

# LIMIT CYCLES FOR A CLASS OF DISCONTINUOUS GENERALIZED LIENARD POLYNOMIAL DIFFERENTIAL EQUATIONS

JAUME LLIBRE AND ANA C. MEREU

ABSTRACT. We divide  $\mathbb{R}^2$  in  $l$  sectors  $S_1, \dots, S_l$ , with  $l > 1$  even. We define in  $\mathbb{R}^2$  a discontinuous differential system such that in each sector  $S_k$ , for  $k = 1, \dots, l$ , is defined a smooth generalized Lienard polynomial differential equation  $\ddot{x} + f_i(x)\dot{x} + g_i(x) = 0$ ,  $i = 1, 2$  alternatively, where  $f_i$  and  $g_i$  are polynomials of degree  $n-1$  and  $m$  respectively. We apply the averaging theory of first order for discontinuous differential systems to this class of non-smooth generalized Lienard polynomial differential systems and we show that for any  $n$  and  $m$  there are such non-smooth Lienard polynomial equations having at least  $\max\{n, m\}$  limit cycles. Note that this number is independent of  $l$ .

Roughly speaking this result shows that the non-smooth classical ( $m = 1$ ) Lienard polynomial differential systems can have at least the double number of limit cycles than the smooth ones, and that the non-smooth generalized Lienard polynomial differential systems can have at least one more limit cycle than the smooth ones. Of course, these comparisons are done with the present known results.

## 1. INTRODUCTION

A large number of problems from mechanics and electrical engineering, theory of automatic control, economy, impact systems among others cannot be described with smooth dynamical systems. This fact has motivated many researchers to the study of qualitative aspects of the phase space of non-smooth dynamical systems.

One of the main problems in the qualitative theory of real planar continuous and discontinuous differential systems is the determination of their limit cycles. The non-existence, existence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc (see for instance the books [2, 5, 24]). This problem restricted to continuous planar polynomial differential equations is the well known 16th Hilbert's problem [10]. Since this Hilbert's problem turned out a strongly difficult one Smale [23] particularized it to Lienard polynomial differential equations in his list of problems for the present century.

The classical Lienard polynomial differential equations

$$(1) \quad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where  $f(x)$  and  $g(x) = x$  goes back to [11]. The dot denotes differentiation with respect to the time  $t$ . This second order differential equation (1) can be written as

---

2010 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

*Key words and phrases.* limit cycles, non-smooth Liénard systems, averaging theory.

the following first order differential system in  $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \quad \text{where } F(x) = \int_0^x f(s)ds.$$

Many results on the number of limit cycles has been obtained for the continuous generalized polynomial differential equations (1) being  $f(x)$  and  $g(x)$  polynomials in the variable  $x$  of degrees  $n - 1$  and  $m$  respectively. The continuous classical Lienard polynomial differential equations (1) were studied in 1977 by Lins, de Melo and Pugh [12] who stated the following conjecture: *if  $f(x)$  has degree  $n - 1 > 0$  and  $g(x) = x$ , then (1) has at most  $[(n - 1)/2]$  limit cycles.* Here  $[z]$  denotes the integer part function of  $z \in \mathbb{R}$ . They also proved the conjecture for  $n = 2, 3$ . For  $n = 4$  this conjecture has been proved in 2012 (see [13]). For  $n \geq 7$  Dumortier, Panazzolo and Roussarie proved that this conjecture is not true in [7], they show that these differential equations can have  $[(n - 1)/2] + 1$  limit cycles. Recently De Maesschalck and Dumortier proved in [22] that the classical Lienard equation of degree  $n \geq 6$  can have  $[(n - 1)/2] + 2$  limit cycles. The conjecture for  $n = 5$  is still open.

Results on the number of limit cycles for continuous generalized Lienard polynomial differential equations can be found in [14] where the authors show that there are differential equations (1) having at least  $[(n + m - 2)/2]$  limit cycles. See also [1, 3, 6, 8, 18, 19, 20, 21, 25].

The objective of this work is to start the study of the number of limit cycles for a kind of discontinuous generalized Lienard polynomial differential systems. Here we shall play with many straight lines of discontinuities through the origin of coordinates and with two different continuous generalized Lienard polynomial differential systems located alternatively in the sectors defined by these straight lines.

