LIMIT SOLUTIONS OF THE CHERN-SIMONS EQUATION

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Abstract. Given a bounded domain Ω in $\mathbb{R}^2$, we investigate the scalar Chern-Simons equation

$$-\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu \quad \text{in } \Omega,$$

in cases where there is no solution for a given nonnegative finite measure μ. Approximating μ by a sequence $(\mu_n)_{n\in\mathbb{N}}$ of nonnegative L^1 functions or finite measures for which this equation has a solution, we show that the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the Dirichlet problem converges to the solution with largest possible datum $\mu^\# \leq \mu$ and we derive an explicit formula of $\mu^\#$ in terms of μ. The counterpart for the Chern-Simons system with datum (μ, ν) behaves differently and the conclusion depends on how much the measures μ and ν charge singletons.

1. Introduction and main results

In this paper we investigate a question concerning convergence and stability of solutions of the scalar Chern-Simons problem

$$\begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and μ is a finite Borel measure — equivalently a Radon measure — in Ω. By a solution of (1.1), we mean a function $u \in W_0^{1,1}(\Omega)$ such that $\mathrm{e}^u(\mathrm{e}^u - 1) \in L^1(\Omega)$ and satisfying the equation in the sense of distributions.

Using for instance a minimization argument in $W_0^{1,2}(\Omega)$, one shows that the scalar Chern-Simons equation always has a solution with datum $\mu \in L^p(\Omega)$ for any $1 < p \leq \infty$ [16, Chapter 2]. Existence in the case of datum $\mu \in L^1(\Omega)$ can be obtained by approximation using L^∞ data [5, Corollary 12; 16, Chapter 3].

The case of nonlinear Dirichlet problems with measure data is more subtle. This issue has been discovered by Bénilan and Brezis [2–4] in a pioneering work concerning polynomial nonlinearities in dimension greater than 2.

The case of exponential nonlinearities in dimension 2 has been investigated by Vázquez [19]. For instance, if $\mu = \alpha\delta_a$ for some $a \in \Omega$, then for every $\alpha > 2\pi$ the Dirichlet problem (1.1) has no solution with datum μ. The counterexample above gives the only possible obstruction in the case of exponential nonlinearities: μ is a good measure — that is the Dirichlet problem (1.1) has a solution — if and only if for every $x \in \Omega$,

$$\mu(\{x\}) \leq 2\pi.$$

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We want to understand what happens when one forces the Dirichlet problem to have a solution when no solution is available. For instance, if μ is a measure for which (1.1) has no solution, then one could approximate μ by a sequence $(\rho_n * \mu)_{n\in\mathbb{N}}$ of convolutions of μ — for which we know the Dirichlet problem has a solution — and then investigate the limit of the sequence of solutions $(u_n)_{n\in\mathbb{N}}$.

This program has been proposed and implemented by Brezis, Marcus and Ponce [6] in the case where μ is approximated via convolution. They have proved that for any sequence of nonnegative mollifiers $(\rho_n)_{n\in\mathbb{N}}$, if u_n satisfies

$$\begin{cases} -\Delta u_n + \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1) = \rho_n * \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

then the sequence $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the largest subsolution u^* of the scalar Chern-Simons problem with datum μ [6, Theorem 4.11].

The result in [6] concerns more general convex nonlinearities and holds in any dimension, but strongly relies on the fact that the approximating sequence $(\rho_n * \mu)_{n\in\mathbb{N}}$ is constructed via convolution of μ [6, Example 4.1].

Our first result shows that for the Chern-Simons equation the conclusion of Brezis, Marcus and Ponce is always true regardless of the sequences of functions — or even measures — $(\mu_n)_{n\in\mathbb{N}}$ converging to μ.

Theorem 1.1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a nonnegative sequence of measures in Ω such that for every $n \in \mathbb{N}$ and for every $x \in \Omega$,*

$$\mu_n(\{x\}) \le 2\pi$$

and let u_n satisfy the scalar Chern-Simons problem

$$\begin{cases} -\Delta u_n + \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1) = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

If the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to a measure μ in the sense of measures in Ω, then the sequence $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the solution of the scalar Chern-Simons problem with datum $\mu^\#$, where $\mu^\#$ is the largest measure less than or equal to μ such that for every $x \in \Omega$,

$$\mu^\#(\{x\}) \le 2\pi.$$

A sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ in the sense of measures in Ω, if for every continuous function $\zeta : \overline{\Omega} \to \mathbb{R}$ such that $\zeta = 0$ on $\partial\Omega$,

$$\lim_{n\to\infty} \int_\Omega \zeta \,\mathrm{d}\mu_n = \int_\Omega \zeta \,\mathrm{d}\mu.$$

We denote this convergence by $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the vector space of (finite) measures in Ω equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega) = \int_\Omega \mathrm{d}|\mu|.$$

Applying Theorem 1.1 we deduce an explicit formula of $\mu^\#$ in terms of μ. Indeed, if we write μ as a sum of nonatomic part $\overline{\mu}$ and an atomic part

$$\mu = \overline{\mu} + \sum_{i=0}^{\infty} \alpha_i \delta_{a_i},$$

where $\alpha_i \geq 0$ and the points a_i are distinct, then

$$\mu^{\#} = \overline{\mu} + \sum_{i=0}^{\infty} \min\{\alpha_i, 2\pi\}\delta_{a_i}.$$

Since μ is a finite measure, there can only be finitely many indices i such that $\alpha_i > 2\pi$. In particular, the measure $\mu - \mu^{\#}$ is supported in a finite set and for every $a \in \Omega$,

$$\mu^{\#}(\{a\}) = \min\{\mu(\{a\}), 2\pi\}.$$

We may recover the result of Brezis, Marcus and Ponce using their notion of reduced measure μ^*. By definition, the reduced measure is the unique locally finite measure in Ω such that

$$\mu^* = -\Delta u^* + \mathrm{e}^{u^*}(\mathrm{e}^{u^*} - 1)$$

in the sense of distributions in Ω, where u^* is the largest subsolution of the Dirichlet problem (1.1). The fundamental property of reduced measures [6, Theorem 4.1] asserts that μ^* is a (finite) measure in Ω, u^* satisfies the Dirichlet problem (1.1) with datum μ^* and μ^* is the largest good measure less than or equal to μ. According to Vázquez's result such largest good measure is precisely $\mu^{\#}$. Therefore,

$$\mu^{\#} = \mu^*.$$

As an application of the tools we use to prove Theorem 1.1, we investigate what happens to the approximation scheme for the Chern-Simons system

$$\begin{cases} -\Delta u + \mathrm{e}^{v}(\mathrm{e}^{u} - 1) = \mu & \text{in } \Omega, \\ -\Delta v + \mathrm{e}^{u}(\mathrm{e}^{v} - 1) = \nu & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

By a result of Lin, Ponce and Yang [12, Theorem 1.1], the system above has a solution for nonnegative measures μ and ν in Ω if and only if for every $x \in \Omega$,

$$\mu(\{x\}) + \nu(\{x\}) \leq 4\pi.$$

A first result in this direction consists in identifying the sum of the components of the *reduced limit* $(\mu^{\#}, \nu^{\#})$.

