R}^N} |x|^{-2_* b}|\tilde{w}_\varepsilon|^{2_*} dx = (B_{a,b,\mu})^{\frac{2_*}{2_*-2}},$$

where $B_{a,b,\mu}$ is the best constant,

$$B_{a,b,\mu} := \inf_{u \in H_\mu(\mathbb{R}^N)\setminus\{0\}} E_{a,b,\mu}(u) = E_{a,b,\mu}(\tilde{w}_\varepsilon). \tag{7}$$

By using (5) and (7), we define

$$S_{a,b,\mu} := \begin{cases} A_{a,b,\mu}, & \text{if } (a,\mu) \in]-1,0[\times]0,\bar{\mu}_a - b^2[\cup[0,\frac{N-2}{2}[\times]a(a-N+2),\bar{\mu}_a - b^2[, \\ B_{a,b,\mu}, & \text{if } (a,\mu) \in [0,\frac{N-2}{2}[\times[0,\bar{\mu}_a[. \end{cases} \tag{8}$$

We will need of the first eigenvalue associated to the eigenvalue problem of the operator $L_{\mu,a}u := -\operatorname{div}(|x|^{-2a}\nabla u) - \mu|x|^{-2(a+1)}u$:

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{|x|^{2a}}\right) - \mu\frac{u}{|x|^{2(a+1)}} &= \lambda h(x)|x|^{-c}u \quad \text{in } \Omega\setminus\{0\}, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where Ω is an open bounded domain, $N \ge 3$, $0 \in \Omega$, $a < \frac{N-2}{2}$, $c < 2(a+1)$, $0 \le \mu < \bar{\mu}_a := \frac{(N-2(a+1))^2}{4}$, λ is a positive parameter, and h is a positive continuous function defined in $\bar{\Omega}$. As in Wu [17], we conclude that the first eigenvalue to problem (9) is given by

$$\lambda_1 = \inf\left\{\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx; u \in H_\mu, \int_\Omega |x|^{-c}h(x)u^2 dx = 1\right\}. \tag{10}$$

and it is positive.

In our work we prove the existence of a nontrivial nonnegative solution to problem (1) in two cases, when $2 < q < 2_*$ and when $q = 2$. In both cases we use the Mountain Pass Theorem. For the case $q = 2$, we also prove a nonexistence result.

As in the work of Bouchekif and Matallah [1] we consider the following assumptions:

(H) h is a continuous function defined in $\bar{\Omega}$ and there exist h_0 and ρ_0 positive constants such that $h(x) \geq h_0$ for all $x \in B(0, 2\rho_0)$, where $B(a, r)$ is a ball centered at a with radius r;

(K) k is a continuous function defined in $\bar{\Omega}$ and satisfies $k(0) = \max_{x\in\bar{\Omega}} k(x) > 0$, $k(x) = k(0) + O(x^\beta)$ for $x \in B(0, 2\rho_0)$ with $\beta > 2_*\sqrt{\bar{\mu}_a - \mu}$;

and one of the following two assumptions

(A1) $N \geq 3$ and

$$(a, \mu) \in]-1, 0[\times]0, \bar{\mu}_a - b^2[\cup[0, \tfrac{N-2}{2}[\times]a(a - N + 2), \bar{\mu}_a - b^2[,$$

(A2) $N \geq 3$ and $(a, \mu) \in [0, \frac{N-2}{2}[\times[0, \bar{\mu}_a[$.

The aim of our work is to prove the following theorems:

Theorem 1.1 *Suppose that* $a < (N - 2)/2$, $a \leq b < a + 1$, $2 < q < 2_*$, $c < q(a + 1) + N(1 - q/2)$, (H) *and* (K) *hold,* $k(x) > 0$ *for all* $x \in \Omega$, *and* (A1) *or* (A2) *is satisfied. Then, for all* $\lambda > 0$, *the problem (1) has at least one nontrivial nonnegative solution in* H_μ.

Theorem 1.2 *Suppose that* $a < (N - 2)/2$, $a \leq b < a + 1$, $q = 2$, $c < 2(a + 1)$, $0 \leq \mu < \bar{\mu}_a := \frac{(N-2(a+1))^2}{4}$, (H) *and* (K) *hold,* $h(x) > 0$ *for all* $x \in \Omega$, *and* (A1) *or* (A2) *is satisfied. If* λ_1 *is the first eigenvalue to problem (9) then the problem (1) has at least one nontrivial nonegative solution in* H_μ *for all* $\lambda \in (0, \lambda_1)$.

Theorem 1.3 *Suppose that* $a < (N - 2)/2$, $a \leq b < a + 1$, $q = 2$, $c < 2(a + 1)$, $0 \leq \mu < \bar{\mu}_a := \frac{(N-2(a+1))^2}{4}$, (H) *and* (K) *hold,* $h(x) > 0$ *for all* $x \in \Omega$, *and* $k(x) > 0$ *for all* $x \in \Omega$. *If* λ_1 *is the first eigenvalue of the problem (9) then the problem (1) has not nontrivial nonegative solution in* H_μ, *for all* $\lambda > \lambda_1$.

This paper is organized as follows. In Section 2 we give some preliminaries. Section 3 is dedicated to Theorem 1.1 and Section 4 to Theorems 1.2 and 1.3.

2 Preliminary results

Since our aprouch is variational, we define the Euler-Lagrange functional $I_{\lambda,\mu}$ as

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2_{\mu,a} - \frac{\lambda}{q}\int_\Omega h(x)|x|^{-c}u_+^q\,dx - \frac{1}{2_*}\int_\Omega k(x)|x|^{-2_*b}u_+^{2_*}\,dx,$$

for all $u \in H_\mu$. The functional $I_{\lambda,\mu}$ is well defined in H_μ and $I_{\lambda,\mu} \in C^1(H_\mu, \mathbb{R})$.

We say that $u \in H_\mu$ is a weak solution of the problem (1) if it satisfies

$$\int_\Omega (|x|^{-2a}\nabla u \nabla v - \mu|x|^{-2(a+1)}uv - \lambda h(x)|x|^{-c}u_+^{q-1}v - k(x)|x|^{-2_* b}u_+^{2_*-1}v)dx = 0$$

for all $v \in H_\mu$.