A similar work but with only one classical Lienard polynomial differential system and only one straight line of discontinuity was studied in [16] obtaining  $[n/2]$  limit cycles, instead of the  $[(n - 1)/2]$  of the continuous classical Lienard polynomial differential equation obtained in [12].

Now we shall define the discontinuous generalized Lienard polynomial differential system that we will study. We consider the function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$h(x, y) = \prod_{k=0}^{\frac{l}{2}-1} \left( y - \tan \left( \alpha + \frac{2k\pi}{l} \right) x \right),$$

where  $l > 1$  even. The set

$$h^{-1}(0) = \bigcup_{k=0}^{\frac{l}{2}-1} \left\{ (x, y) : y = \tan \left( \alpha + \frac{2k\pi}{l} \right) x \right\},$$

divides  $\mathbb{R}^2$  into  $l$  sectors,  $S_1, S_2, \dots, S_l$ , i.e.  $h^{-1}(0)$  is the product of  $l/2$  straight lines passing through the origin of coordinates dividing the plane in sectors of angle  $2\pi/l$ .

In this work we study the maximum number of limit cycles given by the averaging theory of first order, which can bifurcate from the periodic orbits of the linear

center  $\dot{x} = y, \dot{y} = -x$ , perturbed inside the following class of discontinuous Lienard polynomial differential systems:

$$(2) \quad \dot{X} = Z(x, y) = \begin{cases} Y_1(x, y) & \text{if } h(x, y) > 0, \\ Y_2(x, y) & \text{if } h(x, y) < 0, \end{cases}$$

where

$$(3) \quad Y_1(x, y) = \begin{pmatrix} y - \varepsilon F_1(x) \\ -x - \varepsilon g_1(x) \end{pmatrix}, \quad Y_2(x, y) = \begin{pmatrix} y - \varepsilon F_2(x) \\ -x - \varepsilon g_2(x) \end{pmatrix},$$

$\varepsilon$  is a small parameter, and  $F_i(x)$  and  $g_i(x)$ , for  $i = 1, 2$  are polynomials in the variable  $x$  of degrees  $n$  and  $m$  respectively. System (2) can be written using the sign function as

$$(4) \quad \dot{X} = Z(x, y) = G_1(x, y) + \text{sign}(h(x, y))G_2(x, y),$$

where  $G_1(x, y) = \frac{1}{2}(Y_1(x, y) + Y_2(x, y))$  and  $G_2(x, y) = \frac{1}{2}(Y_1(x, y) - Y_2(x, y))$ .

Our main result is the following one.

**Theorem 1.** *Assume that for  $i = 1, 2$  the polynomials  $F_i(x)$  and  $g_i(x)$  have degree  $n \geq 1$  and  $m \geq 1$  respectively, and that  $l > 1$  is even. Then for  $|\varepsilon|$  sufficiently small there are discontinuous Lienard polynomial differential systems (2) having at least  $\max\{n, m\}$  limit cycles bifurcating from the periodic orbits of the linear center  $\dot{x} = y, \dot{y} = -x$ .*

In short, taking into account Theorem 1 we can say roughly speaking that the non-smooth classical Lienard polynomial differential systems can have at least  $\max\{n, 1\}$  limit cycles, i.e. roughly speaking the double number of limit cycles than the smooth ones which at least have  $[(n-1)/2] + 2$  for  $n \geq 6$ . Comparing the mentioned result from [14], that smooth generalized Lienard polynomial differential systems have at least  $[(n+m-1)/2]$  limit cycles with Theorem 1, we can say that the non-smooth generalized Lienard polynomial differential systems can have at least one more limit cycle than the smooth ones. Of course all these comparisons are done with the present known results.

## 2. AVERAGING THEORY OF FIRST ORDER FOR DISCONTINUOUS DIFFERENTIAL SYSTEMS

The first-order averaging theory developed for discontinuous differential systems in [15] is presented in this section. It is summarized as follows.

