EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO STRONGLY INDEFINITE HAMILTONIAN SYSTEM INVOLVING CRITICAL HARDY-SOBOLEV EXPONENTS

FRANCISCO ODAIR DE PAIVA, RODRIGO DA SILVA RODRIGUES

Abstract. In this article, we study the existence and multiplicity of nontrivial solutions for a class of Hamiltoniam systems with weights and nonlinearity involving the Hardy-Sobolev exponents. Results are proved using variational methods for strongly indefinite functionals.

1. Introduction

Elliptic problems involving general operators, such as the degenerate quasilinear elliptic equation $-\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{\zeta} f(u)$, were motivated by the Caffarelli, Kohn, and Nirenberg's inequality [5]

$$\Big(\int_{\mathbb{R}^N} |x|^{-2_a^* e_1}|u|^{2_a^*}dx\Big)^{2/2_a^*} \leq C_{a,e_1}\Big(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx\Big), \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 3$, $-\infty < a < (N-2)/2$, $a \leq e_1 \leq a+1$, $2_a^* := 2N/(N-2d_a)$, $d_a = 1 + a - e_1$, and $C_{a,e_1} > 0$. Note that several papers have appeared on this subject. Mainly, the works about the existence of solution for quasilinear equations and systems of the gradient type with nonlinearity involving critical growth. See, for instance, [1, 7, 16, 17, 22] and references therein. In particular, for $a = e_1 = 0$, Smets, Willem, and Su [19] studied the existence of non-radial ground states for the Hénon equation

$$\begin{gathered} -\Delta u = |x|^{\zeta} u^{l-1} \quad \text{in } B, \\ u = 0 \quad \text{on } \partial B, \end{gathered}$$

where B denotes the unit ball in $\mathbb{R}^N$ with $\zeta \geq 0$ and $l \in (2, 2^*)$. More general Hénon-Type problems has been studied by Carrião, de Figueiredo, and Miyagaki [6], for example. Also, we would like to refer to [18] for Hénon equation, and to [11] for Hardy-Hénon system.

In this article, we study the following class of quasilinear elliptic systems with weights and nonlinearity involving critical Hardy-Sobolev exponent

$$\begin{gathered} -\operatorname{div}(|x|^{-2a}\nabla u) = \mu_1 \frac{|u|^{\tau-2}u}{|x|^{\beta_0}} + \frac{H_u(x,u,v)}{|x|^{\beta_2}} + \alpha \frac{|u|^{\alpha-2}|v|^{\gamma}u}{|x|^{2_a^* e_1}} \quad \text{in } \Omega, \\ -\operatorname{div}(|x|^{-2b}\nabla v) = -\mu_2 \frac{|v|^{\xi-2}v}{|x|^{\beta_1}} - \frac{H_v(x,u,v)}{|x|^{\beta_2}} - \gamma \frac{|u|^{\alpha}|v|^{\gamma-2}v}{|x|^{2_b^* e_2}} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where

- (H1) Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$) with $0 \in \Omega$ and $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the class C^1.
- (H2) The exponents satisfy

$$0 \leq a, \quad b < \frac{N-2}{2}, \quad \xi \in (1, \frac{2N}{N-2}), \quad \tau \in (2, \frac{2N}{N-2}), \quad \alpha, \gamma > 1,$$

$2_a^* = \frac{2N}{N-2d_a}$ and $2_b^* = \frac{2N}{N-2d_b}$ are the Hardy-Sobolev exponents,

$$\begin{gathered} d_a = 1 + a - e_1, \quad 0 \leq a \leq e_1 < a+1, \\ d_b = 1 + b - e_2, \quad 0 \leq b \leq e_2 < b+1, \end{gathered}$$

with $2_a^* e_1 = 2_b^* e_2$ and

$$\frac{\alpha}{2_a^*} + \frac{\gamma}{2_b^*} = 1.$$

2000 *Mathematics Subject Classification.* 35B25, 35B33, 35J55, 35J70.
Key words and phrases. Hamiltonian systems; strongly indefinite variational structure; critical Hard-Sobolev exponents.
Supported by Fapesp-Brazil.

Note that problem (1.2) belongs to the class of Hamiltonian elliptic systems

$$
\begin{gathered}
-L_1(x,u) = \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \Omega,\\
-L_2(x,v) = -\frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where L_1 and L_2 are self-adjoint elliptic operators of second order. It is well known that the Euler-Lagrange functional associated is strongly indefinite. For this type of system with $L_1 = L_2 = \Delta$, the Laplacian operator, and subcritical growth, we would like to refer Benci and Rabinowitz [2], Costa and Magalhães [9], and de Figueiredo and Ding [10]. For critical and supercritical growth, we cite the de Figueiredo and Ding [10] and Hulshof, Mitidieri, and van der Vorst [12]. We also cite the papers [8] and [23]. Our results will be obtained as an application of some critical point results to strongly indefinite functionals proved in [10] and certain Galerkin approximations.

For the rest of this article we will assume that $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the class C^1 and satisfies:

(H4) There exist $K_0, K_1 > 0$, $p_i, q_i, r_i \in (1, 2N/(N-2))$, for $i = 1, 2$, $p_0, q_0 \in (1, 2N/(N-2))$, such that

$$
\begin{gathered}
|H(x,s,t)| \le K_0(|s|^{p_0} + |t|^{q_0}),\\
|H_s(x,s,t)l| \le K_1(|s|^{p_1} + |t|^{q_1} + |l|^{r_1}),\\
|H_t(x,s,t)l| \le K_1(|s|^{p_2} + |t|^{q_2} + |l|^{r_2}),
\end{gathered}
$$

for all $s, t, l \in \mathbb{R}$, $x \in \Omega$;

(H5) β_0, β_1, β_2 satisfy

$$
\begin{gathered}
\beta_0 < (a+1)\tau + N(1 - \frac{\tau}{2}),\\
\beta_1 \le (b+1)\xi + N(1 - \frac{\xi}{2}),\\
\beta_2 < \min\big\{(a+1)p_i + N(1 - \frac{p_i}{2}), (a+1)r_1 + N(1 - \frac{r_1}{2}),\\
(b+1)q_i + N(1 - \frac{q_i}{2}), (b+1)r_2 + N(1 - \frac{r_2}{2}) : i = 1, 2\big\};
\end{gathered}
$$

(H6) for all $s, t \in \mathbb{R}$, almost everywhere $x \in \Omega$,

$$
H(x,s,t) \ge -\Big(\frac{\mu_2}{\xi}|x|^{\beta_2 - \beta_1}|t|^{\xi} + |x|^{\beta_2 - 2_a^* e_1}|s|^{\alpha}|t|^{\gamma}\Big);
$$

(H7) there exist $\theta_1 \in (2, \tau]$ and $\theta_2 \in (1, 2)$ such that, for all $s, t \in \mathbb{R}$, almost everywhere $x \in \Omega$,

$$
\frac{1}{\theta_1} H_u(x,s,t)s + \frac{1}{\theta_2} H_v(x,s,t)t \ge H(x,s,t), \quad \frac{\alpha}{\theta_1} + \frac{\gamma}{\theta_2} \ge 1.
$$

Under the assumptions (H1), (H2)–(H5), $\mu_1 \le 0$, $\mu_2 \ge 0$, and $\gamma H_u(x,s,t)s = \alpha H_v(x,s,t)t$ for all $s, t \in \mathbb{R}$, almost everywhere $x \in \Omega$, we note that system (1.2) does not possess any nontrivial weak solution. Indeed, supposing by contradiction that (u, v) is a nontrivial weak solution, we obtain

$$
\int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}\,dx - \mu_1 \int_\Omega \frac{|u|^\tau}{|x|^{\beta_0}}\,dx = \alpha\Big[\frac{1}{\alpha}\int_\Omega \frac{H_u(x,u,v)u}{|x|^{\beta_2}}\,dx + \int_\Omega \frac{|u|^\alpha |v|^\gamma}{|x|^{2_a^* e_1}}\,dx\Big]
$$

and

$$
\int_\Omega \frac{|\nabla v|^2}{|x|^{2b}}\,dx + \mu_2 \int_\Omega \frac{|v|^\xi}{|x|^{\beta_1}}\,dx = -\gamma\Big[\frac{1}{\gamma}\int_\Omega \frac{H_v(x,u,v)v}{|x|^{\beta_2}}\,dx + \int_\Omega \frac{|u|^\alpha |v|^\gamma}{|x|^{2_a^* e_1}}\,dx\Big].
$$

So, we conclude that

$$
\int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}\,dx - \mu_1 \int_\Omega \frac{|u|^\tau}{|x|^{\beta_0}}\,dx = -\frac{\alpha}{\gamma}\Big(\int_\Omega \frac{|\nabla v|^2}{|x|^{2b}}\,dx + \mu_2 \int_\Omega \frac{|v|^\xi}{|x|^{\beta_1}}\,dx\Big),
$$

hence, $u = v = 0$ almost everywhere in Ω, which is a contradiction.

Before enunciating our results, we recall that Xuan [21], under the assumption (H1), proved that if $0 \le a < (N-2)/2$, and $\beta_2 < 2(a+1)$, then there exists the first eigenvalue $\lambda_{1_{\beta_2}} > 0$ of problem

$$
\begin{gathered}
-\operatorname{div}(|x|^{-2a}\nabla u) = \lambda |x|^{-\beta_2} u \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.3}
$$

which is associated to an eigenfunction $\varphi_{1_{\beta_2}} \in C^{1,\alpha_1}(\Omega \setminus \{0\})$ with $\varphi_{1_{\beta_2}} > 0$ in $\Omega \setminus \{0\}$ for some $\alpha_1 > 0$.

