Semilinear elliptic problems with asymmetric nonlinearities*

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Abstract

In this paper we are concerned on the semilinear elliptic problem

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + b(u^+)^{p-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, $1 < q < 2 < p \leq 2^*$. If a is between two eigenvalues, we get the existence of three nontrivial solutions for the problem above.

KEY WORDS. positive solution, indefinite sublinear nonlinearity, concave-convex nonlinearity, critical growth.

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1 Introduction

We consider the semilinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + b(u^+)^{p-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, $N \geq 3$, $1 < q < 2 < p \leq 2^*$, $a \in I\!R$, $b > 0$, λ is a positive parameter and $u^+ = \max\{u, 0\}$.

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The weak solutions of the problem (P) correspond to critical points of the C^1 functional I_λ, defined on $H_0^1 :\equiv H_0^1(\Omega)$ by

$$I_\lambda(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\lambda}{q}\int |u|^q - \frac{a}{2}\int u^2 - \frac{b}{p}\int (u^+)^p, \quad u \in H_0^1. \tag{1}$$

After the appearance of [1], there has been an increasing concern about multiple solutions of semilinear elliptic problem of the type:

$$-\Delta u = \mu|u|^{q-2}u + g(u) \ \text{ in } \ \Omega. \tag{2}$$

When g is asymmetric and asymptotically linear this problem was considered in [8, 10, 13, 20]. Here asymmetric means that g satisfies an Ambrosetti-Prodi type condition (i.e. $g_- := \lim_{t\to-\infty} g(t)/t < \lambda_k < g_+ := \lim_{t\to+\infty} g(t)/t$). When g is asymmetric and superlinear at $+\infty$, $g_+ = \infty$, this problem was approached in [8, 13, 17]. In [8] a Neumann problem was considered and in [17] the authors studied a problem involving the p-Laplace operator. In [13], one was assumed that $g(t)/t$ crosses an eigenvalue of the Laplacian when the t varies from 0 to $-\infty$ (i.e. $g'(0) < \lambda_k < g_-$). Similar hypotheses also appears in [20]. Assumptions involving the first eigenvalue, as $g'(0), g_- \leq \lambda_1$, were considered in [8, 10, 17]. It is known that crossing eigenvalues, in particular the first one, is related to existence and multiplicity for such problems. Notice that the nonlinearity $g(t) = at + b(t^+)^{p-1}$, with $a > \lambda_1$, is not included in the cases count on the previous works. Moreover, similar problems with $\mu = 0$ were studied in [16] for Dirichlet problems, and in [2, 19] for Neumann problems.

Our problem is also closely related to the class of superlinear Ambrosetti-Prodi problem:

$$-\Delta u = au + (u^+)^p + f(x) \ \text{ in } \ \Omega, \tag{3}$$

with $f \in L^2$. For instance, this problem have a solution if $||f||_{L^2}$ is small enough (see [12]). Further results and references for the above problem can be found in [5, 6, 11, 18, 21, 22].

For the critical case, our main motivation to (P) is the Brezis-Nirenberg pioneering work [4], where the following critical problem was considered

$$\begin{cases} -\Delta u = au + |u|^{2^*-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $a < \lambda_1$. They noticed that the problem had a breaking of compactness at the value $\frac{S^{N/2}}{N}$, so that they constructed minimax levels for the energy functional associated below this value. Such ideas have been permeating many later works as well as ours. One of them, it was the Capozzi, Fortunato and Palmieri work [7]. They basically studied the problem above with a between two eigenvalues. They showed that the problem above has a nontrivial solution for all $a > 0$ when $N \geq 5$ and for a different from eigenvalues when $N = 4$.

We are denoting by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ the eigenvalues of $(-\Delta, H_0^1(\Omega))$ and by φ_j the correspondents eigenfunctions. The $H_0^1(\Omega)$ norm and $L^p(\Omega)$ norm are represented by $\|\cdot\|$ and $|\cdot|_p$ and we denote these spaces by H_0^1 and L^p, for simplicity, respectively.

