LIMIT CYCLES FOR DISCONTINUOUS QUADRATIC DIFFERENTIAL SYSTEMS WITH TWO ZONES

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Abstract. In this paper we study the maximum number of limit cycles given by the averaging theory of first order for discontinuous differential systems, which can bifurcate from the periodic orbits of the quadratic isochronous centers $\dot{x} = -y + x^2$, $\dot{y} = x + xy$ and $\dot{x} = -y + x^2 - y^2$, $\dot{y} = x + 2xy$ when they are perturbed inside the class of all discontinuous quadratic polynomial differential systems with the straight line of discontinuity $y = 0$.

Comparing the obtained results for the discontinuous with the results for the continuous quadratic polynomial differential systems, this work shows that the discontinuous systems have at least 3 more limit cycles surrounding the origin than the continuous ones.

1. Introduction

One of the main problems in the qualitative theory of continuous planar polynomial differential systems is the study of their limit cycles, see for instance [13]. The limit cycles of continuous planar quadratic polynomial differential systems has been studied intensively, see for instance the books [9, 19] and the hundreds of references quoted therein.

The classification of the quadratic polynomial differential systems having an isochronous center is due to Loud [17]. He proved that after an affine change of variables and a rescaling of the independent variable any quadratic isochronous center can be written as one of the four systems of Table 1.

Chicone and Jacobs proved in [8] that at most 2 limit cycles bifurcate from the periodic orbits of the isochronous center

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy, \tag{1}$$

and that at most 1 limit cycle bifurcate from the isochronous center

$$\dot{x} = -y + x^2 - y^2, \quad \dot{y} = x + 2xy, \tag{2}$$

when these quadratic centers are perturbed inside the class of all quadratic polynomial differential systems. Their study is based in the displacement function using some results of Bautin [1]. In [4] the authors reproved in an easier way, using the averaging theory, the existence of at least 2 limit cycles

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bifurcating from the periodic orbits of the center (1) when this is perturbed inside the class of all quadratic polynomial differential systems.

Stimulated by discontinuous phenomena in the real world (see for instance the book [2] and the references quoted therein), a big interest has appeared for studying the limit cycles of discontinuous differential systems, mainly for discontinuous piecewise linear differential systems, see also the paper [15] and the references quoted there.

Our objective is to study the number of limit cycles of the discontinuous quadratic differential systems with two zones separated by a straight line. As far as we know for discontinuous quadratic differential systems only the center problem and the Hopf bifurcation has been studied partially, see [10, 11, 12]. Related studies about the number of limit cycles bifurcating from center and isochronous centers some of them perturbed in discontinuous quadratic systems can be found in [6, 7].

Using the averaging theory of first order we study the maximum number of limit cycles which can bifurcate from the periodic orbits of the isochronous centers (1) and (2) perturbed inside the following class of discontinuous quadratic polynomial differential systems

$$\dot{X}_i = Z_i(x,y) = \begin{cases} Y_1^i(x,y) & \text{if } y>0, \\ Y_2^i(x,y) & \text{if } y<0, \end{cases} \tag{3}$$

$i=1,2$, where

$$\begin{aligned} Y_1^1(x,y) &= \begin{pmatrix} -y+x^2+\varepsilon p_1(x,y) \\ x+xy+\varepsilon q_1(x,y) \end{pmatrix}, \\ Y_2^1(x,y) &= \begin{pmatrix} -y+x^2+\varepsilon p_2(x,y) \\ x+xy+\varepsilon q_2(x,y) \end{pmatrix}, \\ Y_1^2(x,y) &= \begin{pmatrix} -y+x^2-y^2+\varepsilon p_1(x,y) \\ x+2xy+\varepsilon q_1(x,y) \end{pmatrix}, \\ Y_2^2(x,y) &= \begin{pmatrix} -y+x^2-y^2+\varepsilon p_2(x,y) \\ x+2xy+\varepsilon q_2(x,y) \end{pmatrix}, \end{aligned}$$

ε is a small parameter, and

$$\begin{aligned} p_1(x,y) &= a_1x+a_2y+a_3xy+a_4x^2+a_5y^2, \\ q_1(x,y) &= b_1x+b_2y+b_3xy+b_4x^2+b_5y^2, \\ p_2(x,y) &= c_1x+c_2y+c_3xy+c_4x^2+c_5y^2, \\ q_2(x,y) &= d_1x+d_2y+d_3xy+d_4x^2+d_5y^2. \end{aligned} \tag{4}$$

In other words, in some sense we extend the work done by Chicone and Jacobs [8] for the continuous quadratic polynomial differential systems to the discontinuous ones with the straight line of discontinuity $y=0$.