Theorem 1.1. *Assume* (H1), (H2)–(H7), *and* $\theta_2 \in (1,2)\cap(1,\xi]$. *Then system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 > 0$ *and* $\mu_2 \geq 0$, *provided that one of the following conditions is satisfied*

(i) $p_0 \in (2, \frac{2N}{N-2})$;

(ii) $p_0 = 2$ *and* $K_0 \in (0, \frac{\lambda_{1_{\beta_2}}}{2})$.

Moreover, if $p_0 \in (1,2)$ *there exists* $\bar{\mu}_0 > 0$ *such that system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 \in (0, \bar{\mu}_0)$ *and* $\mu_2 \geq 0$.

Theorem 1.2. *Assume* (H1), (H2)–(H7), $\xi < 2$, $\beta_1 < (b+1)\xi + N[1-(\xi/2)]$, *and* $\theta_2 \in [\xi, 2)$. *Then system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 > 0$ *and* $\mu_2 < 0$, *provided that one of the following conditions is satisfied*

(i) $p_0 \in (2, \frac{2N}{N-2})$;

(ii) $p_0 = 2$ *and* $K_0 \in (0, \frac{\lambda_{1_{\beta_2}}}{2})$.

Moreover, if $p_0 \in (1,2)$ *there exists* $\bar{\mu}_0 > 0$ *such that system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 \in (0, \bar{\mu}_0)$ *and* $\mu_2 < 0$.

Theorem 1.3. *In addition to* (H1), (H2)–(H7), $\theta_2 \in (1,2)\cap(1,\xi]$, *and* H *even in the variables* s, t, *suppose either* $H(x,s,0) \leq 0$ *for all* $s \in \mathbb{R}$, $x \in \Omega$ *or* $p_0 = \tau$. *Then system* (1.2) *possesses a sequence* $\{(u_n, v_n)\}$ *of nontrivial weak solutions with energies* $I(u_n, v_n) \to \infty$ *as* $n \to \infty$ *for each* $\mu_1 > 0$ *and* $\mu_2 \geq 0$. *Moreover, this result still held if* $\xi < 2$, $\beta_1 < (b+1)\xi + N[1-(\xi/2)]$, $\theta_2 \in [\xi, 2)$, $\mu_1 > 0$, *and* $\mu_2 < 0$. *See the definition of* I *in* (2.2).

Now we present some complementary results, for which we use following condition:

(H9) Assume that

$$H(x,s,t) \geq -(|x|^{\beta_2 - 2_a^* e_1}|s|^\alpha |t|^\gamma),$$

instead of the condition (H6). Notice that if $\mu_2 < 0$ then (H6) is more restrictive than (H9). To obtain similar results we will impose that $-\mu_2$ is small or that $\xi < 2$.

Theorem 1.4. *Assume* (H1), (H2)–(H5), (H7), (H9), $\xi < 2$, $\beta_1 < (b+1)\xi + N[1-(\xi/2)]$, *and* $\theta_2 \in [\xi, 2)$. *Then, there exists* $\tilde{\mu}_0 > 0$ *such that system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 > 0$ *and* $\mu_2 \in (-\tilde{\mu}_0, 0)$, *provided that one of the following conditions is satisfied*

(i) $p_0 \in (2, \frac{2N}{N-2})$;

(ii) $p_0 = 2$ *and* $K_0 \in (0, \frac{\lambda_{1_{\beta_2}}}{2})$.

Moreover, if $p_0 \in (1,2)$ *there exist* $\tilde{\mu}_0, \bar{\mu}_0 > 0$ *such that system* (1.2) *possesses a nontrivial weak solution for each* $\mu_1 \in (0, \bar{\mu}_0)$ *and* $\mu_2 \in (-\tilde{\mu}_0, 0)$.

Theorem 1.5. *In addition to* (H1), (H2)–(H5), (H7), (H9), $\xi < 2$, $\beta_1 < (b+1)\xi + N[1-(\xi/2)]$, $\theta_2 \in [\xi, 2)$, *and* H *even in the variables* s, t, *suppose either* $H(x,s,0) \leq 0$ *for all* $s \in \mathbb{R}$, $x \in \Omega$ *or* $p_0 = \tau$. *Then, system* (1.2) *possesses a sequence* $\{(u_n, v_n)\}$ *of nontrivial weak solution with energies* $I(u_n, v_n) \to \infty$ *as* $n \to \infty$ *for each* $\mu_1 > 0$ *and* $\mu_2 < 0$. *See the definition of* I *in* (2.2).

Remark 1.6. The Theorems 1.1–1.5 still hold for system (1.2) with subcritical growth; that is, when $\xi \in (1, 2N/(N-2))$, $\beta_1 < (b+1)\xi + N[1-(\xi/2)]$, and we consider β instead $2_a^* e_1 = 2_b^* e_2$, where $\beta < \min\{(a+1)p_3 + N[1-(p_3/2)], (b+1)p_4 + N[1-(p_4/2)]\}$ for some $p_3, p_4 \in (1, \frac{2N}{N-2})$ with

$$\frac{\alpha}{p_3} + \frac{\gamma}{p_4} = 1.$$

2. Preliminaries

Consider Ω a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$) with $0 \in \Omega$. If $\alpha \in \mathbb{R}$ and $l \in (0, +\infty)$, let $L^l(\Omega, |x|^\alpha)$ be the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\|u\|_{L^l(\Omega, |x|^\alpha)} := \Big(\int_\Omega |x|^\alpha |u|^l dx\Big)^{1/l} < \infty.$$

If $-\infty < a < (N-2)/2$, we define $W_0^{1,2}(\Omega, |x|^{-2a})$ as being the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_a$ defined by

$$\|u\|_a = \|u\|_{W_0^{1,2}(\Omega, |x|^{-2a})} := \Big(\int_\Omega |x|^{-2a} |\nabla u|^2 dx\Big)^{1/2},$$

which is induced by inner product

$$\langle u, w\rangle_{W_0^{1,2}(\Omega,|x|^{-2a})} := \int_\Omega |x|^{-2a}\nabla u\nabla w\,dx.$$

First of all, by using inequality (1.1) and the boundedness of Ω, in [22] was proved that there exists $C > 0$ such that

$$\Big(\int_\Omega |x|^{-\delta}|u|^r dx\Big)^{2/r} \le C\Big(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\Big), \quad \forall u \in W_0^{1,2}(\Omega, |x|^{-2a}), \tag{2.1}$$

where $1 \le r \le 2N/(N-2)$ and $\delta \le (a+1)r + N[1-(r/2)]$, which is the Caffarelli, Kohn, Nirenberg's inequality. In other words, the embedding $W_0^{1,2}(\Omega,|x|^{-2a}) \hookrightarrow L^r(\Omega,|x|^{-\delta})$ is continuous if $1 \le r \le 2N/(N-2)$ and $\delta \le (a+1)r + N[1-(r/2)]$. Moreover, this embedding is compact if $1 \le r < 2N/(N-2)$ and $\delta < (a+1)r + N[1-(r/2)]$, see [22, Theorem 2.1].

Due to Theorem 4.3, see Appendix, we can consider

$$\{\frac{\varphi_{a,n}}{\sqrt{\lambda_{a,n}}}\} \subset C^1(\overline{\Omega}\setminus\{0\}) \cap C^0(\overline{\Omega})$$

and

$$\{\frac{\varphi_{b,n}}{\sqrt{\lambda_{b,n}}}\} \subset C^1(\overline{\Omega}\setminus\{0\}) \cap C^0(\overline{\Omega})$$

the Hilbertian bases of spaces $W_0^{1,2}(\Omega,|x|^{-2a})$ and $W_0^{1,2}(\Omega,|x|^{-2b})$, respectively. We define

$$E := W_0^{1,2}(\Omega,|x|^{-2a}) \times W_0^{1,2}(\Omega,|x|^{-2b}),$$

endowed with the norm $\|(u,v)\| := \|u\|_a + \|v\|_b$. We will denote $\varphi_n^a = (\varphi_{a,n}, 0)$, and $\varphi_n^b = (0, \varphi_{b,n})$. Evidently, $\{\varphi_n^a\}$ (resp. $\{\varphi_n^b\}$) is a basis for space $E^+ := W_0^{1,2}(\Omega,|x|^{-2a}) \times \{0\}$ (resp. $E^- := \{0\} \times W_0^{1,2}(\Omega,|x|^{-2b})$) and $E = E^- \oplus E^+$. We define the spaces

$$X^m := \operatorname{span}\{\varphi_1^a, \ldots, \varphi_m^a\} \oplus E^-, \quad X_n := E^+ \oplus \operatorname{span}\{\varphi_1^b, \ldots, \varphi_n^b\},$$

and we denote by $(X^m)^\perp$ (resp. $(X_n)^\perp$) the complement of X^m (resp. X_n) in E.