In the sequel, we set up precisely the results obtained

Theorem 1. *Let $N \geq 3$ and $\lambda_k < a < \lambda_{k+1}$. If $2 < p < 2^*$, then, for λ small enough, (P) has at least three nontrivial solutions.*

Theorem 2. *Let $N \geq 4$ and $\lambda_k < a < \lambda_{k+1}$. If $p = 2^*$, for λ small enough, (P) has at least three nontrivial solutions.*

The major arguments of the proofs of our theorems are based on variational methods. As it is well-known, we have to show some geometric conditions and prove a compactness condition. Provided us with these tools, we obtain a negative and a positive solution and the third one cames from linking theorem. In order to do that, we follows some tricks used in [6, 14]. In the next section, we show the (PS) condition for the energy functional. In the third section, we present the proofs of theorems above.

2 The (PS) condition

We begin by showing the (PS) condition for I_λ.

Lemma 1. *Let $\lambda_1 < a$, $2 < p \leq 2^*$ and $\lambda > 0$. Then every (PS) sequence of I_λ is bounded.*

Proof. Let (u_n) be a (PS) sequence for I_λ, i.e., it satisfies

$$\left|\tfrac{1}{2}\int|\nabla u_n| + \tfrac{\lambda}{q}\int|u_n|^q - \tfrac{a}{2}\int {u_n}^2 - \tfrac{b}{p}\int(u_n^+)^p\right| \leq C, \tag{4}$$

$$\left|\int \nabla u_n \nabla h + \lambda\int|u_n|^{q-2}u_n h - a\int u_n h - b\int(u_n^+)^{p-1}h\right| \leq \epsilon_n\|h\|, \quad \forall\, h \in H_0^1, \tag{5}$$

where $\epsilon_n \to 0$ as $n \to \infty$. By (4) and (5) we have

$$\begin{aligned} C + \epsilon_n\|u_n\| &\geq \left|I_\lambda(u_n) - \frac{1}{2}\langle I_\lambda'(u_n), u_n\rangle\right| \\ &= \left|\left(\frac{\lambda}{q} - \frac{\lambda}{2}\right)\int|u_n|^q + \left(\frac{b}{2} - \frac{b}{p}\right)\int(u_n^+)^p\right| \\ &\geq \left(\frac{b}{2} - \frac{b}{p}\right)\int(u_n^+)^p. \end{aligned}$$

Since $p > 2$ we get

$$\int (u_n^+)^p \le C + \epsilon_n \|u_n\|. \tag{6}$$

We also have by (5),

$$|\langle I'_\lambda(u_n), u_n^-\rangle| = \left| \|u_n^-\|^2 + \lambda |u_n^-|_q^q - a|u_n^-|_2^2 \right| \le \epsilon_n \|u_n^-\|, \tag{7}$$

with $u^- = \max\{-u, 0\}$. It follows from (4), (6) and (7) that

$$\begin{aligned} \frac{1}{2}\|u_n^+\|^2 &\le \left(\frac{\lambda}{2} - \frac{\lambda}{q}\right) \int |u_n|^q + \frac{a}{2} \int (u_n^+)^2 + \frac{b}{p} \int (u_n^+)^p + \frac{1}{2} |\langle I'_\lambda(u_n), u_n^-\rangle| + C \\ &\le C \int (u_n^+)^p + \epsilon_n \|u_n^-\| + C \le \epsilon_n \|u_n\| + \epsilon_n \|u_n^-\| + C. \end{aligned} \tag{8}$$

Suppose by contradiction that $\|u_n\| \to \infty$. We first show that (u_n^+) is bounded in H_0^1, so that assume also that $\|u_n^+\| \to \infty$. By (8), (u_n^-) is also unbounded. Let $v_n = u_n / \|u_n\|$. Since (v_n) is bounded in H_0^1, there exists $v \in H_0^1$ such that

$$v_n \rightharpoonup v \ \text{ in } H_0^1, \quad v_n \to v \text{ in } L^r, \ \forall\, 1 \le r < 2^* \quad \text{and} \quad v_n \to v \ \text{ a.e. in } \Omega.$$