Theorem 2.1 *If u is a weak solution of the problem (1), then $u \geq 0$ almost everywhere in Ω.*

Proof. Let u be a weak solution of the problem (1). As $u = u_+ - u_-$ and $u_+u_- = 0$, we have

$$0 = \left\langle I'_{\lambda,\mu}(u), u_- \right\rangle = -\|u_-\|^2_{\mu,a},$$

which means that $u_- = 0$ almost everywhere in Ω. Thus, $u = u_+ \geq 0$ almost everywhere in Ω. ■

Definition. *Let E be a Banach space and consider a functional $I \in C^1(E, \mathbb{R})$. We say that $(u_n) \subset E$ is a Palais-Smale sequence at level l, ($(PS)_l$, in short) if $I(u_n) \to l$ and $I'(u_n) \to 0$ in E^{-1} (dual of E), as $n \to \infty$.*

Theorem 2.2 *Let (u_n) be a $(PS)_l$ sequence and $u_n \rightharpoonup u$ weakly in H_μ, as $n \to \infty$. Then, u is a weak solution of the problem (1).*

Proof. Since $u_n \rightharpoonup u$ weakly in H_μ, as $n \to \infty$, we have that (u_n) is bounded in H_μ. Now, as $\|u_n\|_{2_*,b} \leq C\|u_n\|_{\mu,a}$, we have (u_n) bounded in $L_{2_*}(\Omega, |x|^{-2_* b})$. Since $L_{2_*}(\Omega, |x|^{-2_* b})$ is reflexive,

$$u_n \rightharpoonup u \text{ in } L_{2_*}(\Omega, |x|^{-2_* b}), \text{ as } n \to \infty.$$

As the embedding $H_\mu \hookrightarrow L_q(\Omega, |x|^{-c})$ is compact, we have

$$u_n \to u \text{ in } L_q(\Omega, |x|^{-c}), \text{ as } n \to \infty.$$

Then

$$\|u_n - u\|_{q,c} \to 0, \text{ as } n \to \infty.$$

Passing to a subsequence if necessary, it follows that

$$u_n(x) \to u(x), \text{ almost everywhere } (a.e.) \text{ in } \Omega, \text{ as } n \to \infty,$$

and there exists a function $g(x) \in L_q(\Omega, |x|^{-c})$ such that $|u_n(x)| \leq g(x)$, $a.e.$ in Ω for all $n \in \mathbb{N}$.

We consider for $w \in H_\mu$ the functional $F_w : H_\mu \longrightarrow \mathbb{R}$, given by

$$F_w(u) = \int_\Omega \left(|x|^{-2a}\nabla u \nabla w - \mu|x|^{-2(a+1)}uw\right) dx, \quad \forall u \in H_\mu.$$

Evidently, we have $F_w \in H_\mu^{-1}$ (dual of H_μ) and it follows from weak convergence definition that $F_w(u_n) \to F_w(u)$, as $n \to \infty$. Then we obtain

$$\lim_{n\to\infty} \int_\Omega \left(|x|^{-2a}\nabla u_n \nabla w - \mu|x|^{-2(a+1)}u_n w\right) dx = \int_\Omega \left(|x|^{-2a}\nabla u \nabla w - \mu|x|^{-2(a+1)}uw\right) dx. \tag{11}$$

Considering the function $g(x) \in L_q(\Omega, |x|^{-c})$ such that $|u_n(x)| \le g(x)$, *a.e.* in Ω, for all $n \in \mathbb{N}$, we have $g^q \in L_1(\Omega, |x|^{-c})$ and

$$|h(x)|x|^{-c}u_{n_+}^{q-1}w| \le |h^+|_\infty |x|^{-c}|u_n|^{q-1}|w| \le |h^+|_\infty |x|^{-c}|g|^{q-1}|w|,$$

for all $w \in H_\mu$. Using the Hölder's inequality

$$\int_\Omega |h^+|_\infty |x|^{-c}|g|^{q-1}|w|dx \le |h^+|_\infty \left(\int_\Omega |x|^{-c}|w|^q dx\right)^{1/q} \left(\int_\Omega |x|^{-c}|g|^q dx\right)^{(q-1)/q} < \infty.$$

Thus $|h^+|_\infty |x|^{-c}|g|^{q-1}|w| \in L_1(\Omega, |x|^{-c})$ and from the Dominated Convergence Theorem we have

$$\lim_{n\to\infty} \int_\Omega h(x)|x|^{-c}u_{n_+}^{q-1}wdx = \int_\Omega h(x)|x|^{-c}u_+^{q-1}wdx. \tag{12}$$

Observe that

$$\left(\int_\Omega |x|^{-2_*b}|u_{n_+}^{2_*-1}|^{\frac{2_*}{2_*-1}}dx\right)^{\frac{2_*-1}{2_*}} = \left(\int_\Omega |x|^{-2_*b}|u_n|^{2_*}dx\right)^{\frac{2_*-1}{2_*}} < \infty,$$

which implies $u_{n_+}^{2_*-1} \in L_{\frac{2_*}{2_*-1}}(\Omega, |x|^{-2_*b})$. Since $L_{\frac{2_*}{2_*-1}}(\Omega, |x|^{-2_*b})$ is the dual of $L_{2_*}(\Omega, |x|^{-2_*b})$ and $u_n \rightharpoonup u$ in $L_{2_*}(\Omega, |x|^{-2_*b})$, as $n \to \infty$, we have

$$u_{n_+}^{2_*-1} \rightharpoonup u_+^{2_*-1}, \text{ as } n \to \infty.$$

We define for all $w \in H_\mu$ the continuous linear functional $f_w : L_{\frac{2_*}{2_*-1}}(\Omega, |x|^{-2_*b}) \longrightarrow \mathbb{R}$ given by $f_w(u) = \int_\Omega k(x)|x|^{-2_*b}wudx$. Thus we have $f_w(u_{n_+}^{2_*-1}) \to f_w(u_+^{2_*-1})$, as $n \to \infty$, and we obtain

$$\lim_{n\to\infty} \int_\Omega k(x)|x|^{-2_*b}u_{n_+}^{2_*-1}wdx = \int_\Omega k(x)|x|^{-2_*b}u_+^{2_*-1}wdx. \tag{13}$$

It follows from (11), (12), and (13) that

$$\left\langle I'_{\lambda,\mu}(u_n), w\right\rangle \to \left\langle I'_{\lambda,\mu}(u), w\right\rangle, \text{ as } n \to \infty, \quad \forall w \in H_\mu.$$

Since $\langle I'_{\lambda,\mu}(u_n), w\rangle \to 0$, as $n \to \infty$, for all $w \in H_\mu$, we obtain

$$\left\langle I'_{\lambda,\mu}(u), w\right\rangle = 0, \quad \forall w \in H_\mu,$$

that is, u is a weak solution of the problem (1). ■

3 Case $2 < q < 2_*$

In this section, we will study the problem (1) in the case $2 < q < 2_*$. Adding the hypothesis k positive in Ω we will show the existence of at least one nontrivial weak solution to this problem by Mountain Pass Theorem.

