EXISTENCE OF TRAVELING WAVES FOR DIFFUSIVE-DISPERSIVE CONSERVATION LAWS

CEZAR I. KONDO[1] AND ALEX F. ROSSINI[2]

ABSTRACT. In this work we obtain the existence and uniqueness of traveling waves for diffusive-dispersive conservation laws with flux function $C^1(\mathbb{R})$ by the phase plane analysis. Our investigation also gives a estimate of domain of attraction of equilibrium point attractor corresponding to the right-hand state. The equilibrium point corresponding to the left-hand state is a saddle. According to the analysis of phase portrait close to the saddle point there are exactly two semi-orbits of system and we establish that only one semi-orbit come in the domain of attraction that converge to $(u_-, 0)$ as $y \to -\infty$. This gives the connection saddle-attractor desered.

1. Introduction

In this paper we investigate the existence and uniqueness of traveling waves solutions, which are smooth solutions of the form,

$$u(x,t) = s(x - ct)$$

where c is a constant, for the partial differential equation

$$u_t + f(u)_x = (a(u)u_x)_x + (b(u)u_x)_{xx} \quad x \in \mathbb{R}, t > 0, \tag{1}$$

where the diffusion function $a : \mathbb{R} \to \mathbb{R}$ and dispersion function $b : \mathbb{R} \to \mathbb{R}$ are given smooth functions. Furthermore, we assume $a(u)\,, b(u) > 0$ for $u \in \mathbb{R}$ and the flux function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

For the case $f(u) = u^2/2$, where a and b real constants with $ab \neq 0$, the equation (1) reduces to the Korteweg-de Vries-Burgers equation

$$u_t + uu_x = au_{xx} + bu_{xxx}. \tag{2}$$

It is usually considered as a combination of the Burgers equation and KdV equation since in the limit $b \to 0$ the equation reduces to the Burgers equation

$$u_t + uu_x = au_{xx} \tag{3}$$

which is named after its use by Burgers [1] for studying the turbulence, and if the limit $a \to 0$ is taken, then the equation becomes the KdV equation

$$u_t + uu_x = bu_{xxx} \tag{4}$$

1991 *Mathematics Subject Classification.* Primary: 35L65. Secondary: 76N10.

Key words and phrases. scalar conservation law, diffusive-dispersive , weak solution , traveling wave, phase portrait .

which was first suggested by Korteweg and de Vries [5], who used it as a nonlinear model to study the change of forms of long waves advancing in a rectangular channel.

The Korteweg-de Vries-Burgers equation (2) is the simplest form of the wave equation in which the non-linearity (uu_x), the dispersion u_{xxx} and the dissipation u_{xx} occur.

The existence of traveling waves with linear diffusion and dispersion were studied by Bona and Schonbek [3]; and Jacobs, McKinney and Shearer [4]. In [3] was of interest the limiting behaviour of these waves when the coefficients a, b tends to zero while the ratio $\frac{b}{a^2}$ remains bounded. In 1993, Jacobs, McKinney, and Shearer [4] rigorously characterized all week solutions profiles of the single conservation law $u_t + (u^3)_x = 0$.

In the works of N. Bedjaoui and P.G. LeFloch [9]-[10] and M.D Thanh [7] have been studied the equation of the type

$$u_t + f(u)_x = (R(u, \beta u_x))_x + \gamma(c_1(u)(c_2(u)u_x)_x)_x. \tag{5}$$

In [10] the authors considered

$$R(u, v) = b(u, v)|v|^p \text{ for } p > 0.$$

In [7] Thanh estudied the case in that $R = R(u, v)$ satisfies:

$$R_v(u, 0) = R_u(u, 0) = 0, \ R(u, v)v > 0, \ \forall\ v \neq 0, \ \forall\ u.$$

This assumption fails in our case. In [9] Bedjaoui/LeFloch estudied the case $R(u, v) = b(u, v)v$ with $b(u, v) = a(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth mapping satisfying

$$uf''(u) > 0 \text{ for all } u \neq 0, \ \lim_{\pm\infty} f' = +\infty. \quad (*)$$

The associated non-linear system (5) admits exactly three equilibrium point for a certain speed interval that depends the kinetic function (see [11]), thanks the hypothesis $(*)$ on the flux-function $f : \mathbb{R} \to \mathbb{R}$. For such equilibrium points are investigated the existence and uniqueness of traveling waves when a certain speed wave is fixed in this interval, however the connected points do not satisfy the Oleinik entropy criterion, see Theorem 3.3 in [9]. Observe that the work [9] establishes the existence of traveling waves associated with nonclassical shocks. When the equilibrium point satisfies such criterion this implies the existence of the trajectories between them for each speed wave c in a certain interval, see Theorem 5.1 in [9].

However, in our paper we are interested in traveling waves associated with a classical shock (see Definition 5). In the current paper are given two state u_- and u_+ where both u_- and u_+ are arbitrary constants and we investigate the existence and uniqueness of traveling waves when the speed wave is given by $c = \frac{f(u_+)-f(u_-)}{u_+-u_-}$, under only the hypothesis that f is smooth. In this context, the associated system has at least two equilibrium points $(u_-, 0)$ and $(u_+, 0)$ satisfying the Oleinik entropy criterion. This differs of the Theorem 3.3 in [9].

In our main result (Theorem 21) ensures that the points $(u_\pm, 0)$ can be connected by an analysis of the phase portrait of the saddle $(u_-, 0)$ and estimates of the domain attraction of the node $(u_+, 0)$. This technique is different of the Theorem 5.1 in [9], without the use of the existence of the kinetic function.

In the case of traveling waves our manuscript can be used to establish existence and uniqueness of traveling waves when the flux-function associated is given by $f(u) = u^2/2$ and when the diffusive-dispersive coefficients are positive constants. Moreover, we can adapt our work to establish existence and uniqueness of traveling waves for the equation

$$u_t + f(u)_x = au_{xx} + bu_{xxt} \tag{6}$$

where $a > 0$ and $b \neq 0$, see in Appendix II.

