

# BORSUK-ULAM THEOREMS AND THEIR PARAMETRIZED VERSIONS FOR SPACES OF TYPE $(a, b)$

DENISE DE MATTOS<sup>1</sup>, PEDRO L. Q. PERGHER<sup>2</sup> AND EDIVALDO L. DOS SANTOS<sup>3</sup>

**ABSTRACT.** Let  $X$  be a space of type  $(a, b)$  equipped with a free  $G$ -action, with  $G = \mathbb{Z}_2$  or  $S^1$ . In this paper, we prove some theorems of Borsuk-Ulam type and the corresponding parametrized versions for such  $G$ -spaces.

*Key words:* Space of type  $(a, b)$ , parametrized Borsuk-Ulam theorem, Leray-Serre spectral sequence, Borel construction, characteristic polynomial, free action, equivariant map.

## 1. INTRODUCTION

Following [11] and [13], we define a space of type  $(a, b)$  as follow. Let  $X$  be a simply connected finite CW complex with  $\mathbb{Z}$ -cohomology groups satisfying  $H^j(X; \mathbb{Z}) = \mathbb{Z}$ , if  $j = 0, n, 2n$  or  $3n$ , and  $H^j(X; \mathbb{Z}) = 0$ , otherwise ( $n > 1$ ). Let  $u_i$  generate  $H^{in}(X; \mathbb{Z})$ , for  $i = 0, 1, 2$  and  $3$ . Then the structure of the  $\mathbb{Z}$ -cohomology ring of  $X$  is determined by the two integers  $a$  and  $b$  for which  $u_1^2 = au_2$  and  $u_1u_2 = bu_3$ . In this case,  $X$  is said to be of type  $(a, b)$ . These spaces include certain products of spheres and projective spaces, and were first studied by James [6] and Toda [16].

In [11], Pergher et al. proved that  $G = \mathbb{Z}_2$  cannot act freely on a space of type  $(a, b)$  if  $a$  is odd and  $b$  is even, and  $G = S^1$  cannot act freely on a space of type  $(a, b)$  if  $a \neq 0$ . For the remaining  $(a, b)$ , we may have free actions, for example,  $S^3 \times S^6$  is of type  $(0, 1)$  and admits free  $G$ -actions for  $G = \mathbb{Z}_2$  and  $S^1$  (for other examples, see [3] and [13]), and also in [11] the possible  $\mathbb{Z}_2$ -cohomology rings of orbit spaces  $X/G$  of free actions of  $G = \mathbb{Z}_2$  on spaces of type  $(a, b)$ , where  $a$  and  $b$  are even, and of free actions of  $G = S^1$  on spaces of type  $(0, b)$ , were determined. For  $G = S^1$ , one has two possibilities for the ring structure of the  $\mathbb{Z}_2$ -cohomology of  $X/G$ , which are described in Theorem 2.5. We denote by  $\Lambda_1$ , (respectively  $\Lambda_2$ ), the collection of all free  $G = S^1$ -actions on  $X$  for which  $H^*(X/G; \mathbb{Z}_2)$  has the structure described in Theorem 2.5( $\Lambda_1$ ), (respectively, in Theorem 2.5( $\Lambda_2$ )).

The first aim of this paper is to prove results of Borsuk-Ulam type involving spaces  $X$  of type  $(a, b)$ . For general information about the Borsuk-Ulam theorem, including many of the concepts in this paper, the book [8] of Matoušek is

---

2000 *Mathematics Subject Classification.* Primary 55M20 Secondary 55R91, 55R25 .

<sup>1</sup>Supported by FAPESP of Brazil, Grant 2011/18758-1.

<sup>2</sup>Supported by FAPESP and CNPq.

<sup>3</sup>Supported by FAPESP of Brazil, Grant 2011/18761-2.

recommended. In this direction, the results below concern to the existence of equivariant maps.

**Theorem 1.1.** (i) *Let  $X$  be a space of type  $(a, b)$ , characterized by a natural number  $n > 1$ , where  $a$  and  $b$  are even, and let  $Y$  be a Hausdorff, pathwise connected and paracompact space. Suppose that  $X$  and  $Y$  are equipped with free  $\mathbb{Z}_2$ -actions and  $H^{k+1}(Y/G; \mathbb{Z}_2) = 0$ , for some  $k$ ,  $1 \leq k < 3n$ . Then, there is no  $\mathbb{Z}_2$ -equivariant map  $f : X \rightarrow Y$ .*

(ii) *Let  $X$  be a space of type  $(0, b)$ , characterized by a natural odd number  $n > 1$ , and let  $Y$  be a Hausdorff, pathwise connected and paracompact space. Suppose that  $X$  is equipped with a free  $S^1$ -action  $\rho \in \Lambda_1$ , (respectively  $\rho \in \Lambda_2$ ), and  $Y$  is equipped with a free  $S^1$ -action; further, suppose  $H^{k+1}(Y/G; \mathbb{Z}_2) = 0$ , for some  $k$ , with  $1 \leq k < 3n$ , (respectively  $1 \leq k < n$ ). Then, there is no  $S^1$ -equivariant map  $f : X \rightarrow Y$ .*

**Remark 1.2.** In the above direction, some related results were obtained in [11], concerning the existence of equivariant maps  $S^m \rightarrow X$ , where  $S^m$  is equipped with standard  $G$ -actions ( $G = \mathbb{Z}_2$  or  $S^1$ ) and  $X$  is a space of type  $(a, b)$  equipped with arbitrary free  $G$ -actions.

Note that, in Theorem 1.1,  $Y$  can be taken as the  $k$ -dimensional sphere  $S^k$ .

In addition, the following Borsuk-Ulam type theorems will be obtained.

**Theorem 1.3.** *Let  $X$  be a space of type  $(a, b)$ , characterized by a natural number  $n > 1$ , where  $a$  and  $b$  are even. Suppose  $X$  is equipped with a free  $\mathbb{Z}_2$ -action, determined by a free involution  $T : X \rightarrow X$ .*

(i) *Then, for every continuous map  $f : X \rightarrow \mathbb{R}^k$ ,*

$$\text{cov.dim } A(f) \geq 3n - k, \text{ if } 3n \geq k,$$

*where  $A(f)$  denotes the  $\mathbb{Z}_2$ -coincidence set of  $f$  (that is,  $A(f) = \{x \in X \mid f(x) = f(T(x))\}$ ).*

(ii) *If  $Y$  is a finite  $k$ -dimensional CW complex and  $3n \geq 2k$ , then for every continuous map  $f : X \rightarrow Y$ ,  $A(f)$  is non-empty.*

**Remark 1.4.** Theorem 1.3(i) is the Yang version of the Borsuk-Ulam theorem for spaces of type  $(a, b)$ . In particular, we will compute the  $\mathbb{Z}_2$ -index of Yang for these  $\mathbb{Z}_2$ -spaces. Theorem 1.3(ii) has the spirit of the results of Izydorek and Jaworowski [5], with spheres being replaced by spaces of type  $(a, b)$ .