Lemma 3.1 *For all $\lambda > 0$ there exists a $(PS)_c$ sequence $(u_n) \subset H_\mu$ for $I_{\lambda,\mu}$ where*

$$0 < c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1]); \gamma(0) = 0, \gamma(1) = t_0 u_0\},$$

and $u_0 \in H_\mu$ is such that $\int_\Omega k(x)|x|^{-2_* b} u_{0_+}^{2_*}\, dx > 0$.

Proof. We will verify that $I_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass Theorem without the Palais-Smale condition. Indeed:
1) $I_{\lambda,\mu}(0) = 0$.
2) Let $u \in H_\mu$ and $\lambda > 0$. Then, by using (3) and (8), we obtain

$$\begin{aligned} I_{\lambda,\mu}(u) \geq \|u\|_{\mu,a}^2 \Bigg(& \frac{1}{2} - \frac{\lambda}{q}|h^+|_\infty C_1 (S_{a,b,\mu})^{-q/2} \|u\|_{\mu,a}^{q-2} \\ & - \frac{1}{2_*}|k^+|_\infty (S_{a,b,\mu})^{-2_*/2} \|u\|_{\mu,a}^{2_*-2} \Bigg). \end{aligned} \tag{14}$$

We define $H : (0, +\infty) \longrightarrow \mathbb{R}$ given by

$$H(s) = \frac{\lambda}{q}|h^+|_\infty C_1 (S_{a,b,\mu})^{\frac{-q}{2}} s^{q-2} + \frac{1}{2_*}|k^+|_\infty (S_{a,b,\mu})^{\frac{-2_*}{2}} s^{2_*-2}, \quad \forall s \in (0, +\infty).$$

Note that $H(s) \to 0$, as $s \to 0$. Thus, there exists $\rho > 0$ such that $H(s) < \dfrac{1}{2}$ for all $0 < s \leq \rho$. Using (14), we obtain $\delta > 0$ satisfying $I_{\lambda,\mu}(u) \geq \delta > 0$ if $\|u\|_{\mu,a} = \rho$, $u \in H_\mu$.
3) Let $t > 0$ and $u_0 \in H_\mu$ such that $\int_\Omega k(x)|x|^{-2_* b} u_{0_+}^{2_*}\, dx > 0$. We have

$$I_{\lambda,\mu}(tu_0) = \frac{t^2}{2}\|u_0\|_{\mu,a}^2 - \lambda \frac{t^q}{q} \int_\Omega h(x)|x|^{-c} u_{0_+}^q\, dx - \frac{t^{2_*}}{2_*} \int_\Omega k(x)|x|^{-2_* b} u_{0_+}^{2_*}\, dx.$$

Since $2 < q < 2_*$ we have $I_{\lambda,\mu}(tu_0) \to -\infty$, as $t \to +\infty$. Then there exists $t_0 > 0$ satisfying $\|t_0 u_0\|_{\mu,a} > \rho$ and $I_{\lambda,\mu}(t_0 u_0) < 0$.

It follows from 1), 2), and 3) that $I_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass Theorem without Palais-Smale condition and the Lemma is proved. ■

Lemma 3.2 *Suppose (K) holds and $k(x) > 0$ for all $x \in \Omega$. Let (u_n) be a $(PS)_l$ sequence with $u_n \rightharpoonup u$ weakly in H_μ as $n \to \infty$. Then*

$$I'_{\lambda,\mu}(u) = 0 \text{ in } H_\mu^{-1} \text{ and } I_{\lambda,\mu}(u) \geq 0.$$

Proof. Since (u_n) is a $(PS)_l$ sequence with $u_n \rightharpoonup u$ weakly in H_μ, as $n \to \infty$, it follows from Theorem 2.2 that u is a weak solution of the problem (1). So we obtain that $I'_{\lambda,\mu}(u) = 0$ in H_μ^{-1}.

Observe now that

$$I_{\lambda,\mu}(u) - \frac{1}{q}\langle I'_{\lambda,\mu}(u), u\rangle = \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|_{\mu,a}^2 + \left(\frac{1}{q} - \frac{1}{2_*}\right)\int_\Omega k(x)|x|^{-2_* b} u_+^{2_*}\, dx.$$

Since $I'_{\lambda,\mu}(u) = 0$, we have $\langle I'_{\lambda,\mu}(u), u\rangle = 0$ which implies

$$I_{\lambda,\mu}(u) = \left(\frac{1}{2} - \frac{1}{q}\right) \|u\|^2_{\mu,a} + \left(\frac{1}{q} - \frac{1}{2_*}\right) \int_\Omega k(x)|x|^{-2_* b} u_+^{2_*} dx.$$

Moreover, since $2 < q < 2_*$ and $k(x) > 0$ for all $x \in \Omega$, we obtain $I_{\lambda,\mu}(u) \geq 0$. ■

Theorem 3.1 *Suppose (K) holds and $k(x) > 0$ for all $x \in \Omega$. Let (u_n) be a sequence in H_μ such that*

$$I_{\lambda,\mu}(u_n) \to l < l^* = (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty^{\frac{-2}{2_*-2}} (S_{a,b,\mu})^{2_*/(2_*-2)} \tag{15}$$

and

$$I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1}, \text{ as } n \to \infty. \tag{16}$$

Then there exists a subsequence strongly convergent in H_μ.

Proof. From (15) and (16), we have $I_{\lambda,\mu}(u_n) = l + o_n(1)$ and $I'_{\lambda,\mu}(u_n) = o_n(1)$. Moreover,

$$-\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle \leq |\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle| \leq |I'_{\lambda,\mu}(u_n)| \cdot \|u_n\|_{\mu,a}.$$

Then we have

$$\begin{aligned} l + o_n(1) + o_n(1)\frac{1}{q}\|u_n\|_{\mu,a} &\geq I_{\lambda,\mu}(u_n) - \frac{1}{q}\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2_{\mu,a} + \left(\frac{1}{q} - \frac{1}{2_*}\right)\int_\Omega k(x)|x|^{-2_* b} u_{n_+}^{2_*} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2_{\mu,a}, \end{aligned} \tag{17}$$

which implies

$$\frac{l + o_n(1)}{\|u_n\|_{\mu,a}} + o_n(1)\frac{1}{q} \geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_{\mu,a}.$$

So we have (u_n) bounded in H_μ.