System (3) can be written using the sign function in the form

$$\dot{X}_i = Z_i(x,y) = G_1^i(x,y) + \operatorname{sign}(y) G_2^i(x,y), \tag{5}$$

where $G_1^i(x,y) = \dfrac{1}{2}(Y_1^i(x,y)+Y_2^i(x,y))$ and $G_2^i(x,y) = \dfrac{1}{2}(Y_1^i(x,y)-Y_2^i(x,y))$, for $i = 1, 2$.

Our main results are the following ones.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small there are discontinuous quadratic polynomial differential systems* (3) *with $i = 1$ having at least* 5 *limit cycles bifurcating from the periodic orbits of the isochronous center* (1).

Theorem 2. *For $|\varepsilon| \neq 0$ sufficiently small there are discontinuous quadratic polynomial differential systems* (3) *with $i = 2$ having at least* 4 *limit cycles bifurcating from the periodic orbits of the isochronous center* (2).

We recall that the perturbation of the periodic orbits of the isochronous centers (1) and (2) inside the class of continuous quadratic polynomial differential systems produce at most 2 and 1 limit cycles, respectively (see [8]). So comparing the obtained results for the discontinuous with the results for the continuous quadratic polynomial differential systems, this work shows that the discontinuous systems have at least 3 more limit cycles surrounding the origin than the continuous systems when we perturbed the centers (1) and (2), respectively.

In short, the results on the number of limit cycles which can bifurcate from the periodic orbits of the quadratic isochronous centers when they are perturbed inside the class of all continuous or discontinuous quadratic polynomial differential systems (QPDS) of the form (3) are summarized in Table 1. The numbers of column 2 of Table 1 are the maximum number of limit cycles for the continuous QPDS which where obtained by Chicone and Jacobs in [8]. The numbers of column 3 of Table 1 are the maximum number of limit cycles for the discontinuous QPDS which can be obtained using the averaging theory of first order. This result only says that the maximum number of limit cycles which can bifurcate from the periodic solutions of the unperturbed isochronous center when they are perturbed inside the class of discontinuous QPDS is at least the ones which appear in this third column. The exact maximum number of these limit cycles is unknown.

2. Preliminary Results

In this section we summarize the main results that we will use to study the discontinuous quadratic differential systems (3). The next theorem is the first-order averaging theory developed for discontinuous differential systems in [14].

Theorem 3 ([14])**.** *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{6}$$

with

$$F(t,x) = F_1(t,x) + \operatorname{sign}(h(t,x))F_2(t,x),$$
$$R(t,x,\varepsilon) = R_1(t,x,\varepsilon) + \operatorname{sign}(h(t,x))R_2(t,x,\varepsilon),$$

Table 1. The number of limit cycles for the QPDS.

Quadratic isochronous centers	The number of limit cycles for the continuous QPDS	discontinuous QPDS
$\dot{x} = -y + x^2$ $\dot{y} = x + xy$	2	5
$\dot{x} = -y + x^2 - y^2$ $\dot{y} = x + 2xy$	1	4
$\dot{x} = -y - \frac{4}{3}x^2$ $\dot{y} = x - \frac{16}{3}xy$	2	?
$\dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2$ $\dot{y} = x + \frac{8}{3}xy$	2	?

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions,* T*–periodic in the variable* t *and* D *is an open subset of* $\mathbb{R}^n$. *We also suppose that* h *is a* C^1 *function having* 0 *as a regular value. Denote by* $\mathcal{M} = h^{-1}(0)$, *by* $\Sigma = \{0\} \times D \nsubseteq \mathcal{M}$, *by* $\Sigma_0 = \Sigma \backslash \mathcal{M} \neq \varnothing$, *and its elements by* $z \equiv (0, z) \notin \mathcal{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(x) = \int_0^T F(t, x)dt. \tag{7}$$

We assume the following three conditions.

(i) F_1, F_2, R_1, R_2 *and* h *are locally* L*–Lipschitz with respect to* x;

(ii) *for* $a \in \Sigma_0$ *with* $f(a) = 0$, *there exist a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and* $d_B(f, V, a) \neq 0$, *(i.e. the Brouwer degree of* f *at* a *is not zero).*

(iii) *If* $\partial h / \partial t(t_0, z_0) = 0$ *for some* $(t_0, z_0) \in \mathcal{M}$, *then* $\big(\langle \nabla_x h, F_1 \rangle^2 - \langle \nabla_x h, F_2 \rangle^2\big)(t_0, z_0) > 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $x(\cdot, \varepsilon)$ *of system* (6) *such that* $x(t, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

Remark 1. *We note that if the function* $f(z)$ *is* C^1 *and the Jacobian of* f *at* a *is not zero, then* $d_B(f, V, a) \neq 0$. *For more details on the Brouwer degree see* [3] *and* [18].