The second general goal of this paper is to prove parametrized Borsuk-Ulam theorems for spaces of type  $(a, b)$ . Jaworowski [7], Dold [2], Nakaoka [10] and others extended the Borsuk-Ulam problem to a fibrewise setting, by considering continuous maps  $f : S(E) \rightarrow E'$  which preserve fibres, where  $E$  and  $E'$  are total spaces of vector bundles over a space  $B$  and  $S(E)$  means the associated sphere bundle. In this direction, related results were proved in [9] (for bundles whose fibre has the same cohomology (mod  $p$ ) of a product of spheres, with any free

$\mathbb{Z}_p$ -action, and for bundles whose fibre has the same rational cohomology ring as a product of spheres, with any free  $S^1$ -action), and in [14] (for bundles whose fibre has the mod 2 cohomology algebra of a real or complex projective space, with any free involution).

In this paper, we obtain results of this nature, for bundles whose fibre is a space of type  $(a, b)$  with any free  $\mathbb{Z}_2$ -action and  $a, b$  even (or free  $S^1$ -action with  $a = 0$ ). Specifically, we will prove the following two theorems.

**Theorem 1.5.** *Let  $X$  be a space of type  $(a, b)$ , characterized by a natural number  $n > 1$ , where  $a$  and  $b$  are even. Given a paracompact space  $B$ , let  $\pi : X \hookrightarrow E \rightarrow B$  be a fibre bundle equipped with a fibrewise free  $\mathbb{Z}_2$ -action, such that the quotient bundle  $\hat{\pi} : \hat{E} \rightarrow B$  has the cohomology extension property (see [15, Chapter 5 §7] and [1, p. 372]). Also, consider  $\pi' : E' \rightarrow B$ , a  $k$ -dimensional vector bundle, equipped with a fibrewise  $\mathbb{Z}_2$ -action on  $E'$ , which is free on  $E' - \{0\}$  ( $\{0\}$  is the image of the zero-section). Let  $f : E \rightarrow E'$  be a fibre preserving equivariant map and set  $Z_f = f^{-1}(\{0\})$ . If  $3n \geq k$ , then*

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } B + 3n - k.$$

**Theorem 1.6.** *Let  $X$  be a space of type  $(0, b)$ , characterized by a natural odd number  $n > 1$ . Given a paracompact space  $B$ , let  $\pi : X \hookrightarrow E \rightarrow B$  be a fibre bundle with a fibrewise free  $S^1$ -action, such that the quotient bundle  $\hat{\pi} : \hat{E} \rightarrow B$  has the cohomology extension property. Let  $\pi' : E' \rightarrow B$  be a  $k$ -dimensional vector bundle, where  $k$  is even, with fibrewise  $S^1$ -action on  $E'$ , which is free on  $E' - \{0\}$ . Consider  $f : E \rightarrow E'$ , a fibre preserving equivariant map, and set  $Z_f = f^{-1}(\{0\})$ .*

(i) *If the free  $S^1$ -action  $\rho$  on  $X$  belongs to  $\Lambda_1$  and  $3n \geq k$ , then*

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } B + 3n - k.$$

(ii) *If the free  $S^1$ -action  $\rho$  on  $X$  belongs to  $\Lambda_2$  and  $n \geq k$ , then*

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } B + n - k.$$

Finally, in the next result, we estimate the size of the  $\mathbb{Z}_2$ -coincidence set of a fibre preserving map.

**Theorem 1.7.** *Let  $X$  be a space of type  $(a, b)$ , characterized by a natural number  $n > 1$ , where  $a$  and  $b$  are even. Given a paracompact space  $B$ , let  $\pi : X \hookrightarrow E \rightarrow B$  be a fibre bundle, equipped with a fibrewise free  $\mathbb{Z}_2$ -action, such that the quotient bundle  $\hat{\pi} : \hat{E} \rightarrow B$  has the cohomology extension property. Consider  $\pi'' : E'' \rightarrow B$ , a  $k$ -dimensional vector bundle, and  $f : E \rightarrow E''$  be a fibre preserving map. As before, consider  $A(f) = \{x \in E \mid f(x) = f(Tx)\}$ , the  $\mathbb{Z}_2$ -coincidence set of  $f$ , where  $T : E \rightarrow E$  is the generator of the free  $\mathbb{Z}_2$ -action on  $E$ . If  $3n \geq k$ , then*

$$\text{cohom.dim } A(f) \geq \text{cohom.dim } B + 3n - k.$$

**Remark 1.8.** If  $B$  is a point, Theorem 1.7 reduces to Theorem 1.3(i).

The paper is organized as follows. In Section 2, we recall the required definitions and results, and establish notations. In Section 3, we compute a numerical index for spaces of type  $(a, b)$ , which is related to the  $\mathbb{Z}_2$ -index of Yang. By using these indices, we prove Theorems 1.1 and 1.3. In Section 4, we present some lemmas involving the  $H^*(B)$ -algebra of  $H^*(\hat{E})$ . In Section 4.2, we prove such lemmas and Theorems 1.5, 1.6 and 1.7, using characteristic polynomials (these characteristic polynomials are presented in Section 4.1).

## 2. PRELIMINARIES

We start by introducing some basic facts and establishing some notations. We assume that all spaces under consideration are paracompact and Hausdorff spaces. Here  $H^*$  denotes Čech cohomology, unless otherwise indicated. The symbol “ $\cong$ ” denotes an appropriate isomorphism between algebraic objects.

Suppose that  $G$  is a compact Lie group. Write  $B_G$ , as usual, for the classifying space of  $G$  and  $E_G \rightarrow B_G$  for the universal  $G$ -bundle. Given a  $G$ -space  $X$ , there is an associated fibration  $p_X : X_G \rightarrow B_G$ , with fibre  $X$ , where  $X_G = (E_G \times X)/G$  is the Borel construction. There is also a natural map  $\eta : X_G \rightarrow X/G$  which is a homotopy equivalence if  $G$  acts freely on  $X$ , and thus in this case the cohomology rings  $H^*(X_G)$  and  $H^*(X/G)$  are isomorphic. Associated to the fibration  $p_X : X_G \rightarrow B_G$ , one has the cohomological Leray-Serre spectral sequence. This spectral sequence has

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; R)),$$

as its  $E_2$ -term and converges to  $H^*(X_G; R)$  as an algebra in the sense of Bredon, where  $R$  is a commutative ring with unit; here,  $H^k(B_G; \mathcal{H}^l(X; R))$  is the cohomology of  $B_G$  with local coefficients in the cohomology of  $X$ .

Suppose that  $X$  is connected. Then the local coefficients system  $\mathcal{H}^0(X; R)$  over  $B_G$  is trivial and

$$E_2^{*,0} = H^*(B_G; H^0(X; R)) = H^*(B_G; R).$$

We say that the index of the  $G$ -space  $X$  is  $s$ , which depends on  $R$ , and we write  $i(X; R) = s$ , if the following condition is satisfied:

$$E_2^{*,0} = \dots = E_s^{*,0} \neq E_{s+1}^{*,0}.$$

If  $E_2^{*,0} = \dots = E_\infty^{*,0}$ , we say that  $i(X; R) = \infty$ .