Since H_μ is reflexive, passing to a subsequence if necessary, we have

$$u_n \rightharpoonup u_\lambda \text{ in } H_\mu, \text{ as } n \to \infty.$$

It follows from Theorem 2.2 that u is a weak solution of the problem (1).

We denote $v_n = u_n - u$. Since $k(x)$ is continuous in Ω, it follows from Brézis-Lieb [2] that

$$\int_\Omega k(x)\frac{u_{n_+}^{2_*}}{|x|^{2_* b}}dx = \int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_* b}}dx + \int_\Omega k(x)\frac{u_+^{2_*}}{|x|^{2_* b}}dx + o_n(1) \tag{18}$$

and

$$\|u_n\|^2_{\mu,a} = \|v_n\|^2_{\mu,a} + \|u\|^2_{\mu,a} + o_n(1). \tag{19}$$

The Dominated Convergence Theorem implies

$$\lim_{n\to\infty}\int_\Omega h(x)\frac{u_{n_+}^q}{|x|^c}dx=\int_\Omega h(x)\frac{u_+^q}{|x|^c}dx. \tag{20}$$

From (18), (19), and (20), we obtain

$$I_{\lambda,\mu}(u_n)=I_{\lambda,\mu}(u)+\frac{1}{2}\|v_n\|_{\mu,a}^2-\frac{1}{2_*}\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx+o_n(1) \tag{21}$$

and

$$\langle I'_{\lambda,\mu}(u_n),u_n\rangle=\langle I'_{\lambda,\mu}(u),u\rangle+\|v_n\|_{\mu,a}^2-\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx+o_n(1).$$

Then

$$\|v_n\|_{\mu,a}^2-\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx=\langle I'_{\lambda,\mu}(u_n),u_n\rangle-\langle I'_{\lambda,\mu}(u),u\rangle+o_n(1).$$

Since u is a weak solution of (1), we have

$$\left\langle I'_{\lambda,\mu}(u),u\right\rangle=0. \tag{22}$$

Also, since $I'_{\lambda,\mu}(u_n)\to 0$ in H_μ^{-1}, as $n\to\infty$, and (u_n) is bounded in H_μ, we have

$$|\left\langle I'_{\lambda,\mu}(u_n),u_n\right\rangle|\le|I'_{\lambda,\mu}(u_n)|\|u_n\|_{\mu,a}\to 0,\ \text{as } n\to\infty. \tag{23}$$

It follows from (22) and (23) that

$$\langle I'_{\lambda,\mu}(u_n),u_n\rangle-\langle I'_{\lambda,\mu}(u),u\rangle=o_n(1).$$

Then

$$\|v_n\|_{\mu,a}^2-\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx=o_n(1). \tag{24}$$

Moreover, as (v_n) is bounded we have that $\lim\limits_{n\to\infty}\|v_n\|_{\mu,a}$ exists. It follows from (24) that

$$\lim_{n\to\infty}\|v_n\|_{\mu,a}^2=\lim_{n\to\infty}\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx.$$

Then there exists $\theta\ge 0$ such that

$$\|v_n\|_{\mu,a}^2\to\theta\quad\text{and}\quad\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx\to\theta,\ \text{as } n\to\infty.$$

By definition of $S_{a,b,\mu}$, we have

$$\begin{aligned}\|v_n\|_{\mu,a}^2&\ge S_{a,b,\mu}\Big(\int_\Omega\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx\Big)^{2/2_*}\\&\ge S_{a,b,\mu}\Big(\int_\Omega k(x)\frac{v_{n_+}^{2_*}}{|x|^{2_*b}}dx\Big)^{2/2_*}|k^+|_\infty^{-2/2_*}.\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\theta \geq S_{a,b,\mu} |k^+|_\infty^{-2/2_*} \theta^{2/2_*}.$$

Assume $\theta \neq 0$, then

$$\theta^{(2_*-2)/2_*} \geq S_{a,b,\mu} |k^+|_\infty^{-2/2_*}$$

which implies

$$\theta \geq S_{a,b,\mu}^{2_*/(2_*-2)} |k^+|_\infty^{-2/(2_*-2)}.$$

We get by (21) and Lemma 3.2 that

$$\begin{aligned} l &= I_{\lambda,\mu}(u) + \left(\frac{1}{2} - \frac{1}{2_*}\right)\theta \\ &\geq \left(\frac{1}{2} - \frac{1}{2_*}\right) |k^+|_\infty^{\frac{-2}{2_*-2}} (S_{a,b,\mu})^{2_*/(2_*-2)} = l^* \end{aligned}$$

which is a contradiction. So, $\theta = 0$ and we obtain $u_n \to u$ strongly in H_μ, as $n \to \infty$. ■

In the following, we shall give some estimates for the extremal functions defined in (4) and (6). Set $u_\varepsilon(x) = z(\varepsilon) v_\varepsilon(x)$ where

$$v_\varepsilon(x) = \begin{cases} \left(\varepsilon^{\frac{2\sqrt{\bar\mu_a-\mu}}{\sqrt{\bar\mu_a-\mu}-b}} |x|^{\frac{2_*-2}{2}(\sqrt{\bar\mu_a}-\sqrt{\bar\mu_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar\mu_a}+\sqrt{\bar\mu_a-\mu})}\right)^{-\frac{2}{2_*-2}}, \\ \text{if (A1) holds,} \\ \left(\varepsilon^2 |x|^{\frac{2_*-2}{2}(\sqrt{\bar\mu_a}-\sqrt{\bar\mu_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar\mu_a}+\sqrt{\bar\mu_a-\mu})}\right)^{-\frac{2}{2_*-2}}, \\ \text{if (A2) holds,} \end{cases}$$

and

$$z(\varepsilon) = \begin{cases} C_0 \varepsilon^{\frac{2}{2_*-2}}, & \text{if (A1) holds,} \\ \left(2.2_* \varepsilon^2 (\bar\mu_a - \mu)\right)^{\frac{1}{2_*-2}}, & \text{if (A2) holds,} \end{cases} \tag{25}$$

with C_0 given in (4).