Theorem 1.2. *Let $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ be sequences of nonnegative measures such that for every $n \in \mathbb{N}$ and $x \in \Omega$,*

$$\mu_n(\{x\}) + \nu_n(\{x\}) \leq 4\pi$$

and let (u_n, v_n) satisfy the Chern-Simons system

$$\begin{cases} -\Delta u_n + \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) = \mu_n & \text{in } \Omega, \\ -\Delta v_n + \mathrm{e}^{u_n}(\mathrm{e}^{v_n} - 1) = \nu_n & \text{in } \Omega, \\ u_n = v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

If the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ converge to μ and ν in the sense of measures in Ω, and if $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to u and v in $L^1(\Omega)$, then (u, v) satisfies the Chern-Simons system with datum $(\mu^{\#}, \nu^{\#})$, where

(i) $0 \leq \mu^{\#} \leq \mu$,
(ii) $0 \leq \nu^{\#} \leq \nu$,

(*iii*) *$\mu^\# + \nu^\#$ is the largest measure less than or equal to $\mu + \nu$ such that for every $x \in \Omega$,*

$$\mu^\#(\{x\}) + \nu^\#(\{x\}) \le 4\pi.$$

The assumption concerning the convergence of the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $L^1(\Omega)$ is not restrictive since if the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ are bounded $\mathcal{M}(\Omega)$, then both $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are compact in $L^1(\Omega)$. This is a consequence of a contraction estimate for Dirichlet problems with an absorption nonlinearity and Stampacchia's linear regularity theory [16, Chapter 3 and Chapter 4; 18, Théorème 9.1].

By Theorem 1.2, the measure $\mu^\# + \nu^\#$ can be expressed only in terms of the measure $\mu + \nu$ as we did in the case of the scalar Chern-Simons problem. We deduce that the nonatomic parts of $\mu^\#$ and $\nu^\#$ coincide with the nonatomic parts of μ and ν, so we are still left to identify the atomic parts of $\mu^\#$ and $\nu^\#$.

In order to give a hint of the type of conclusion one may get from Theorem 1.2, let us consider some examples:

Model case 1. $\mu = \alpha\delta_a$ and $\nu = \beta\delta_b$ with $a, b \in \Omega$, $a \ne b$, and $\alpha, \beta \ge 0$.

In this case there is no interaction in the limit and everything happens as if the measures did not feel each other. It follows from Theorem 1.2 that

$$\mu^\# = \min\{\alpha, 4\pi\}\delta_a \quad \text{and} \quad \nu^\# = \min\{\beta, 4\pi\}\delta_b.$$

Model case 2. $\mu = 3\pi\delta_a$ and $\nu = 2\pi\delta_a$ with $a \in \Omega$.

Model case 3. $\mu = 5\pi\delta_a$ and $\nu = 2\pi\delta_a$ with $a \in \Omega$.

In both cases, Theorem 1.2 ensures that

$$\mu^\# + \nu^\# = 4\pi\delta_a,$$

but gives no information on how the measure $4\pi\delta_a$ is shared by $\mu^\#$ and $\nu^\#$.

This raises the following question: are the measures $\mu^\#$ and $\nu^\#$ independent of the choice of sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ converging weakly in measure to μ and ν?

We have been able to identify two cases where the answer is affirmative.

Theorem 1.3. *Let $a \in \Omega$. Under the assumptions of Theorem 1.2,*

(*i*) *if $\mu(\{a\}) = 0$ or $\nu(\{a\}) = 0$, then*

$$\mu^\#(\{a\}) = \min\{\mu(\{a\}), 4\pi\},$$
$$\nu^\#(\{a\}) = \min\{\nu(\{a\}), 4\pi\};$$

(*ii*) *if $\mu(\{a\}) \le 4\pi$ and $\nu(\{a\}) \le 4\pi$, then*

$$\mu^\#(\{a\}) = \min\left\{\mu(\{a\}),\ \mu(\{a\}) - \frac{\mu(\{a\}) + \nu(\{a\}) - 4\pi}{2}\right\},$$
$$\nu^\#(\{a\}) = \min\left\{\nu(\{a\}),\ \nu(\{a\}) - \frac{\mu(\{a\}) + \nu(\{a\}) - 4\pi}{2}\right\}.$$

Assertion (*i*) above covers Model case 1. Applying Assertion (*ii*) above, we deduce that in Model case 2 the loss of mass is equally shared by $\mu^\#$ and $\nu^\#$ and we have

$$\mu^\# = \tfrac{5\pi}{2}\delta_a \quad \text{and} \quad \nu^\# = \tfrac{3\pi}{2}\delta_a.$$

Despite of what happens in the scalar case, in general the reduced limit depends on the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$. In Section 5.3 we show that two possible reduced limits in Model case 3 are

$$\left(\tfrac{7\pi}{2}\delta_a, \tfrac{\pi}{2}\delta_a\right) \quad \text{and} \quad (3\pi\delta_a, \pi\delta_a).$$

A counterpart of Theorem 1.3 in this case would have predicted only the first possibility. Using Cantor's diagonal argument we may trick the system by pretending we are under the regime of Assertion (i) above, and this gives the second reduced limit. We do not know whether the first reduced limit is the only possibility when the sequence $\big((\mu_n, \nu_n)\big)_{n\in\mathbb{N}}$ arises as a convolution. For instance, $\mu_n = 5\pi\rho_n$ and $\nu_n = 2\pi\rho_n$ where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of mollifiers.

We have restricted ourselves to domains in dimension 2. In dimension greater than or equal to 3, the characterization of measures for which the *scalar* Chern-Simons problem (1.1) has a solution is not completely understood [1, 15, 20].