This index has the following property.

**Proposition 2.1.** [17, Property (iii), p. 917] *If  $G = \mathbb{Z}_2$  and  $X$  is a free  $G$ -space, then  $i(X; \mathbb{Z}_2) = i(X)$  exceeds the  $\mathbb{Z}_2$ -index of Yang of [18] by unity, i.e.,*

$$(2.1) \quad i(X) = 1 + \mathbb{Z}_2\text{-Yang-index}(X).$$

Others results related to  $i(X)$ :

**Theorem 2.2.** [17, Theorem 2.2, p. 918,  $G = \mathbb{Z}_2$ ] *Let  $X$  be a compact and connected  $\mathbb{Z}_2$ -space such that  $i(X; \mathbb{Z}_2) \geq 2m + 1$ . Let  $Y$  be a CW complex of dimension  $m$  and  $f : X \rightarrow Y$  a continuous map. In addition, if  $i(X; \mathbb{Z}_2) = 2m + 1$ , assume that  $f^* : H^m(Y) \rightarrow H^m(X)$  is trivial. Then  $A(f)$  is non-empty.*

**Theorem 2.3.** [4, Theorem 1.1] *Let  $G$  be a compact Lie group and  $X, Y$  be Hausdorff, pathwise connected and paracompact free  $G$ -spaces. Suppose that for some natural  $m \geq 1$ ,  $i(X; R) \geq m + 1$  and  $H^{k+1}(Y/G; R) = 0$ , for some  $1 \leq k \leq m$ .*

(i) If  $k = m$  and  $\beta_m(X; R) < \beta_{m+1}(B_G; R)$ , there is no  $G$ -equivariant map  $f : X \rightarrow Y$ .

(ii) If  $1 \leq k < m$  and  $0 < \beta_{k+1}(B_G; R)$ , there is no  $G$ -equivariant map  $f : X \rightarrow Y$ ; here,  $\beta_i(\ ; R)$  denotes the  $i$ -th Betti number.

We recall the following well known facts:

$$(2.2) \quad H^*(B_G; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2[s], & \deg s = 1, \quad G = \mathbb{Z}_2 \\ \mathbb{Z}_2[t], & \deg t = 2, \quad G = S^1 \end{cases}$$

**2.1. The cohomology rings of some orbit spaces .** In [11], Pergher et al. determined the possible  $\mathbb{Z}_2$ -cohomology rings of orbit spaces  $X/G$  of free actions of  $G = \mathbb{Z}_2$  on spaces of type  $(a, b)$ , where  $a$  and  $b$  are even, and of free actions of  $G = S^1$  on spaces of type  $(0, b)$ . This is described below.

**Theorem 2.4.** [11, Theorem 4.1] *Let  $G = \mathbb{Z}_2$  act freely on a space  $X$  of type  $(a, b)$ , characterized by a natural number  $n > 1$ , where both  $a$  and  $b$  are even. Then, as a graded commutative algebra,*

$$H^*(X/G; \mathbb{Z}_2) = \mathbb{Z}_2[x, z] / \langle x^{3n+1}, z^2 + \alpha x^n z + \beta x^{2n}, x^{n+1} z \rangle,$$

where  $\alpha, \beta \in \mathbb{Z}_2$ ,  $\deg x = 1$  and  $\deg z = n$ .

**Theorem 2.5.** [11, Theorem 4.2] *Let  $G = S^1$  act freely on a space  $X$  of type  $(0, b)$ , characterized by a natural number  $n > 1$ . Then  $H^*(X/G; \mathbb{Z}_2)$  is isomorphic to one of the following graded commutative algebras:*

$$(\Lambda_1) \quad \mathbb{Z}_2[y, z] / \langle y^{\frac{3n+1}{2}}, z^2 + \alpha y^n, y^{\frac{n+1}{2}} z \rangle, \text{ where } \alpha \in \mathbb{Z}_2, \deg y = 2, \text{ and } \deg z = n,$$

$$(\Lambda_2) \quad \mathbb{Z}_2[y, z] / \langle y^{\frac{n+1}{2}}, z^2 \rangle, \text{ where } \deg y = 2, \deg z = 2n, \text{ and } b \text{ is odd.}$$

### 3. PROOFS OF THE THEOREMS OF BORSUK-ULAM TYPE

In this section, we prove Theorems 1.1 and 1.2. We need the following lemma, where the  $G$ -spaces  $X$  are understood as those described in the statements of these two theorems.

**Lemma 3.1.** (i) If  $G = \mathbb{Z}_2$  or  $G = S^1$ , with  $\rho \in \Lambda_1$ , then

$$i(X; \mathbb{Z}_2) = 3n + 1.$$

(ii) If  $G = S^1$ , with  $\rho \in \Lambda_2$ , then

$$i(X; \mathbb{Z}_2) = n + 1.$$

*Proof.* In the case  $G = \mathbb{Z}_2$ , for the generators of  $H^*(X; \mathbb{Z}_2)$ , we have the relations  $u_1^2 = 0$  and  $u_1 u_2 = 0$ . For the corresponding spectral sequence, one has that  $E_2^{k,l} \cong H^k(BG) \otimes H^l(X)$ , the sequence does not collapse at the  $E_2$ -term and no line can survive to infinity (see [11, proof of Theorem 4.1]). By the multiplicative properties of the spectral sequence, we have  $d_{n+1}(1 \otimes u_1) = 0$ ,  $d_{n+1}(1 \otimes u_3) = 0$  and  $d_{n+1}(1 \otimes u_2) \neq 0$ . Therefore, we get that  $E_{n+2}^{k,l} = \mathbb{Z}_2$ , for every  $k$ , if  $l = 0$  or  $l = 3n$ . Also, we have  $E_{n+2}^{k,l} = \mathbb{Z}_2$ , for  $k = 0, 1, 2, \dots, n$ , if  $l = n$ . In the remaining cases,  $E_{n+2}^{k,l} = 0$ . Again, the multiplicative properties show that  $d_{n+2}(1 \otimes u_i) = 0$ , for  $i = 1, 2, 3$ , and  $d_{n+3}(1 \otimes u_3) \neq 0$ .

Then, the differential

$$d_{3n+1} : E_{3n+1}^{k,3n} \rightarrow E_{3n+1}^{k+3n+1,0}$$

is an isomorphism and for all  $k \geq 0$ ,

$$E_{3n+2}^{k+3n+1,0} = \frac{\ker d_{3n+1}}{\text{im } d_{3n+1}} = \frac{E_{3n+1}^{k+3n+1,0}}{E_{3n+1}^{k+3n+1,0}} = 0 \neq E_{3n+1}^{k+3n+1,0}.$$

Thus,

$$E_2^{*,0} = \dots = E_{3n+1}^{*,0} \neq E_{3n+2}^{*,0},$$

which implies  $i(X; \mathbb{Z}_2) = 3n + 1$ .