Again by (8) there exists $\delta > 0$ satisfying

$$\|u_n^-\| \ge \delta \|u_n^+\|^2 \tag{9}$$

whenever n is large. Since

$$v_n^+ = \frac{u_n^+}{\|u_n\|} = \frac{u_n^+}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}} \le \frac{u_n^+}{(\|u_n^+\|^2 + \delta^2 \|u_n^+\|^4)^{1/2}},$$

we deduce that $v \le 0$. Moreover, by

$$v_n^- = \frac{u_n^-}{\|u_n\|} = \frac{u_n^-}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}} = \frac{u_n^-}{\|u_n^-\|} \cdot \frac{\|u_n^-\|}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}},$$

and (9), we have $\|v_n^-\| \to 1$. Thus, by (7),

$$-\lambda \frac{|u_n^-|_q^q}{\|u_n^-\|^2} + a \frac{|u_n^-|_2^2}{\|u_n^-\|^2} \to 1. \tag{10}$$

We also note that by (9) and $\|v_n^-\| \to 1$ in H_0^1,

$$\frac{u_n^-}{\|u_n^-\|} - \frac{u_n^-}{\|u_n\|} = \frac{u_n^-}{\|u_n\|} \left(\frac{\|u_n\|}{\|u_n^-\|} - 1 \right) \to 0 \quad \text{in } H_0^1.$$

Hence we may exchange $\|u_n^-\|$ for $\|u_n\|$ in (10). Recalling that $q < 2$, we obtain $|v_n^-|_2 \to 1/\sqrt{a}$, then $v \neq 0$. We then take $h = \varphi_1$ in (5) to obtain

$$\int \nabla v_n \nabla \varphi_1 + \frac{\lambda}{\|u_n\|} \int |u_n|^{q-2} u_n \varphi_1 - a \int v_n \varphi_1 - \frac{b}{\|u_n\|} \int (u_n^+)^{p-1} \varphi_1 \to 0,$$

that is

$$(\lambda_1 - a) \int v_n \varphi_1 + \frac{\lambda}{\|u_n\|^{2-q}} \int |v_n|^{q-2} v_n \varphi_1 - \frac{b}{\|u_n\|} \int (u_n^+)^{p-1} \varphi_1 \to 0.$$

The second and third terms above go to zero, consequently,

$$(\lambda_1 - a) \int v \varphi_1 = 0,$$

which is a contradiction, because $v \leq 0$, $v \neq 0$ and $\lambda_1 < a$, so that (u_n^+) is bounded. Finally, suppose that $\|u_n\| \to \infty$ and $\|u_n^+\| \leq C$ for all $n \in \mathbb{N}$. Since $p \leq 2^*$,

$$\frac{1}{\|u_n\|} \int_\Omega (u_n^+)^p \to 0.$$

On the other hand, by taking $h = v_n$ in (5) we obtain

$$a |v_n|_2^2 \to 1,$$

so that $v_n \to v$ in L^2 with $v \neq 0$. Then by (5) we get

$$\int_\Omega \nabla v \nabla h - a \int_\Omega v h = 0 \quad \text{for all } h \in H_0^1,$$

with $v \neq 0$ and $v \leq 0$, which is a contradiction, because a is not the first eigenvalue. Therefore, we conclude that (u_n) must be bounded in H_0^1. □

In the subcritical case, $1 \leq p < 2^*$, it is well-known that the lemma above implies that I_λ satisfies the (PS) condition at every level.