An outline of this paper follows. In Section 2, initially we recall the concept of traveling wave solution connecting the states $u\pm$ together with the concept of week solution and we close the section the method of linearization for differential equations. In section 3, we begin by recalling well-known concepts and results. Also, the stability of equilibrium point of the associated differential equation is established. After declaring an invariance theorem we establish a result about traveling waves, which is essential to the existence of trajectories connecting the states $u\pm$. In section 4, an estimate of domain of attraction is provided. Finally, in Section 5 of the analysis of phase portrait close to the saddle point we have that there are exactly two semi-orbits of system and we establish that only one semi-orbit enters the attraction domain of the attracting equilibrium point $(u_+, 0)$ that converge to $(u_-, 0)$ as $y \to -\infty$. This gives the connection saddle-attractor desered.

2. traveling Waves : Weak Solution

We consider an partial differential equations of the form

$$u_t + f(u)_x = (a(u)u_x)_x + (b(u)u_x)_{xx} \tag{7}$$

where the flux function $f : \mathbb{R} \to \mathbb{R}$, the functions $a = a(u) > 0, b = b(u) > 0$ are $C^1(\mathbb{R})$ and $C^2(\mathbb{R})$ respectively. We seek the existence of *traveling waves* solution $u(x, t) = s(x - ct)$, for some constant speed $c \in \mathbb{R}$, satisfying the following conditions at infinity:

$$\lim_{y\to\pm\infty} s^{(j)}(y) = 0, \quad j = 1, 2 \quad \text{and} \quad \lim_{y\to\pm\infty} s(y) = u_\pm, \quad u_- \neq u_+. \tag{8}$$

Note that of (7) the function $s = s(y)$ satisfies the ordinary differential equation

$$(f(s(y)) - cs(y))' = (a(s(y))s'(y))' + (b(s(y))s'(y))'' \tag{9}$$

where

$$(\cdot)' = \frac{d}{dy}(\cdot), \ y \in \mathbb{R}.$$

Integrating (9) on $]-\infty, y[$ and using the conditions at infinity (8) we have:

$$\begin{aligned} -c(s(y) - u_-) + f(s(y)) - f(u_-) &= a(s(y))s'(y) - \lim_{y\to-\infty} a(s(y))s'(y) \\ &+ (b(s(y))s'(y))' - \lim_{y\to-\infty} (b(s(y))s'(y))' \end{aligned} \tag{10}$$

Define the function $h(s) = -c(s-u_-) + f(s) - f(u_-),\ s \in \mathbb{R}$, since

$$\lim_{y\to-\infty} a(s(y))s'(y) = \lim_{y\to-\infty} b'(s(y))(s'(y))^2 = 0,$$

we can re-write (10) of the following form

$$h(s(y)) = a(s(y))s'(y) + (b(s(y))s'(y))',\ y \in \mathbb{R}. \tag{11}$$

We obtain

Lemma 1. *Let $u(x,t) = s(x-ct)$, $c \in \mathbb{R}$, be a* traveling waves *solution of (7) satisfying (8). Then, in equation (11), let $y \to +\infty$ to obtain*

$$-c(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{12}$$

i.e, the triple (u_-, u_+, c) satisfies the Rankine-Hugoniot *relation .*

Remark 2. In agreement with the Lemma 1, it will be assumed that

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Setting $w(y) = b(s(y))s'(y)$ in (11) we have to the following second-order system

$$\begin{cases} s'(y) = \dfrac{w(y)}{b(s(y))} \\ \\ w'(y) = h(s(y)) - \dfrac{a(s(y))}{b(s(y))} w(y). \end{cases} \tag{13}$$

Let F be vector field

$$F(s,w) = \left(\frac{w}{b(s)}, h(s) - \frac{a(s)}{b(s)} w \right),$$

a point in the (s,w)-plane is an equilibrium point of the (13) if and only if it has of the form $(s,0)$ with $h(s) = 0$. That is,

Proposition 3. *A point (s,w) is an equilibrium point of the (13) if and only if $w = 0$ and the triple (s, u_-, c) satisfies the Rankine-Hugoniot relation for the associate conservation law $u_t + f(u)_x = 0$.*

Remark 4. Denote by Γ the set equilibrium point the of (13) and let $(u_i, 0) \in \Gamma$ then

$$h(s) = -c(s-u_i) + f(s) - f(u_i).$$

Geometrically, Γ is the intersection of the straight line connecting $(u_\pm, 0)$ and the graph of f .

Definition 5. (Weak Solution) A discontinuous function of the form

$$u(x,t) = \begin{cases} u_-, & x - ct \le 0 \\ u_+, & x - ct > 0 \end{cases} \tag{14}$$

is said to be an weak solution of the conservation law $u_t + f(u)_x = 0$ if and only if satisfies the *Rankine-Hugoniot* relation

$$-c(u_+ - u_-) + f(u_+) - f(u_-) = 0\,.$$

We know that weak solutions are not unique. To choose the only physically relevant solution we use the *Oleinik* entropy criterion, that requires

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} < \frac{f(u) - f(u_-)}{u - u_-},\ \ \forall\ u \in (u_+,\, u_-)\,, \tag{15}$$

or equivalently,

$$\frac{f(u) - f(u_+)}{u - u_+} < \frac{f(u_+) - f(u_-)}{u_+ - u_-},\ \ \forall\ u \in (u_+,\, u_-). \tag{16}$$

In view of the proposition 3, assuming that $(s, 0)$ is a equilibrium point, the function

$$u(x,t) = \left\{ \begin{array}{ll} u_-, & x - ct \leq 0 \\ s, & x - ct > 0 \end{array} \right. \tag{17}$$

is a weak solution of the conservation law $u_t + f(u)_x = 0$. Conversely, if $u(x, t)$ is as in (17) and also a weak solution then the points $(s, 0)$ e $(u_-, 0)$ are equilibrium points of the F.