As in [15] and [11] cited before, u_ε is a weak solution to problem:

$$-\operatorname{div}\left(|x|^{-2a}\nabla u\right) - \mu |x|^{-2(a+1)} u = |x|^{-2_* b} |u|^{2_*-2} u \quad \text{in } \mathbb{R}^N \backslash \{0\}$$

and u_ε satisfies

$$\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u_\varepsilon|^2 dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} u_\varepsilon^2 dx = \int_{\mathbb{R}^N} |x|^{-2_* b} |u_\varepsilon|^{2_*} dx = (S_{a,b,\mu})^{\frac{2_*}{2_*-2}}, \tag{26}$$

where

$$S_{a,b,\mu} = \inf_{u \in H_\mu(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} u^2 dx}{\left(\int_{\mathbb{R}^N} |x|^{-2_* b} |u|^{2_*} dx\right)^{2/2_*}}.$$

From (26) we have

$$\begin{aligned}(S_{a,b,\mu})^{\frac{2_*}{2_*-2}} &= \int_{\mathbb{R}^N} |x|^{-2a}|\nabla(z(\varepsilon)v_\varepsilon(x))|^2dx - \mu\int_{\mathbb{R}^N} |x|^{-2(a+1)}(z(\varepsilon)v_\varepsilon(x))^2dx \\ &= [z(\varepsilon)]^2\left(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla v_\varepsilon(x)|^2dx - \mu\int_{\mathbb{R}^N} |x|^{-2(a+1)}(v_\varepsilon(x))^2dx\right)\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla v_\varepsilon(x)|^2 - \mu|x|^{-2(a+1)}(v_\varepsilon(x))^2dx = [z(\varepsilon)]^{-2}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}}.$$

So

$$\int_{\mathbb{R}^N} |x|^{-2_*b}|v_\varepsilon(x)|^{2_*}dx = [z(\varepsilon)]^{-2_*}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}}.$$

Let $\Psi(x) \in C_0^\infty(\Omega)$ such that $0 \le \Psi(x) \le 1$, $\Psi(x) = 1$ for $|x| \le \rho_0$ and $\Psi(x) = 0$ for $|x| \ge 2\rho_0$, where ρ_0 is given as in (K). Set $\tilde{v}_\varepsilon(x) = \Psi(x)v_\varepsilon(x)$.

By a straightforward computation, if (K) and (A1) or (A2) are hold, we obtain

$$\|\tilde{v}_\varepsilon\|^2_{\mu,a} = [z(\varepsilon)]^{-2}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(1) \tag{27}$$

and

$$\int_\Omega k(x)|x|^{-2_*b}|\tilde{v}_\varepsilon|^{2_*}dx = [z(\varepsilon)]^{-2_*}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(\varepsilon^{\frac{-2\cdot 2_*\gamma}{2_*-2}}), \tag{28}$$

where $O(\varepsilon^\zeta)$ denotes $|O(\varepsilon^\zeta)|/\varepsilon^\zeta \le C$ and $\gamma = \frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b} > 0$ if (A1) holds and $\gamma = 2$ if (A2) holds.

Lemma 3.3 *If (K) holds, $k(x) > 0$ for all $x \in \Omega$, and (A1) or (A2) is satisfied, then*

$$\lim_{\varepsilon\to 0}\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_*b}|\tilde{v}_\varepsilon|^{2_*}dx\right)^{2/2_*}} = 0. \tag{29}$$

Proof. From (27) and (28), we obtain

$$\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_*b}|\tilde{v}_\varepsilon|^{2_*}dx\right)^{2/2_*}} = \frac{[z(\varepsilon)]^{-2}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(1)}{\left([z(\varepsilon)]^{-2_*}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(\varepsilon^{\frac{-2\cdot 2_*\gamma}{2_*-2}})\right)^{2/2_*}}.$$

Since that $z(\varepsilon) = C\varepsilon^{\frac{2}{2_*-2}}$, where $C > 0$ is given in (25), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_*b}|\tilde{v}_\varepsilon|^{2_*}dx\right)^{2/2_*}} &= \lim_{\varepsilon\to 0}\frac{[z(\varepsilon)]^{-2}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(1)}{\left([z(\varepsilon)]^{-2_*}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(\varepsilon^{\frac{-2\cdot 2_*\gamma}{2_*-2}})\right)^{2/2_*}} \\ &= \lim_{\varepsilon\to 0}\frac{[z(\varepsilon)]^{-2}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(1)}{\left([z(\varepsilon)]^{-2_*}\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}} + O(1)\varepsilon^{\frac{-2\cdot 2_*\gamma}{2_*-2}}\right)^{2/2_*}} \\ &= \lim_{\varepsilon\to 0}\frac{\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}}C\varepsilon^{\frac{4}{2_*-2}(\gamma-1)} + O(1)\varepsilon^{\frac{4\gamma}{2_*-2}}}{\left(C\left(S_{a,b,\mu}\right)^{\frac{2_*}{2_*-2}}\varepsilon^{\frac{2\cdot 2_*}{2_*-2}(\gamma-1)} + O(1)\right)^{2/2_*}}.\end{aligned}$$

If (A1) holds, we have $\gamma = \frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b} > 0$ and $|b| < \sqrt{\bar{\mu}_a - \mu}$, which implies

$$\gamma - 1 = \frac{\sqrt{\bar{\mu}_a - \mu} + b}{\sqrt{\bar{\mu}_a - \mu} - b} > 0.$$

If (A2) holds, $\gamma = 2$ and we have $\gamma - 1 > 0$. In both cases, we have

$$\lim_{\varepsilon \to 0} \frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_* b}|\tilde{v}_\varepsilon|^{2_*} dx\right)^{2/2_*}} = 0.$$

■

Lemma 3.4 *Suppose (H) and (K) hold, $k(x) > 0$ for all $x \in \Omega$, and (A1) or (A2) is satisfied. Let l^* be defined in Theorem 3.1. Then there exists $\varepsilon_1 \in (0,1)$ such that $\sup\limits_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_{\varepsilon_1}) < l^*$ for all $\lambda > 0$.*

Proof. Let $\lambda > 0$. We consider the following functions

$$f(t) = I_{\lambda,\mu}(t\tilde{v}_\varepsilon) = \frac{t^2}{2}\|\tilde{v}_\varepsilon\|^2_{\mu,a} - \frac{t^{2_*}}{2_*}\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_+} dx - \lambda\frac{t^q}{q}\int_\Omega h(x)|x|^{-c}\tilde{v}^q_{\varepsilon_+} dx$$

and

$$\tilde{f}(t) = \frac{t^2}{2}\|\tilde{v}_\varepsilon\|^2_{\mu,a} - \frac{t^{2_*}}{2_*}\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_+} dx.$$

Since $2 < q < 2_*$ and $\tilde{v}_\varepsilon(x) = 0$ if $x \notin B(0, 2\rho_0)$, $k(x) > 0$ for all $x \in \Omega$, $\Psi(x) = 1$ if $x \in B(0,\rho_0)$, and $v_\varepsilon > 0$, we have $\lim\limits_{t\to+\infty} f(t) = -\infty$. As $f(0) = 0$, we have that $\sup\limits_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon)$ is achieved, that is, for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) = I_{\lambda,\mu}(t_\varepsilon\tilde{v}_\varepsilon).$$