Our approach is variational, so we will study the critical points of the Euler-Lagrange functional $I : E \to \mathbb{R}$ given by

$$\begin{aligned} I(u,v) = \frac{1}{2}(\|u\|_a^2 - \|v\|_b^2) - \frac{\mu_1}{\tau}\int_\Omega |x|^{-\beta_0}|u|^\tau\,dx - \frac{\mu_2}{\xi}\int_\Omega |x|^{-\beta_1}|v|^\xi\,dx \\ - \int_\Omega |x|^{-\beta_2}H(x,u,v)\,dx - \int_\Omega |x|^{-2_a^* e_1}|u|^\alpha|v|^\gamma\,dx, \end{aligned} \tag{2.2}$$

which belongs to the class C^1.

Now, we will proof that I' is weakly sequentially continuous.

Theorem 2.1. *Let $\{(u_j,v_j)\} \subset E$ be a sequence and $(u,v) \in E$ such that $(u_j,v_j) \rightharpoonup (u,v)$ weakly in E as $j \to \infty$. Assume* (H1), (H2)–(H5). *Then, $I'(u_j,v_j) \rightharpoonup I'(u,v)$ weakly in E^* as $j \to \infty$.*

Proof. By definition of weak convergence in E, we have for $(w,z) \in E$ that

$$\lim_{j\to\infty}\int_\Omega |x|^{-2a}\nabla u_j\nabla w\,dx = \int_\Omega |x|^{-2a}\nabla u\nabla w\,dx, \tag{2.3}$$

$$\lim_{j\to\infty}\int_\Omega |x|^{-2b}\nabla v_j\nabla z\,dx = \int_\Omega |x|^{-2b}\nabla v\nabla z\,dx. \tag{2.4}$$

By compact embedding, we have

$$u_j \to u \text{ strongly in } L^\tau(\Omega,|x|^{-\beta_0}) \text{ and } L^{p_1}(\Omega,|x|^{-\beta_2}) \text{ as } j \to \infty,$$

$$v_j \to v \text{ strongly in } L^{q_1}(\Omega,|x|^{-\beta_2}) \text{ as } j \to \infty.$$

In particular, there exist functions $h \in L^\tau(\Omega,|x|^{-\beta_0})$, $f \in L^{p_1}(\Omega,|x|^{-\beta_2})$, and $g \in L^{q_1}(\Omega,|x|^{-\beta_2})$ such that $|u_j|(x) \le \min\{f(x), h(x)\}$ and $|v_j|(x) \le g(x)$ almost everywhere $x \in \Omega$. Passing to a subsequence, if necessary, we obtain $u_j(x) \to u(x)$ and $v_j(x) \to v(x)$, as $j \to \infty$, for almost everywhere $x \in \Omega$. Therefore, we obtain

$$[H_u(x,u_j,v_j)w](x) \to [H_u(x,u,v)w](x) \quad \text{as } j \to \infty \text{ almost everywhere } x \in \Omega,$$

$$(|u_j|^{\tau-2}u_jw)(x) \to (|u|^{\tau-2}uw)(x) \quad \text{as } j \to \infty \text{ almost everywhere } x \in \Omega,$$

$$\begin{aligned} |H_u(x,u_j,v_j)w| &\le K_1(|u_j|^{p_1} + |v_j|^{q_1} + |w|^{r_1}) \\ &\le K_1(f^{p_1} + g^{q_1} + |w|^{r_1}) \in L^1(\Omega,|x|^{-\beta_2}), \end{aligned}$$

$$\begin{aligned}||u_j|^{\tau-2}u_jw| &\le h^{\tau-1}|w| \\ &\le \frac{\tau-1}{\tau}h^{\tau}+\frac{1}{\tau}|w|^{\tau}\in L^1(\Omega,|x|^{-\beta_0}).\end{aligned}$$

Consequently, the Lebesgue Theorem implies that

$$\lim_{j\to\infty}\int_\Omega |x|^{-\beta_2}H_u(x,u_j,v_j)w\,dx=\int_\Omega |x|^{-\beta_2}H_u(x,u,v)w\,dx,$$

$$\lim_{j\to\infty}\int_\Omega |x|^{-\beta_0}|u_j|^{\tau-2}u_jw\,dx=\int_\Omega |x|^{-\beta_0}|u|^{\tau-2}uw\,dx.$$

Analogously, we obtain

$$\lim_{j\to\infty}\int_\Omega |x|^{-\beta_2}H_v(x,u_j,v_j)z\,dx=\int_\Omega |x|^{-\beta_2}H_v(x,u,v)z\,dx. \tag{2.5}$$

Due to weak convergence, $\{(u_j,v_j)\}$ is bounded in E. Also, since that $(\alpha/2^*_a)+(\gamma/2^*_b)=1$, we obtain

$$\frac{\alpha-1}{2^*_a-1}+\frac{2^*_a\,\gamma}{2^*_b(2^*_a-1)}=\frac{\gamma-1}{2^*_b-1}+\frac{2^*_b\,\alpha}{2^*_a(2^*_b-1)}=1,\quad \frac{2^*_a-1}{\alpha-1},\frac{2^*_b-1}{\gamma-1}>1\,.$$

Then, by Hölder's inequality,

$$\begin{aligned}&\Big|\int_\Omega |x|^{-2^*_ae_1}(|u_j|^{\alpha-2}|v_j|^{\gamma}u_j)^{\frac{2^*_a}{2^*_a-1}}\,dx\Big| \\ &\le \Big(\|u_j\|_{L^{2^*_a}(\Omega,|x|^{-2^*_ae_1})}\Big)^{\frac{2^*_a(\alpha-1)}{2^*_a-1}}\Big(\|v_j\|_{L^{2^*_b}(\Omega,|x|^{-2^*_be_2})}\Big)^{\frac{2^*_a\,\gamma}{2^*_a-1}}.\end{aligned}$$

Then $\{|u_j|^{\alpha-2}|v_j|^{\gamma}u_j\}$ is a bounded sequence in $L^{\frac{2^*_a}{2^*_a-1}}(\Omega,|x|^{-2^*_ae_1})$. Also, the sequence $\{|v_j|^{\xi-2}v_j\}$ is bounded in $L^{\frac{\xi}{\xi-1}}(\Omega,|x|^{-\beta_1})$. Moreover,

$$(|u_j|^{\alpha-2}|v_j|^{\gamma}u_j)(x)\to(|u|^{\alpha-2}|v|^{\gamma}u)(x)\quad\text{and}\quad(|v_j|^{\xi-2}v_j)(x)\to(|v|^{\xi-2}v)(x),$$

as $j\to\infty$, for almost everywhere $x\in\Omega$. Then, by [13, Lemma 4.8], we obtain

$$|u_j|^{\alpha-2}|v_j|^{\gamma}u_j\rightharpoonup|u|^{\alpha-2}|v|^{\gamma}u\quad\text{weakly in }L^{\frac{2^*_a}{2^*_a-1}}(\Omega,|x|^{-2^*_ae_1})\text{ as }j\to\infty,$$

$$|v_j|^{\xi-2}v_j\rightharpoonup|v|^{\xi-2}v\quad\text{weakly in }L^{\frac{\xi}{\xi-1}}(\Omega,|x|^{-\beta_1})\text{ as }j\to\infty.$$

In particular, we have

$$\lim_{n\to\infty}\int_\Omega |x|^{-2^*_ae_1}|u_j|^{\alpha-2}|v_j|^{\gamma}u_jw\,dx=\int_\Omega |x|^{-2^*_ae_1}|u|^{\alpha-2}|v|^{\gamma}uw\,dx, \tag{2.6}$$

$$\lim_{j\to\infty}\int_\Omega |x|^{-\beta_1}|v_j|^{\xi-2}v_jz\,dx=\int_\Omega |x|^{-\beta_1}|v|^{\xi-2}vz\,dx. \tag{2.7}$$

Similarly, we obtain

$$\lim_{j\to\infty}\int_\Omega |x|^{-2^*_ae_1}|u_j|^{\alpha}|v_j|^{\gamma-2}v_jz\,dx=\int_\Omega |x|^{-2^*_ae_1}|u|^{\alpha}|v|^{\gamma-2}vz\,dx. \tag{2.8}$$

By combining the limits (2.3)-(2.8), we conclude that

$$\lim_{j\to\infty}\langle I'(u_j,v_j),(w,z)\rangle=\langle I'(u,v),(w,z)\rangle,\ \ \forall\,(w,z)\in E.$$

□

Definition 2.2. We say that $\{(u_j,v_j)\}\subset E$ is a $(PS)^*_c$-sequence with relation to the functional I if $(u_j,v_j)\in X_{n_j}$, $n_j\to\infty$ as $j\to\infty$, $I(u_j,v_j)\to c$, and $\|I'|_{X_{n_j}}(u_j,v_j)\|_{(X_{n_j})^*}\le\epsilon_{n_j}$, $\epsilon_{n_j}\to 0$ as $j\to\infty$. Moreover, if all $(PS)^*_c$-sequence be precompact, we say that functional I satisfies the $(PS)^*_c$-condition.