Another fact that follows from the existence of weak solution physically relevant is that $h(s) < 0$ in (u_+, u_-), we assume without loss of generality that $u_+ < u_-$. We can also conclude that $f'(u_+) \leq c \leq f'(u_-)$.

2.1. **Linearization.** The Jacobian matrix $DF(s, w)$ is given by

$$DF(s,w) = \begin{pmatrix} -w\left(\dfrac{b'(s)}{b^2(s)}\right) & \dfrac{1}{b(s)} \\ & \\ (f'(s) - c) - w\left(\dfrac{a'(s)b(s) - a(s)b'(s)}{b^2(s)}\right) & -\dfrac{a(s)}{b(s)} \end{pmatrix}$$

We choose $(u_i, 0)$ an arbitrary equilibrium point we obtain

$$DF(u_i, 0) = \begin{pmatrix} 0 & \dfrac{1}{b(u_i)} \\ & \\ f'(s) - c & -\dfrac{a(u_i)}{b(u_i)} \end{pmatrix}.$$

Therefore, the characteristic equation of $DF(u_i, 0)$ is given by

$$|DF(u_i, 0) - \lambda Id| = \lambda^2 + \frac{a(u_i)}{b(u_i)}\lambda - \frac{1}{b(u_i)}(f'(u_i) - c) = 0,$$

which admits two roots as

(18)

$$\lambda_1 = -\frac{a(u_i)}{2b(u_i)} - \sqrt{\frac{(a(u_i))^2}{4(b(u_i))^2} + \frac{f'(u_i) - c}{b(u_i)}}\,, \quad \lambda_2 = -\frac{a(u_i)}{2b(u_i)} + \sqrt{\frac{(a(u_i))^2}{4(b(u_i))^2} + \frac{f'(u_i) - c}{b(u_i)}}$$

where $i = \pm$.

We recall some concepts and theorems that will be useful for classification (see subsection 3.1) of the equilibrium points of field F.

3. Definitions and Results

Consider the nonlinear system

$$X' = G(X), \tag{19}$$

where $G : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and D is a neighborhood of the $X = X_0$.

Definition 6. The equilibrium point $X = X_0$ of (19) is

1) *stable*, if for each $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that $\|X(t) - X_0\| < \epsilon, \ \forall \ t \geq 0$ whenever $\|X(0) - X_0\| < \delta$.
2) *unstable*, if not stable.
3) *asymptotically stable*, if it is stable and δ can be chosen such that

$$\lim_{t\to+\infty} \|X(t) - X_0\| = 0$$

whenever $\|X(0) - X_0\| < \delta$.

Theorem 7. *Let $X = X_0$ be an equilibrium point for the nonlinear system (19). Let $A = DG(X_0)$ then*

1) If $Re(\lambda_i) < 0$ for all eigenvalues λ_i of A then X_0 is asymptotically stable *relative to the nonlinear system.*
2) If $Re(\lambda_i) > 0$ for one or more of the eigenvalues λ_i then X_0 is unstable *relative to the nonlinear system. Where $i = 1, 2$.*

Proof. See [8]. □

3.1. **Nonlinear Classification System.** It follows of the Theorem 7 together with (18) the following classification:

1) If $f'(u_i) - c > 0$ then $(u_i, 0)$ is a saddle point (of the linearized system). Thus, the point $(u_i, 0)$ is unstable.
2) If $f'(u_i) - c < 0$ and since $a(u_i) > 0$ then $(u_i, 0)$ is asymptotically stable.

In the remainder of this paper we will devote to the case when we have $f'(u_+) < c < f'(u_-)$ and thus shall be ensured the existence and uniqueness of traveling wave connecting the states $u_\pm$. Now follow with some more definitions and results of the theory of differential equations.

Definition 8. (Invariant set)

1) A set $M \subset D$ is said to be *invariant* set with respect to (19) if $X(0) \in M$ then $X(t) \in M, \;\forall\; t \in \mathbb{R}$.
2) A set $M \subset D$ dito ser *positively invariant* set (negatively invariant set) with respect to $X' = G(X)$, if $X(0) \in M$ then $X(t) \in M, \;\forall\, t \geq 0\,(t \leq 0)$.

Definition 9. A trajectory $X(t)$ of (19) approaches a set $M \subset D$ as $t \to +\infty$, if for every $\epsilon > 0$ there is $T > 0$ such that

$$dist(X(t), M) \doteq \inf_{p \in M} \|X(t) - p\| < \epsilon, \;\forall\; t > T.$$

Theorem 10. (LaSalle's invariance principle, see [8]) *Let $V : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that*

$$\dot{V}(s, w) \doteq \nabla V(s, w) \cdot G(s, w) \leq 0$$

for all $X = (s, w) \in \Omega$, with Ω compact positively invariant in D. Let E be the set of all points in Ω where $\dot{V}(s, w) = 0$. Let M be the largest invariant set in E. Then every trajectory of $X' = G(X)$ starting Ω, i.e, $X(0) \in \Omega$, approaches of M as $t \to +\infty$.

Now turning our attention to the system (13). Consider the function

$$V(s, w) = -\int_{u_+}^{s} h(x)b(x)\,dx + \frac{w^2}{2}, \quad (s, w) \in \mathbb{R}^2. \tag{20}$$

Note that

$$\dot{V}(s, w) = \frac{w^2}{b(s)}(-a(s)) \leq 0 \;\text{ and }\; \dot{V}(s, w) = 0 \Leftrightarrow w = 0.$$

Proposition 11. *Suppose that there is a compact $\Omega \subset \mathbb{R}^2$ positively invariant with respect to system (13) with vector field*

$$F(s, w) = \left(\frac{w}{b(s)}, h(s) - \frac{a(s)}{b(s)}w\right).$$

Then every solution of this system starting in Ω approaches the set $\Gamma \cap \Omega$.