For  $G = S^1$ , in both cases, the proof is analogous by using the properties of the corresponding spectral sequence given in [11, Proof of Theorem 4.2].  $\square$

*Proof of Theorem 1.1.* For (i), by Lemma 3.1(i), we have that  $i(X; \mathbb{Z}_2) = 3n + 1$ . Since  $H^{k+1}(Y/\mathbb{Z}_2; \mathbb{Z}_2) = 0$  for some  $1 \leq k < 3n$  and  $\beta_{k+1}(B\mathbb{Z}_2) = 1$ , it follows from Theorem 2.3(ii) that there is no equivariant map  $X \rightarrow Y$ . The argument is analogous for (ii).  $\square$

*Proof of Theorem 1.3.* By Lemma 3.1(i),  $i(X; \mathbb{Z}_2) = 3n + 1$ . It follows from Proposition 2.1 that the  $\mathbb{Z}_2$ -index of  $X$  is  $3n$ . Thus, from [18, Theorem 4.1, p. 270],

$$\text{cov.dim } A(f) \geq 3n - k,$$

which proves (i). For (ii), since  $i(X; \mathbb{Z}_2) = 3n + 1 \geq 2k + 1$  and additionally if  $i(X; \mathbb{Z}_2) = 2k + 1$ ,  $f^* : H^k(Y) \rightarrow H^k(X)$  is trivial, it follows from Theorem 2.2 that  $A(f)$  is non-empty.  $\square$

4. PROOF OF PARAMETRIZED BORSUK-ULAM THEOREMS FOR SPACES OF TYPE  $(a, b)$

In this section, we prove Theorems 1.5, 1.6 and 1.7. First we develop a technical discussion on the objects involved in these theorems, for which will be assumed the hypotheses described in their statements. We will need some lemmas involving the  $H^*(B)$ -algebra of  $H^*(\hat{E})$ , where  $\hat{E}$  is the total space of quotient bundle  $\hat{\pi} : \hat{E} \rightarrow B$ .

Given a topological space  $X$  of type  $(a, b)$ , where  $a$  and  $b$  are even (respectively, a topological space  $X$  of type  $(0, b)$ ), let  $\pi : X \hookrightarrow E \rightarrow B$  be a fibre bundle equipped with a fibrewise free  $\mathbb{Z}_2$ -action (respectively, fibrewise free  $S^1$ -action) such that the quotient bundle  $\hat{\pi} : \hat{E} \rightarrow B$  has the cohomology extension property. Consider  $\pi' : E' \rightarrow B$  a  $k$ -dimensional vector bundle equipped with a fibrewise  $G$ -action ( $G = \mathbb{Z}_2$  or  $S^1$ ), which is free on  $E' - \{0\}$ . If  $f : E \rightarrow E'$  is a fibre preserving  $G$ -equivariant map, write  $Z_f = f^{-1}(\{0\})$  and  $\hat{Z}_f = Z_f/G$ .

Let  $H^*(B)[x, z]$  be the polynomial ring over  $H^*(B)$  in the indeterminates  $x$  and  $z$ . For  $G = \mathbb{Z}_2$ , in Section 4.1 we will introduce certain characteristic polynomials belonging to  $H^*(B)[x, z]$ , denoted by  $W_1(x, z)$ ,  $W_2(x, z)$  and  $W_3(x, z)$ , and will show that  $H^*(\hat{E})$  and  $H^*(B)[x, z]/\langle W_1(x, z), W_2(x, z), W_3(x, z) \rangle$  are isomorphic as  $H^*(B)$ -modules. Therefore, each polynomial  $q(x, z)$  in  $H^*(B)[x, z]$  determines an element of  $H^*(\hat{E})$ , which will be denoted by  $q(x, z)|_{\hat{E}}$ . We will write  $q(x, z)|_{\hat{Z}_f}$  for the image of  $q(x, z)|_{\hat{E}}$  by the  $H^*(B)$ -homomorphism

$$i^* : H^*(\hat{E}) \rightarrow H^*(\hat{Z}_f),$$

where  $i^*$  is the homomorphism induced by the natural inclusion.

Similarly, for  $G = S^1$ , we will show that if the free  $S^1$ -action on  $X$  is in  $\Lambda_1$  (respectively in  $\Lambda_2$ ), then  $H^*(B)[y, z]/\langle W_1(y, z), W_2(y, z), W_3(y, z) \rangle$  (respectively  $H^*(B)[y, z]/\langle W_1(y, z), W_2(y, z) \rangle$ ) and  $H^*(\hat{E})$  are isomorphic as  $H^*(B)$ -modules; again,  $W_1(y, z)$ ,  $W_2(y, z)$  and  $W_3(y, z)$  will be certain special characteristic polynomials belonging to  $H^*(B)[y, z]$ . Therefore, each polynomial  $q(y, z)$  in  $H^*(B)[y, z]$  yields elements  $q(y, z)|_{\hat{E}}$  and  $q(y, z)|_{\hat{Z}_f}$  in  $H^*(\hat{E})$  and  $H^*(\hat{Z}_f)$ , respectively.

Also, we will recall the known characteristic polynomial  $W'(x)$ , used by Dold [2] (and called there the Stiefel-Whitney polynomial), which is a characteristic polynomial in the indeterminate  $x$  of degree 1, associated to the vector bundle  $\pi' : E' \rightarrow B$ . With these objects in hand, we have the following lemmas.

**Lemma 4.1** (Case  $G = \mathbb{Z}_2$ ). *Suppose that  $q(x, z) \in H^*(B)[x, z]$  is a polynomial satisfying  $q(x, z)|_{\hat{Z}_f} = 0$ . Then, there are polynomials  $r_1(x, z)$ ,  $r_2(x, z)$  and  $r_3(x, z)$  in  $H^*(B)[x, z]$  so that*

$$q(x, z)W'(x) = r_1(x, z)W_1(x, z) + r_2(x, z)W_2(x, z) + r_3(x, z)W_3(x, z).$$

**Lemma 4.2** (Case  $G = S^1$ ). *Suppose that  $q(y, z) \in H^*(B)[y, z]$  is a polynomial satisfying  $q(y, z)|_{\hat{Z}_f} = 0$ .*

(i) *If the free  $S^1$ -action on  $X$  is in  $\Lambda_1$ , then there are polynomials  $r_1(y, z)$ ,  $r_2(y, z)$  and  $r_3(y, z)$  in  $H^*(B)[y, z]$  so that*

$$q(y, z)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z) + r_3(y, z)W_3(y, z).$$

(ii) *If the free  $S^1$ -action on  $X$  is in  $\Lambda_2$ , then there are polynomials  $r_1(y, z)$  and  $r_2(y, z)$  in  $H^*(B)[y, z]$  so that*

$$q(y, z)W'(y) = r_1(y, z)W_1(y) + r_2(y, z)W_2(y, z).$$

**4.1. Characteristic polynomials.** As announced above and using the Dold technique, in this section we introduce the characteristic polynomials associated to the fibre bundle  $(X, E, \pi, B)$ . Since the quotient bundle  $(X/G, \hat{E}, \hat{\pi}, B)$  ( $G$  is  $\mathbb{Z}_2$  or  $S^1$ ) has the cohomology extension property, the Leray-Hirsch theorem can be applied (see [1, Chapter VII, Theorem 1.4]). There are two cases to consider.