Lemma 2.3. *Assume* (H1), (H2)–(H5). *Then, all $(PS)^*_c$-sequence is bounded in E, if one of the following conditions occurs:*

(i) *$\mu_1>0$, $\mu_2\ge 0$, and (H7) are satisfied with $\theta_2\in(1,2)\cap(1,\xi]$;*
(ii) *$\xi<2$, $\beta_1<(b+1)\xi[1-(\xi/2)]$, $\mu_1>0$, $\mu_2<0$, and* (H7) *are satisfied with $\theta_2\in[\xi,2)$.*

Proof. Let $\{(u_j, v_j)\}$ be a $(PS)_c^*$-sequence with relation to the functional I. We consider $\theta_1 \in (2, \tau]$ and $\theta_2 \in (1, 2) \cap (1, \xi]$ if (i) is satisfied and, for (ii), we consider $\theta_1 \in (2, \tau]$ and $\theta_2 \in [\xi, 2)$. In both cases, we obtain

$$\begin{aligned} c + o(1)\|(u_j, v_j)\| + o(1) &\geq I(u_j, v_j) - \langle I'|_{X_{n_j}}(u_j, v_j), (\frac{1}{\theta_1}u_j, \frac{1}{\theta_2}v_j)\rangle \\ &\geq (\frac{1}{2} - \frac{1}{\theta_1})\|u_j\|_a^2 + (\frac{1}{\theta_2} - \frac{1}{2})\|v_j\|_b^2, \end{aligned}$$

so, $\{(u_j, v_j)\}$ is bounded in E. □

Theorem 2.4. *Assume* (H1), (H2)–(H5). *Let* $\{(u_j, v_j)\} \subset E$ *be a* $(PS)_c^*$*-sequence with relation to the functional* I *such that* $(u_j, v_j) \rightharpoonup (u, v)$ *weakly in* E *as* $j \to \infty$. *Then,* (u, v) *is a weak solution of system* (1.2) *and* $(u_j, v_j) \to (u, v)$ *strongly in* E *as* $j \to \infty$, *provided that one of the following conditions is satisfied*

(i) $\mu_1 > 0$ *and* $\mu_2 \geq 0$*;*
(ii) $\xi < 2$, $\beta_1 < (b+1)\xi + [1 - (\xi/2)]$, $\mu_1 > 0$, *and* $\mu_2 < 0$.

Proof. Due to weak convergence, $\{(u_j, v_j)\}$ is bounded in E.
Step I. We will prove that $(u_j, v_j) \to (u, v)$ strongly in E as $j \to \infty$. For each $z \in W_0^{1,2}(\Omega, |x|^{-2b})$, we can write $z = \sum_{k=1}^{\infty} a_k \varphi_k^b$. Thus, we have the projection $P_{n_j}^0 : W_0^{1,2}(\Omega, |x|^{-2b}) \to \text{span}\{\varphi_1^b, \ldots, \varphi_{n_j}^b\}$ given by $P_{n_j}^0(z) = \sum_{k=1}^{n_j} a_k \varphi_k^b$. Moreover, it is easy to see that $P_{n_j}^0(z) \to z$ strongly in $W_0^{1,2}(\Omega, |x|^{-2b})$ as $j \to \infty$.

By definition of $(PS)_c^*$-sequence, we obtain

$$\begin{aligned} &\int_\Omega |x|^{-2b} \nabla v_j \nabla (v - v_j)\, dx \\ &= \langle I'|_{X_{n_j}}(u_j, v_j), (0, v_j - P_{n_j}^0(v))\rangle - \langle I'(u_j, v_j), (0, v - P_{n_j}^0(v))\rangle \\ &\quad + \mu_2 \int_\Omega |x|^{-\beta_1} |v_j|^{\xi-2} v_j (v_j - v)\, dx + \int_\Omega |x|^{-\beta_2} H_v(x, u_j, v_j)(v_j - v)\, dx \\ &\quad + \gamma \int_\Omega |x|^{-2_a^* e_1} |u_j|^\alpha |v_j|^{\gamma-2} v_j (v_j - v)\, dx. \end{aligned} \tag{2.9}$$

Since that $(0, v_j - P_{n_j}^0(v)) \in X_{n_j}$ and $\{(0, v_j - P_{n_j}^0(v))\}$ is bounded in E, we have

$$\langle I'|_{X_{n_j}}(u_j, v_j), (0, v_j - P_{n_j}^0(v))\rangle \to 0 \quad \text{as } j \to \infty. \tag{2.10}$$

From $P_{n_j}^0(v) \to v$ strongly in $W_0^{1,2}(\Omega, |x|^{-2b})$ as $j \to \infty$ and boundedness of $\{(u_j, v_j)\}$ in E follow that

$$\langle I'(u_j, v_j), (0, v - P_{n_j}^0(v))\rangle \to 0 \quad \text{as } j \to \infty. \tag{2.11}$$

Similarly to proof of Theorem 2.1, we obtain

$$\lim_{j\to\infty} \int_\Omega |x|^{-\beta_1} |v_j|^{\xi-2} v_j v\, dx = \int_\Omega |x|^{-\beta_1} |v|^\xi\, dx, \tag{2.12}$$

$$\lim_{j\to\infty} \int_\Omega |x|^{-\beta_2} H_v(x, u_j, v_j)(v_j - v)\, dx = 0, \tag{2.13}$$

$$\lim_{j\to\infty} \int_\Omega |x|^{-2_a^* e_1} |u_j|^\alpha |v_j|^{\gamma-2} v_j v\, dx = \int_\Omega |x|^{-2_a^* e_1} |u|^\alpha |v|^\gamma\, dx. \tag{2.14}$$

By compact embedding, $u_j(x) \to u(x)$ and $v_j(x) \to v(x)$, as $j \to \infty$, for almost everywhere $x \in \Omega$. Then $|x|^{-2_a^* e_1} |u_j|^\alpha(x) |v_j|^\gamma(x) \to |x|^{-2_a^* e_1} |u|^\alpha(x) |v|^\gamma(x)$, as $j \to \infty$, for almost everywhere $x \in \Omega$. Hence, we obtain by Fatou's Lemma that

$$\int_\Omega |x|^{-2_a^* e_1} |u|^\alpha |v|^\gamma\, dx \leq \liminf_{j\to\infty} \int_\Omega |x|^{-2_a^* e_1} |u_j|^\alpha |v_j|^\gamma\, dx. \tag{2.15}$$

Hence, taking the lower limit in equation (2.9) and by using (2.10)-(2.15), we obtain

$$\begin{aligned} \|v\|_b^2 - \limsup_{j\to\infty} \|v_j\|_b^2 &= \liminf_{j\to\infty} \int_\Omega |x|^{-2a} \nabla v_j \nabla (v - v_j)\, dx \\ &\geq \liminf_{j\to\infty} \Big(\mu_2 \int_\Omega |x|^{-\beta_1} |v_j|^{\xi-2} v_j (v_j - v)\, dx \Big) \\ &\geq \liminf_{j\to\infty} \Big(\mu_2 \int_\Omega |x|^{-\beta_1} |v_j|^\xi\, dx \Big) - \mu_2 \int_\Omega |x|^{-\beta_1} |v|^\xi\, dx. \end{aligned} \tag{2.16}$$

Consider $\mu_2 \geq 0$. Then, since $|x|^{-\beta_1}|v_j|^{\xi}(x) \to |x|^{-\beta_1}|v|^{\xi}(x)$ as $j \to \infty$ for almost everywhere $x \in \Omega$, we obtain by Fatou's Lemma that

$$\int_\Omega |x|^{-\beta_1}|v|^{\xi}\, dx \leq \liminf_{j\to\infty} \int_\Omega |x|^{-\beta_1}|v_j|^{\xi}\, dx;$$

therefore, from (2.16) we obtain

$$\|v\|_b^2 - \limsup_{j\to\infty} \|v_j\|_b^2 \geq 0.$$

But, if $\mu_2 < 0$, $\xi < 2$, and $\beta_1 < (b+1)\xi + [1-(\xi/2)]$, then the embedding $W_0^{1,2}(\Omega, |x|^{-2b}) \hookrightarrow L^{\xi}(\Omega, |x|^{-\beta_1})$ is compact. Therefore,

$$\int_\Omega |x|^{-\beta_1}|v|^{\xi}\, dx = \lim_{j\to\infty} \int_\Omega |x|^{-\beta_1}|v_j|^{\xi}\, dx,$$

and from (2.16) it follows that

$$\|v\|_b^2 - \limsup_{j\to\infty} \|v_j\|_b^2 \geq 0.$$

Then, in both cases, we have

$$\limsup_{j\to\infty} \|v_j\|_b^2 \leq \|v\|_b^2 \leq \liminf_{j\to\infty} \|v_j\|_b^2,$$

so, $v_j \to v$ strongly in $W_0^{1,2}(\Omega, |x|^{-2b})$ as $j \to \infty$.

Define $\tilde{u}_j := u_j - u$ and $\tilde{v}_j := v_j - v$. From definition of $(PS)_c^*$-sequence and by Brezis-Lieb's Lemma follow

$$\begin{aligned} &\|\tilde{u}_j\|_a^2 - \alpha \int_\Omega |x|^{-2_a^* e_1}|\tilde{u}_j|^{\alpha}|\tilde{v}_j|^{\gamma} dx \\ &= \langle I'|_{X_{n_j}}(u_j, v_j), (u_j, 0)\rangle - \langle I'(u,v), (u,0)\rangle + o(1), \end{aligned} \tag{2.17}$$

where $o(1) \to 0$ as $j \to \infty$.