Proof. With the quantities (20) at hands, we can rewrite this context $E = \{(s, w) \in \Omega\,;\, w = 0\}$. Using Theorem 10 every solution starting in Ω approaches the set M the largest invariant set in E. It is easy to see that $\Gamma \cap \Omega$ is invariant, here Γ is defined in Remark 4, so $\Gamma \cap \Omega \subset M$. Let us show that $M \subset \Gamma \cap \Omega$. Indeed, given $(u, 0) \in M$, let $(s(y), w(y))$ be the solution of the system with initial data $(s(0), w(0)) = (u, 0)$. We have that $(s(y), w(y)) \in M, \forall y \in \mathbb{R}$, since it is M invariant, so $w(y) = 0$, $y \in \mathbb{R}$. On the other hand, follow of the system that

$$s'(y) = \frac{w(y)}{b(s(y))} = 0,$$

which implies $s(y) = u$. Thus, $(u, 0)$ is an equilibrium point of the system (13), i.e, $(u, 0) \in \Gamma \cap \Omega$. ∎

4. Traveling wave solution: domain of attraction

The goal of this section is to determine a compact Ω positively invariant in $\mathbb{R}^2$ and estimate the *domain of attraction* of the equilibrium point $(u_+, 0)$. Initially we give the definition of the domain of attraction .

Definition 12. (Domain of attraction) Let $X = X_0$ be an equilibrium point asymptotically stable of the system $X' = G(X)$. Denote by $\phi(t, \overline{X})$ the solution starting at $\overline{X}$ in $t = 0$. Then, the *domain attraction* corresponding to X_0 is the set from $\overline{X}$ such that

$$\lim_{t \to +\infty} \|\phi(t, \overline{X}) - X_0\| = 0.$$

4.1. **Estimation of Domain Attraction.** Let $m = \min_{(s,w) \in \partial(D \cup R)} V(s, w)$, where the set D and R are given by

$$D = \left\{ (s, w) \in \mathbb{R}^2 | (s - u_+)^2 + \frac{w^2}{\gamma^2} \leq (u_+ - q)^2 \text{ e } u_+ \leq s \leq q \right\}$$

$$R = \left\{ (s, w) \in \mathbb{R}^2 | (s - u_+)^2 + \frac{(u_+ - p)^2}{(\gamma |u_+ - q|)^2} w^2 \leq (u_+ - p)^2 \text{ e } p \leq s \leq u_+ \right\}$$

(see figure 1),

$$\text{with} \qquad \gamma^2 > (\mathrm{Lip}_{[p,u_-]} f + |c|) \max_{[p,u_-]} b(s) \tag{21}$$

and

$$\int_p^{u_-} h(x) b(x)\, dx > 0 \text{ and } h > 0 \text{ in } (p, u_+). \tag{22}$$

where p was chosen satisfying the inequality $2u_+ - u_- \leq p < u_+$.

The condition (22) makes sense since we have $f'(u_+) < c$. Indeed, from $f'(u_+) < c$, there is a $t > 0$ such that

$$\frac{f(u) - f(u_+)}{u - u_+} < c, \ \forall \ u \in (u_+ - t, u_+).$$

Hence, $h(u) > 0, \ \forall \ u \in (u_+ - t, u_+)$ thus

$$\int_u^{u_+} h(x) b(x)\, dx > 0, \ \forall \ u \in (u_+ - t, u_+) \, .$$

Choose $p \in (u_+ - t, u_+)$ and being $I(v) = \int_p^v h(x) b(x)\, dx > 0$ continuous positive in $(p, u_+]$ there is some $u_+ < q \leq u_-$ such that $I(q) > 0$, i.e, $\int_p^q h(x) b(x)\, dx > 0$. In this work we are assuming that holds $q = u_-$.

Lemma 13. *Let $\Sigma = D \cup R$ be defined above and $\partial\Sigma$ denotes its boundary. Let γ be given in (21) then we have*

$$0 < m = \min_{(s,w) \in \partial\Sigma} V(s, w) = V(u_-, 0) = \int_{u_+}^{u_-} -h(x) b(x)\, dx.$$

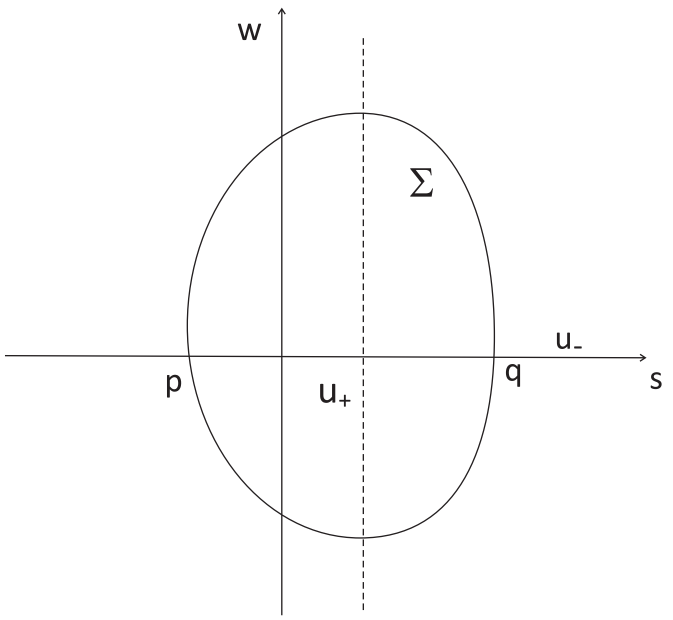

FIGURE 1

Proof. Let $E = \partial D \cap \partial\Sigma$ be then

$$E = \{(s,w) \in \mathbb{R}^2 | (s-u_+)^2 + \frac{w^2}{\gamma^2} = (u_+ - u_-)^2 \text{ e } u_+ \le s \le u_-\}$$

and we have

$$w^2 = \gamma^2((u_+ - u_-)^2 - (s-u_+)^2),$$

we replace w in $V(\cdot,\cdot)$ in (20) we have

$$\min_{(s,w)\in E} V(s,w) = \min_{s\in[u_+,u_-]} \left[-\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}((u_+ - u_-)^2 - (s-u_+)^2) \right].$$