**Theorem 2.** *We consider the following discontinuous differential system*

$$(5) \quad x'(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon),$$

with

$$\begin{aligned} F(t, x) &= F_1(t, x) + \text{sign}(h(t, x))F_2(t, x), \\ R(t, x, \varepsilon) &= R_1(t, x, \varepsilon) + \text{sign}(h(t, x))R_2(t, x, \varepsilon), \end{aligned}$$

where  $F_1, F_2 : \mathbb{R} \times D \rightarrow \mathbb{R}^n$ ,  $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$  and  $h : \mathbb{R} \times D \rightarrow \mathbb{R}$  are continuous functions,  $T$ -periodic in the variable  $t$  and  $D$  is an open subset of  $\mathbb{R}^n$ . We also suppose that  $h$  is a  $C^1$  function having 0 as a regular value. Denote by  $\mathcal{M} = h^{-1}(0)$ , by  $\Sigma = \{0\} \times D \not\subseteq \mathcal{M}$ , by  $\Sigma_0 = \Sigma \setminus \mathcal{M} \neq \emptyset$ , and its elements by  $z \equiv (0, z) \notin \mathcal{M}$ .

Define the averaged function  $f : D \rightarrow \mathbb{R}^n$  as

$$f(x) = \int_0^T F(t, x) dt.$$

We assume the following three conditions.

- (i)  $F_1, F_2, R_1, R_2$  and  $h$  are locally  $L$ -Lipschitz with respect to  $x$ ;
- (ii) for  $a \in \Sigma_0$  with  $f(a) = 0$ , there exist a neighborhood  $V$  of  $a$  such that  $f(z) \neq 0$  for all  $z \in \bar{V} \setminus \{a\}$  and  $d_B(f, V, a) \neq 0$ , (i.e. the Brouwer degree of  $f$  at  $a$  is not zero).
- (iii) If  $\partial h / \partial t(t_0, z_0) = 0$  for some  $(t_0, z_0) \in \mathcal{M}$ , then  $(\langle \nabla_x h, F_1 \rangle^2 - \langle \nabla_x h, F_2 \rangle^2)(t_0, z_0) > 0$ .

Then, for  $|\varepsilon| > 0$  sufficiently small, there exists a  $T$ -periodic solution  $x(\cdot, \varepsilon)$  of system (5) such that  $x(t, \varepsilon) \rightarrow a$  as  $\varepsilon \rightarrow 0$ .

**Remark 1.** We note that if the function  $f(z)$  is  $C^1$  and the Jacobian of  $f$  at  $a$  is not zero, then  $d_B(f, V, a) \neq 0$ . For more details on the Brouwer degree see [4] and [17].

### 3. PROOF OF THEOREM 1

The discontinuous Lienard differential systems (2) in polar coordinates  $(r, \theta)$  become

$$\begin{aligned} \dot{r} &= -\varepsilon (\cos \theta F_i(r \cos \theta) + \sin \theta g_i(r \cos \theta)), \\ \dot{\theta} &= -1 + \frac{\varepsilon}{r} (\sin \theta F_i(r \cos \theta) - \cos \theta g_i(r \cos \theta)), \end{aligned}$$

with  $i = 1$  if  $\text{sign}(h(r \cos \theta, r \sin \theta)) > 0$  and  $i = 2$  if  $\text{sign}(h(r \cos \theta, r \sin \theta)) < 0$ . Taking the angle  $\theta$  as new independent variable the discontinuous differential systems become

$$(6) \quad \dot{r} = \varepsilon (\cos \theta F_i(r \cos \theta) + \sin \theta g_i(r \cos \theta)) + O(\varepsilon^2).$$

The discontinuous differential system (6) is under the assumptions of Theorem 2 taking

$$t = \theta, \quad T = 2\pi, \quad x = r, \quad \mathcal{M} = h^{-1}(0) = \bigcup_{k=0}^{\frac{l}{2}-1} \left\{ (\theta, r) : \theta = \alpha + \frac{2k\pi}{l}, r > 0 \right\}.$$