As $\{(u_j, 0)\}$ is bounded in E, $(u_j, 0), (w, 0) \in X_{n_j} := E^+ \oplus \text{span}\{\varphi_1^b, \ldots, \varphi_{n_j}^b\}$ where $E^+ := W_0^{1,2}(\Omega, |x|^{-2a}) \times \{0\}$, we have by definition of $(PS)_c^*$-sequence that $\langle I'|_{X_{n_j}}(u_j, v_j), (u_j, 0)\rangle \to 0$ and $\langle I'|_{X_{n_j}}(u_j, v_j), (w, 0)\rangle \to 0$ as $j \to \infty$ for all $w \in W_0^{1,2}(\Omega, |x|^{-2a})$. On the other hand, by Theorem 2.1, $\langle I'|_{X_{n_j}}(u_n, v_n), (w, 0)\rangle \to \langle I'(u,v), (w,0)\rangle$ as $j \to \infty$ for all $w \in W_0^{1,2}(\Omega, |x|^{-2a})$. Then,

$$\langle I'(u,v), (w,0)\rangle = 0, \quad \forall w \in W_0^{1,2}(\Omega, |x|^{-2a}). \tag{2.18}$$

Thus, we obtain by Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, boundedness of $\{\tilde{u}_n\}$ in $W_0^{1,2}(\Omega, |x|^{-2a})$, and (2.17) that

$$\|\tilde{u}_j\|_a^2 = \alpha \int_\Omega |x|^{-2_a^* e_1}|\tilde{u}_j|^{\alpha}|\tilde{v}_j|^{\gamma} dx + o(1) \leq M\|v\|_b^{\gamma} + o(1), \tag{2.19}$$

so, as $\tilde{v}_j \to 0$ strongly in E as $j \to \infty$, it follows that $\tilde{u}_j \to 0$ strongly in $W_0^{1,2}(\Omega, |x|^{-2a})$ as $j \to \infty$. Hence, we conclude that $(u_j, v_j) \to (u, v)$ strongly in E as $n \to \infty$.

Step II. We will prove that (u, v) is a weak solution of system (1.2). Consider $z \in W_0^{1,2}(\Omega, |x|^{-2b})$. Then, we have

$$\langle I'(u_j, v_j), (0, z)\rangle = \langle I'|_{X_{n_j}}(u_j, v_j), (0, P_{n_j}^0(z))\rangle + \langle I'(u_j, v_j), (0, z - P_{n_j}^0(z))\rangle. \tag{2.20}$$

However, as $(0, P_{n_j}^0(z)) \in X_{n_j}$ and $\{(0, P_{n_j}^0(z))\}$ is bounded in E, we have

$$\langle I'|_{X_{n_j}}(u_j, v_j), (0, P_{n_j}^0(z))\rangle \to 0 \text{ as } j \to \infty. \tag{2.21}$$

Also, follows similar to (2.11) that

$$\langle I'(u_j, v_j), (0, z - P_{n_j}^0(z))\rangle \to 0 \quad \text{as } j \to \infty. \tag{2.22}$$

Hence, by (2.20), (2.21), and (2.22), we obtain

$$\langle I'(u_j, v_j), (0, z)\rangle \to 0 \quad \text{as } j \to \infty.$$

But, by Theorem 2.1, we have $\langle I'(u_j, v_j), (0, z)\rangle \to \langle I'(u,v), (0,z)\rangle$ as $j \to \infty$ for all $z \in W_0^{1,2}(\Omega, |x|^{-2b})$. Then,

$$\langle I'(u,v), (0,z)\rangle = 0, \quad \forall z \in W_0^{1,2}(\Omega, |x|^{-2b}). \tag{2.23}$$

Hence, by using (2.18) and (2.23), we conclude that (u, v) is a weak solution of system (1.2). □

3. Proof of main results

Lemma 3.1. *Assume* (H1), (H2)–(H5), (H7), $\mu_1 > 0$, *and* $\mu_2 \in \mathbb{R}$. *Then, there exist* $r, \sigma > 0$ *such that*

$$\inf I(\partial B_r(E^+)) \geq \sigma, \tag{3.1}$$

provided that one of the following conditions is satisfied

(i) $p_0 \in (2, \frac{2N}{N-2})$;

(ii) $p_0 = 2$ *and* $K_0 \in (0, \frac{\lambda_{1_{\beta_2}}}{2})$.

Moreover, if $p_0 \in (1,2)$, *there exist* $\bar{\mu}_0, r, \sigma > 0$ *such that* (3.1) *is held for each* $\mu_1 \in (0, \bar{\mu}_0)$ *and* $\mu_2 \in \mathbb{R}$.

Proof. If (i) is satisfied, we obtain

$$\begin{aligned} I(u,0) &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau - K_0 \int_\Omega |x|^{-\beta_2} |u|^{p_0}\, dx \\ &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau - K_0 C^{\frac{p_0}{2}} \|u\|_a^{p_0}, \end{aligned}$$

so, as $\mu_1 > 0$ and $\tau, p_0 > 2$, there exist $r, \sigma \in (0,1)$ such that $I(u,0) \geq \sigma$ for all $(u,0) \in E^+$ with $\|(u,0)\| = r$.

Assuming (ii), we obtain

$$\begin{aligned} I(u,0) &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau - K_0 \int_\Omega |x|^{-\beta_2} |u|^2\, dx \\ &= (\frac{1}{2} - \frac{K_0}{\lambda_{1_{\beta_2}}}) \|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau, \end{aligned}$$

so, as $\mu_1 > 0$, $K_0 \in (0, \frac{\lambda_{1_{\beta_2}}}{2})$, and $\tau > 2$, there exist $r, \sigma \in (0,1)$ such that $I(u,0) \geq \sigma$ for all $(u,0) \in E^+$ with $\|(u,0)\| = r$.

Now, for $p_0 \in (1,2)$, we have

$$\begin{aligned} I(u,0) &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau - K_0 C^{\frac{p_0}{2}} \|u\|_a^{p_0} \\ &= (\frac{1}{4}\|u\|_a^2 - K_0 C^{\frac{p_0}{2}} \|u\|_a^{p_0}) + (\frac{1}{4}\|u\|_a^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} \|u\|_a^\tau). \end{aligned}$$

Since $p_0 \in (1,2)$, there exist $r, \sigma > 0$ such that

$$(\tfrac{1}{4} r^2 - K_0 C^{\frac{p_0}{2}} r^{p_0}) \geq \sigma.$$

We choose $\bar{\mu}_0 > 0$ such that

$$(\frac{1}{4} r^2 - \frac{\mu_1}{\tau} C^{\frac{\tau}{2}} r^\tau) \geq 0$$

for all $\mu_1 \in (0, \bar{\mu}_0)$.

Hence, we conclude that $I(u,0) \geq \sigma$ for all $(u,0) \in E^+$ with $\|(u,0)\| = r$, provided that $\mu_1 \in (0, \bar{\mu}_0)$ and $\mu_2 \in \mathbb{R}$. □

Consider $(e,0) \in E^+$ with $\|(e,0)\| = r$. We define the sets

$$\begin{aligned} M = M(\rho) &:= \{(se, v) : v \in W_0^{1,2}(\Omega, |x|^{-2b}), \|(se,v)\| \leq \rho\}, \\ M_0 = M_0(\rho) &:= \{(se, v) : v \in W_0^{1,2}(\Omega, |x|^{-2b}), \|(se,v)\| = \rho \\ &\qquad \text{and } s > 0 \text{ or } \|v\|_b \leq \rho \text{ and } s = 0\}. \end{aligned}$$

Lemma 3.2. *Assume* (H1) and (H2)-(H7). *Then, there exists* $\rho > r > 0$ *such that* $I(u,v) \leq 0$ *for all* $(se,v) \in M_0$, *for each* $\mu_1 > 0$ *and* $\mu_2 \in \mathbb{R}$.

Proof. If $(se,v) \in M_0$, then, by using (H6), we obtain

$$I(se,v) \leq \frac{r^2}{2} s^2 - s^\tau \frac{\mu_1}{\tau} \int_\Omega |x|^{-\beta_0} |e|^\tau\, dx - \frac{1}{2}\|v\|_b^2. \tag{3.2}$$

Fix $\rho_0 > r > 0$ such that

$$\frac{r^2}{2} s^2 - s^\tau \frac{\mu_1}{\tau} \int_\Omega |x|^{-\beta_0} |e|^\tau\, dx \leq 0, \forall s \geq \rho_0, \tag{3.3}$$

and, define

$$0 \,<\, b_0 := \max_{s \geq 0} (\frac{r^2}{2} s^2 - s^\tau \frac{\mu_1}{\tau} \int_\Omega |x|^{-\beta_0} |e|^\tau\, dx) < \infty.$$

Then, we choose $\rho > \max\{\rho_0, r\rho_0\} > r$ such that

$$\frac{1}{2}\|v\|_b^2 \geq b_0, \quad \text{for all } v \text{ with } \|v\|_b \geq \rho - \rho_0 r. \tag{3.4}$$

Thus, if $s = 0$ and $\|v\|_b \leq \rho$, follows by (3.2) that $I(0, v) \leq 0$.