Lemma 2. *Let $\lambda_1 < a$ and $p = 2^*$. For every $\lambda > 0$, I_λ satisfies the (PS) condition at level c with $c < \frac{b^{\frac{2-N}{2}} S^{\frac{N}{2}}}{N}$.*

Proof. Let $(u_n) \subset H_0^1$ be a sequence satisfying

$$I_\lambda(u_n) \to c \quad \text{and} \quad |\langle I'_\lambda(u_n), h\rangle| \leq \epsilon_n \|h\|, \ \forall h \in H_0^1, \tag{11}$$

with $\epsilon_n \to 0$ as $n \to \infty$. By Lemma 1 we have that (u_n) is bounded. Hence, by passing to a subsequence, we may suppose that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1, & u_n &\to u \quad \text{in } L^2, \\ u_n &\to u \quad \text{in } L^q, & u_n &\to u \quad \text{a.e. in } \Omega. \end{aligned} \tag{12}$$

Since (u_n^+) is bounded in H_0^1, from Gagliardo-Nirenberg Inequality it follows that (u_n^+) is also bounded in L^{2^*}. By passing to a subsequence again, we have $u_n^+ \rightharpoonup u^+$ in L^{2^*}. Thus, u solves

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + b(u^+)^{2^*-1} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Note that by (13) we obtain

$$I_\lambda(u) = \left(\frac{\lambda}{q} - \frac{\lambda}{2}\right)|u|_q^q + \left(\frac{b}{2} - \frac{b}{2^*}\right)|u^+|_{2^*}^{2^*} \geq 0. \tag{14}$$

We denote $v_n = u_n - u$. By the Brezis-Lieb's Lemma

$$\lim_{n\to\infty} (|u_n^+|_p^p - |v_n^+|_p^p) = |u^+|_p^p \quad \text{for all } 1 \leq p \leq 2^*.$$

Moreover, by (12) we have $v_n \to 0$ in L^q and L^2, so that

$$\begin{aligned} \lim_{n\to\infty}[I_\lambda(u) + I_\lambda(v_n)] = \lim_{n\to\infty}\Bigg[\|u\|^2 - \int_\Omega \nabla u_n \nabla u + \frac{1}{2}\|u_n\|^2 + \frac{\lambda}{q}(|u|_q^q + |v_n|_q^q) \\ - \frac{a}{2}(|u|_2^2 + |v_n|_2^2) - \frac{b}{2^*}(|u^+|^{2^*} + |v_n^+|^{2^*})\Bigg] \\ = \lim_{n\to\infty}\left[\frac{1}{2}\|u_n\|^2 + \frac{\lambda}{q}|u_n|_q^q - \frac{a}{2}|u_n|_2^2 - \frac{b}{2^*}|u_n^+|^{2^*}\right] = c. \end{aligned} \tag{15}$$

On the other hand, by (11) and again Brezis-Lieb's Lemma,

$$\begin{aligned} &\lim_{n\to\infty}\left[\|v_n\|^2 + \lambda|v_n|_q^q - a|v_n|_2^2 - b|v_n^+|_{2^*}^{2^*}\right] \\ &= \lim_{n\to\infty}\left[\langle I_\lambda'(u_n), u_n\rangle - 2\int_\Omega \nabla u_n \nabla u + 2\|u\|^2 - \langle I_\lambda'(u), u\rangle\right] = 0. \end{aligned} \tag{16}$$

Since $v_n \to 0$ in L^q and L^2, we may suppose that

$$\|v_n\|^2 \to d \quad \text{and} \quad b|v_n^+|_{2^*}^{2^*} \to d.$$

By Sobolev's Inequality, $\|v_n\|^2 \geq S|v_n^+|_{2^*}^2$, consequently, $d \geq S(d/b)^{2/2^*}$. If $d = 0$ the proof is concluded. Otherwise, $d \geq S^{\frac{N}{2}} b^{\frac{2-N}{2}}$. Then by (14), (15) and (16) we conclude

$$\frac{S^{\frac{N}{2}} b^{\frac{2-N}{2}}}{N} \leq \left(\frac{1}{2} - \frac{1}{2^*}\right) d \leq c < \frac{b^{\frac{2-N}{2}} S^{\frac{N}{2}}}{N},$$

which is a contradiction. □

3 Proof of Theorems

3.1 Existence of the nonnegative solution

Consider the functional $I_\lambda^+ : H_0^1 \to \mathbb{R}$ given by

$$I_\lambda^+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{q}\int_\Omega (u^+)^q - \frac{1}{2}\int_\Omega (u^+)^2 - \frac{1}{p}\int_\Omega (u^+)^p,\ u \in H_0^1. \tag{17}$$

It follows that $I_\lambda^+ \in C^1$ and the critical points u_+ of I_λ^+ satisfy $u_+ \geq 0$ and so are critical points of I_λ as well, actually, $(I_\lambda^+)'(u_+)[(u_+)^-] = -\int_\Omega |\nabla (u_+)^-|^2 = 0$.