So we have

$$\begin{aligned}\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|\tilde{v}_\varepsilon\|^2_{\mu,a} - \frac{t_\varepsilon^{2_*}}{2_*}\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_+} dx - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega h(x)|x|^{-c}\tilde{v}^q_{\varepsilon_+} dx \\ &= \tilde{f}(t_\varepsilon) - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega h(x)|x|^{-c}\tilde{v}^q_{\varepsilon_+} dx.\end{aligned}$$

Moreover, $t_0 = \left(\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_+} dx}\right)^{\frac{1}{2_*-2}}$ is a maximum of $\tilde{f}(t)$ and

$$\tilde{f}(t_0) = \left(\frac{1}{2} - \frac{1}{2_*}\right)\left(\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_+} dx\right)^{2/2_*}}\right)^{\frac{2_*}{2_*-2}}.$$

So we obtain

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) \leq \tilde{f}(t_0) - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega h(x)|x|^{-c}\tilde{v}^q_{\varepsilon_+} dx.$$

Again, since that $\tilde{v}_\varepsilon(x) = 0$ if $x \notin B(0, 2\rho_0)$, $h(x) \geq h_0 > 0$, for all $x \in B(0, 2\rho_0)$, and $\tilde{v}_{\varepsilon_+} = |\tilde{v}_\varepsilon|$ we obtain

$$\begin{aligned}\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) &\leq \tilde{f}(t_0) - \frac{\lambda t_\varepsilon^q}{q} h_0 \int_{B(0,2\rho_0)} |x|^{-c}|\tilde{v}_\varepsilon|^q dx \\ &\leq \tilde{f}(t_0) \\ &= \left(\frac{1}{2} - \frac{1}{2_*}\right)\left(\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_* b}|\tilde{v}_\varepsilon|^{2_*} dx\right)^{2/2_*}}\right)^{\frac{2_*}{2_*-2}}.\end{aligned}$$

Lemma 3.3 implies

$$\lim_{\varepsilon\to 0} \frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_* b}|\tilde{v}_\varepsilon|^{2_*} dx\right)^{2/2_*}} = 0.$$

Then there exists $0 < \varepsilon_1 < 1$ such

$$\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\left(\int_\Omega k(x)|x|^{-2_* b}|\tilde{v}_\varepsilon|^{2_*} dx\right)^{2/2_*}} < |k^+|_\infty^{\frac{-2}{2_*}}(S_{a,b,\mu}),$$

for all $\varepsilon \leq \varepsilon_1$, which implies

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_{\varepsilon_1}) < \left(\frac{1}{2} - \frac{1}{2_*}\right)|k^+|_\infty^{\frac{-2}{2_*-2}}(S_{a,b,\mu})^{\frac{2_*}{2_*-2}} = l^*.$$

■

Proof of Theorem 1.1. Given $\lambda > 0$, since $\int_\Omega k(x)|x|^{-2_* b}\tilde{v}^{2_*}_{\varepsilon_{1_+}} dx > 0$, by Lemma 3.1 there exists $t_0 > 0$ and a sequence $(u_n) \subset H_\mu$ such that

$$I_{\lambda,\mu}(u_n) \to c \quad \text{and} \quad I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1}, \text{ as } n \to \infty, \tag{30}$$

where

$$0 < c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t))$$

and
$$\Gamma = \{\gamma \in C([0,1], H_\mu); \gamma(0) = 0, \gamma(1) = t_0\tilde{v}_{\varepsilon_1}\}.$$

From (30) we have $I_{\lambda,\mu}(u_n) = c + o_n(1)$ and $I'_{\lambda,\mu}(u_n) = o_n(1)$. Using that

$$-\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle \leq |\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle| \leq |I'_{\lambda,\mu}(u_n)| \cdot \|u_n\|_{\mu,a},$$

we have

$$\begin{aligned}c + o_n(1) + o_n(1)\frac{1}{q}\|u_n\|_{\mu,a} &\geq I_{\lambda,\mu}(u_n) - \frac{1}{q}\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2_{\mu,a}.\end{aligned}$$

So we obtain (u_n) bounded in H_μ. Since H_μ is reflexive, we obtain a subsequence (u_n) such that

$$u_n \rightharpoonup v_\lambda \quad \text{in } H_\mu, \text{ as } n \to \infty.$$

It follows from Theorem 2.2 that v_λ is a weak solution of the problem (1).

We consider the curve γ_1 given by $\gamma_1(t) = t(t_0\tilde{v}_{\varepsilon_1})$, where $\tilde{v}_{\varepsilon_1}$ is given in Lemma 3.4. Since that $\gamma_1 \in C([0,1], H_\mu)$, $\gamma_1(0) = 0$, and $\gamma_1(1) = t_0\tilde{v}_{\varepsilon_1}$, we have $\gamma_1 \in \Gamma$. So, by Lemma 3.4, we have for all $\lambda > 0$ that

$$0 < c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)) \leq \sup_{s\geq 0} I_{\lambda,\mu}(s\tilde{v}_{\varepsilon_1}) < l^*. \tag{31}$$

From (30), (31), and Theorem 3.1, passing to a subsequence if necessary, we have that $u_n \to v_\lambda$ strongly in H_μ, as $n \to \infty$. Then we obtain $I_{\lambda,\mu}(u_n) \to I_{\lambda,\mu}(v_\lambda)$, as $n \to \infty$ and consequently $I_{\lambda,\mu}(v_\lambda) = c$. This implies $I_{\lambda,\mu}(v_\lambda) > 0$. Since that $I_{\lambda,\mu}(0) = 0$ we have $v_\lambda \neq 0$. Moreover, it follows from Theorem 2.1 that v_λ is nonnegative.

We conclude that for each $\lambda > 0$ the problem (1) has at least one nontrivial nonnegative solution. ■

4 Case $q = 2$

Ending our study of the problem (1), we deal with the case $q = 2$. Again we will make use of the Mountain Pass Theorem to find at least one nontrivial and nonnegative weak solution of the problem (1). Also we are interested in find a nonexistence result in this case.

Lemma 4.1 *Suppose (H') holds and $h(x) > 0$ for all $x \in \Omega$. Let λ_1 be the first eigenvalue of the problem (9). If $\lambda \in (0, \lambda_1)$ then there exists a $(PS)_c$ sequence $(u_n) \subset H_\mu$ for $I_{\lambda,\mu}$, where*

$$0 < c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1]); \gamma(0) = 0, \gamma(1) = t_0u_0\},$$

and $u_0 \in H_\mu$ is such that $\int_\Omega k(x)|x|^{-2_*b}u_{0_+}^{2_*}dx > 0$.