Define $g(s) = -\displaystyle\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}((u_+ - u_-)^2 - (s-u_+)^2)$, for $s \in [u_+, u_-]$. It follows that

$$\begin{aligned} g'(s) &= -h(s)b(s) - \gamma^2(s-u_+) \\ &= -(s-u_+)\left(\gamma^2 + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c\right)\right) < 0, \;\; s \in (u_+, u_-) \end{aligned}$$

where the inequality above comes from (21), because, since $-(s-u_+) < 0$ we can deduce that $\left(\gamma^2 + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c\right)\right) > 0$, in fact

$$
\begin{aligned}
cb(s) - b(s)\left(\frac{f(s)-f(u_+)}{s-u_+}\right) &\le |c|b(s) + b(s)\left|\frac{f(s)-f(u_+)}{s-u_+}\right| \\
&\le (\mathrm{Lip}_{[p,u_-]} f + |c|) \max_{[p,u_-]} b(s) \\
&< \gamma^2.
\end{aligned} \tag{23}
$$

Therefore, the function g is strictly decreasing in $[u_+, u_-]$ and realizes its minimum value in $s = u_-$. Thus,

$$\min_{(s,w)\in E} V(s,w) = V(u_-, 0).$$

On the other hand, on $F = \partial R \cap \partial(D \cup R)$

$$F = \left\{ (s,w) \in \mathbb{R}^2 | (s-u_+)^2 + \frac{(u_+-p)^2}{(\gamma|u_+-u_-|)^2} w^2 = (u_+-p)^2 \text{ e } p \le s \le u_+ \right\}$$

one has

$$w^2 = \gamma^2 \left((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} \right),$$

we replace w in $V(\cdot,\cdot)$ in (20) we have

$$\min_{(s,w)\in F} V(s,w) = \min_{s\in[p,u_+]} \left[-\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}\left((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} \right) \right].$$

Setting $G(s) = -\displaystyle\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}\left((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} \right)$,
for $s \in [p, u_+]$ we have

$$
\begin{aligned}
G'(s) &= -h(s)b(s) - \gamma^2 (s-u_+) \frac{(u_+-u_-)^2}{(u_+-p)^2} \\
&= -(s-u_+)\left(\gamma^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c \right) \right), \quad s \in (p, u_+).
\end{aligned}
$$

Note that

$$\frac{(u_+-u_-)^2}{(u_+-p)^2} > 1 \quad \text{and} \quad -(s-u_+) > 0$$

so

$$\gamma^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c \right) > \gamma^2 + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c \right).$$

But, we show that $\gamma^2 + b(s)\left(\frac{f(s)-f(u_+)}{s-u_+} - c\right) > 0$ similarly as in (23). Therefore G is strictly decreasing in $[p, u_+]$ and

$$\min_{(s,w)\in F} V(s,w) = V(p,0)\,.$$

Now let us compare the values $V(u_-,0)$ e $V(p,0)$:
Clearly from (22) it follows that

$$V(p,0) = \int_p^{u_+} h(x)b(x)\,dx = \int_p^{u_-} h(x)b(x)\,dx - \int_{u_+}^{u_-} h(x)b(x)\,dx > V(u_-,0).$$

So the lemma is proved. □

We are now able to build a compact positively invariant. Consider $l \in (0, m)$, where m is given in Lemma 13 , and define a set Ω_l as follow:

$$\Omega_l = \{(s, w) \in D \cup R : V(s, w) \leq l\}.$$

Assertion 14. $\Omega_l \subset$ *int* $(D \cup R)$.

Proof. To prove this, we suppose to the contrary, then there exists $(s_0, w_0) \in \Omega_l \cap \partial(D \cup R)$, therefore $V(s_0, w_0) \geq m > l$ producing a contradiction. □

Assertion 15. *Let Ω_l be o set above. Then, it is compact positively invariant.*

For proof of the assertion 15 we need of Lemma 16, whose proof can be found at [8].

Lemma 16. *Suppose that there exists a compact set $W \subset \mathbb{R}^2$ such that every local solution of $X' = G(X)$, $y > 0$, $X(0) = X_0 = (s(0), w(0)) \in W$, lies intirely in W. Then, there is a unique solution passing through X_0 defined in $(0, \infty)$.*

Proof. (Assertion 15)

We clearly have that Ω_l is compacty, since $\Omega_l = (D \cup R) \cap V^{-1}((-\infty, l])$. Let $(s(y), w(y))$ be a solution for system starting in Ω_l, i.e, $(s(0), w(0)) \in \Omega_l$. We saw earlier that $\dot{V}(s, w) \leq 0$, then

$$\frac{d}{dy}(V(s(y), w(y))) \leq 0$$

so the function $V(s(y), w(y))$ is decreasing for $y \in J = (0, \omega_+)$, we denote by J the maximal interval associated to the maximal solution $(s(y), w(y))$. Thus,

$$V(s(y), w(y)) \leq V(s(0), w(0)) \leq l(< m).$$

Consequently,

$$(s(y), w(y)) \in \Omega_l \quad \text{for} \quad y \in J,$$

provided that $(s(y), w(y)) \in \Sigma$ for all $y \in J$, since $m = \min_{(s,w)\in\partial\Sigma} V(s, w)$. It then follows from Lemma 16 that $\omega_+ = \infty$. So the assertion 15 follows. □

We have proven $(u_+, 0) \in \Omega_l$, $\forall\, l \in (0, m)$ and is the only equilibrium point of the system (13) in Ω_l, once Ω_l lies entirely in the interior of $D \cup R$ and the function h is positive in (p, u_+) and negative in (u_+, u_-).