We consider a planar system

$$\dot{x} = P(x, y) \qquad \dot{y} = Q(x, y), \tag{8}$$

where P, $Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. Assume that (8) has a continuous family of oval

$$\{\Gamma_h\} \subset \{(x, y) : H(x, y) = h,\ h_1 < h < h_2\}$$

where H is a first integral of (8).

We perturbed systems (8) as follows

$$\begin{aligned} \dot{x} &= \quad P(x,y) + \varepsilon\, p(x,y), \\ \dot{y} &= \quad Q(x,y) + \varepsilon\, q(x,y), \end{aligned} \tag{9}$$

where $p,\, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In order to apply the averaging method for studying limit cycles of (9) for ε sufficiently small, we need write system (9) in the standard form (6) for applying the averaging theory. The following result of [4] provides a way for transforming (9) in this standard form.

Theorem 4 ([4]). *Consider system* (8) *and its first integral H. Assume that $xQ(x,y) - yP(x,y) \neq 0$ for all (x,y) in the period annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ be a continuous function such that*

$$H(\rho(R,\varphi)\cos\varphi,\ \rho(R,\varphi)\sin\varphi) = R^2, \tag{10}$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\varphi \in [0, 2\pi)$. Then the differential equation which describes the dependence between the square root of the energy $R = \sqrt{h}$ and the angle φ for system (9) *is*

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2), \tag{11}$$

where $\mu = \mu(x,y)$ is the integrating factor of system (8) *corresponding to the first integral H, and $x = \rho(R,\varphi)\cos\varphi$ and $y = \rho(R,\varphi)\sin\varphi$.*

In order to study the number of zeros of the averaged function (7) we will use the following result proved in [16].

Let A be a set and let $f_1, f_2, ..., f_n : A \to \mathbb{R}$. We say that f_1, ..., f_n are *linearly independent* functions if and only if we have that

$$\sum_{i=1}^{n} \alpha_i f_i(a) = 0 \text{ for all } a \in A \ \Rightarrow\ \alpha_1 = \alpha_2 = ... = \alpha_n = 0.$$

Proposition 5 ([16]). *If $f_1, f_2, ..., f_n : A \to \mathbb{R}$ are linearly independent then there exist a_1,..., $a_{n-1} \in A$ and α_1, ..., $\alpha_n \in \mathbb{R}$ such that for every $i \in \{1, ..., n-1\}$*

$$\sum_{k=1}^{n} \alpha_k f_k(a_i) = 0.$$

3. Proof of Theorem 1

We recall that the *period annulus* of a center is the topological annulus formed by all the periodic orbits surrounding the center, and no other singular points.

A first integral H and an integrating factor μ in the period annulus of the center of the quadratic differential system (1) have the expressions $H(x,y) = (x^2+y^2)/(1+y)^2$ and $\mu(x,y) = 2/(1+y)^3$, respectively; for more details see

[5]. For this system we note that $h_1 = 0$, $h_2 = 1$, and that the function ρ that satisfies the hypotheses of Theorem 4 is given by $\rho(R, \varphi) = R/(1 - R\sin\varphi)$ for all $0 < R < 1$ and $\varphi \in [0, 2\pi)$.

Then using Theorem 4 we transform system (3) into the form

$$\frac{dR}{d\varphi} = \begin{cases} \varepsilon \dfrac{A(\varphi; a, b)R + B(\varphi; a, b)R^2 + C(\varphi; a, b)R^3}{2(1 - R\sin\varphi)} + \mathcal{O}(\varepsilon^2) & \text{if } y > 0, \\ \varepsilon \dfrac{A(\varphi; c, d)R + B(\varphi; c, d)R^2 + C(\varphi; c, d)R^3}{2(1 - R\sin\varphi)} + \mathcal{O}(\varepsilon^2) & \text{if } y < 0, \end{cases} \tag{12}$$

where

$$\begin{aligned} A(\varphi; a, b) = {} & a_1\cos^2\varphi + (a_2 + b_1)\cos\varphi\sin\varphi + b_2\sin^2\varphi, \\ B(\varphi; a, b) = {} & (a_4 - b_1)\cos^3\varphi + (-a_1 + a_3 - b_2 + b_4)\cos^2\varphi\sin\varphi + \\ & (-a_2 + a_5 - b_1 + b_3)\cos\varphi\sin^2\varphi + (-b_2 + b_5)\sin^3\varphi, \\ C(\varphi; a, b) = {} & -b_4\cos^4\varphi + b_1\cos^3\varphi\sin\varphi + (b_2 - b_4 - b_5)\cos^2\varphi\sin^2\varphi + \\ & (b_1 - b_3)\cos\varphi\sin^3\varphi + (b_2 - b_5)\sin^4\varphi, \end{aligned}$$

and $a = (a_1, ..., a_5)$, $b = (b_1, ..., b_5)$, $c = (c_1, ..., c_5)$ and $d = (d_1, ..., d_5)$.