So according with Theorem 2 we must study the zeros of the averaged function

$$(7) \quad f(r) = \sum_{k=1}^l \left[ \int_{\alpha + \frac{2(k-1)\pi}{l}}^{\alpha + \frac{(2k-1)\pi}{l}} (\cos \theta F_1(r \cos \theta) + \sin \theta g_1(r \cos \theta)) d\theta + \int_{\alpha + \frac{(2k-1)\pi}{l}}^{\alpha + \frac{2k\pi}{l}} (\cos \theta F_2(r \cos \theta) + \sin \theta g_2(r \cos \theta)) d\theta \right].$$

If we denote by  $F_1(x) = \sum_{i=0}^n a_i x^i$ ,  $F_2(x) = \sum_{i=0}^n b_i x^i$ ,  $g_1(x) = \sum_{i=0}^m c_i x^i$  and  $g_2(x) = \sum_{i=0}^m d_i x^i$  we have

$$f(r) = \sum_{k=1}^l \left[ \int_{\alpha + \frac{2(k-1)\pi}{l}}^{\alpha + \frac{(2k-1)\pi}{l}} \left( \sum_{i=0}^n a_i r^i \cos^{i+1} \theta + \sum_{i=0}^m c_i r^i \cos^i \theta \sin \theta \right) d\theta + \int_{\alpha + \frac{(2k-1)\pi}{l}}^{\alpha + \frac{2k\pi}{l}} \left( \sum_{i=0}^n b_i r^i \cos^{i+1} \theta + \sum_{i=0}^m d_i r^i \cos^i \theta \sin \theta \right) d\theta \right].$$

In order to calculate the exact expression of  $f(r)$  we use the following formulae [2.513 3] and [2.513 4] proved in [9]:

$$\int \cos^{2m} \theta d\theta = \frac{1}{2^{2m}} \binom{2m}{m} \theta + \frac{1}{2^{2m-1}} \sum_{j=0}^{m-1} \binom{2m}{j} \frac{\sin(2m-2j)\theta}{2m-2j},$$

$$\int \cos^{2m+1} \theta d\theta = \frac{1}{2^{2m}} \sum_{j=0}^m \binom{2m+1}{j} \frac{\sin(2m-2j+1)\theta}{2m-2j+1}.$$

Thus we have

$$f(r) = f_1(r) + f_2(r) + f_3(r) + f_4(r),$$

where

$$\begin{aligned} f_1(r) &= \sum_{k=1}^l \int_{\alpha + \frac{2(k-1)\pi}{l}}^{\alpha + \frac{(2k-1)\pi}{l}} \left( \sum_{i=0}^n a_i r^i \cos^{i+1} \theta \right) d\theta = \\ &= \sum_{k=1}^l \left[ \sum_{\substack{i=1 \\ i \text{ odd}}}^n a_i r^i \left[ \frac{1}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l} + \frac{1}{2^i} \sum_{j=0}^{\frac{i-1}{2}} \binom{i+1}{j} \varphi_{i,j,k} \right] + \right. \\ &\quad \left. \sum_{\substack{i=0 \\ i \text{ even}}}^n \frac{a_i r^i}{2^i} \sum_{j=0}^{\frac{i}{2}} \binom{i+1}{j} \varphi_{i,j,k} \right] = \\ &= \sum_{k=1}^l \left[ \sum_{\substack{i=1 \\ i \text{ odd}}}^n \frac{a_i r^i}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l} + \sum_{i=0}^n \frac{a_i r^i}{2^i} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i+1}{j} \varphi_{i,j,k} \right], \end{aligned}$$

with

$$\varphi_{i,j,k} = \frac{\sin\left((i-2j+1)\left(\alpha + \frac{(2k-1)\pi}{l}\right)\right) - \sin\left((i-2j+1)\left(\alpha + \frac{2(k-1)\pi}{l}\right)\right)}{i-2j+1} \neq 0;$$

$$f_2(r) = \sum_{k=1}^l \int_{\alpha + \frac{2(k-1)\pi}{l}}^{\alpha + \frac{(2k-1)\pi}{l}} \left( \sum_{i=0}^m c_i r^i \cos^i \theta \sin \theta \right) d\theta = - \sum_{k=1}^l \sum_{i=0}^m \frac{c_i r^i}{i+1} \phi_{i,k},$$