2. Proof of Theorem 1.1

By a standard property of elliptic equations with absorption term [16, Chapter 7], for every $n \in \mathbb{N}$,

$$\|\mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1)\|_{L^1(\Omega)} \le \|\mu_n\|_{\mathcal{M}(\Omega)}. \tag{2.2}$$

Thus, by the triangle inequality,

$$\|\Delta u_n\|_{\mathcal{M}(\Omega)} \le 2\|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Since the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, the sequence $(\Delta u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathcal{M}(\Omega)$. From Stampacchia's linear regularity theory [16, Chapter 3; 18, Théorème 9.1], the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$ for every $1 \le q < 2$. By the Rellich-Kondrachov compactness theorem, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to some function u in $L^1(\Omega)$ and a.e. in Ω. By (2.2), the sequence $(\mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1))_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. Passing to a further subsequence if necessary, we may assume that there exists a finite measure τ in Ω such that

$$\mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^{u}(\mathrm{e}^{u} - 1) + \tau \quad \text{in } \mathcal{M}(\Omega).$$

Thus, u satisfies the scalar Chern-Simons problem

$$\begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu - \tau & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the set

$$A = \big\{x \in \Omega : \mu(\{x\}) \ge 2\pi\big\}.$$

Since μ is a finite measure, the set A is finite. We first prove that τ is supported in A.

For this purpose, let $N(\mu_n)$ be the Newtonian potential generated by μ_n,

$$N(\mu_n)(x) = \frac{1}{2\pi}\int_\Omega \log\left(\frac{d}{|x - y|}\right) \mathrm{d}\mu_n(y),$$

where $d \geq \operatorname{diam}\Omega$. Given $b \in \Omega$ and $r > 0$, we first write the Newtonian potential of μ_n as

$$N(\mu_n) = N(\mu_n\lfloor_{B_r(b)}) + N(\mu_n\lfloor_{\Omega\setminus B_r(b)}).$$

Assume for the moment that there exist $\epsilon > 0$ and $m \in \mathbb{N}$ such that for every $n \geq m$,

$$\mu_n(B_r(b)) \leq 2\pi - \epsilon. \tag{2.3}$$

By the Brezis-Merle inequality [7, Theorem 1; 16, Lemma 8.2], there exist $p > 1$ and $C_1 > 0$ such that for every $n \geq m$,

$$\left\|e^{2N(\mu_n\lfloor_{B_r(b)})}\right\|_{L^p(\Omega)} \leq C_1.$$

Since the functions $N(\mu_n\lfloor_{\Omega\setminus B_r(b)})$ are harmonic in $B_r(b)$ and have a uniformly bounded L^1 norm in $B_r(b)$, the sequence $\big(N(\mu_n\lfloor_{\Omega\setminus B_r(b)})\big)_{n\in\mathbb{N}}$ is uniformly bounded in $B_{r/2}(b)$. We conclude that there exists $C_2 > 0$ such that for every $n \geq m$,

$$\left\|e^{2N(\mu_n)}\right\|_{L^p(B_{r/2}(b))} \leq C_2. \tag{2.4}$$

Note that if $b \in \Omega \setminus A$, then there exist $\epsilon > 0$ and $r > 0$ satisfying (2.3). Indeed, let $\overline{\epsilon} > 0$ and $R > 0$ such that

$$\mu(B_R(b)) \leq 2\pi - \overline{\epsilon}.$$

Then, by weak convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$ [11, Section 1.9], property (2.3) holds for every $0 < r < R$ and for every $0 < \epsilon < \overline{\epsilon}$.

Let U_n be the solution of the linear Dirichlet problem

$$\begin{cases} -\Delta U_n = \mu_n & \text{in } \Omega,\\ \quad U_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

By the comparison estimate between the solution U_n of the linear Dirichlet problem (2.5) and the solution u_n of the nonlinear Dirichlet problem [16, Chapter 7], for every $n \in \mathbb{N}$ we have

$$u_n \leq U_n \quad \text{in } \Omega.$$

By the weak maximum principle [16, Chapter 5], $U_n \leq N(\mu_n)$ in Ω. Hence,

$$u_n \leq N(\mu_n) \quad \text{in } \Omega.$$

It follows from (2.4) that the sequence $(e^{u_n}(e^{u_n}-1))_{n\in\mathbb{N}}$ is uniformly bounded in $L^p(B_{r/2}(b))$. Since $u_{n_k} \to u$ a.e. in $B_{r/2}(b)$, by Egorov's theorem we get

$$e^{u_{n_k}}(e^{u_{n_k}} - 1) \to e^u(e^u - 1) \quad \text{in } L^1(B_{r/2}(b)).$$

We deduce that $\tau = 0$ in $B_{r/2}(b)$. Since $b \in \Omega \setminus A$ is arbitrary, we conclude that τ is supported in A.

If the set A is empty, the conclusion of the theorem follows with $\mu^{\#} = \mu$. We may assume that A is nonempty. Recalling that A is a finite set, we may write

$$A = \{x_1, \ldots, x_l\},$$

where the points $x_i \in \Omega$ are distinct.

Given $i \in \{1, \dots, l\}$, let $r > 0$ be such that $B_r(x_i) \cap A = \{x_i\}$. For every

$$0 \le \alpha < \frac{2\pi}{\mu(\{x_i\})},$$

let v_k be a function satisfying the scalar Chern-Simons problem

$$\begin{cases} -\Delta v_k + \mathrm{e}^{v_k}(\mathrm{e}^{v_k} - 1) = \alpha\mu_{n_k} & \text{in } B_r(x_i), \\ v_k = 0 & \text{on } \partial B_r(x_i). \end{cases}$$

The existence of v_k follows from [19, Theorem 2]; alternatively, one may apply the method of sub and supersolution [14, Corollary 5.4; 16, Chapter 6] with subsolution 0 and supersolution u_{n_k}. In particular,

$$0 \le v_k \le u_{n_k} \quad \text{in } B_r(x_i).$$

Since for every $x \in B_r(x_i)$,

$$\alpha\mu(\{x\}) < 2\pi,$$

the sequence $(v_k)_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the unique solution v of scalar Chern-Simons problem in $B_r(x_i)$ with datum $\alpha\mu$.