If $s > 0$ and $\|(se, v)\| = \rho$, we have $\|v\|_b = \rho - s\|e\|_a = \rho - sr$. Then, for $s \geq \rho_0$, we obtain by (3.2) and (3.3) that $I(0, v) \leq 0$. However, if $s < \rho_0$, we have $\|v\|_b = \rho - sr \geq \rho - \rho_0 r$, so, by (3.2) and (3.4), we obtain $I(se, v) \leq 0$. Note that $\frac{1}{2}\|v\|_b^2 \leq b_0$ and $s > 0$ imply $s \geq (\rho - \sqrt{2b_0})/r > \rho_0$. □

Proof of Theorems 1.1 and 1.2. We have

$$X_n = E^+ \oplus \operatorname{span}\{\varphi_1^b, \cdots, \varphi_n^b\}.$$

We define

$$M_n := M \cap X_n, \quad M_{0,n} := M_0 \cap X_n, \quad N_n := \partial B_r(E^+),$$
$$c_n := \inf_{h \in \Gamma_n} \max I(h(M_n)),$$

where

$$\Gamma_n := \{h \in C(M_n, X_n) : h|_{M_{0,n}} \equiv id_{M_{0,n}}\}.$$

Similar to the proof of [20, Theorem 2.12], we obtain

$$h(M_n) \cap \partial B_r(E^+) \neq \emptyset, \quad \forall h \in \Gamma_n.$$

Then, by using Lemmata 3.1 and 3.2, we obtain

$$\sup I(M_{0,n}) \leq 0 < \sigma \leq \inf I(\partial B_r E^+) \leq c_n \leq k_0 := \sup I(M_n) < \infty.$$

In particular, we obtain a subsequence $\{c_{n_j}\}$ of $\{c_n\}$ and $c \in [\sigma, k_0]$ such that $c_{n_j} \to c$ as $j \to \infty$.

Then, by applying [20, Theorem 2.8], we obtain $(u_n, v_n) \in X_n$ with $|I(u_n, v_n) - c_n| \leq 1/n$ and $\|I'|_{X_n}(u_n, v_n)\|_{(X_n)^*} \leq 1/n$ for each $n \in \mathbb{N}$. Thus, $\{(u_{n_j}, v_{n_j})\}$ is a $(PS)_c^*$-sequence with relation to the functional I. Due to Lemma 2.3, $\{(u_{n_j}, v_{n_j})\}$ is bounded in E. Therefore, there exists $(u, v) \in E$ such that $(u_{n_j}, v_{n_j}) \rightharpoonup (u, v)$ weakly in E as $j \to \infty$. Hence, by Theorem 2.4, we conclude that (u, v) is a weak solution of system (1.2) and $(u_{n_j}, v_{n_j}) \to (u, v)$ strongly in E as $j \to \infty$. In particular, $I(u, v) = c > 0$, then (u, v) is nontrivial. □

Lemma 3.3. *Assume* (H1), (H2)–(H6), $\mu_1 > 0$, *and* $\mu_2 \in \mathbb{R}$. *Then, there exists* $R_m > 0$ *such that* $I(u, v) \leq 0$ *for all* $(u, v) \in X^m$ *with* $\|(u, v)\| \geq R_m$.

Proof. We recall that $X^m \approx \operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\} \times W_0^{1,2}(\Omega, |x|^{-2b})$. Thus, as $\operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\}$ has finite dimension, all norms in this space are equivalent. From Caffarelli, Kohn, and Nirenberg's inequality $\|w\|_{L^\tau(\Omega, |x|^{-\beta_0})} \leq C^{1/2}\|w\|_a$ for all $w \in W_0^{1,2}(\Omega, |x|^{-2a})$ and $\|z\|_{L^\xi(\Omega, |x|^{-\beta_1})} \leq C^{1/2}\|z\|_b$ for all $z \in W_0^{1,2}(\Omega, |x|^{-2b})$. In particular, $\|\cdot\|_{L^\tau(\Omega, |x|^{-\beta_0})}$ define a norm on the space $\operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\}$. Then, there exists $K_m > 0$ such that

$$\|w\|_{L^\tau(\Omega, |x|^{-\beta_0})} \geq K_m\|w\|_a, \quad \forall w \in \operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\}.$$

Hence, we obtain

$$I(u, v) \leq (\frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau}K_m^\tau\|u\|_a^\tau) - \frac{1}{2}\|v\|_b^2 \leq 0,$$

for all $(u, v) \in X^m$, $\|(u, v)\| \geq R_m$, for some $R_m > 0$ large enough, because $\tau > 2$. □

Lemma 3.4. *In addition to* (H1), (H2)–(H5), $\mu_1 > 0$, *and* $\mu_2 \in \mathbb{R}$, *suppose either* $H(x, s, 0) \leq 0$ *for all* $s \in \mathbb{R}$, $x \in \Omega$ *or* $p_0 = \tau$. *Then, there exist* $r_m, a_m > 0$ *such that* $a_m \to \infty$ *as* $m \to \infty$ *and* $I(u, v) \geq a_m$ *for all* $(u, v) \in (X^{m-1})^\perp$ *with* $\|(u, v)\| = r_m$.

Proof. We have $(X^{m-1})^\perp \approx \operatorname{span}\{\varphi_{a,j} : j \geq m\} \times \{0\} \approx \operatorname{span}\{\varphi_{a,j} : j \geq m\}$. Thus, we can consider $(X^{m-1})^\perp \subset W_0^{1,2}(\Omega, |x|^{-2a})$. Let

$$\sigma_m := \sup_{u \in (X^{m-1})^\perp,\, \|u\|_a = 1} \|u\|_{L^\tau(\Omega, |x|^{-\beta_0})},$$
$$\rho_m := \sup_{u \in (X^{m-1})^\perp,\, \|u\|_a = 1} \|u\|_{L^{p_0}(\Omega, |x|^{-\beta_2})}.$$

Since that $(X^m)^\perp \subset (X^{m-1})^\perp$, it follows that $\sigma_m \geq \sigma_{m+1}$ for all $m \in \mathbb{N}$. Thus, $\sigma_m \searrow \sigma \geq 0$ as $m \to \infty$. We will prove that $\sigma = 0$. By definition of σ_m, for each $m \in \mathbb{N}$, there exists $u_m \in (X^{m-1})^\perp$ with $\|u_m\|_a = 1$ and

$$\|u_m\|_{L^\tau(\Omega, |x|^{-\beta_0})} \geq \frac{\sigma_m}{2}.$$

Moreover, as $(X^{m-1})^\perp \approx \text{span}\{\varphi_{a,j} : j \geq m\}$, we obtain $u_m \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega, |x|^{-2a})$ as $m \to \infty$. We have, from compact embedding $W_0^{1,2}(\Omega, |x|^{-2a}) \hookrightarrow L^\tau(\Omega, |x|^{-\beta_0})$, that $u_m \to 0$ strongly in $L^\tau(\Omega, |x|^{-\beta_0})$ as $m \to \infty$. Then, $\sigma_m \searrow \sigma = 0$ as $m \to \infty$. Similarly, we have that $\rho_m \searrow 0$ as $m \to \infty$.

If $H(x, s, 0) \leq 0$ for all $s \in \mathbb{R}$, $x \in \Omega$, then, for each $(u, 0) \in (X^{m-1})^\perp$, we obtain

$$I(u, 0) \geq \frac{1}{2}\|u\|_a^2 - \mu_1(\sigma_m)^\tau \|u\|_a^\tau,$$

therefore, taking $r_m := [\mu_1(\sigma_m)^\tau]^{\frac{1}{2-\tau}}$ and $a_m := (\frac{1}{2} - \frac{1}{\tau})r_m^2$, we conclude that

$$I(u, 0) \geq a_m,$$

where $a_m \to \infty$ as $m \to \infty$, for all $(u, 0) \in (X^{m-1})^\perp$ with $\|(u, 0)\| = r_m$.

However, if $p_0 = \tau$, we define $l := \max\{\mu_1/\tau, K_0\}$ and $\eta_m := \max\{\sigma_m, \rho_m\}$. Then, for each $(u, 0) \in (X^{m-1})^\perp$, we obtain

$$\begin{aligned} I(u, 0) &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau}(\sigma_m)^\tau \|u\|_a^\tau - K_0 \int_\Omega |x|^{-\beta_2}|u|^\tau \, dx \\ &\geq \frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau}(\sigma_m)^\tau \|u\|_a^\tau - K_0(\rho_m)^\tau \|u\|_a^\tau \\ &\geq \frac{1}{2}\|u\|_a^2 - 2l(\eta_m)^\tau \|u\|_a^\tau, \end{aligned}$$

so, taking $r_m := [2\,\tau\, l\,(\eta_m)^\tau]^{\frac{1}{2-\tau}}$ and $a_m := (\frac{1}{2} - \frac{1}{\tau})r_m^2$, we conclude that

$$I(u, 0) \geq a_m,$$

where $a_m \to \infty$ as $m \to \infty$, for all $(u, 0) \in (X^{m-1})^\perp$ with $\|(u, 0)\| = r_m$. □

Proof of Theorem 1.3. We remark that I is an even functional in the variables u and v. By using Lemma 3.3, we obtain

$$\sup_{X^m} I < \infty. \tag{3.5}$$

Then, by Lemmata 2.3, 3.3, and 3.4, Theorem 2.4, and by (3.5), we have the hypotheses of [10, Proposition 2.1], which concludes Theorem 1.3. □

Proof of Theorem 1.4. The proof is similar to prove of Theorem 1.2, the difference is that we apply the next lemma instead of Lemma 3.2. □