We will show that I_λ^+ satisfies the assumptions of the mountain pass theorem. In a similar argument to proofs of Lemmas 1 and 2, we show the (PS) condition for I_λ^+.

Lemma 3. *Let $2 < p \leq 2^*$. For all $\lambda > 0$, I_λ^+ satisfies the (PS) condition at level c with $c < \frac{b^{\frac{2-N}{2}} S^{\frac{N}{2}}}{N}$.*

Lemma 4. *The trivial solution $u \equiv 0$ is a local minimizer for I_λ^+, for all $\lambda > 0$.*

Proof. It suffices to show that 0 is a local minimizer of I_λ^+ in the C^1 topology (see [3]). Then, for $u \in C_0^1(\overline{\Omega})$ we have

$$\begin{aligned} I_\lambda^+(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{q}\int_\Omega (u^+)^q - \frac{a}{2}\int_\Omega (u^+)^2 - \frac{b}{p}\int_\Omega (u^+)^p \\ &\geq \frac{\lambda}{q}\int_\Omega |u|^q - \frac{a}{2}\int_\Omega u^2 - \frac{b}{p}\int_\Omega |u|^p \\ &\geq \left(\frac{\lambda}{q} - \frac{a}{2}|u|_{C^0}^{2-q} - \frac{b}{p}|u|_{C^0}^{p-q}\right)\int_\Omega |u|^q \geq 0 \end{aligned}$$

whenever $\frac{a}{2}|u|_{C_0^1}^{2-q} + \frac{b}{p}|u|_{C_0^1}^{p-q} \leq \frac{\lambda}{q}$. □

Lemma 5. *There exists $t_0 > 0$ such that $I_\lambda^+(t_0\varphi_1) \leq 0$, for all λ in a bounded set.*

Proof. Denoting by φ_1 the positive eigenfunction associated to λ_1, we have, for $t > 0$,

$$\begin{aligned} I_\lambda^+(t\varphi_1) &= \frac{t^2}{2}\int_\Omega |\nabla \varphi_1|^2 + \frac{t^q\lambda}{q}\int_\Omega \varphi_1^q - \frac{at^2}{2}\int_\Omega \varphi_1^2 - \frac{t^p}{p}\int_\Omega \varphi_1^p \\ &= \frac{1}{2}t^2(\lambda_1 - a)\int_\Omega \varphi_1^2 + \frac{t^q\lambda}{q}\int_\Omega \varphi_1^q - \frac{t^p}{p}\int_\Omega \varphi_1^p \end{aligned}$$

and, since $\lambda_k < a < \lambda_{k+1}$ and $q < 2 < p$, there exists a choice of $t_0 > 0$ which proves the lemma. □

Finally, define

$$c_\lambda^+ = \inf_{\gamma\in\Gamma^+} \sup_{t\in[0,1]} I_\lambda^+(\gamma(t)),$$

where

$$\Gamma^+ = \{\gamma \in \mathcal{C}([0,1], H_0^1);\ \gamma(0) = 0, \gamma(1) = t_0\varphi_1\}.$$

On the other hand, by the proof of the previous lemma we obtain

$$I_\lambda^+(t\varphi_1) \le \frac{t^q\lambda}{q}\int_\Omega \varphi_1^q.$$

Then, if λ is small enough, $c_\lambda^+ < \frac{b^{\frac{2-N}{2}}S^{\frac{N}{2}}}{N}$, consequently, by the Mountain Pass Theorem c_λ^+ is a critical value of I_λ^+.