Since $v \le u$ and since points have zero $W^{1,2}$ capacity in $\mathbb{R}^2$, by the Inverse maximum principle [10, Theorem 3; 16, Chapter 5] we have for every $x \in B_r(x_i)$,

$$-\Delta v(\{x\}) \le -\Delta u(\{x\}).$$

Computing in particular both measure in the set $\{x_i\}$, we get

$$\alpha\mu(\{x_i\}) \le (\mu - \tau)(\{x_i\}) = \mu^{\#}(\{x_i\}),$$

where

$$\mu^{\#} = \mu - \tau.$$

Taking the supremum over α, we deduce that

$$2\pi \le \mu^{\#}(\{x_i\}).$$

On the other hand, by Vázquez's nonexistence result [1, Section 5; 19, Section 5], we also have $\mu^{\#}(\{x_i\}) \le 2\pi$. We conclude that

$$\mu = \mu^{\#} \quad \text{in } \Omega \setminus \{x_1, \dots, x_l\}$$

and for every $i \in \{1, \dots, l\}$,

$$\mu^{\#}(\{x_i\}) = 2\pi.$$

In particular, the measure $\mu^{\#}$ does not depend on the subsequence $(u_{n_k})_{k\in\mathbb{N}}$. Since the solution of the Chern-Simons problem is unique for nonnegative datum, we deduce that the entire sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(\Omega)$. The proof of the theorem is complete. □

3. Proof of Theorem 1.2

We first show that

$$\mu^{\#} \le \mu.$$

Recall $u_n \in W_0^{1,1}(\Omega)$ and that for every $\varphi \in C_c^\infty(\Omega)$,

$$-\int_\Omega u_n \Delta\varphi + \int_\Omega \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1)\varphi = \int_\Omega \varphi \,\mathrm{d}\mu_n. \tag{3.6}$$

The nonlinear term in the equation verified by u_n satisfies the sign condition: for every $t \in \mathbb{R}$,

$$\mathrm{e}^{v_n}(\mathrm{e}^t - 1)\operatorname{sign} t \ge 0.$$

From the comparison estimate [16, Corollary 7.9], $\mu_n \ge 0$ implies that $u_n \ge 0$. Since the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to u and v in $L^1(\Omega)$, if the test function satisfies $\varphi \ge 0$, then by Fatou's lemma,

$$\int_\Omega \mathrm{e}^{v}(\mathrm{e}^{u} - 1)\varphi \le \liminf_{n\to\infty} \int_\Omega \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1)\varphi.$$

As we let n tend to infinity in (3.6), we get

$$\int_\Omega \varphi \,\mathrm{d}\mu^{\#} = -\int_\Omega u\Delta\varphi + \int_\Omega \mathrm{e}^{v}(\mathrm{e}^{u} - 1)\varphi \le \int_\Omega \varphi \,\mathrm{d}\mu.$$

Since this property holds for every $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \ge 0$, we deduce that $\mu^{\#} \le \mu$.

We now show that

$$\mu^{\#} \ge 0.$$

This property is proved in [13, Theorem 1.3] in the case of semilinear equations with nonlinearities without dependence on the domain variable. The fundamental ingredient is a counterpart of the Inverse maximum principle for concentrated limits with respect to the $W^{1,2}$ capacity [13, Theorem 4.2] which we explain below.

We recall that by the Biting lemma [8; 13, Section 2], up to a subsequence every bounded sequence of measures $(\lambda_n)_{n\in\mathbb{N}}$ may be decomposed as

$$\lambda_n = \lambda_n\lfloor_{A_n} + \lambda_n\lfloor_{\Omega\setminus A_n},$$

where the sequence $(\lambda_n\lfloor_{A_n})_{n\in\mathbb{N}}$ is equidiffuse — in the sense that the measure of sets with small capacity are small — and $\operatorname{cap}_{W^{1,2}}(\Omega \setminus A_n) \to 0$ as n tends to infinity. Assuming that the sequence $(\lambda_n\lfloor_{\Omega\setminus A_n})_{n\in\mathbb{N}}$ converges weakly* to some measure γ, we call γ the concentrated limit of the sequence $(\lambda_n)_{n\in\mathbb{N}}$.

Taking a subsequence if necessary we may assume that $(\Delta u_n)_{n\in\mathbb{N}}$ has a concentrated limit γ. Since for every $n \in \mathbb{N}$, $u_n \ge 0$, it follows from [13, Theorem 4.2] that

$$\gamma \le 0,$$

which is the counterpart of the Inverse maximum principle we mentioned. As we shall see, the inequality $\mu^{\#} \ge 0$ follows from this one by computing the measure γ in terms of $\mu^{\#}$.

For this purpose, we note that for every $n \in \mathbb{N}$,

$$\Delta u_n = \mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) - \mu_n$$

and

$$e^{v_n}(e^{u_n}-1) \overset{*}{\rightharpoonup} e^{v}(e^{u}-1)+\mu-\mu^{\#} \quad \text{in } \mathcal{M}(\Omega). \tag{3.7}$$

We show that $\mu-\mu^{\#}$ is the concentrated limit of a subsequence $\big(e^{v_{n_k}}(e^{u_{n_k}}-1)\big)_{k\in\mathbb{N}}$.

For every $k\in\mathbb{N}$ let $\Phi_k:\mathbb{R}\to\mathbb{R}$ be a continuous function such that

(a) $0\le\Phi_k\le 1$,
(b) for every $t\le k$, $\Phi_k(t)=1$,
(c) for every $t\ge k+1$, $\Phi_k(t)=0$,

Then, for every $k\in\mathbb{N}$,

$$e^{v_n}(e^{u_n}-1)\Phi_k(u_n)\Phi_k(v_n)\to e^{v}(e^{u}-1)\Phi_k(u)\Phi_k(v) \quad \text{in } L^1(\Omega)$$

as n tends to infinity and

$$e^{v}(e^{u}-1)\Phi_k(u)\Phi_k(v)\to e^{v}(e^{u}-1) \quad \text{in } L^1(\Omega)$$

as k tends to infinity. Using Cantor's diagonal argument, we may find an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that

$$e^{v_{n_k}}(e^{u_{n_k}}-1)\Phi_k(u_{n_k})\Phi_k(v_{n_k})\to e^{v}(e^{u}-1) \quad \text{in } L^1(\Omega).$$

In particular,

$$e^{v_{n_k}}(e^{u_{n_k}}-1)\chi_{A_k}\to e^{v}(e^{u}-1) \quad \text{in } L^1(\Omega), \tag{3.8}$$

where

$$A_k=\{u_{n_k}\le k\}\cap\{v_{n_k}\le k\}.$$

Hence, the sequence $\big(e^{v_{n_k}}(e^{u_{n_k}}-1)\chi_{A_k}\big)_{k\in\mathbb{N}}$ is equidiffuse with respect to the capacity $\operatorname{cap}_{W^{1,2}}$.