4.1.1. *Case  $G = \mathbb{Z}_2$ .* From Theorem 2.4, one has that  $H^*(X/G; \mathbb{Z}_2)$  is a free graded module generated by the elements

$$1, a, a^2, \dots, a^{3n-1}, a^{3n}, c, ac, \dots, a^n c,$$

subject to the relations  $a^{3n+1} = 0$ ,  $c^2 + \alpha a^n c + \beta a^{2n} = 0$  and  $a^{n+1}c = 0$ , where  $a \in H^1(X/G; \mathbb{Z}_2)$ ,  $c \in H^n(X/G; \mathbb{Z}_2)$  and  $\alpha, \beta \in \mathbb{Z}_2$ .

It follows from the Leray-Hirsch theorem that there exist elements  $\mathbf{a} \in H^1(\hat{E})$  and  $\mathbf{c} \in H^n(\hat{E})$  such that the natural homomorphism  $j^* : H^*(\hat{E}) \rightarrow H^*(X/G)$  maps  $\mathbf{a}$  into  $a$  and  $\mathbf{c}$  into  $c$ . Further, via the induced homomorphism  $\hat{\pi}^*$ ,  $H^*(\hat{E})$  is an  $H^*(B)$ -module generated by

$$(4.1) \quad 1, \mathbf{a}, \mathbf{a}^2, \dots, \mathbf{a}^{3n-1}, \mathbf{a}^{3n}, \mathbf{c}, \mathbf{a}\mathbf{c}, \dots, \mathbf{a}^n \mathbf{c}.$$

We can express the elements  $\mathbf{a}^{3n+1} \in H^{3n+1}(\hat{E})$ ,  $\mathbf{a}^{n+1}\mathbf{c} \in H^{2n+1}(\hat{E})$  and  $\mathbf{c}^2 + \alpha\mathbf{a}^n\mathbf{c} + \beta\mathbf{a}^{2n} \in H^{2n}(\hat{E})$  in terms of the basis (4.1), that is, there exist unique elements  $\omega_i, \bar{\omega}_i, \nu_i, \bar{\nu}_i, \mu_i, \bar{\mu}_i \in H^i(B)$  such that

$$\begin{aligned} \mathbf{a}^{3n+1} &= \omega_{3n+1} + \omega_{3n}\mathbf{a} + \dots + \omega_1\mathbf{a}^{3n} + \bar{\omega}_{2n+1}\mathbf{c} + \bar{\omega}_{2n}\mathbf{a}\mathbf{c} + \dots + \bar{\omega}_{n+1}\mathbf{a}^n\mathbf{c}, \\ \mathbf{a}^{n+1}\mathbf{c} &= \nu_{2n+1} + \nu_{2n}\mathbf{a} + \dots + \nu_1\mathbf{a}^{2n} + \gamma\mathbf{a}^{2n+1} + \bar{\nu}_{n+1}\mathbf{c} + \bar{\nu}_n\mathbf{a}\mathbf{c} + \dots + \bar{\nu}_1\mathbf{a}^n\mathbf{c}, \\ \mathbf{c}^2 + \alpha\mathbf{a}^n\mathbf{c} + \beta\mathbf{a}^{2n} &= \mu_{2n} + \mu_{2n-1}\mathbf{a} + \dots + \mu_1\mathbf{a}^{2n-1} + \beta'\mathbf{a}^{2n} + \bar{\mu}_n\mathbf{c} + \dots \\ &\quad + \bar{\mu}_1\mathbf{a}^{n-1}\mathbf{c} + \alpha'\mathbf{a}^n\mathbf{c}, \end{aligned}$$

where  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{Z}_2$ . The announced characteristic polynomials in the indeterminates  $x$  and  $z$  of degrees 1 and  $n$ , respectively, associated to the fibre bundle



$(X, E, \pi, B)$ , are then defined by the following formulas:

$$\begin{aligned} W_1(x, z) &= \omega_{3n+1} + \omega_{3n}x + \cdots + \omega_1x^{3n} + x^{3n+1} + \bar{\omega}_{2n+1}z + \cdots + \bar{\omega}_{n+1}x^n z, \\ W_2(x, z) &= \nu_{2n+1} + \nu_{2n}x + \cdots + \nu_1x^{2n} + \gamma x^{2n+1} + \bar{\nu}_{n+1}z + \cdots + \bar{\nu}_1x^n z + x^{n+1}z, \\ W_3(x, z) &= \mu_{2n} + \mu_{2n-1}x + \cdots + \mu_1x^{2n-1} + (\beta + \beta')x^{2n} + \bar{\mu}_n z + \cdots \\ &\quad + \bar{\mu}_1x^{n-1}z + (\alpha + \alpha')x^n z + z^2. \end{aligned}$$

Consider the homomorphism of  $H^*(B)$ -algebras,

$$(4.2) \quad \sigma : H^*(B)[x, z] \rightarrow H^*(\hat{E}) \text{ determined by } (x, z) \mapsto (\mathbf{a}, \mathbf{c}).$$

We have that  $\text{Ker}(\sigma)$  is the ideal generated by the characteristic polynomials  $W_1(x, z)$ ,  $W_2(x, z)$  and  $W_3(x, z)$  and, consequently,

$$(4.3) \quad H^*(B)[x, z]/\langle W_1(x, z), W_2(x, z), W_3(x, z) \rangle \cong H^*(\hat{E}).$$