Therefore, according to Proposition 11 every solution starting in Ω_l tends toward $(u_+, 0)$ when $y \to +\infty$. Furthermore, sets Ω_l are an approximation of the domain of attraction of the point $(u_+, 0)$ which is the subject of the following lemma.

Lemma 17. *The domain of attraction of the equilibrium point $(u_+, 0)$ contains the set*

$$W = \{(s, w) \in D \cup R \; : V(s, w) < V(u_-, 0)\}.$$

Moreover, the line segment $[u_+, u_-) \times \{0\}$ *is contained in* W.

Proof. It is enough to prove that $W = \bigcup_{0<l<m} \Omega_l$. It is easy to see that $\Omega_l \subset W$. It remains to check that $W \subset \bigcup_{0<l<m} \Omega_l$. For this, let (l_n) be a sequence in $(0, m)$ such that $l_n \to m$ quando $n \to \infty$. As $V(u_-, 0) - V(s, w) > 0$ for $(s, w) \in W$ follows from the definition of limit that

$$l_n > \int_{u_+}^{s} -h(x)b(x)\,dx + \frac{w^2}{2}$$

for some $n = n(V(u_-, 0) - V(s, w))$. Thus, $(s, w) \in \Omega_{l_n}$ and the identity $W = \bigcup_{0<l<m} \Omega_l$ holds. For $u_+ < u < u_-$ we have

$$V(u, 0) = \int_{u_+}^{u} -h(x)b(x)\,dx < \int_{u_+}^{u_-} -h(x)b(x)\,dx = V(u_-, 0) = m$$

and then $(u, 0) \in \omega$. □

5. Existence of Semi-orbits

In this section we recall the basic results and concepts listed below. The reader is referred to [12] and [13].

Definition 18. Let X_0 a equilibrium point of a (s, w)-planar C^r vector field $G = (G_1, G_2)$. We say that

$$DG(X_0) = \begin{pmatrix} \frac{\partial G_1}{\partial s}(X_0) & \frac{\partial G_1}{\partial w}(X_0) \\ \frac{\partial G_2}{\partial s}(X_0) & \frac{\partial G_2}{\partial w}(X_0) \end{pmatrix}$$

is the *linear part* of the vector field G at the equilibrium point X_0. The equilibrium point X_0 is called *hyperbolic* if the two eigenvalues de $DG(X_0)$ have real part different from 0.

Theorem 19. *(The Stable Manifold Theorem, see* [13]*). Assume* $A = DG(X_0)$ *has eigenvalues* λ_1, λ_2 *with* $\lambda_1 < 0 < \lambda_2$*. Then there are two orbits of* $X' = G(X)$ *that go to* X_0 *as* $y \to +\infty$ *along a smooth curve tangent at* X_0 *to the eigenvectors for* λ_1 *and two orbits that go to* X_0 *as* $y \to -\infty$ *along a smooth curve tangent at* X_0 *to the eigenvectors of* λ_2.

5.1. **Semi-orbits.** We now return to the system (13). We recall that the equilibrium point $(u_-, 0)$ is a saddle point, so it is *hyperbolic*. It follows from Theorem 19 that there are two orbits of (13), that go to $(u_-, 0)$ as $y \to -\infty$ along a smooth curve tangent at $(u_-, 0)$ to the eigenvectors of λ_2 given in (18).

More specifically, according to the analysis of phase portrait close to the equilibrium point $(u-, 0)$ [see Appendix I] there are exactly two semi-orbits of system (13) that converge to $(u_-, 0)$ as $y \to -\infty$, one orbit approaches the $(u-, 0)$ from region $S = \{(s, w) \in \mathbb{R}^2 : s < u_-, w < 0\}$, while the other approaches from region $T = \{(s, w) \in \mathbb{R}^2 : s > u_-, w > 0\}$[see figure 2].

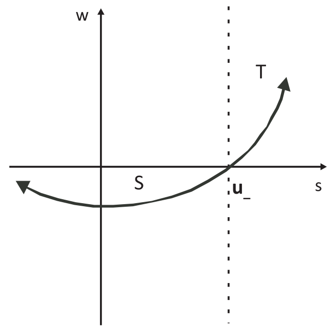

FIGURE 2

Proposition 20. *Let $(s(y), w(y))$ be a trajectory of system (13) globally defined entering the saddle point $(u_-, 0)$ from the region T as $y \to -\infty$, then such trajectory cannot tend to $(u_+, 0)$ as $y \to +\infty$.*

Proof. Multiply (11) by $b(s(y))s'(y)$ and integrate on $(-\infty, z)$, i.e,

$$\int_{-\infty}^{z} h(s(y)b(s(y))s'(y)\,dy = \int_{-\infty}^{z} a(s(y)b(s(y))(s'(y))^2\,dy + \frac{1}{2}\int_{-\infty}^{z} [(b(s(y))s'(y))^2]'\,dy$$

equivalently,

$$\int_{u_-}^{s(z)} h(x)b(x)\,dx = \int_{-\infty}^{z} a(s(y)b(s(y))(s'(y))^2\,dy + \frac{1}{2}[b(s(z))s'(z)]^2. \tag{24}$$

Suppose now that $(s(y), w(y)) \to (u_+, 0)$ as $y \to \infty$ then there is some z_0 such that $s(z_0) = u_-$. Choice $z = z_0$ and replacing in (24) we have

$$0 = \int_{-\infty}^{z_0} a(s(y)b(s(y))(s'(y))^2\,dy + \frac{1}{2}[b(u_-)s'(z_0)]^2\,.$$

As each term is non-negative we must have

$$\int_{-\infty}^{z_0} a(s(y)b(s(y))(s'(y))^2\,dy = 0$$

and by continuity it follows that $s'(y) = 0$ for $y \in (-\infty, z_0]$. Then $s(y) = u_-$ for $y \in (-\infty, z_0]$. Thus, $(s(z_0), w(z_0)) = (u_-, 0)$ and by uniqueness of solutions $(s(y), w(y)) = (u_-, 0)$ for every $y \in \mathbb{R}$. Therefore, $(s(y), w(y)) \to (u_-, 0)$ as $y \to \infty$ which contradicts our assumption since $u_- \neq u_+$. □

We conclude that the unique semi-orbit that can tend to $(u_+, 0)$ as $y \to \infty$ is the one that enters the region S. In the next section we will prove that this semi-orbit is the traveling wave desired.