Note that system (1) has the invariant straight line $y = -1$. So the minimal distance of the external boundary of the period annulus of the center to the origin is 1.

The discontinuous differential system (14) is under the assumptions of Theorem 3. So we must study the zeros of the averaged function $f : (0, 1) \to \mathbb{R}$,

$$\begin{aligned} f(R) = {} & \int_0^\pi \frac{A(\varphi; a, b)R + B(\varphi; a, b)R^2 + C(\varphi; a, b)R^3}{2(1 - R\sin\varphi)} d\varphi + \\ & \int_\pi^{2\pi} \frac{A(\varphi, c, d)R + B(\varphi, c, d)R^2 + C(\varphi, c, d)R^3}{2(1 - R\sin\varphi)} d\varphi. \end{aligned}$$

We compute these integrals and we obtain

$$f(R) = (a_1 + c_1)g_1 + a_3g_2 + b_2g_3 + b_4g_4 + b_5g_5 + c_3g_6 + d_2g_7 + d_4g_8 + d_5g_9,$$

where

$$\begin{aligned} g_1 = {} & \frac{\pi R}{4}, \\ g_2 = {} & 1 + \frac{\pi}{2R} - \frac{\pi R}{4} + \frac{1}{4R\sqrt{1 - R^2}}\Big(- 2\pi + 2\pi R^2 - \\ & (4 + 4R^2)\arcsin R\Big), \\ g_3 = {} & \frac{\pi R}{4} - R^2 + \frac{1}{4R\sqrt{1 - R^2}}\Big(2\pi + 4\,\mathrm{arccosh}\, R - \\ & 4\arctan\frac{R}{\sqrt{1 - R^2}}\Big), \end{aligned}$$

$$
\begin{aligned}
g_4 =& 1+\frac{\pi}{2R}-\frac{3\pi R}{4}-R^2+\frac{1}{4R\sqrt{1-R^2}}\Big(-2\pi+4\pi R^2-\\
& 2\pi R^4+(-4+8R^2-4R^4)\arcsin R\Big),\\
g_5 =& -1-\frac{\pi}{2R}+\frac{\pi R}{4}+R^2+\frac{1}{4R\sqrt{1-R^2}}\Big(2\pi-2\pi R^2+\\
& (4-4R^2)\arcsin R\Big),\\
g_6 =& -1+\frac{\pi}{2R}-\frac{\pi R}{4}-\frac{\sqrt{1-R^2}\arccos R}{R},\\
g_7 =& \frac{\pi R}{4}+R^2,\\
g_8 =& -1+\frac{\pi}{2R}-\frac{3\pi R}{4}+R^2+\left(R-\frac{1}{R}\right)\sqrt{1-R^2}\arccos R,\\
g_9 =& 1-\frac{\pi}{2R}+\frac{\pi R}{4}-R^2+\frac{\sqrt{1-R^2}\arccos R}{R}.
\end{aligned} \tag{13}
$$

We have checked the results of these integrals with algebraic manipulators as Mathematica and Mapple.

We note the equalities

$$
\begin{aligned}
g_5 =& \ g_1-g_2-g_3,\\
g_7 =& \ 2g_1-g_3,\\
g_9 =& \ -g_1+g_3-g_6.
\end{aligned}
$$

Thus the function f can be written as

$$
\begin{aligned}
f(R) = & \ (a_1+b_5+c_1+2d_2-d_5)g_1+(a_3-b_5)g_2+\\
& (b_2-b_5-d_2+d_5)g_3+b_4g_4+(c_3-d_5)g_6+d_4g_8.
\end{aligned}
$$

The six functions $g_i:(0,1)\to\mathbb{R}$, $i\in\{1,2,3,4,6,8\}$ given in (13) are linearly independent. Indeed, we obtain the following Taylor expansions in the variable R around $R=0$ for the functions g_1, g_2, g_3, g_4, g_6 and g_8:

$$
\begin{aligned}
g_1(R) =& \frac{\pi}{4}R+\mathcal{O}(R^7),\\
g_2(R) =& \frac{1}{3}R^2+\frac{\pi}{16}R^3+\frac{2}{15}R^4+\frac{\pi}{32}R^5+\frac{8}{105}R^6+\mathcal{O}(R^7),\\
g_3(R) =& \frac{\pi}{4}R-R^2+\mathcal{O}(R^7),\\
g_4(R) =& \frac{1}{3}R^2-\frac{3\pi}{16}R^3-\frac{1}{5}R^4-\frac{\pi}{32}R^5-\frac{2}{35}R^6+\mathcal{O}(R^7),\\
g_6(R) =& -\frac{1}{3}R^2+\frac{\pi}{16}R^3-\frac{2}{15}R^4+\frac{\pi}{32}R^5-\frac{8}{105}R^6+\mathcal{O}(R^7),\\
g_8(R) =& -\frac{1}{3}R^2-\frac{3\pi}{16}R^3+\frac{1}{5}R^4-\frac{\pi}{32}R^5+\frac{2}{35}R^6+\mathcal{O}(R^7).
\end{aligned}
$$

The determinant of the coefficient matrix of the variables R, R^2, R^3, R^4, R^5, R^6 is $-\pi^3/33600$.

By Proposition 5 since the six functions g_1, g_2, g_3, g_4, g_6 and g_8 are linearly independent, then there exists a linear combination of their with at least 5 zeros. Moreover the coefficients of the functions g_i, for $i = 1,2,3,4,6,8$ in the expression of f are linear functions of the variables a_1, a_3, b_2, b_4, b_5, c_1, c_3, d_2, d_4, d_5. The rank of the Jacobian matrix of the coefficients of g_1, g_2, g_3, g_4, g_6 and g_8 in $f(R)$ in the variables a_1, a_3, b_2, b_4, b_5, c_1, c_3, d_2, d_4, d_5 is 6. Thus there exist R_1, R_2, R_3, R_4, $R_5 \in (0,1)$ and coefficients a_j, b_j c_j, $d_j \in \mathbb{R}$, $j = 1, ..., 5$ such that $f(R_i) = 0$ for $i = 1, ..., 5$.

In summary, there are discontinuous quadratic polynomial differential systems (3) having at least 5 limit cycles bifurcating from the periodic orbits of the isochronous center $\dot{x} = -y+x^2$, $\dot{y} = x+xy$, using the averaging theory of first order for discontinuous differential systems. This completes the proof of Theorem 1.

4. Example

In this section we illustrate Theorem 1 by studying a particular discontinuous quadratic polynomial differential systems (3) which has the maximum number of limit cycles under the assumptions of Theorem 1, i.e. 5 limit cycles.

We consider system (3) with

$$\begin{aligned} a_1 &= 16.0642220739, & a_3 &= -0.1985389328,\\ b_2 &= -15,9170992052, & b_4 &= -0,6584831027,\\ c_3 &= 39,2749291386, & d_4 &= 10, \end{aligned}$$

$$a_2 = a_4 = a_5 = b_1 = b_3 = b_5 = c_1 = c_2 = c_4 = c_5 = d_1 = d_2 = d_3 = d_5 = 0.$$

Thus we have

$$Y_1(x,y) = \begin{pmatrix} -y + x^2 + \varepsilon(16.0642220739x - 0.1985389328xy) \\ x + xy + \varepsilon(-15,9170992052y - 0,6584831027x^2) \end{pmatrix},$$

$$Y_2(x,y) = \begin{pmatrix} -y + x^2 + \varepsilon(39,2749291386xy) \\ x + xy + \varepsilon(10x^2) \end{pmatrix}.$$

Computing the averaged function f for this system we obtain

$$\begin{aligned} f(R) = &\frac{1}{4R\sqrt{1-R^2}}\Big(- 63.6683968209 \operatorname{arccosh} R - 195.5597397036 + \\ &91.4122844881R^2 + 4.1373713562R^4 + \\ &\sqrt{1-R^2}\Big(304.2186825148 - 200.5278046967R - \\ &210.3416228492R^2 + 106.30232923194566R^3\Big) \\ &\arccos R \left(-132.8428282587 + 172.8428282587R^2 - 40R^4\right) + \end{aligned}$$

$$\arcsin R\,\big(67.6849764380 - 70.3189088490R^2 + 2.6339324110R^4\big) + 63.6683968209 \arctan \frac{R}{\sqrt{1-R^2}}\Big)$$

The zeros of $f(R) = 0$ are

$$R_1 = \frac{9}{10},\ R_2 = \frac{8}{10},\ R_3 = \frac{7}{10},\ R_4 = \frac{6}{10} \text{ and } R_5 = \frac{5}{10},$$

and

$$f'\left(\frac{9}{10}\right) = -0.0149101309, \quad f'\left(\frac{8}{10}\right) = 0.0017276939,$$
$$f'\left(\frac{7}{10}\right) = -0.000665467, \quad f'\left(\frac{6}{10}\right) = 0.0006333899,$$
$$f'\left(\frac{5}{10}\right) = -0.0016764687,$$

i.e., $f'(R_i) \neq 0$ for $i = 1, ..., 5$. Hence, by Theorem 3 it follows that for $\varepsilon \neq 0$ sufficiently small this discontinuous differential system has 5 periodic solutions.