### The characteristic polynomial for the bundle $\pi' : E' \rightarrow B$ .

Following [2] and [10], we first recall the characteristic polynomial associated to a  $k$ -dimensional vector bundle  $\pi' : E' \rightarrow B$ , equipped with a fibrewise  $\mathbb{Z}_2$ -action which is free on  $E' - (\{0\})$ . Write  $SE'$  for the total space of the sphere bundle associated to  $\pi' : E' \rightarrow B$ . Since  $\mathbb{Z}_2$  acts freely on  $SE'$ , we obtain the projective bundle  $(\mathbb{R}P^{k-1}, \hat{S}E', \hat{\pi}', B)$  and the principal  $\mathbb{Z}_2$ -bundle  $SE' \rightarrow \hat{S}E'$ . We have that

$$H^*(\mathbb{R}P^{k-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2[a']/\langle a'^k \rangle,$$

where  $a' = (i')^*(s)$ ,  $s \in H^1(B\mathbb{Z}_2)$  is the generator and  $i' : \mathbb{R}P^{k-1} \rightarrow B\mathbb{Z}_2$  is a classifying map for the principal  $\mathbb{Z}_2$ -bundle  $S^{k-1} \rightarrow \mathbb{R}P^{k-1}$ . Consider the class  $\mathbf{a}' = h^*(s) \in H^1(\hat{S}E')$ , where  $h : \hat{S}E' \rightarrow B\mathbb{Z}_2$  is a classifying map for the principal  $\mathbb{Z}_2$ -bundle  $SE' \rightarrow \hat{S}E'$ . The  $\mathbb{Z}_2$ -module homomorphism  $\theta : H^*(\mathbb{R}P^{k-1}) \rightarrow H^*(\hat{S}E')$  defined by  $a' \mapsto \mathbf{a}'$  is a cohomology extension of the fibre. Then, it follows from the Leray-Hirsch theorem that  $H^*(\hat{S}E')$  is an  $H^*(B)$ -module, via the induced homomorphism  $(\hat{\pi}')^*$ , generated by the elements

$$1, \mathbf{a}', (\mathbf{a}')^2, \dots, (\mathbf{a}')^{k-1}.$$

We can express  $(\mathbf{a}')^k \in H^k(\hat{S}E')$  as

$$(\mathbf{a}')^k = \omega'_k + \omega'_{k-1}\mathbf{a}' + \cdots + (\mathbf{a}')^{k-1},$$

for unique elements  $\omega'_i \in H^i(B)$ . Following the usual pattern, the characteristic polynomial in the indeterminate  $x$  of degree 1, associated to the vector bundle  $\pi' : E' \rightarrow B$ , is defined as

$$W'(x) = \omega'_k + \omega'_{k-1}x + \cdots + \omega'_1x^{k-1} + x^k.$$

As before, we then have the isomorphism of  $H^*(B)$ -algebras

$$H^*(B)[x]/\langle W'(x) \rangle \cong H^*(\hat{S}E'),$$

which comes from the rule  $x \mapsto \mathbf{a}'$ .

4.1.2. *Case  $G = S^1$ .* Taking the previously considered fibre bundle  $(X, E, \pi, B)$ , let us now consider the quotient bundle  $(X/G, \hat{E}, \hat{\pi}, B)$ . It follows from Theorem 2.5 and Leray-Hirsch theorem that  $H^*(\hat{E})$  is  $H^*(B)$ -isomorphic to one of the following  $H^*(B)$ -algebras:

(i) If the free  $S^1$ -action  $\rho$  on  $X$  is in  $\Lambda_1$ ,

$$(4.4) \quad H^*(B)[y, z]/\langle W_1(y, z), W_2(y, z), W_3(y, z) \rangle,$$

where the characteristic polynomials associated to the fibre bundle  $(X, E, \pi, B)$ , in the indeterminates  $y$  and  $z$ , of degrees 2 and  $n$ , respectively, are given by:

$$\begin{aligned} W_1(y, z) &= \omega_{3n+1} + \omega_{3n-1}y + \cdots + \omega_2 y^{\frac{3n-1}{2}} + y^{\frac{3n+1}{2}} + \bar{\omega}_{2n+1}z + \cdots \\ &\quad + \bar{\omega}_{n+2} y^{\frac{n-1}{2}} z, \\ W_2(y, z) &= \nu_{2n+1} + \nu_{2n-1}y + \cdots + \nu_2 y^{\frac{2n-1}{2}} + \bar{\nu}_{n+1}z + \bar{\nu}_{n-1}yz + \cdots \\ &\quad + \bar{\nu}_2 y^{\frac{n-1}{2}} z + y^{\frac{n+1}{2}} z, \\ W_3(y, z) &= \mu_{2n} + \mu_{2n-2}y + \cdots + \mu_2 y^{n-1} + \alpha' y^n + \bar{\mu}_n z + \bar{\mu}_{n-2}yz + \cdots \\ &\quad + \bar{\mu}_1 y^{\frac{n-1}{2}} z + z^2, \end{aligned}$$

with  $\omega_i, \bar{\omega}_i, \nu_i, \bar{\nu}_i, \mu_i, \bar{\mu}_i \in H^i(B)$  and  $\alpha' \in \mathbb{Z}_2$ .

(ii) If the free  $S^1$ -action  $\rho$  on  $X$  is in  $\Lambda_2$ ,

$$(4.5) \quad H^*(B)[y, z]/\langle W_1(y), W_2(y, z) \rangle,$$

where the characteristic polynomials in the indeterminates  $y$  and  $z$ , of degrees 2 and  $2n$ , respectively, are given by:

$$\begin{aligned} W_1(y) &= \omega_{n+1} + \omega_{n-1}y + \cdots + \omega_2 y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}} \\ W_2(y, z) &= \nu_{4n} + \nu_{4n-2}y + \cdots + \nu_{3n+1} y^{\frac{n-1}{2}} + \bar{\nu}_{2n}z + \bar{\nu}_{2n-2}yz + \cdots \\ &\quad + \bar{\nu}_{n+1} y^{\frac{n-1}{2}} z + z^2, \end{aligned}$$

with  $\omega_i, \nu_i, \bar{\nu}_i \in H^i(B)$ .

### Characteristic polynomial for the bundle $\pi' : E' \rightarrow B$ with $S^1$ -action.

Similarly to the  $\mathbb{Z}_2$ -case, we recall the characteristic polynomial associated to a  $k$ -dimensional vector bundle  $\pi' : E' \rightarrow B$ , equipped with a fibrewise  $S^1$ -action which is free on  $E' - (\{0\})$ , with  $k$  even. Denote by  $SE'$  the total space of the sphere bundle associated to  $\pi' : E' \rightarrow B$ . Since  $S^1$  acts freely on  $SE'$ , we obtain the complex projective-bundle  $(P^{\frac{k-2}{2}}(\mathbb{C}), \hat{S}E', \hat{\pi}', B)$  and the principal  $S^1$ -bundle

$SE' \rightarrow \widehat{SE}'$ , where  $P^{\frac{k-2}{2}}(\mathbb{C}) = S^{k-1}/S^1$  denotes the  $(k-2)$ -dimensional complex projective space. We have

$$H^*(P^{\frac{k-2}{2}}(\mathbb{C}); \mathbb{Z}_2) \cong \mathbb{Z}_2[b'] / ((b')^{\frac{k}{2}}),$$

with  $b' = i^*(t) \in H^2(P^{\frac{k-2}{2}}(\mathbb{C}); \mathbb{Z}_2)$ , where  $t \in H^2(BS^1; \mathbb{Z}_2)$  is the generator and  $i : P^{\frac{k-2}{2}}(\mathbb{C}) \rightarrow BS^1$  is a classifying map for the principal  $S^1$ -bundle  $S^{k-1} \rightarrow P^{\frac{k-2}{2}}(\mathbb{C})$ .