with

$$\phi_{i,k} = \cos^{i+1} \left( \alpha + \frac{(2k-1)\pi}{l} \right) - \cos^{i+1} \left( \alpha + \frac{2(k-1)\pi}{l} \right) \neq 0;$$

$$f_3(r) = \sum_{k=1}^l \int_{\alpha + \frac{2(k-1)\pi}{l}}^{\alpha + \frac{(2k-1)\pi}{l}} \left( \sum_{i=0}^n b_i r^i \cos^{i+1} \theta \right) d\theta =$$

$$\sum_{k=1}^l \left[ \sum_{\substack{i=1 \\ i \text{ odd}}}^n \frac{b_i r^i}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l} + \sum_{i=0}^n \frac{b_i r^i}{2^i} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i+1}{j} \psi_{i,j,k} \right],$$

with

$$\psi_{i,j,k} = \frac{\sin \left( (i-2j+1) \left( \alpha + \frac{2k\pi}{l} \right) \right) - \sin \left( (i-2j+1) \left( \alpha + \frac{(2k-1)\pi}{l} \right) \right)}{i-2j+1} \neq 0;$$

$$f_4(r) = \sum_{k=1}^l \int_{\alpha + \frac{(2k-1)\pi}{l}}^{\alpha + \frac{2k\pi}{l}} \left( \sum_{i=0}^m d_i r^i \cos^i \theta \sin \theta \right) d\theta = - \sum_{k=1}^l \sum_{i=0}^m \frac{d_i r^i}{i+1} \zeta_{i,k},$$

with

$$\zeta_{i,k} = \cos^{i+1} \left( \alpha + \frac{2k\pi}{l} \right) - \cos^{i+1} \left( \alpha + \frac{(2k-1)\pi}{l} \right) \neq 0.$$

Thus

$$f(r) = \sum_{k=1}^l \left[ \sum_{\substack{i=1 \\ i \text{ odd}}}^n \frac{r^i}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l} (a_i + b_i) + \right.$$

$$\left. \sum_{i=0}^n \frac{r^i}{2^i} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i+1}{j} (a_i \varphi_{i,j,k} + b_i \psi_{i,j,k}) - \sum_{i=0}^m \frac{r^i}{i+1} (c_i \phi_{i,k} + d_i \zeta_{i,k}) \right].$$

The function  $f(r)$  is a polynomial in the variable  $r$  of degree  $\max\{n, m\}$  therefore  $f(r)$  has at most  $\max\{n, m\}$  positive roots. If  $r^*$  is a simple zero of  $f(r)$ , i.e.  $f(r^*) = 0$  and  $\left. \frac{df}{dr} \right|_{r=r^*} \neq 0$ , then the Brouwer degree  $d_B(f, V, r^*) \neq 0$  being  $V$  a convenient open neighborhood of  $r^*$  (see Remark 1). We can choose the coefficients  $a_i, b_i, c_i$  e  $d_i$  in such a way that  $f(r)$  has exactly  $\max\{n, m\}$  simple positive roots. Hence Theorem 1 is proved.

## 4. EXAMPLES

In this section we illustrate Theorem 1 by studying the existence of  $2\pi$ -periodic solutions for two non-smooth Lienard polynomial differential systems.

**Example 1.** We consider  $l = 2$  and  $\alpha = 0$ . Thus the function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $h(x, y) = y$  and  $h^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ . System (2) becomes

$$(8) \quad \dot{X} = Z(x, y) = \begin{cases} Y_1(x, y) & \text{if } y > 0, \\ Y_2(x, y) & \text{if } y < 0, \end{cases}$$

where  $F_1 = 1 + x + x^2 + \left(\frac{1}{9\pi} - 1\right)x^3$ ,  $F_2 = 1 + \left(\frac{11}{12\pi} - 1\right)x + x^2 + x^3$ ,  $g_1 = \frac{7}{8} + x + \frac{5}{8}x^2$  and  $g_2 = 1 + x + x^2$ . Thus we have

$$Y_1(x, y) = \begin{pmatrix} y - \varepsilon \left(1 + x + x^2 + \left(\frac{1}{9\pi} - 1\right)x^3\right) \\ -x - \varepsilon \left(\frac{7}{8} + x + \frac{5}{8}x^2\right) \end{pmatrix},$$

$$Y_2(x, y) = \begin{pmatrix} y - \varepsilon \left(1 + \left(\frac{11}{12\pi} - 1\right)x + x^2 + x^3\right) \\ -x - \varepsilon(1 + x + x^2) \end{pmatrix}.$$