5. Proof of Theorem 2

A first integral H and an integrating factor μ in the period annulus of the center at the origin of the quadratic differential system (2) have the expressions $H(x, y) = (x^2 + y^2)/(1 + 2y)$ and $\mu(x, y) = 2/(1 + 2y)^2$, respectively; for more details see [5]. For this system we note that $h_1 = 0$, $h_2 = 1$, and that the function ρ that satisfies the hypotheses of Theorem 4 is given by $\rho(R, \varphi) = R^2 \sin\varphi + R\sqrt{R^2 \sin^2\varphi + 1}$ for all $R > 0$ and $\varphi \in [0, 2\pi)$.

Then using Theorem 4 we transform system (3) into the form

$$\frac{dR}{d\varphi} = \begin{cases} \dfrac{\varepsilon}{K(\varphi)}\Big(D(\varphi,a,b)R + E(\varphi,a,b)R^2 + \\ F(\varphi,a,b)R^3 + G(\varphi,a,b)R^4 + H(\varphi,a,b)R^5 + \\ I(\varphi,a,b)R^6 + J(\varphi,a,b)R^7\Big) + \mathcal{O}(\varepsilon^2) & \text{if } y > 0, \\ \dfrac{\varepsilon}{K(\varphi)}\Big(D(\varphi,c,d)R + E(\varphi,c,d)R^2 + \\ F(\varphi,c,d)R^3 + G(\varphi,c,d)R^4 + H(\varphi,c,d)R^5 + \\ I(\varphi,c,d)R^6 + J(\varphi,c,d)R^7\Big) + \mathcal{O}(\varepsilon^2) & \text{if } y < 0, \end{cases} \tag{14}$$

where

$$D(\varphi,a,b) = a_1\cos^2\varphi + (a_2 + b_1)\cos\varphi\sin\varphi + b_2\sin^2\varphi,$$
$$E(\varphi,a,b) = \sqrt{1 + R^2\sin^2\varphi}\,\big((a_4 - b_1)\cos^3\varphi + (4a_1 + a_3 - b_2 + b_4)\cos^2\varphi\sin\varphi + (4a_2 + a_5 + 3b_1 + b_3)\cos\varphi\sin^2\varphi + (3b_2 + b_5)\sin^3\varphi\big),$$
$$F(\varphi,a,b) = b_4\cos^4\varphi + (5a_4 - 3b_1 - b_3)\cos^3\varphi\sin\varphi +$$

$$
\begin{aligned}
&(8a_1+5a_3-3b_2+4b_4-b_5)\cos^2\varphi\sin^2\varphi+\\
&(8a_2+5a_5+5b_1+4b_3)\cos\varphi\sin^3\varphi+(5b_2+4b_5)\sin^4\varphi,\\
G(\varphi,a,b)=&\sqrt{1+R^2\sin^2\varphi}\,\big(4b_4\cos^4\varphi\sin\varphi+\\
&4(3a_4-b_1-b_3)\cos^3\varphi\sin^2\varphi+4(2a_1+3a_3-b_2+2b_4-b_5)\\
&\cos^2\varphi\sin^3\varphi+4(2a_2+3a_5+b_1+2b_3)\cos\varphi\sin^4\varphi+\\
&4(b_2+2b_5)\sin^5\varphi\big)\,,\\
H(\varphi,a,b)=&8b_4\cos^4\varphi\sin^2\varphi+4(5a_4-b_1-2b_3)\cos^3\varphi\sin^3\varphi+\\
&4(2a_1+5a_3-b_2+3b_4-2b_5)\cos^2\varphi\sin^4\varphi+\\
&4(2a_2+5a_5+b_1+3b_3)\cos\varphi\sin^5\varphi+4(b_2+3b_5)\sin^6\varphi,\\
I(\varphi,a,b)=&\sqrt{1+R^2\sin^2\varphi}\,\big(8b_4\cos^4\varphi\sin^3\varphi+8(2a_4-b_3)\cos^3\varphi\sin^4\varphi+\\
&8(2a_3+b_4-b_5)\cos^2\varphi\sin^5\varphi+8(2a_5+b_3)\cos\varphi\sin^6\varphi+8b_5\sin^7\varphi\big)\,,\\
J(\varphi,a,b)=&8b_4\cos^4\varphi\sin^4\varphi+8(2a_4-b_3)\cos^3\varphi\sin^5\varphi+\\
&8(2a_3+b_4-b_5)\cos^2\varphi\sin^6\varphi+\\
&8(2a_5+b_3)\cos\varphi\sin^7\varphi+8b_5\sin^8\varphi,\\
K(\varphi)=&(1+R^2\sin^2\varphi+R\sin\varphi\sqrt{1+R^2\sin^2\varphi})\\
&(1+2R^2\sin^2\varphi+2R\sin\varphi\sqrt{1+R^2\sin^2\varphi})^2,
\end{aligned}
$$

and $a=(a_1,...,a_5)$, $b=(b_1,...,b_5)$, $c=(c_1,...,c_5)$ and $d=(d_1,...,d_5)$.