6. Existence of traveling wave solution

Let $(s(z), w(z))$ be a solution (13) starting in $D \cup R$ and defined at least for values of y sufficiently negative such that

$$(s(y), w(y)) \to (u_-, 0), \qquad y \to -\infty$$

and that $s(y) < u_-$, $w(y) < 0$ for values of y sufficiently negative , that corresponds to semi-orbit that approach the $(u_-, 0)$ from region S. We can assure that $(s(y), w(y))$ belong $D \cup R$ for y sufficiently negative, this follows from convergence mentioned above .

Let us multiply the second equation of the system (13) by $w(z) = b(s(z))s'(z)$ and integrate from $-\infty$ to y

$$\int_{-\infty}^{y} w(z)w'(z)\,dz = \int_{-\infty}^{y} w(z)h(s(z))\,dz - \int_{-\infty}^{y} \frac{a(s(z))}{b(s(z))}(w(z))^2\,dz. \tag{25}$$

Note that the second term on the right of (25) is nonegative. We prove that

$$I = \int_{-\infty}^{y} \frac{a(s(z))}{b(s(z))}(w(z))^2\,dz > 0.$$

Suppose, by contradiction, that $I = 0$ then, by continuity, $w(z) = 0$ for every $z \in (-\infty, y]$, so $s'(z) = 0$, which means precisely that $s(z)$ is constant for $z \in (-\infty, y]$. However, we have $s(z) \to u_-$ as $z \to -\infty$, it then follows that $s(z) = u_-$ on $(-\infty, y]$ and thus $(s(y), w(y)) = (u_-, 0)$ which contradicts the uniqueness of solutions.

From this fact it follows that

$$\frac{(w(y))^2}{2} < \int_{-\infty}^{y} b(s(z))s'(z)h(s(z))\,dz = \int_{s(y)}^{u_-} -b(x)h(x)\,dx\,. \tag{26}$$

We can write

$$\int_{s(y)}^{u_-} -b(x)h(x)\,dx = \int_{u_+}^{u_-} -b(x)h(x)\,dx - \int_{u_+}^{s(y)} -b(x)h(x)\,dx$$

so rewriting (26) we have

$$\int_{u_+}^{s(y)} -b(x)h(x)\,dx + \frac{w^2(y)}{2} < \int_{u_+}^{u_-} -b(x)h(x)\,dx.$$

Thus,

$$V(s(y), w(y)) < V(u_-, 0).$$

Consequently, we have $(s(y), w(y)) \in W$, and

$$(s(z), w(z)) \to (u_+, 0) \quad \text{quando} \quad z \to \infty.$$

Using the fact that $(s(y), w(y)) \in W$, then $(s(y), w(y)) \in \Omega_l$, for some $l \in (0, m)$. We formulate the following the Cauchy problem

$$\left\{ \begin{array}{rl} X'(t) = & F(X(t)) \\ X(y) = & (s(y), w(y)). \end{array} \right. \tag{27}$$

Thus, $(s(t), w(t))$ is a solution and $(s(t), w(t)) \in \Omega_l$, for $t \geq y$, recalling once Ω_l is invariant positively. Moreover,

$$(s(t), w(t)) \to (u_+, 0) \quad \text{as} \quad t \to \infty \ ,$$

recalling once again the fact $(s(y), w(y))$ belong to domain of attraction of the equilibrium point $(u_+, 0)$.

Therefore, we prove the existence of a single orbit (up to translation of the independent variable) connecting the states $(u_\pm, 0)$ with flux function $f : \mathbb{R} \to \mathbb{R}$ de classe C^1. We then checked our main result:

Theorem 21. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose there is a weak solution connecting the states $u_\pm$ with constant speed c, given by (12). Further, assume (15) and $f'(u_+) < c < f'(u_-)$. Then there is (up to translation of the independent variable y) a unique solution $s(y)$ of (9) satisfying (8).*

7. Appendix I

Here we will carry out an analysis of the phase portrait close to the equilibrium point $(u-, 0)$. Let's eliminate the possibilities for the behavior of semi-orbits near the saddle point.

Case 1:
If $w(y) > 0$ and $s(y) < u_-$ for $y \in (-\infty, 0] = I$, then of the (13) we have $s'(y) > 0$ in I, so $s(y)$ is increscent. Consequently, $\lim_{y \to -\infty} s(y) \neq u_-$.

Therefore $(s(y), w(y)) \nrightarrow (u_-, 0)$ as $y \to -\infty$ and the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : \, s < u_-, w > 0\}$.

Case 2:
If $w(y) < 0$ and $s(y) > u_-$ for $y \in (-\infty, 0] = I$, then of the (13) we have $s'(y) < 0$ in I, so $s(y)$ is decrescent in I. Consequently, $\lim_{y \to -\infty} s(y) \neq u_-$. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s > u_-, w < 0\}$.

Case 3:
If $w(y) = 0$ and $s(y) > u_-$ in I. It follows from (13) that $s'(y) = 0$ in I, so $s(y)$ constant. If $\lim_{y \to -\infty} s(y) = u_-$, we have $s(y) = u_-$, contradiction. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s > u_-, w = 0\}$.