The averaging function (7) is given by

$$f(r) = \int_0^\pi (\cos \theta F_1(r \cos \theta) + \sin \theta g_1(r \cos \theta)) d\theta +$$

$$\int_\pi^{2\pi} (\cos \theta F_2(r \cos \theta) + \sin \theta g_2(r \cos \theta)) d\theta =$$

$$-6 + 11r - 6r^2 + r^3.$$

The zeros of  $f(r)$  are  $r = 1$ ,  $r = 2$  and  $r = 3$ , and they are simple. Hence, by Theorem 1 it follows that for  $\varepsilon \neq 0$  sufficiently small the discontinuous differential system (8) has three periodic solutions.

**Example 2.** We consider  $l = 4$  and  $\alpha = \pi/4$ . Thus the function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $h(x, y) = (y - x)(y + x)$  and  $h^{-1}(0) = \{(x, y) : y = x\} \cup \{(x, y) : y = -x\}$ . System (2) becomes

$$(9) \quad \dot{X} = Z(x, y) = \begin{cases} Y_1(x, y) & \text{if } (y - x)(y + x) > 0, \\ Y_2(x, y) & \text{if } (y - x)(y + x) < 0, \end{cases}$$

where  $F_1 = x^2$ ,  $F_2 = 12\sqrt{2}\pi + \frac{72\sqrt{2}}{5}x^2$ ,  $g_1 = 1 + x + x^2 + x^3$  and  $g_2 = -88\pi x - \frac{32\pi}{3}x^3$ .

Thus we have

$$Y_1(x, y) = \begin{pmatrix} y - \varepsilon x^2 \\ -x - \varepsilon(1 + x + x^2 + x^3) \end{pmatrix}$$

$$Y_2(x, y) = \begin{pmatrix} y - \varepsilon \left(12\sqrt{2}\pi + \frac{72\sqrt{2}}{5}x^2\right) \\ -x - \varepsilon \left(-88\pi x - \frac{32\pi}{3}x^3\right) \end{pmatrix}.$$

The averaging function (7) is given by

$$\begin{aligned}
 f(r) = & \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos \theta F_1(r \cos \theta) + \sin \theta g_1(r \cos \theta)) d\theta + \\
 & \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} (\cos \theta F_2(r \cos \theta) + \sin \theta g_2(r \cos \theta)) d\theta + \\
 & \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} (\cos \theta F_1(r \cos \theta) + \sin \theta g_1(r \cos \theta)) d\theta + \\
 & \int_{\frac{7\pi}{4}}^{\frac{9\pi}{4}} (\cos \theta F_2(r \cos \theta) + \sin \theta g_2(r \cos \theta)) d\theta = \\
 & -6 + 11r - 6r^2 + r^3.
 \end{aligned}$$

The zeros of  $f(r)$  are  $r = 1$ ,  $r = 2$  and  $r = 3$ , and they are simple. Hence, by Theorem 1 it follows that for  $\varepsilon \neq 0$  sufficiently small the discontinuous differential system (9) has three periodic solutions.

#### ACKNOWLEDGMENTS

The first author is partially supported by a MICINN/FEDER grant MTM 2008–03437, an AGAUR grant number 2009SGR-0410, an ICREA Academia, and FP7-PEOPLE-2012-IRSES-316338 and 318999. The second author is partially supported by a FAPESP-BRAZIL grant 2012/20884-8. Both authors are also supported by the joint project CAPES–MECD grant PHB-2009-0025-PC.