Lemma 3.5. *Assume* (H1), (H2)–(H5), (H7), (H9), $\xi < 2$, $\beta_1 < (b+1)\xi + N[1 - (\xi/2)]$, *and* $\theta_2 \in [\xi, 2)$. *Then, there exist* $\tilde{\mu}_0 > 0$ *and* $\rho > r > 0$ *such that* $\sup I(M_0) < \sigma$ *for all* $\mu_1 > 0$ *and* $\mu_2 \in (-\tilde{\mu}_0, 0)$, *where* $\sigma, r > 0$ *are coming from Lemma 3.1.*

Proof. If $(se, v) \in M_0$, then

$$\begin{aligned} I(se, v) &\leq \frac{r^2}{2}s^2 - s^\tau \, \frac{\mu_1}{\tau}\int_\Omega |x|^{-\beta_0}|e|^\tau \, dx - \frac{1}{2}\|v\|_b^2 + \frac{|\mu_2|}{\xi}\int_\Omega |x|^{-\beta_1}|v|^\xi \, dx \\ &\leq \Big(\frac{r^2}{2}s^2 - s^\tau \, \frac{\mu_1}{\tau}\int_\Omega |x|^{-\beta_0}|e|^\tau \, dx\Big) + \Big(\frac{|\mu_2|}{\xi}C^{\frac{\xi}{2}}\|v\|_b^\xi - \frac{1}{2}\|v\|_b^2\Big). \end{aligned} \tag{3.6}$$

It is easy verify that

$$t_{\mu_1} := \Big(\frac{r^2}{\mu_1 \int_\Omega |x|^{-\beta_0}|e|^\tau \, dx}\Big)^{\frac{1}{\tau-2}}, \quad t_{\mu_2} := \big(|\mu_2| C^{\frac{\xi}{2}}\big)^{\frac{1}{2-\xi}}$$

are the respective maximum points of the functions $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to \mathbb{R}$ given by

$$f(s) := \frac{r^2}{2}s^2 - s^\tau \frac{\mu_1}{\tau}\int_\Omega |x|^{-\beta_0}|e|^\tau \, dx, \quad g(t) := \frac{|\mu_2|}{\xi}C^{\frac{\xi}{2}}t^\xi - \frac{t^2}{2}.$$

Moreover, we have

$$f(t_{\mu_1}) = (\frac{1}{2} - \frac{1}{\tau})(\mu_1 \int_\Omega |x|^{-\beta_0}|e|^\tau \, dx)^{\frac{-2}{\tau-2}}(r^2)^{\frac{\tau}{\tau-2}},$$

$$g(t_{\mu_2}) = (\frac{1}{\xi} - \frac{1}{2})(|\mu_2| C^{\frac{\xi}{2}})^{\frac{2}{2-\xi}}.$$

Let us fix $\tilde{\mu}_0 > 0$ and $\rho_0 > r > 0$ such that

$$g(t_{\mu_2}) < \sigma \quad \text{if } 0 < |\mu_2| < \tilde{\mu}_0, \tag{3.7}$$

$$f(s) \leq 0 \quad \text{for all } s \geq \rho_0. \tag{3.8}$$

Also, we choose $\rho > \max\{\rho_0, r\rho_0\} > r$ such that

$$g(\|v\|_b) + f(t_{\mu_1}) \leq 0, \quad \forall \|v\|_b \geq \rho - \rho_0 r. \tag{3.9}$$

Thus, if $s = 0$ and $\|v\|_b \leq \rho$, it follows by (3.6) and (3.7) that $I(0, v) \leq g(t_{\mu_2}) < \sigma$ if $0 < |\mu_2| < \tilde{\mu}_0$.

If $s > 0$ and $\|(se, v)\| = \rho$, we have $\|v\|_b = \rho - s\|e\|_a = \rho - sr$. Then, for $s \geq \rho_0$, we obtain by (3.6), (3.7), and (3.8) that $I(se, v) \leq g(t_{\mu_2}) < \sigma$ if $0 < |\mu_2| < \tilde{\mu}_0$. However, if $s < \rho_0$, we have $\|v\|_b = \rho - sr \geq \rho - \rho_0 r$, so, by (3.6) and (3.9), we obtain $I(se, v) \leq f(t_{\mu_1}) + g(\|v\|_b) \leq 0 < \sigma$. Note that $\|v\|_b \leq \rho - r\rho_0$ and $s > 0$ imply $s \geq \rho_0$. □

To prove Theorem 1.5, we will need of the following lemma.

Lemma 3.6. *Assume* (H1), (H2)–(H5), (H7), (H9), $\xi < 2$, $\beta_1 < (b+1)\xi + N[1 - (\xi/2)]$, $\theta_2 \in [\xi, 2)$, $\mu_1 > 0$, *and* $\mu_2 < 0$. *Then, there exists* $R_m > 0$ *such that* $I(u, v) \leq 0$ *for all* $(u, v) \in X^m$ *with* $\|(u, v)\| \geq R_m$.

We remark that I is an even functional in the variables u and v. By using Lemma 3.6, we obtain

$$\sup_{X^m} I < \infty. \tag{3.10}$$

Then, by Lemmata 3.6 and 3.4, Theorem 2.4, and (3.10), we have the hypotheses of [10, Proposition 2.1], which concludes Theorem 1.5.

Proof of Lemma 3.6. We have $X^m \approx \operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\} \times W_0^{1,2}(\Omega, |x|^{-2b})$. Following as in Lemma 3.3, we obtain that there exists $K_m > 0$ such that

$$\|w\|_{L^\tau(\Omega, |x|^{-\beta_0})} \geq K_m \|w\|_a, \quad \forall w \in \operatorname{span}\{\varphi_{a,1}, \ldots, \varphi_{a,m}\}.$$

Hence, we obtain

$$\begin{aligned}
I(u, v) &\leq (\frac{1}{2}\|u\|_a^2 - \frac{\mu_1}{\tau} K_m^\tau \|u\|_a^\tau) - \frac{1}{2}\|v\|_b^2 + \frac{|\mu_2|}{\xi} C^{\frac{\xi}{2}} \|v\|_b^\xi \\
&\leq -(\mu_1 K^\tau \|u\|_a^{\tau-2} - \frac{1}{2})\|u\|_a^2 - (\frac{1}{2}\|v\|_b^{2-\xi} - |\mu_2| C^{\frac{\xi}{2}})\|v\|_b^\xi \\
&\leq 0, \quad \forall (u, v) \in X^m, \ \|(u, v)\| \geq R_m,
\end{aligned}$$

for some $R_m > 0$ large enough, because $\tau > 2 > \xi$. □

4. Appendix: Basis for $L^2(\Omega, |x|^{-2a})$ and $W_0^{1,2}(\Omega, |x|^{-2a})$

The next result was proved in [4].

Theorem 4.1. *For each* $f \in L^2(\Omega, |x|^{-2a})$, *the problem*

$$\begin{aligned}
-\operatorname{div}(|x|^{-2a}\nabla u) &= |x|^{-2(a+1)+c} f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

has an unique weak solution $u \in W_0^{1,2}(\Omega, |x|^{-2a})$ *for each* $c > 0$. *Moreover, the operator* $T_c : L^2(\Omega, |x|^{-2a}) \to L^2(\Omega, |x|^{-2a})$, $T_c f = u$ *is continuous and nondecreasing.*

Lemma 4.2. *If* $c = 2$, *then the operator* $T := T_2$ *is a compact self-adjoint operator and* $N(T) = \{0\}$.

Proof. Let $\{f_n\} \subset L^2(\Omega, |x|^{-2a})$ a bounded sequence and $f \in L^2(\Omega, |x|^{-2a})$ such that $f_n \rightharpoonup f$ weakly in $L^2(\Omega, |x|^{-2a})$ as $n \to \infty$. By using definition of T, we obtain

$$\begin{aligned}
\int_\Omega |x|^{-2a} |\nabla(Tf_n)|^2 \, dx &= \int_\Omega |x|^{-2a} f_n (Tf_n) \, dx \\
&\leq M(\int_\Omega |x|^{-2a} (Tf_n)^2 \, dx)^{1/2} \\
&\leq M(\int_\Omega |x|^{-2a} |\nabla(Tf_n)|^2 \, dx)^{1/2},
\end{aligned}$$

where M is a positive constant; therefore

$$\|Tf_n\|_{W_0^{1,2}(\Omega, |x|^{-2a})} \leq M, \quad \forall n \in \mathbb{N}.$$

Consequently, there exists $g \in W_0^{1,2}(\Omega, |x|^{-2a})$ such that $Tf_n \rightharpoonup g$ weakly in $W_0^{1,2}(\Omega, |x|^{-2a})$ as $n \to \infty$. Hence form the compact embedding $W_0^{1,2}(\Omega, |x|^{-2a}) \hookrightarrow L^2(\Omega, |x|^{-2a})$ we conclude that $Tf_n \to g$ strongly in $L^2(\Omega, |x|^{-2a})$ as $n \to \infty$, in other words, T is compact.

Now, we prove that T is self-adjoint. Let $f, g \in L^2(\Omega, |x|^{-2a})$. Then, we have

$$\int_\Omega |x|^{-2a} g(Tf)\, dx = \int_\Omega |x|^{-2a} \nabla(Tf)\nabla(Tg)\, dx = \int_\Omega |x|^{-2a} f\,(Tg)\, dx;$$

that is,

$$\langle Tf, g\rangle_{L^2(\Omega,|x|^{-2a})} = \langle f, Tg\rangle_{L^2(\Omega,|x|^{-2a})},$$

so, T is self-adjoint.