Proof. As in Lemma 3.1, we will verify that $I_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass Theorem without the Palais-Smale condition. Indeed

1) $I_{\lambda,\mu}(0) = 0$.

2) Let $u \in H_\mu$ and $\lambda \in (0, \lambda_1)$. We have by variational characterization of λ_1 that

$$\lambda_1 \leq \|v\|^2_{\mu,a} = \frac{\|u\|^2_{\mu,a}}{\int_\Omega h(x)|x|^{-c}u^2dx}$$

and

$$\int_\Omega h(x)|x|^{-c}u^2dx \leq \frac{\|u\|^2_{\mu,a}}{\lambda_1}.$$

This implies

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2_{\mu,a} - \frac{\lambda}{2}\int_\Omega h(x)|x|^{-c}u_+^2\,dx - \frac{1}{2_*}\int_\Omega k(x)|x|^{-2_*b}u_+^{2_*}\,dx$$
$$\geq \|u\|^2_{\mu,a}\left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right) - \frac{1}{2_*}|k^+|_\infty(S_{a,b,\mu})^{-2_*/2}\|u\|^{2_*-2}_{\mu,a}\right]. \tag{32}$$

Since $\lambda \in (0,\lambda_1)$ we have $1-\frac{\lambda}{\lambda_1} > 0$.

We define $H : (0,+\infty) \longrightarrow \mathbb{R}$ given by

$$H(s) = \frac{1}{2_*}|k^+|_\infty(S_{a,b,\mu})^{-2_*/2}s^{2_*-2}.$$

Note that $H(s) \to 0$, as $s \to 0$. Then there exists $\rho > 0$ such that $H(s) < \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)$ for all $0 < s \leq \rho$. It follows from (32) that there exists $\delta > 0$ satisfying $I_{\lambda,\mu}(u) \geq \delta > 0$ if $\|u\|_{\mu,a} = \rho$, $u \in H_\mu$.

3) Let $t > 0$ and $u_0 \in H_\mu$ be such that $\int_\Omega k(x)|x|^{-2_*b}u_{0_+}^{2_*}\,dx > 0$. Then we have

$$I_{\lambda,\mu}(tu_0) = \frac{t^2}{2}\|u_0\|^2_{\mu,a} - \lambda\frac{t^2}{2}\int_\Omega h(x)|x|^{-c}u_{0_+}^2\,dx - \frac{t^{2_*}}{2_*}\int_\Omega k(x)|x|^{-2_*b}u_{0_+}^{2_*}\,dx.$$

Since $2 < 2_*$ we obtain $I_{\lambda,\mu}(tu_0) \to -\infty$ as $t \to +\infty$ which implies that there exists $t_0 > 0$ satisfying $\|t_0u_0\|_{\mu,a} > \rho$ and $I_{\lambda,\mu}(t_0u_0) < 0$.

From 1), 2), and 3), we conclude that $I_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass Theorem without the Palais-Smale condition. ∎

Lemma 4.2 *Suppose (H) holds and $h(x) > 0$ for all $x \in \Omega$. Let (u_n) be a $(PS)_l$ sequence with $u_n \rightharpoonup u$ weakly in H_μ, as $n \to \infty$, and let λ_1 be the first eigenvalue to problem (9). If $\lambda \in (0,\lambda_1)$ then*

$$I'_{\lambda,\mu}(u) = 0 \text{ in } H_\mu^{-1} \text{ and } I_{\lambda,\mu}(u) \geq 0.$$

Proof. Since (u_n) is a $(PS)_l$ sequence with $u_n \rightharpoonup u$ weakly in H_μ, as $n \to \infty$, it follows from Theorem 2.2 that u is a weak solution of the problem (1). Then we obtain $I'_{\lambda,\mu}(u) = 0$ in H_μ^{-1}.

Note that

$$I_{\lambda,\mu}(u) - \frac{1}{2_*}\left\langle I'_{\lambda,\mu}(u), u\right\rangle = \left(\frac{1}{2}-\frac{1}{2_*}\right)\|u\|^2_{\mu,a} - \lambda\left(\frac{1}{2}-\frac{1}{2_*}\right)\int_\Omega h(x)|x|^{-c}u_+^2\,dx.$$

As $I'_{\lambda,\mu}(u) = 0$ in H_μ^{-1}, we have $\langle I'_{\lambda,\mu}(u), u\rangle = 0$. Thus

$$I_{\lambda,\mu}(u) \geq \|u\|^2_{\mu,a}\left[\left(\frac{1}{2}-\frac{1}{2_*}\right)\left(1-\frac{\lambda}{\lambda_1}\right)\right].$$

Since $2 < 2_*$ and $\lambda \in (0,\lambda_1)$, we obtain $I_{\lambda,\mu}(u) \geq 0$. ∎

Theorem 4.1 *Suppose (H) holds and $h(x) > 0$ for all $x \in \Omega$. Let (u_n) be a sequence in H_μ such that*

$$I_{\lambda,\mu}(u_n) \to l < l^* = \left(\frac{1}{2} - \frac{1}{2_*}\right) |k^+|_\infty^{\frac{-2}{2_*-2}} (S_{a,b,\mu})^{2_*/(2_*-2)}$$

and

$$I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1}, \text{ as } n \to \infty.$$

If λ_1 is the first eigenvalue of the problem (9) and $\lambda \in (0, \lambda_1)$, then there exists a subsequence strongly convergent in H_μ.

Proof. The proof is essentially the same as in Theorem 3.1, replacing q by 2_* in (17) and using the variational characterization of λ_1. ■

Lemma 4.3 *Suppose (H) and (K) hold, $h(x) > 0$ for all $x \in \Omega$, (A1) or (A2) is satisfied and $\mu \geq 0$. Let l^* be as in Theorem 4.1 and let λ_1 be the first eigenvalue of the problem (9). Then there exists $\varepsilon_1 \in (0,1)$ such that*

$$\sup_{t \geq 0} I_{\lambda,\mu}(t\tilde{v}_{\varepsilon_1}) < l^*,$$

for all $\lambda \in (0, \lambda_1)$.