By subadditivity of the capacity,

$$\operatorname{cap}_{W^{1,2}}(\Omega\setminus A_k)\le \operatorname{cap}_{W^{1,2}}(\{u_{n_k}>k\})+\operatorname{cap}_{W^{1,2}}(\{v_{n_k}>k\}).$$

We recall the capacitary estimate satisfied by u_n [9, Remark 2.18; 13, Lemma 3.2; 16, Lemma 9.4],

$$\operatorname{cap}_{W^{1,2}}(\{u_n>k\})\le \frac{1}{k^2}\int_\Omega |\nabla T_k(u_n)|^2\le \frac{1}{k}\|\Delta u_n\|_{\mathcal{M}(\Omega)}.$$

By the equation satisfied by u_n and by the absorption estimate (2.2),

$$\|\Delta u_n\|_{\mathcal{M}(\Omega)}\le 2\|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Therefore,

$$\operatorname{cap}_{W^{1,2}}(\{u_n>k\})\le \frac{2}{k}\|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Since the function v_n satisfies a similar estimate, we conclude that

$$\operatorname{cap}_{W^{1,2}}(\Omega\setminus A_k)\le \frac{2}{k}\big(\|\mu_{n_k}\|_{\mathcal{M}(\Omega)}+\|\nu_{n_k}\|_{\mathcal{M}(\Omega)}\big)\le\frac{C}{k}, \tag{3.9}$$

for some constant $C\ge 0$ independent of k. In particular, the $W^{1,2}$ capacity of the set $\Omega\setminus A_k$ converges to zero as k tends to infinity. Combining (3.7)–(3.9), we deduce that $\mu-\mu^{\#}$ is the concentrated limit of the sequence $\big(e^{v_{n_k}}(e^{u_{n_k}}-1)\big)_{k\in\mathbb{N}}$.

Passing to a further subsequence if necessary, we may assume that $(\mu_{n_k})_{k\in\mathbb{N}}$ has a concentrated limit λ. We conclude that the concentrated limit of the sequence $(\Delta u_{n_k})_{k\in\mathbb{N}}$ is $\mu - \mu^{\#} - \lambda$. Thus,

$$\mu - \mu^{\#} - \lambda = \gamma \le 0.$$

Since the sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ is nonnegative and converges weakly* to μ in $\mathcal{M}(\Omega)$,

$$0 \le \lambda \le \mu.$$

Therefore,

$$\mu^{\#} \ge \mu - \lambda \ge 0,$$

which gives the inequality we claimed.

We have proved that $0 \le \mu^{\#} \le \mu$. Reverting the roles of u and v, we obtain $0 \le \nu^{\#} \le \nu$.

It remains to establish Assertion (iii). For this purpose, let τ_1 and τ_2 be finite measures such that

$$\begin{aligned} &\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^{v}(\mathrm{e}^{u} - 1) + \tau_1 \\ &\mathrm{e}^{u_n}(\mathrm{e}^{v_n} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^{u}(\mathrm{e}^{v} - 1) + \tau_2 \end{aligned} \quad \text{in } \mathcal{M}(\Omega). \tag{3.10}$$

Thus, (u, v) solves the Chern-Simons problem

$$\begin{cases} -\Delta u + \mathrm{e}^{v}(\mathrm{e}^{u} - 1) = \mu - \tau_1 & \text{in } \Omega, \\ -\Delta v + \mathrm{e}^{u}(\mathrm{e}^{v} - 1) = \nu - \tau_2 & \text{in } \Omega, \\ \qquad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proceeding as in the proof of the previous theorem, we can apply the Brezis-Merle inequality and comparison estimates to show that τ_1 and τ_2 are supported in the finite set

$$B = \big\{x \in \Omega : \mu(\{x\}) + \nu(\{x\}) \ge 4\pi\big\}.$$

Assuming that B is nonempty, we may write

$$B = \{x_1, \dots, x_l\}$$

where the points $x_i \in \Omega$ are distinct. Given $i \in \{1, \dots, l\}$, let $r > 0$ be such that $B_r(x_i) \cap B = \{x_i\}$. Adding the equations satisfied by u_n and v_n, we have

$$-\Delta(u_n + v_n) + 2(\mathrm{e}^{u_n + v_n} - 1) \ge \mu_n + \nu_n - 2 \quad \text{in } \Omega.$$

Note that for every

$$0 \le \alpha < \frac{4\pi}{\mu(\{x_i\}) + \nu(\{x_i\})},$$

there exists w_n satisfying the equation

$$\begin{cases} -\Delta w_n + 2(\mathrm{e}^{w_n} - 1) = \alpha(\mu_n + \nu_n) - 2 & \text{in } B_r(x_i), \\ \qquad w_n = 0 & \text{on } \partial B_r(x_i). \end{cases}$$

The existence of w_n follows from [16, Chapter 8; 19, Theorem 2]; alternatively, one may apply the method of sub and supersolution [14, Corollary 5.4; 16, Chapter 6] with subsolution 0 and supersolution $u_n + v_n$. In particular,

$$0 \le w_n \le u_n + v_n \quad \text{in } \Omega.$$

By a variant of Theorem 1.1 with nonlinearity $\mathrm{e}^t(\mathrm{e}^t-1)$ replaced by e^t-1, the sequence $(w_n)_{n\in\mathbb{N}}$ converges in $L^1(B_r(y_i))$ to the solution of

$$\begin{cases} -\Delta w + 2(\mathrm{e}^w - 1) = \alpha(\mu+\nu) - 2 & \text{in } B_r(x_i),\\ w = 0 & \text{on } \partial B_r(x_i). \end{cases}$$

In particular, $w \le u + v$ in $B_r(x_i)$. By the Inverse maximum principle [10, Theorem 3; 16, Chapter 5], we deduce that

$$\alpha(\mu(\{x_i\}) + \nu(\{x_i\})) \le \mu^\#(\{x_i\}) + \nu^\#(\{x_i\}).$$

Taking the supremum over α, we conclude that

$$4\pi \le \mu^\#(\{x_i\}) + \nu^\#(\{x_i\}).$$

Since the reverse inequality holds, equality follows for every $i \in \{1, \dots, l\}$. The proof of the theorem is complete. □

4. Proof of Theorem 1.3

If $\mu(\{a\}) + \nu(\{a\}) \le 4\pi$, then by Theorem 1.2,

$$\mu^\#(\{a\}) + \nu^\#(\{a\}) = \mu(\{a\}) + \nu(\{a\}).$$

Since $\mu^\# \le \mu$ and $\nu^\# \le \nu$, we deduce that

$$\mu^\#(\{a\}) = \mu(\{a\}) \quad \text{and} \quad \nu^\#(\{a\}) = \nu(\{a\}).$$

We now assume that $\mu(\{a\})+\nu(\{a\}) > 4\pi$. In this case, by Theorem 1.2,

$$\mu^\#(\{a\}) + \nu^\#(\{a\}) = 4\pi. \tag{4.11}$$

Recall that $0 \le \mu^\# \le \mu$. Thus, if $\mu(\{a\}) = 0$, then $\mu^\#(\{a\}) = 0$, whence $\nu^\#(\{a\}) = 4\pi$ by the above identity. Similarly, if $\nu(\{a\}) = 0$, then $\nu^\#(\{a\}) = 0$ and $\mu^\#(\{a\}) = 4\pi$. This concludes the proof of Assertion (i).