#### REFERENCES

- [1] S. BADI AND A. MAKHLOUF, *Limit cycles of the generalized Liénard differential equation via averaging theory*, Electron. J. Differential Equations **2012**, No. 68, 11 pp.
- [2] N. N. BOGOLIUBOV, *On some statistical methods in mathematical physics*, Izv. vo Akad. Nauk Ukr. SSR, Kiev, 1945.
- [3] I. BOUSSAADA AND A. R. CHOUIKHA, *Existence of periodic solution for perturbed generalized Liénard equations*, Electron. J. Differential Equations **2006**, No. 140, 10 pp.
- [4] F. BROWDER, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.
- [5] S.N. CHOW, C. LI AND D. WANG, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge Univ. Press., 1994.
- [6] C.J. CHRISTOPHER AND S. LYNCH, *Small-amplitude limit cycle bifurcations for Liénard systems with quadratic or cubic dumping or restoring forces*, Nonlinearity **12** (1999), 1099–1112.
- [7] F. DUMORTIER, D. PANAZZOLO AND R. ROUSSARIE, *More limit cycles than expected in Liénard systems*, Proc. Amer. Math. Soc. **135** (2007), 1895–1904.
- [8] A. GASULL AND J. TORREGROSA, *Small-amplitude limit cycles in Liénard systems via multiplicity*, J. Differential Equations **159** (1998), 1015–1039.
- [9] I. S. GRADSHTEYN AND I. M. RYSHIK, *Table of Integrals, Series, and Products*, Edited by A. Jeffrey. Academic Press, New York, 5th edition, 1994.
- [10] D. HILBERT, *Mathematische Probleme (lecture)*, Second Internat. Congress Math. Paris, 1900, Nach. Ges. Wiss. Gottingen Math.–Phys. Kl., 1900, 253–297.
- [11] A. LIÉNARD, *Étude des oscillations entretenues*, Revue Générale de l'Électricité **23** (1928), 946–954.
- [12] A. LINS, W. DE MELO AND C.C. PUGH, *On Liénard's Equation*, Lecture Notes in Math **597**, Springer, Berlin, (1977), pp. 335–357.

- [13] C. LI Y J. LLIBRE, *Uniqueness of limit cycle for Liénard equations of degree four*, J. Differential Equations **252** (2012), 3142–3162.
- [14] J. LLIBRE, A.C. MEREU, M.A. TEIXEIRA, *Limit cycles of the generalized polynomial Liénard differential equations*, Math. Proc. Cambridge Philos. Soc. **148** (2010) 363–383.
- [15] J. LLIBRE, D.D. NOVAES AND M.A. TEIXEIRA, *Averaging methods for studying the periodic orbits of discontinuous differential systems*, <http://arxiv.org/pdf/1205.4211.pdf>
- [16] J. LLIBRE AND M.A. TEIXEIRA, *Periodic solutions of discontinuous second order differential systems*, preprint, 2012
- [17] N.G. LLOYD, *Degree Theory*, Cambridge University Press, 1978.
- [18] S. LYNCH, *Limit cycles of generalized Liénard equations*, Applied Math. Letters **8** (1995), 15–17.
- [19] S. LYNCH, *Generalized quadratic Liénard equations*, Applied Math. Letters **11** (1998), 7–10.
- [20] S. LYNCH, *Generalized cubic Liénard equations*, Applied Math. Letters **12** (1999), 1–6.
- [21] S. LYNCH AND C. J. CHRISTOPHER, *Limit cycles in highly non-linear differential equations*, J. Sound Vib. **224** (1999), 505–517
- [22] P. DE MAESSCHALCK AND F. DUMORTIER, *Classical Liénard equation of degree  $n \geq 6$  can have  $\lfloor \frac{n-1}{2} \rfloor + 2$  limit cycles*, J. Differential Equations **250** (2011), 2162–2176.
- [23] S. SMALE, *Mathematical Problems for the Next Century*, Mathematical Intelligencer **20** (1998), 7–15.
- [24] YE YANQIAN, *Theory of Limit Cycles*, Translations of Math. Monographs **66** (Providence, RI Amer. Math. Soc.), 1986.
- [25] P. YU AND M. HAN, *Limit cycles in generalized Liénard systems*, Chaos, Solitons and Fractals **30** (2006), 1048–1068.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

*E-mail address:* `jllibre@mat.uab.cat`

DEPARTMENT OF PHYSICS, CHEMISTRY AND MATHEMATICS. UFSCAR. 18052-780, SOROCABA, SP, BRAZIL

*E-mail address:* `anamereu@ufscar.br`