Note that system (2) has the invariant straight line $y=-1/2$. So the minimal distance of the external boundary of the period annulus of the center to the origin is $1/2$.

The discontinuous differential system (14) is under the assumptions of Theorem 3. So we must study the zeros of the averaged function $f:(0,\frac{1}{2})\to\mathbb{R}$,

$$
\begin{aligned}
f(R)=\quad&\int_0^\pi\frac{1}{K(\varphi)}\Big(D(\varphi,a,b)R+E(\varphi,a,b)R^2+F(\varphi,a,b)R^3+\\
&G(\varphi,a,b)R^4+H(\varphi,a,b)R^5+I(\varphi,a,b)R^6+J(\,a,b)R^7\Big)d\varphi+\\
&\int_\pi^{2\pi}\frac{1}{K(\varphi)}\Big(D(\varphi,c,d)R+E(\varphi,c,d)R^2+F(\varphi,c,d)R^3+\\
&G(\varphi,c,d)R^4+H(\varphi,c,d)R^5+I(\varphi,c,d)R^6+J(\varphi,c,d)R^7\Big)d\varphi.
\end{aligned}
$$

We compute these integrals and we obtain

$$
f(R)=a_1g_1+a_3g_2+b_2g_3+b_4g_4+b_5g_5+c_1g_6+c_3g_7+d_2g_8+d_4g_9+d_5g_{10},
$$

where

$$
\begin{aligned}
g_1 =& 1+\frac{\pi}{2}R-\frac{1}{R}\arctan R-R\arctan R,\\
g_2 =& -1+\frac{1}{R}\arctan R+R\arctan R,\\
g_3 =& -1+\frac{\pi}{2}R-2R^2+\pi R^3+\arctan R\left(\frac{1}{R}-R-2R^3\right),\\
g_4 =& -1-R^2-\frac{\pi}{2}R^3+\arctan R\left(\frac{1}{R}+2R+R^3\right),\\
g_5 =& 1+R^2-\frac{\pi}{2}R^3+\arctan R\left(-\frac{1}{R}+R^3\right),\\
g_6 =& -1+\frac{\pi}{2}R+\arctan R\left(\frac{1}{R}+R\right),\\
g_7 =& 1+\arctan R\left(-\frac{1}{R}-R\right),\\
g_8 =& 1+\frac{\pi}{2}R+2R^2+\pi R^3+\arctan R\left(-\frac{1}{R}+R+2R^3\right),\\
g_9 =& 1+R^2-\frac{\pi}{2}R^3+\arctan R\left(-\frac{1}{R}-2R-R^3\right),\\
g_{10} =& -1-R^2-\frac{\pi}{2}R^3+\arctan R\left(\frac{1}{R}-R^3\right).
\end{aligned}
\tag{15}
$$

We have the equalities

$$
\begin{aligned}
g_5 =& \tfrac{1}{2}(g_1-g_3),\\
g_6 =& g_1+2g_2,\\
g_7 =& -g_2,\\
g_9 =& \tfrac{1}{4}(4g_1+4g_2-2g_3-4g_4-2g_8),\\
g_{10} =& g_2+\frac{1}{2}(g_1-g_8),
\end{aligned}
$$

so we can rewrite the function f as

$$
\begin{aligned}
f(R) =& (a_1+c_1+d_4+\tfrac{1}{2}(b_5+d_5))g_1+(a_3+2c_1-c_3+d_4+d_5)g_2+\\
& (b_2-\tfrac{1}{2}(b_5+d_4))g_3+(b_4-d_4)g_4+(d_2-\tfrac{1}{2}(d_4+d_5))g_8.
\end{aligned}
$$