In order to complete the proof of Assertion (ii), we assume that in addition to (4.11), we have

$$\mu(\{a\}) \le 4\pi \quad \text{and} \quad \nu(\{a\}) \le 4\pi. \tag{4.12}$$

Using the notation of the proof of Theorem 1.2, we show that

$$\tau_1(\{a\}) = \tau_2(\{a\}).$$

Since $v_n \ge 0$ in Ω,

$$\mathrm{e}^{v_n}(\mathrm{e}^{u_n} - 1) \ge \mathrm{e}^{u_n} - 1 \quad \text{in } \Omega.$$

In particular, u_n is a subsolution of the Dirichlet problem

$$\begin{cases} -\Delta w + \mathrm{e}^w - 1 = \lambda & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{4.13}$$

with datum $\lambda = \mu_n$. Since $\nu_n \ge 0$ and for every $x \in \Omega$, $\mu_n(\{x\}) + \nu_n(\{x\}) \le 4\pi$, we have for every $x \in \Omega$, $\mu_n(\{x\}) \le 4\pi$. By Vázquez existence result [16, Chapter 8; 19, Theorem 2], there exists $\overline{u}_n$ satisfying the Dirichlet problem above with datum $\lambda = \mu_n$. By a comparison principle between the subsolution and the solution of the Dirichlet problem [16, Chapter 5], $u_n \le \overline{u}_n$ in Ω.

It follows from a variant of Theorem 1.1 with nonlinearity $e^t(e^t - 1)$ replaced by $e^t - 1$ that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the function $\overline{u}$ satisfying the Dirichlet problem (4.13) with datum $\lambda = \tilde{\mu}$, where $\tilde{\mu}$ is the largest measure less than or equal to μ such that for every $x \in \Omega$,

$$\tilde{\mu}(\{x\}) \leq 4\pi.$$

In particular, since $\mu(\{a\}) \leq 4\pi$, we have

$$\tilde{\mu}(\{a\}) = \mu(\{a\}).$$

We also observe that the measure $\mu - \tilde{\mu}$ is supported in a finite set, thus there exists $r_1 > 0$ such that

$$\tilde{\mu} = \mu \quad \text{in } B_{r_1}(a).$$

Hence,

$$e^{\overline{u}_n} \overset{*}{\rightharpoonup} e^{\overline{u}} \quad \text{in } \mathcal{M}(B_{r_1}(a)). \tag{4.14}$$

Similarly, if $\overline{v}_n$ denotes the solution of the Dirichlet problem (4.13) with datum $\lambda = \nu_n$, then $v_n \leq \overline{v}_n$ in Ω and the sequence $(\overline{v}_n)_{n\in\mathbb{N}}$ converges to the solution of the Dirichlet problem (4.13) with datum $\lambda = \tilde{\nu}$ where the measure $\tilde{\nu}$ satisfies $\tilde{\nu}(\{a\}) = \nu(\{a\})$ and $\nu - \tilde{\nu}$ is supported in a finite set. In particular, there exists $r_2 > 0$ such that

$$e^{\overline{v}_n} \overset{*}{\rightharpoonup} e^{\overline{v}} \quad \text{in } \mathcal{M}(B_{r_2}(a)). \tag{4.15}$$

On the other hand, writing

$$e^{u_n} - e^{v_n} = -e^{u_n}(e^{v_n} - 1) + e^{v_n}(e^{u_n} - 1),$$

it follows from (3.10) that

$$e^{u_n} - e^{v_n} \overset{*}{\rightharpoonup} e^{u} - e^{v} + \tau_1 - \tau_2 \quad \text{in } \mathcal{M}(\Omega).$$

We observe that for every $n \in \mathbb{N}$,

$$-e^{\overline{v}_n} \leq e^{u_n} - e^{v_n} \leq e^{\overline{u}_n} \quad \text{in } \Omega.$$

As we let n tend to infinity, we deduce from (4.14) and (4.15) that for every $0 < r \leq \min\{r_1, r_2\}$,

$$-e^{\overline{v}} \leq e^{u} - e^{v} + \tau_1 - \tau_2 \leq e^{\overline{u}} \quad \text{in } B_r(a).$$

Since the measure $\tau_1 - \tau_2$ is supported in a finite set — in particular is singular with respect to the Lebesgue measure — we conclude that $\tau_1 = \tau_2$.

Let $\tau = \tau_1 = \tau_2$. By Theorem 1.2, we have

$$\mu(\{a\}) + \nu(\{a\}) - 2\tau(\{a\}) = \mu^{\#}(\{a\}) + \nu^{\#}(\{a\}) = 4\pi.$$

Thus,

$$\tau(\{a\}) = \frac{\mu(\{a\}) + \nu(\{a\}) - 4\pi}{2},$$

from which the conclusion follows. □

5. Concluding remarks

5.1. **Connection to reduced limits.** We have restricted ourselves to solutions of the Dirichlet problem, but we could also ask what happens to nonnegative solutions of the scalar Chern-Simons equation

$$-\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu \quad \text{in } \Omega,$$

without taking into account the Dirichlet boundary condition.

Solutions of the equation depend on the boundary data, but the approximation scheme does not. More precisely, for each $n \in \mathbb{N}$ take a solution u_n of the equation with datum μ_n without prescribing any boundary condition. If the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a function u in $L^1(\Omega)$ and if the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to some measure μ, one shows that u satisfies the scalar Chern-Simons equation with some datum $\mu^\#$, possibly different from μ. If we now take another sequence of solutions $(v_n)_{n\in\mathbb{N}}$ with the same data $(\mu_n)_{n\in\mathbb{N}}$ converging to another function v in $L^1(\Omega)$, then v satisfies the scalar Chern-Simons equation with the same datum $\mu^\#$. This remarkable property has been recently discovered by Marcus and Ponce [13], where they introduce the concept of reduced limit $\mu^\#$.