Following the same argument of the  $\mathbb{Z}_2$ -case, we have

$$\frac{H^*(B)[y]}{\langle W'(y) \rangle} \cong H^*(\widehat{SE}'), \text{ where}$$

$$W'(y) = \omega'_{m+1}1 + \omega'_{m-1}y + \cdots + \omega'_2 y^{\frac{k-2}{2}} + y^{\frac{k}{2}}$$

is the characteristic polynomial associated to  $E' \rightarrow B$ .

#### 4.2. Proofs of the announced results .

*Proof of Lemma 4.1.* The arguments follow the pattern developed by Dold [2]. Let  $q(x, z)$  be a polynomial in  $H^*(B)[x, z]$  with  $q(x, z)|_{\widehat{Z}_f} = 0$ . From the continuity property of the Čech cohomology, there is an open subset  $V \subset \widehat{E}$ , with  $V \supset \widehat{Z}_f$  and  $q(x, z)|_V = 0$ . From the exact sequence

$$\cdots \rightarrow H^*(\widehat{E}, V) \xrightarrow{j_1^*} H^*(\widehat{E}) \rightarrow H^*(V) \rightarrow \cdots,$$

there exists  $\mu \in H^*(\widehat{E}, V)$  such that  $j_1^*(\mu) = q(x, z)|_{\widehat{E}}$ , where  $j_1 : \widehat{E} \rightarrow (\widehat{E}, V)$  is the natural inclusion. Now consider the map

$$\widehat{f} : \widehat{E} - \widehat{Z}_f \rightarrow \widehat{E}' - \{0\}$$

induced by the equivariant map  $f : E \rightarrow E'$ . Since  $W'(\mathbf{a}') = 0$  and  $\widehat{f}^*$ , the homomorphism induced in cohomology, is a  $H^*(B)$ -homomorphism, we get

$$W'(x)|_{\widehat{E} - \widehat{Z}_f} = W'(\mathbf{a}) = W'(\widehat{f}^*(\mathbf{a}')) = \widehat{f}^*(W'(\mathbf{a}')) = 0.$$

On the other hand, from the exact sequence

$$\cdots \rightarrow H^*(\widehat{E}, \widehat{E} - \widehat{Z}_f) \xrightarrow{j_2^*} H^*(\widehat{E}) \rightarrow H^*(\widehat{E} - \widehat{Z}_f) \rightarrow \cdots,$$

there exists  $\theta \in H^*(\widehat{E}, \widehat{E} - \widehat{Z}_f)$  such that  $j_2^*(\theta) = W'(x)|_{\widehat{E}}$ , where  $j_2 : \widehat{E} \rightarrow (\widehat{E}, \widehat{E} - \widehat{Z}_f)$  is the inclusion. Hence,

$$q(x, z)W'(x)|_{\widehat{E}} = j_1^*(\mu)j_2^*(\theta) = j^*(\mu \smile \theta)$$

by the naturality of the cup product. Note that

$$\mu \smile \theta \in H^*(\widehat{E}, V \cup (\widehat{E} - \widehat{Z}_f)) = H^*(\widehat{E}, \widehat{E}),$$

which implies  $\mu \smile \theta = 0$ . Thus,  $q(x, z)W'(x)|_{\hat{E}} = 0$ , and from (4.3) we conclude that there exist polynomials  $r_1(x, z)$ ,  $r_2(x, z)$  and  $r_3(x, z)$  in  $H^*(B)[x, z]$  such that

$$q(x, z)W'(x) = r_1(x, z)W_1(x, z) + r_2(x, z)W_2(x, z) + r_3(x, z)W_3(x, z)$$

in the ring  $H^*(B)[x, z]$ . This completes the proof.  $\square$

*Proof of Theorem 1.5.* Let  $q(x) \in H^*(B)[x, z]$  be a nonzero polynomial such that  $\deg q(x) < 3n - k + 1$ . If  $q(x)|_{\hat{Z}_f} = 0$ , consider the equality

$$q(x)W'(x) = r_1(x, z)W_1(x, z) + r_2(x, z)W_2(x, z) + r_3(x, z)W_3(x, z),$$

given by Lemma 4.1. Note that  $\deg W'(x) = k$ ,  $\deg W_1(x, z) = 3n + 1$ ,  $\deg W_2(x, z) = 2n + 1$  and  $\deg W_3(x, z) = 2n$ . Since

$$\deg q(x) + k = \max_i \{\deg r_i(x, y) + \deg W_i(x, y)\},$$

we get

$$\deg q(x) + k \geq \deg r_1(x, y) + 3n + 1 \geq 3n + 1.$$

Therefore,  $\deg q(x) \geq 3n + 1 - k$ , which is a contradiction. Hence  $q(x)|_{\hat{Z}_f} \neq 0$ . Equivalently, the  $H^*(B)$ -homomorphism

$$\bigoplus_{i=0}^{3n-k} H^*(B) \cdot x^i \rightarrow H^*(\hat{Z}_f),$$

given by  $x \mapsto x|_{\hat{Z}_f}$ , is a monomorphism. Thus, if  $3n \geq k$ ,

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } B + 3n - k,$$

since  $\text{cohom.dim } Z_f \geq \text{cohom.dim } \hat{Z}_f$  by [12, Proposition A.11].  $\square$

Next, we prove the results for the case  $G = S^1$ .

*Proof of Lemma 4.2.* For (i), let  $q(y, z)$  be a polynomial in  $H^*(B)[y, z]$  such that  $q(y, z)|_{\hat{Z}_f} = 0$ . From arguments similar to those used in the proof of Lemma 4.1, we conclude that  $q(y, z)W'(y)|_{\hat{E}} = 0$ . Therefore, by (4.4), there are polynomials  $r_1(y, z)$ ,  $r_2(y, z)$  and  $r_3(y, z)$  in  $H^*(B)[y, z]$  such that

$$q(y, z)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z) + r_3(y, z)W_3(y, z).$$

Using (4.5), the proof for (ii) is completely analogous.  $\square$

*Proof of Theorem 1.6.* For (i), let  $q(y) \in H^*(B)[y, z]$  be a nonzero polynomial such that  $\deg q(y) < 3n - k + 1$ . If  $q(y)|_{\hat{Z}_f} = 0$ , one has from Lemma 4.2(i) that

$$q(y)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z) + r_3(y, z)W_3(y, z),$$

where  $\deg W'(y) = k$ ,  $\deg W_1(y, z) = 3n + 1$ ,  $\deg W_2(y, z) = 2n + 1$  and  $\deg W_2(y, z) = 2n$ . Thus, we conclude that  $\deg q(y, z) \geq 3n - k + 1$ , which is a contradiction. Hence  $q(y)|_{\hat{Z}_f} \neq 0$ . As above, the  $H^*(B)$ -homomorphism

$$\bigoplus_{i=0}^{\frac{3n-k-1}{2}} H^*(B).y^i \rightarrow H^*(\hat{Z}_f),$$

given by  $y^i \mapsto y^i|_{\hat{Z}_f}$ , is a monomorphism. Thus, if  $3n \geq k$ ,

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } B + 3n - k.$$

Using Lemma 4.2(ii), the proof for (ii) is completely analogous.  $\square$

Finally, we prove the announced parametrized result.