Proof. The proof is similar to Lemma 3.4. ■

Proof of Theorem 1.2. Given $\lambda \in (0, \lambda_1)$, as $\int_\Omega k(x)|x|^{-2_*b}\tilde{v}^{2_*}_{\varepsilon_{1_+}}\,dx > 0$, it follows from Lemma 4.1 that there exist $t_0 > 0$ and a sequence $(u_n) \subset H_\mu$ such that

$$I_{\lambda,\mu}(u_n) \to c \quad \text{and} \quad I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1}, \text{ as } n \to \infty \tag{33}$$

where

$$0 < c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\mu}(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], H_\mu); \gamma(0) = 0, \gamma(1) = t_0\tilde{v}_{\varepsilon_1}\}.$$

From (33) we obtain $I_{\lambda,\mu}(u_n) = c + o_n(1)$ and $I'_{\lambda,\mu}(u_n) = o_n(1)$. Using that

$$-\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle \leq |\left\langle I'_{\lambda,\mu}(u_n), u_n\right\rangle| \leq |I'_{\lambda,\mu}(u_n)| \cdot \|u_n\|_{\mu,a},$$

we have

$$c + o_n(1) + o_n(1)\frac{1}{2_*}\|u_n\|_{\mu,a} \geq \|u\|^2_{\mu,a}\left[\left(\frac{1}{2} - \frac{1}{2_*}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\right]. \tag{34}$$

So we conclude that (u_n) is bounded in H_μ.

Following analogously to proof of Theorem 1.1 and using Lemma 4.3, (33), and Theorem 4.1, we obtain a nontrivial nonnegative weak solution, $v_\lambda \in H_\mu$, of problem (1) for all $\lambda \in (0, \lambda_1)$. ■

4.1 Proof of Theorem 1.3

Let $\mathcal{M} = \left\{u \in H_\mu : \int_\Omega h(x)|x|^{-c}u^2 dx = 1\right\}$.

Lemma 4.4 *Suppose (H) and (K) hold, $h(x) > 0$ for all $x \in \Omega$, and $k(x) > 0$ for all $x \in \Omega$. If λ_1 is the first eigenvalue to problem (9) then $\lambda_1 = \int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx$ for some $u \in \mathcal{M}$ if and only if u is an eigenfunction associated to λ_1.*

Proof. Suppose $\lambda_1 = ||u||^2_{\mu,a}$ for some $u \in \mathcal{M}$. We define the following functionals $F, G : H_\mu \longrightarrow \mathbb{R}$ given by

$$F(u) = ||u||^2_{\mu,a} \text{ and } G(u) = \int_\Omega h(x)|x|^{-c}u^2 dx, \forall u \in H_\mu.$$

It follows from Theorem 1.2 of [14] that F has a minimum u_0 in $\mathcal{M}$. Thus

$$F(u_0) = ||u_0||^2_{\mu,a} = \lambda_1 = F(u),$$

which shows that u is a minimum of F. Since

$$\langle G'(u), u\rangle = 2\int_\Omega h(x)|x|^{-c}u^2 dx = 2 \neq 0,$$

from Lagrange's multiplier rule there exists a $\tilde{\lambda}_1$ such that

$$\langle F'(u), v\rangle = \tilde{\lambda}_1 \langle G'(u), v\rangle,$$

for all $v \in H_\mu$. We have for $v = u$ that

$$\lambda_1 = \int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx = \tilde{\lambda}_1 \int_\Omega h(x)|x|^{-c}u^2 dx = \tilde{\lambda}_1$$

which means that u is eigenfunction associated to λ_1.

Conversely, let u be an eigenfunction associated to λ_1. Then we have

$$\int_\Omega (|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2) dx = \lambda_1 \int_\Omega h(x)|x|^{-c}u^2 dx$$

and so

$$\lambda_1 = \frac{\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx}{\int_\Omega h(x)|x|^{-c}u^2 dx}.$$

Define $v = \dfrac{u}{\left(\int_\Omega h(x)|x|^{-c}u^2 dx\right)^{1/2}}$. Note that $v \in \mathcal{M}$. Moreover

$$\begin{aligned}\lambda_1 &\leq \int_\Omega \left(|x|^{-2a}|\nabla v|^2 - \mu|x|^{-2(a+1)}v^2\right) dx \\ &= \frac{\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\right) dx}{\int_\Omega h(x)|x|^{-c}u^2 dx} \\ &= \lambda_1.\end{aligned}$$

■

Lemma 4.5 *Suppose (H) and (K) hold, $h(x) > 0$ for all $x \in \Omega$, and $k(x) > 0$ for all $x \in \Omega$. If u is an eigenfunction associated to λ_1, then $|u|$ is also an eigenfunction associated to λ_1.*

Proof. Let u be an eigenfunction associated to λ_1. From Lemma 4.4 we have that u achieves the minimum in (10). Since $\|\nabla|u|\|_{\mu,a} = \|\nabla u\|_{\mu,a}$ and $|u| \in \mathcal{M}$, using Lemma 4.4 again, we have that $|u|$ achieves the minimum in (10) and then $|u|$ is an eigenfunction associated to λ_1. ■

Proof of Theorem 1.3. Suppose, by contradiction, that $u \in H_\mu$ is a nontrivial nonnegative weak solution of the problem (1) for some $\lambda > \lambda_1$ and let φ_1 be an eigenfunction associated to λ_1. It follows from Lemma 4.5 that $|\varphi_1|$ is an eigenfunction associated to λ_1. Then we have

$$\int_\Omega \left(|x|^{-2a}\nabla u \nabla|\varphi_1| - \mu|x|^{-2(a+1)}u|\varphi_1|\right) dx = \lambda \int_\Omega h(x)|x|^{-c}u|\varphi_1|dx + \int_\Omega k(x)|x|^{-2_*b}u^{2_*-1}|\varphi_1|dx \tag{35}$$

and

$$\int_\Omega \left(|x|^{-2a}\nabla u \nabla|\varphi_1| - \mu|x|^{-2(a+1)}u|\varphi_1|\right) dx = \lambda_1 \int_\Omega h(x)|x|^{-c}u|\varphi_1|dx \tag{36}$$

From (35) and (36) we obtain

$$\lambda_1 \int_\Omega h(x)|x|^{-c}u|\varphi_1|dx = \lambda \int_\Omega h(x)|x|^{-c}u|\varphi_1|dx + \int_\Omega k(x)|x|^{-2_*b}u^{2_*-1}|\varphi_1|dx.$$

So we have

$$(\lambda_1 - \lambda)\int_\Omega h(x)|x|^{-c}u|\varphi_1|dx = \int_\Omega k(x)|x|^{-2_*b}u^{2_*-1}|\varphi_1|dx,$$

which is a contradiction since $\lambda > \lambda_1$, $h(x) > 0$ for all $x \in \Omega$, and $k(x) > 0$ for all $x \in \Omega$. ■

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