Case 4:
If $w(y) = 0$ and $s(y) < u_-$ in I, is entirely analogous to the previous item. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s < u_-, w = 0\}$.

Case 5:
Se $s(y) = u_-$ para $y \in (-\infty, 0] = I$, is similar. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : \, s = u_-, w = 0\}$.

Case-possible: The Theorem 19 ensures that there are two semi-orbit along a smooth curve tangent in $(u_-,0)$ to the eigenvectors of λ_2 (up to translation). Therefore, we have a semi-orbit tending to the point $(u_-,0)$ of the region $S=\{(s,w)\in\mathbb{R}^2 : s<u_-, w<0\}$, while the other approaches from region $T=\{(s,w)\in\mathbb{R}^2 : s>u_-, w>0\}$.

8. Appendix II

We consider an partial differential equations of the form

$$u_t+uu_x=au_{xx}+bu_{xxx} \tag{28}$$

where $a>0, b>0$. We seek existence of *traveling waves* solution $u(x,t)=s(x-ct)$, for some constant speed $c\in\mathbb{R}$, satisfying the following conditions at infinity :

$$\lim_{y\to\pm\infty} s^{(j)}(y)=0,\quad j=1,2 \quad \text{and} \quad \lim_{y\to\pm\infty} s(y)=u_\pm,\quad u_-\neq u_+. \tag{29}$$

Note that following [3] the following problem is equivalent (28)-(29), now consider

$$u_t+uu_x=au_{xx}+bu_{xxx} \tag{30}$$

where $a>0, b>0$,

and new conditions at infinity:

$$\lim_{y\to\pm\infty} s^{(j)}(y)=0,\ \ j=1,2,\ \ \lim_{y\to-\infty} s(y)=2\eta>0 \ \textit{and}\ \lim_{y\to\infty} s(y)=0. \tag{31}$$

Now we must only determine p as (22), i.e, let is go to obtain p such that

$$\int_p^{2\eta} bh(x)\,dx>0 \tag{32}$$

with $-2\eta<p<0$ and $h(x)=x^2-2\eta x$. As we have $b>0$ just take

$$\int_p^{2\eta} h(x)\,dx=1/3(p-2\eta)(-p^2+p\eta+2\eta^2)>0. \tag{33}$$

So choosing $p\in(-2\eta,-\eta)$ we have

$$\int_p^{2\eta} bh(x)\,dx>0. \tag{34}$$

Therefore we are able to apply our work and we have existence and uniqueness of traveling waves when the flux-function associated is given by $f(u)=u^2/2$.

For the BBM-Burgers equation

$$u_t+f(u)_x=au_{xx}+bu_{xxt} \tag{35}$$

where $a>0$ and $b\in\mathbb{R}$ we establish the classification of equilibrium points as in section 3.1 keeping $c>0$ and the sign of b was studied separately to establish the desired connection. When $c < 0$ proceed in the same manner. For $b = 0$ we refer to [11] for existence.

For the equation (35) the associated system becomes

$$\begin{cases} s'(y) = \dfrac{w(y)}{-bc} \\ \\ w'(y) = h(s(y)) - \dfrac{a}{-bc}w(y) \end{cases} \tag{36}$$

and $V(s,w) = -\int_{u_+}^{s} -bch(x)\,dx + \dfrac{w^2}{2}$.

Acknowledgement

The second author was supported by Capes, Brazil.

References

[1] Burgers J M. Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. Trans Roy Neth Acad Sci Amsterdam, **17**: 1-53 (1939).
[2] A. V. Kazeykina , Examples of the absence of a traveling wave for generalized-De Vries-Burgers equation, *Facult of Mathematics and Cybernets, Moscow State University,* Vol. 35, N1, pp. 14-21, (2011).
[3] Bona, M. E. Schonbek, traveling waves solutions to the Korteweg-de Vries-Burgers equation, Proc. Roy. Soc. Edinburg, **101** A , 207-226, (1985).
[4] D. Jacobs, W. McKinney, and M. Shearer, Traveling wave solutions of the modified Korteweg-deVries-Burgers equation. J. Differ. Equations, **116**(2), 448-467, (1995).
[5] Korteweg D J, de Vries G. On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves. Phil Mag, **39**: 422-443 (1895).
[6] M. D. Thanh, Global existence of traveling waves for general flux functions, *Nonlinear Anal.: T.M.A* **72**(1), 231-239, (2010).
[7] M. D. Thanh, Existence of traveling waves of conservation laws with singular diffsion and nonlinear dispersion, *Bull. Malays. Math. Sci. Soc.*(2) **35**(2) (2012), 383-398.
[8] H. K. Khalil, Nonlinear Systems. *Prentice Hall*, New Jersey, (2002).
[9] N. Bedjaoui and P.G. LeFloch, Diffusive-dispersive traveling waves and Kinetic relations. I. Nonconvex hyperbolic conservation laws, J. Differential Equations 178 (2002), 574-607.
[10] N. Bedjaoui and P.G. LeFloch, Diffusive-dispersive traveling waves and Kinetic relations. V. Singular diffusion and dispersion terms, Proc. Royal Soc. Edinburgh 134A (2004), 815-844.
[11] P.G. LeFloch, Hyperbolic Systems of Conservation Laws, Lectures in Mathematics, ETH Zurich, Birkhauser Verlag, Basel (2002).
[12] Lawrence Perko, Differential Equations and Dynamical Systems, *Springer-Verlag*, New York, (2000).
[13] J. H. Hubbard and B. H. West, Differential Equations: A Dynamical Systems Approach, Higher-Dimensional Systems, Springer-Verlag, New York, 1995.

Federal University of Sao Carlos, Department of Mathematics, P. O. Box 676 13565-905, Sao Carlos-SP, Brazil.
E-mail address: [1]`dcik@dm.ufscar.br`

E-mail address: [2]`alexrossini@dm.ufscar.br`