*Proof of Theorem 1.7.* Let  $\alpha$  denote the vector bundle  $E'' \rightarrow B$ , and  $V$  denote the total space of  $\alpha \oplus \alpha$ . Then,  $\mathbb{Z}_2$  acts on  $V$  by permuting coordinates in each fibre. This action has the diagonal  $\Delta \subset V$  as its fixed point set. Note that  $\Delta$  is the total space of a  $k$ -dimensional sub-bundle of  $\alpha \oplus \alpha$ , and the orthogonal complement  $\Delta^\perp$  is also the total space of a  $k$ -dimensional sub-bundle of  $\alpha \oplus \alpha$ , which is called the *diagonal bundle*. Note that  $\Delta^\perp$  is invariant under the  $\mathbb{Z}_2$ -action on  $V$ , and this restricted  $\mathbb{Z}_2$ -action on  $\Delta^\perp$  is free outside the zero section. Consider the fibre preserving  $\mathbb{Z}_2$ -equivariant map  $F : E \rightarrow V$  given by

$$F(x) = (f(x), f(Tx)).$$

The linear projection along the diagonal defines an equivariant fibre preserving map  $r : (V, V - \Delta) \rightarrow (\Delta^\perp, \Delta^\perp - 0)$ , where  $0$  is the zero section of  $\Delta^\perp$ . Let  $h = r \circ F$  be the composition

$$(E, E - A(f)) \xrightarrow{F} (V, V - \Delta) \xrightarrow{r} (\Delta^\perp, \Delta^\perp - 0).$$

Note that  $Z_h = h^{-1}(0) = F^{-1}(\Delta) = A(f)$  and  $h : E \rightarrow \Delta^\perp$  is a fibre preserving  $\mathbb{Z}_2$ -equivariant map. Applying Theorem 1.5 to the map  $h$ , if  $3n \geq k$  we obtain

$$\text{cohom.dim } A(f) = \text{cohom.dim } Z_h \geq \text{cohom.dim } B + 3n - k.$$

$\square$

**Remark 4.3.** In Theorem 1.7, we observe that the fibre preserving map  $f : E \rightarrow E''$  is not necessarily  $\mathbb{Z}_2$ -equivariant with respect to the standard fibre-wise  $\mathbb{Z}_2$ -action on  $E'' \rightarrow B$ , where the generating involution of the  $\mathbb{Z}_2$ -action is taken to be the antipodal map (in each fibre)  $x \mapsto -x$ , which is free away from the zero section. In the case that  $f : E \rightarrow E''$  is equivariant with respect to the antipodal action on  $E'' \rightarrow B$ , one has an explicit formula in the proof of Theorem 1.7; indeed, one has  $r(x, y) = ((x - y)/2, (y - x)/2)$  and thus  $h = r \circ F(x) = (f(x), -f(x))$ .

**Acknowledgements.** We are grateful to the referee for his careful reading and helpful comments and corrections concerning the presentation of this article, which led to this present version.

## REFERENCES

- [1] Bredon, G., *Introduction to Compact Transformation Groups*, Academic Press INC. New York and London (1972).
- [2] Dold, A., *Parametrized Borsuk-Ulam theorems*, Comment. Math. Helv. **63** (1988), 275-285.
- [3] Dotzel, R.M., Singh, T. B.,  $\mathbb{Z}_p$  actions on spaces of cohomology type  $(a, 0)$ , Proc. Amer. Math. Soc. **133** (1991), no. 3, 875-878.
- [4] Coelho, F. R. C. ; de Mattos, D. ; dos Santos, E. L., *On the existence of equivariant maps*, Bull. of the Braz. Math. Soc., **43** (2012), no. 3, 407-421.
- [5] Izydorek, M ; Jaworowski, J., *Antipodal coincidence for maps of spheres into complexes*, Proc. Amer. Math. Soc. **123** (1995), no. 6, 1947-1950.
- [6] James, I., *Note on cup products*, Proc. Amer. Math. Soc. **8** (1957), 374-383.
- [7] Jaworowski, J., *A continuous version of the Borsuk-Ulam theorem*, Proc. Amer. Math. Soc. **82** (1981), 112-114.
- [8] Matoušek, J., *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*. Universitext. Springer-Verlag, Berlin, (2003).
- [9] de Mattos, D. ; dos Santos, E. L., *A parametrized Borsuk-Ulam theorem for a product of spheres with free  $\mathbb{Z}_p$ -action and free  $S^1$ -action*, Algebraic and Geometric Topology **7** (2007), 1791-1804.
- [10] Nakaoka, M., *Parametrized Borsuk-Ulam theorems and characteristic polynomials*, Proceedings of the Tianjin Fixed Point Conference 1988, Lectures Notes in Math. 1411, Springer Verlag, Berlin, Heidelberg, New York (1989), 155-170.
- [11] Pergher, P. L. Q. ; Singh, H. K. ; Singh, T. B., *On  $\mathbb{Z}_2$  and  $S^1$  free actions on spaces of cohomology type  $(a, b)$* , Houston J. Math. **36** (2010), no. 1, 137-146.
- [12] Quillen, D., *The spectrum of an equivariant cohomology ring  $I$* , Ann. of Math. **94** (1971), n0.3, 549-572.
- [13] Singh, H., *On the cohomological structure of orbit spaces of certain transformations groups*, PhD Thesis, University of Delhi, 2010.
- [14] Singh, M., *Parametrized Borsuk-Ulam problem for projective space bundles*, Fund. Math. **211** (2011), no. 2, 135-147.
- [15] Spanier, E.H., *Algebraic Topology*, Springer Berlin (1966).
- [16] Toda, H., *Note on cohomology ring of certain spaces*, Proc. Amer. Math. Soc. **14** (1963), 89-95.
- [17] Volovikov, A.Y., *Coincidence points of mappings of  $\mathbb{Z}_p^n$ -spaces*. Izv. Ross. Akad. Nauk Ser. Mat. **69** (2005), no. 5, 53-106 translation in Izv. Math. **69** (2005), no. 5 913-962.
- [18] Yang, C. T., *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I*, Ann. of Math. **60** (1954), 262-282.

Denise de Mattos

*E-mail address* deniseml@icmc.usp.br

Universidade de São Paulo-USP-ICMC, Departamento de Matemática, Caixa Postal 668, 13560-970, São Carlos-SP, Brazil.

Pedro Luiz Queiroz Pergher

*E-mail address* pergher@dm.ufscar.br

Edivaldo Lopes dos Santos

*E-mail address* edivaldo@dm.ufscar.br

Universidade Federal de São Carlos, Departamento de Matemática, Rodovia Washington Luiz, km. 235, Caixa Postal 676, CEP 13.565-905, São Carlos, SP, Brazil.