Let $f \in N(T)$. We have $Tf = 0$ almost everywhere $x \in \Omega$, then

$$0 = \int_\Omega |x|^{-2a} \nabla(Tf)\nabla w\, dx = \int_\Omega |x|^{-2a} f w\, dx,\ \forall w \in W_0^{1,2}(\Omega, |x|^{-2a}).$$

Then, we obtain $f \equiv 0$. $\square$

Theorem 4.3. *The normalized eigenfunctions* $\{\varphi_{a,n}\} \subset C^1(\overline{\Omega} \setminus \{0\}) \cap C^0(\overline{\Omega})$ *of eigenvalue problem*

$$\begin{aligned} -\operatorname{div}(|x|^{-2a}\nabla u) &= \lambda |x|^{-2a} u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

is a Hilbertian basis of space $L^2(\Omega, |x|^{-2a})$*. Moreover, if* $\{\lambda_{a,n}\}$ *are the respective eigenvalues of* $\{\varphi_{a,n}\}$*, the sequence* $\{\frac{\varphi_{a,n}}{\sqrt{\lambda_{a,n}}}\} \subset C^1(\overline{\Omega} \setminus \{0\}) \cap C^0(\overline{\Omega})$ *is a Hilbertian basis for space* $W_0^{1,2}(\Omega, |x|^{-2a})$*.*

Proof. First of all, we recall that Xuan [21] proved that the eigenvalue problem (4.1) has a sequence of eigenfunctions $\{\varphi_{a,n}\}$ associated to eigenvalues $\{\lambda_{a,n}\}$ with $0 < \lambda_{a,1} < \lambda_{a,2} \le \lambda_{a,3} \le \cdots \nearrow +\infty$. Changing $\{\varphi_{a,n}\}$ by $\varphi_{a,n}/\|\varphi_{a,n}\|_{L^2(\Omega,|x|^{-2a})}$, if necessary, we can consider $\|\varphi_{a,n}\|_{L^2(\Omega,|x|^{-2a})} = 1$. Moreover,

$$\{\varphi_{a,n}\} \subset L^\infty(\Omega, |x|^{-2a}) \cap C^1(\Omega \setminus \{0\})$$

and $\varphi_1 > 0$ in $\Omega \setminus \{0\}$. Then, by [4, Theorem 2.1], we obtain $\{\varphi_{a,n}\} \subset C^0(\overline{\Omega})$. Thus, by applying the [14, Theorem 1] follows that $\{\varphi_{a,n}\} \subset C^1(\overline{\Omega} \setminus \{0\})$. Also, by strong maximum principle, see [15, Theorem 2.1], we obtain $\varphi_{a,1} > 0$ in Ω.

By the definition of T, we have

$$\begin{aligned} \int_\Omega |x|^{-2a} \nabla(T\varphi_{a,n})\nabla w\, dx &= \int_\Omega |x|^{-2a} \varphi_{a,n} w\, dx \\ &= \lambda_{a,n}^{-1} \int_\Omega |x|^{-2a} \nabla\varphi_{a,n} \nabla w\, dx \\ &= \int_\Omega |x|^{-2a} \nabla(\lambda_{a,n}^{-1}\varphi_{a,n})\nabla w\, dx \end{aligned}$$

for all $w \in W_0^{1,2}(\Omega, |x|^{-2a})$. Hence, we conclude

$$T\varphi_n = \lambda_{a,n}^{-1}\varphi_{a,n}, \quad \forall n \in \mathbb{N};$$

that is, $\{\lambda_{a,n}^{-1}\}$ and $\{\varphi_{a,n}\}$ are the eigenvalues and eigenfunctions of T, respectively. But, by Lemma 4.2 and [3, Theorem V.I.11], we obtain that the eigenfunctions of T is a Hilbertian basis for space $L^2(\Omega, |x|^{-2a})$.

To prove the second claim, we remark that $\{\frac{\varphi_{a,n}}{\sqrt{\lambda_{a,n}}}\} \subset W_0^{1,2}(\Omega, |x|^{-2a})$. Moreover, from (4.1), we obtain that $\{\frac{\varphi_{a,n}}{\sqrt{\lambda_{a,n}}}\}$ is an orthonormal set with respect to inner product of $W_0^{1,2}(\Omega, |x|^{-2a})$.

Now, we prove that the space spanned by $\{\varphi_{a,n}\}$ is dense in $W_0^{1,2}(\Omega, |x|^{-2a})$. Indeed, if $u \in W_0^{1,2}(\Omega, |x|^{-2a})$ is such that $\langle u, \varphi_{a,n}\rangle_{W_0^{1,2}(\Omega,|x|^{-2a})} = 0$ for all $n \in \mathbb{N}$. Then, by (4.1), we obtain

$$\int_\Omega |x|^{-2a} u\varphi_{a,n}\, dx = 0, \quad \forall n \in \mathbb{N}.$$

Then, since that $\{\varphi_{a,n}\}$ is a Hilbertian basis of $L^2(\Omega, |x|^{-2a})$, we conclude that $u = 0$ for almost everywhere in Ω. Hence, by [3, Corollary I.8] follows that $\{\varphi_{a,n}\}$ is dense in $W_0^{1,2}(\Omega, |x|^{-2a})$. $\square$

References

[1] C. O. Alves, P. C. Carrião, O.H. Miyagaki; *Nontrivial solutions of a class of quasilinear elliptic problems involving critical exponents,* Prog. Nonl. Diff. Eqns. Applic., 54 (2003), 225-238.

[2] V. Benci, P. Rabinowitz; *Critical point theorems for indefinite functionals,* Invent. Math., 52 (1979), 241-273.

[3] H. Brezis; *Analyse fonctionnelle*, Masson, Paris, 1983.

[4] F. Brock, L. Iturriaga, J. Sánchez, P. Ubilla; *Existence of positive solutions for p-laplacian problems with weights,* Commun. Pure Appl. Anal., 5 (2006), 941-952.

[5] L. Caffarelli, R. Kohn, L. Nirenberg; *First order interpolation inequalities with weights,* Compositio Mathematica, 53 (1984), 259-275.

[6] P. C. Carrião, D. G. de Figueiredo, O. H. Miyagaki; *Quasilinear elliptic equations of the Henon-type: existence of nonradial solutions,* Communications in Contemporary Mathematics, 5 (2009), 783-798.

[7] F. Catrina, Z.Q. Wang; *Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in* R^N, Ann. Inst. Henri Poincaré, Analyse Non Linéaire, Vol. 18 (2001), 157-178.

[8] M. Clapp, Y. Ding, S.H. Linares; *Strongly indefinite functionals with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems,* Electronic J. Diff. Eqns., 2004 (100) (2004), 1-18.

[9] D. G. Costa, C. A. Magalhães; *A variational approach to noncooperative elliptic systems,* Nonlinear Anal. TMA, 25 (1995), 669-715.

[10] D. G. de Figueiredo, Y. H. Ding; *Strongly indefinite functionals and multiple solutions of elliptic systems,* Trans. Amer. Math. Soc., 335 (2003), 2973-2989.

[11] L. Fang, Y. Jianfu; *Nontrivial solutions of Hardy-Hénon type elliptic systems,* Acta Mathematica Scientia, 27B (2007), 673-688.

[12] J. Hulshof, E. Mitidieri, R. van der Vorst; *Strongly indefinite systems with critical Sobolev exponents,* Trans. Amer. Math. Soc., 350 (1998), 2349-2365.

[13] O. Kavian; *Introduction à la théorie des points critiques et applications aux problèmes elliptiques,* Springer, Heidelberg, 1993.

[14] G. M. Lieberman; *Boundary regularity for solutions of degenerate elliptic equations,* Nonlinear Anal., 12 (1998), 1203-1219.

[15] O. H. Miyagaki, R. S. Rodrigues; *On positive solutions for a class of singular quasilinear elliptic systems,* J. Math. Anal. Appl., 334 (2007), 818-833.

[16] O. H. Miyagaki, R. S. Rodrigues; *On the existence of weak soluitons for* $p, q-$*Laplacian systems with weights,* Electronic J. Diff. Eqns, 2008(2008), 115, 1-18.

[17] O. H. Miyagaki, R. S. Rodrigues; *On multiple solutions for a singular quasilinear elliptic system involving critical Hardy-Sobolev exponents,* Houston J. Math., 34 (2008), 1271-1293.

[18] W.M. Ni; *A Nonlinear Dirichlet problem on the unit ball and its applications,* Indiana Univ. Math. J., 31 (1982), 801-807.

[19] D. Smets, M. Willem, J. Su; *Non-radial ground states for the Hénon equation,* Comm. Contemporary. Math., 4 (2002), 467-480.

[20] M. Willem; *Minimax theorems,* Birkhäuser, Boston, 1996.

[21] B. Xuan; *The eigenvalue problem for a singular quasilinear elliptic equation,* Electronic J. Diff. Eqns., 2004 (16) (2004), 1-11.

[22] B. Xuan; *The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights,* Nonlinear Anal., 62 (2005), 703-725.

[23] J. Yan, Y. Ye, X. Yu; *Existence results for strongly indefinite elliptic systems,* Electronic J. Diff. Eqns., 2008 (81) (2008), 1-11.