Combining Theorem 1.1 with [13, Theorem 1.2], we deduce the following result.

Corollary 5.1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a nonnegative sequence of measures such that for every $n \in \mathbb{N}$ and for every $x \in \Omega$,*

$$\mu_n(\{x\}) \le 2\pi$$

and let u_n satisfy the scalar Chern-Simons problem

$$-\Delta u_n + \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1) = \mu_n \quad \text{in } \Omega.$$

If the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to a measure μ in the sense of measures in Ω and if the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(\Omega)$, then u is the solution of the scalar Chern-Simons equation with datum $\mu^\#$ defined in Theorem 1.1.

5.2. **Signed measures.** The sign of the measure μ affects substantially the conclusion. If $(\mu_n)_{n\in\mathbb{N}}$ is any sequence of nonpositive measures converging weakly in measure to some measure μ, then μ is nonpositive and the sequence of solutions of the Dirichlet problem for the scalar Chern-Simons equation converge to the solution with datum μ. This case is easier since the solutions u_n are nonpositive, whence the nonlinear term of exponential type is harmless.

The situation is more delicate when the sequence $(\mu_n)_{n\in\mathbb{N}}$ is not assumed to have a fixed sign. In this case, one can show that given a signed measure μ, positive numbers $c_1, \dots, c_m$, and points $x_1, \dots, x_m \in \Omega$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that

- (a) $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$,
- (b) $(f_n)_{n\in\mathbb{N}}$ converges to μ in the sense of measures,
- (c) the solutions u_n of the scalar Chern-Simons problem with datum f_n converge in $L^1(\Omega)$ to a solution of the scalar Chern-Simons problem with datum $\mu - \sum\limits_{i=1}^{m} c_i \delta_{x_i}$.

In particular, it is not possible in this case to have an explicit formula of the measure $\mu^{\#}$ only in terms of μ. We refer to [17, Teorema 4.8] for the proof.

5.3. **Nonuniqueness of the reduced limit of the Chern-Simons system.** The reduced limit $(\mu^{\#}, \nu^{\#})$ of the Chern-Simons system cannot be computed only in terms of the weak* limit (μ, ν) when both conditions

(a) $\mu(\{x\}) = 0$ or $\nu(\{x\}) = 0$,
(b) $\mu(\{x\}) \leq 4\pi$ and $\nu(\{x\}) \leq 4\pi$,

fail for some $x \in \Omega$.

Proposition 5.2. *Let $a \in \Omega$. For every $\alpha > 4\pi$ and for every $\beta > 0$, there exist sequences $(\mu_n^i)_{n\in\mathbb{N}}$ and $(\nu_n^i)_{n\in\mathbb{N}}$ of nonnegative functions in $L^1(\Omega)$ with $i \in \{1, 2\}$ such that*

(i) *$(\mu_n^i)_{n\in\mathbb{N}}$ and $(\nu_n^i)_{n\in\mathbb{N}}$ converge weakly to $\alpha\delta_a$ and $\beta\delta_a$ in the sense of measures in Ω,*
(ii) *there exists a sequence of solutions $\big((u_n^i, v_n^i)\big)_{n\in\mathbb{N}}$ of the Chern-Simons system with datum (μ_n^i, ν_n^i) converging in $L^1(\Omega) \times L^1(\Omega)$ to a solution with datum $(\alpha^{i,\#}\delta_a, \beta^{i,\#}\delta_a)$,*
(iii) *$\alpha^{1,\#} \neq \alpha^{2,\#}$ and $\beta^{1,\#} \neq \beta^{2,\#}$.*

We shall not prove this proposition. Instead, we restrict ourselves to the case where $(\alpha\delta_a, \beta\delta_a)$ is given by

$$(5\pi\delta_a, 2\pi\delta_a)$$

for some $a \in \Omega$ in order to emphasize the main idea of the proof.

For this purpose, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions in $L^1(\Omega)$ such that

$$f_n \overset{*}{\rightharpoonup} \delta_a \quad \text{in } \mathcal{M}(\Omega).$$

We construct the first sequence $\big((\mu_n^1, \nu_n^1)\big)_{n\in\mathbb{N}}$ of the form

$$\big((5\pi f_{m_n}, 2\pi f_n)\big)_{n\in\mathbb{N}}$$

where $(m_n)_{n\in\mathbb{N}}$ is a sequence of positive integers to be chosen below. For fixed $n \in \mathbb{N}$, let $\big((u_{m,n}^1, v_{m,n}^1)\big)_{m\in\mathbb{N}}$ be a solution of the Chern-Simons system with datum $(5\pi f_m, 2\pi f_n)$. Then, as m tends to infinity,

$$(5\pi f_m, 2\pi f_n) \overset{*}{\rightharpoonup} (5\pi\delta_a, 2\pi f_n) \quad \text{in } \mathcal{M}(\Omega) \times \mathcal{M}(\Omega).$$

By a standard property of elliptic equations with absorption term [16, Chapter 7] and by Stampacchia's linear regularity theory [16, Chapter 3], the sequence $\big((u_{m,n}^1, v_{m,n}^1)\big)_{m\in\mathbb{N}}$ is compact in $L^1(\Omega) \times L^1(\Omega)$. It is then possible to extract a subsequence with respect to the index m if necessary such that for every $n \in \mathbb{N}$,

$$(u_{m,n}^1, v_{m,n}^1) \to (u_n^1, v_n^1) \quad \text{in } L^1(\Omega) \times L^1(\Omega)$$

as m tends to infinity. It follows from Theorem 1.3 that (u_n^1, v_n^1) satisfies the Chern-Simons system with datum $(4\pi\delta_a, 2\pi f_n)$.

Note that

$$(4\pi\delta_a, 2\pi f_n) \overset{*}{\rightharpoonup} (4\pi\delta_a, 2\pi\delta_a) \quad \text{in } \mathcal{M}(\Omega) \times \mathcal{M}(\Omega).$$

By compactness of the sequence $\big((u_n^1, v_n^1)\big)_{n\in\mathbb{N}}$ in $L^1(\Omega)\times L^1(\Omega)$, we may extract a subsequence converging to (u^1, v^1). By Theorem 1.3, (u^1, v^1) satisfies the Chern-Simons system with datum $(3\pi\delta_a, \pi\delta_a)$.