We have the following Taylor expansions in the variable R around $R=0$ of the functions g_1, g_2, g_3, g_4 and g_8:

$$
\begin{aligned}
g_1(R) =& \frac{\pi}{2}R-\frac{2}{3}R^2+\frac{2}{15}R^4-\frac{2}{35}R^6+\mathcal{O}(R^7),\\
g_2(R) =& \frac{2}{3}R^2-\frac{2}{15}R^3+\frac{2}{35}R^6+\mathcal{O}(R^7),\\
g_3(R) =& \frac{\pi}{2}R-\frac{10}{3}R^2+\pi R^3-\frac{22}{15}R^4+\frac{34}{105}R^6+\mathcal{O}(R^7),
\end{aligned}
$$

$$g_4(R) = \frac{2}{3}R^2 - \frac{\pi}{2}R^3 + \frac{8}{15}R^4 - \frac{8}{105}R^6 + \mathcal{O}(R^7),$$
$$g_8(R) = \frac{\pi}{2}R + \frac{10}{3}R^2 + \pi R^3 + \frac{22}{15}R^4 - \frac{34}{105}R^6 + \mathcal{O}(R^7),$$

respectively. The determinant of the coefficient matrix of the variables R, R^2, R^3, R^4, R^6 is $-128\pi^2/1575$. So we have that the set of five functions $g_i : \left(0, \frac{1}{2}\right) \to \mathbb{R}$, given by $\{g_1, g_2, g_3, g_4, g_8\}$ is linearly independent.

By Proposition 5 since the five functions g_1, g_2, g_3, g_4, g_8 are linearly independent, then there exists a linear combination of their with at least 4 zeros. Moreover the coefficients of the functions g_i, $i = 1, 2, 3, 4, 8$ in the expression of f are linear functions of the variables a_1, a_3, b_2, b_4, b_5, c_1, c_3, d_2, d_4, d_5. The rank of the Jacobian matrix of the coefficient of g_1, g_2, g_3, g_4 and g_8 in $f(R)$ in the variables a_1, a_3, b_2, b_4, b_5, c_1, c_3, d_2, d_4, d_5 is 5. Thus there exist R_1, R_2, R_3, $R_4 \in (0, \frac{1}{2})$ and coefficients a_j, b_j c_j, $d_j \in \mathbb{R}$, $j = 1, ..., 5$ such that $f(R_i) = 0$ for $i = 1, ..., 4$.

In short, there are discontinuous quadratic polynomial differential systems (3) having at least 4 limit cycles bifurcating from the periodic orbits of the isochronous center $\dot{x} = -y + x^2 - y^2$, $\dot{y} = x + 2xy$, using the averaging theory of first order for discontinuous differential systems. This completes the proof of Theorem 2.

6. Example

In this section we illustrate Theorem 2 by studying a particular discontinuous quadratic polynomial differential systems (3) which has the maximum number of limit cycles under the assumptions of Theorem 2, i.e., 4 limit cycles.

We consider system (3) with

$$\begin{array}{ll} a_1 = -6.7745224606, & a_3 = -0.3706207036, \\ b_2 = 4.7754759853, & b_4 = 11.4284792711, \end{array}$$

$$a_5 = b_3 = c_1 = d_2 = 1,$$

$$a_2 = a_4 = b_1 = b_5 = c_2 = c_3 = c_4 = c_5 = d_1 = d_3 = d_4 = d_5 = 0.$$

Thus we have

$$Y_1(x, y) = \begin{pmatrix} -y + x^2 - y^2 + \varepsilon(-6.7745224606x - 0.370620703xy + y^2) \\ x + 2xy + \varepsilon(4.7754759853y + 11.4284792711x^2 + xy) \end{pmatrix},$$

$$Y_2(x, y) = \begin{pmatrix} -y + x^2 - y^2 + \varepsilon x \\ x + 2xy + \varepsilon y \end{pmatrix}.$$

Computing the averaged function f for this system we obtain

$$f(R) = -22.6078570134 + 0.0014977931R - 18.9794312418R^2 +$$
$$0.1923796666R^3 + \frac{1.3125}{R}\operatorname{arcsinh} R - \frac{0.65625}{R}\operatorname{arctanh}\frac{2R\sqrt{1+R^2}}{1+2R^2} +$$

$$\arctan R\left(\frac{22.6078570134}{R}+26.4853843138R+3.8775273004R^3\right)$$

The zeros of $f(R)=0$ are

$$R_1=\frac{1}{10},\ R_2=\frac{2}{10},\ R_3=\frac{3}{10}\ \text{and}\ R_4=\frac{4}{10},$$

and

$$f'\left(\frac{1}{10}\right)=-0.000402022,\quad f'\left(\frac{2}{10}\right)=0.000282645,$$
$$f'\left(\frac{3}{10}\right)=-0.000439832,\quad f'\left(\frac{4}{10}\right)=0.00179854,$$

i.e., $f'(R_i)\neq 0$ for $i=1,\dots,4$. Hence, by Theorem 3 it follows that for $\varepsilon\neq 0$ sufficiently small this discontinuous differential system has 4 periodic solutions.

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