For every $n \in \mathbb{N}$, take $m_n \in \mathbb{N}$ such that

$$\|u^1_{m_n,n} - u^1_n\|_{L^1(\Omega)} + \|v^1_{m_n,n} - v^1_n\|_{L^1(\Omega)} \le \frac{1}{n+1}.$$

Then, the sequence $\big((u^1_{m_n,n}, v^1_{m_n,n})\big)_{m\in\mathbb{N}}$ converges to (u^1, v^1) in $L^1(\Omega)\times L^1(\Omega)$. We have found a sequence of solutions of the Chern-Simons system with datum $\big((5\pi f_{m_n}, 2\pi f_n)\big)_{n\in\mathbb{N}}$ converging to the solution with datum

$$\boxed{(3\pi\delta_a, \pi\delta_a)}.$$

We construct the second sequence $(\mu_n^2, \nu_n^2)_{n\in\mathbb{N}}$ of the form

$$\big((4\pi f_{m_n} + \pi f_n, 2\pi f_{m_n})\big)_{n\in\mathbb{N}},$$

where $(m_n)_{n\in\mathbb{N}}$ is a sequence of positive integers, possibly different from the previous one. For fixed $n \in \mathbb{N}$, let $\big((u^2_{m,n}, v^2_{m,n})\big)_{m\in\mathbb{N}}$ be a solution of the Chern-Simons system with datum $(4\pi f_m + \pi f_n, 2\pi f_m)$. Then, as m tends to infinity,

$$(4\pi f_m + \pi f_n, 2\pi f_m) \overset{*}{\rightharpoonup} (4\pi\delta_a + \pi f_n, 2\pi\delta_a) \quad \text{in } \mathcal{M}(\Omega)\times\mathcal{M}(\Omega).$$

The sequence $\big((u^2_{m,n}, v^2_{m,n})\big)_{m\in\mathbb{N}}$ is compact in $L^1(\Omega)\times L^1(\Omega)$. It is then possible to extract a subsequence of with respect to the index m if necessary such that for every $n \in \mathbb{N}$,

$$(u^2_{m,n}, v^2_{m,n}) \to (u^2_n, v^2_n) \quad \text{in } L^1(\Omega)\times L^1(\Omega).$$

It follows from Theorem 1.3 that (u_n^2, v_n^2) satisfies the Chern-Simons system with datum $(3\pi\delta_a + \pi f_n, \pi\delta_a)$.

Note that

$$(3\pi\delta_a + \pi f_n, \pi\delta_a) \overset{*}{\rightharpoonup} (4\pi\delta_a, \pi\delta_a) \quad \text{in } \mathcal{M}(\Omega)\times\mathcal{M}(\Omega).$$

By compactness of the sequence $\big((u_n^2, v_n^2)\big)_{n\in\mathbb{N}}$ in $L^1(\Omega)\times L^1(\Omega)$, we may extract a subsequence converging to (u^2, v^2). By Theorem 1.3, (u^2, v^2) satisfies the Chern-Simons system with datum $(\frac{7\pi}{2}\delta_a, \frac{\pi}{2}\delta_a)$.

Proceeding as before, for every $n \in \mathbb{N}$ we may choose $m_n \in \mathbb{N}$ such that $\big((u^2_{m_n,n}, v^2_{m_n,n})\big)_{m\in\mathbb{N}}$ converges to (u^2, v^2) in $L^1(\Omega)\times L^1(\Omega)$. Hence, there exists a sequence of solutions of the Chern-Simons system with datum $\big((4\pi f_{m_n} + \pi f_n, 2\pi f_{m_n})\big)_{n\in\mathbb{N}}$ converging to the solution with datum

$$\boxed{\big(\tfrac{7\pi}{2}\delta_a, \tfrac{\pi}{2}\delta_a\big)}.$$

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References

[1] D. Bartolucci, F. Leoni, L. Orsina, and A. C. Ponce, *Semilinear equations with exponential nonlinearity and measure data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 799–815.

[2] Ph. Bénilan and H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation*, J. Evol. Equ. **3** (2004), 673–770. Dedicated to Ph. Bénilan.

[3] H. Brezis, *Some variational problems of the Thomas-Fermi type*, Variational inequalities and complementarity problems (Proc. Internat. School, Erice, 1978), Wiley, Chichester, 1980, pp. 53–73.

[4] ———, *Problèmes elliptiques et paraboliques non linéaires avec données mesures*, Goulaouic-Meyer-Schwartz Seminar, 1981/1982, École Polytech., Palaiseau, 1982. Exp. No. XX, 13.

[5] H. Brezis and W. A. Strauss, *Semi-linear second-order elliptic equations in L^1*, J. Math. Soc. Japan **25** (1973), 565–590.

[6] H. Brezis, M. Marcus, and A. C. Ponce, *Nonlinear elliptic equations with measures revisited*, Mathematical aspects of nonlinear dispersive equations (J. Bourgain, C. Kenig, and S. Klainerman, eds.), Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ, 2007, pp. 55–110.

[7] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations **16** (1991), 1223–1253.

[8] J. K. Brooks and R. V. Chacon, *Continuity and compactness of measures*, Adv. in Math. **37** (1980), 16–26.

[9] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 741–808.

[10] L. Dupaigne and A. C. Ponce, *Singularities of positive supersolutions in elliptic PDEs*, Selecta Math. (N.S.) **10** (2004), 341–358.

[11] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[12] C.-S. Lin, A. C. Ponce, and Y. Yang, *A system of elliptic equations arising in Chern-Simons field theory*, J. Funct. Anal. **247** (2007), 289–350.

[13] M. Marcus and A. C. Ponce, *Reduced limits for nonlinear equations with measures*, J. Funct. Anal. **258** (2010), 2316–2372.

[14] M. Montenegro and A. C. Ponce, *The sub-supersolution method for weak solution*, Proc. Amer. Math. Soc. **136** (2008), 2429–2438.

[15] A. C. Ponce, *How to construct good measures*, Elliptic and parabolic problems, Progr. Nonlinear Differential Equations Appl., vol. 63, Birkhäuser, Basel, 2005, pp. 375–388.

[16] ———, *Selected problems on elliptic equations involving measures* (2012), available at `http://arxiv.org/abs/1204.0668`.

[17] A. E. Presoto, *Soluções limites para problemas elípticos envolvendo medidas*, Ph.D. Thesis, IMECC—UNICAMP, Campinas, SP, Brazil, 2011.

[18] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.

[19] J. L. Vázquez, *On a semilinear equation in $\mathbf{R}^2$ involving bounded measures*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), 181–202.

[20] L. Véron, *On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data*, Math. Z. **273** (2013), 1–17.

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