3.2 Existence of the nonpositive solution

In order to get the negative solution, consider the following functional $I_\lambda^- : H_0^1 \to \mathbb{R}$ given by

$$I_\lambda^-(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{q}\int_\Omega (u^-)^q - \frac{a}{2}\int_\Omega (u^-)^2. \tag{18}$$

Again, $I_\lambda^- \in C^1$ and the critical points u_- of I_λ^- satisfy $u_- \le 0$ and so are critical points of I_λ as well. We will apply once again the mountain pass theorem to obtain a critical point of I_λ^-.

Lemma 6. *The trivial solution $u \equiv 0$ is a local minimizer for I_λ^-, for all $\lambda > 0$.*

Proof. It suffices to show that 0 is a local minimizer of I_λ^- in the C^1 topology. Then, for $u \in C_0^1(\overline{\Omega})$ we have

$$\begin{aligned} I_\lambda^-(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{q}\int_\Omega (u^-)^q - \frac{a}{2}\int_\Omega (u^-)^2 \ge \frac{\lambda}{q}\int_\Omega (u^-)^q - \frac{a}{2}\int_\Omega (u^-)^2 \\ &\ge \left(\frac{\lambda}{q} - \frac{a}{2}|u^-|_{C^0}^{2-q}\right)\int_\Omega (u^-)^q \ge \left(\frac{\lambda}{q} - \frac{a}{2}|u|_{C^0}^{2-q}\right)\int_\Omega (u^-)^q \ge 0, \end{aligned}$$

whenever $\frac{a}{2}|u|_{C_0^1}^{2-q} \le \frac{\lambda}{q}$. □

Lemma 7. *There exists $t_0 > 0$ such that $I_\lambda^-(-t_0\varphi_1) \le 0$, for all λ in a limited set.*

Proof. We have, for $t > 0$,

$$\begin{aligned} I_\lambda^-(-t\varphi_1) &= \frac{t^2}{2}\int_\Omega |\nabla\varphi_1|^2 dx + \frac{t^q\lambda}{q}\int_\Omega \varphi_1^q dx - \frac{at^2}{2}\int_\Omega \varphi_1^2\, dx \\ &= \frac{1}{2}t^2(\lambda_1 - a)\int_\Omega \varphi_1^2 dx + \frac{t^q\lambda}{q}\int_\Omega \varphi_1^q dx \end{aligned}$$

and, since $\lambda_k < a < \lambda_{k+1}$ and $q < 2$, there exists a choice of $t_0 > 0$ which proves the lemma. □

As in the nonnegative solution case, we obtain a critical value

$$c_\lambda^- = \inf_{\gamma\in\Gamma^-} \sup_{t\in[0,1]} I_\lambda^-(\gamma(t)),$$

where

$$\Gamma^- = \{\gamma \in \mathcal{C}([0,1]);\ \gamma(0) = 0, \gamma(1) = -t_0\varphi_1\}.$$

In view of the proof of Lemma 7, we get the estimate

$$c_\lambda^- \le \max_{s\in[0,1]} I_\lambda^-(-st_0\varphi_1) \le \frac{t_0^q\lambda}{q}\int_\Omega \varphi_1^q.$$

Then, if λ is small enough, $c_\lambda^- < \frac{b^{\frac{2-N}{2}}S^{\frac{N}{2}}}{N}$, consequently, by the Mountain Pass Theorem, c_λ^- is a critical value of I_λ^-.

3.3 Existence of the third solution

Denote $V_k = \langle\varphi_1, \cdots, \varphi_k\rangle$ and $W_k = V_k^\perp$. Consider the functions introduced in [14],

$$\zeta_m = \begin{cases} 0 & \text{if } x \in B_{1/m}, \\ m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m}, \\ 1 & \text{if } x \in \Omega \setminus B_{2/m}, \end{cases}$$

where we may suppose without loss of generality that $0 \in \Omega$. Let $\varphi_i^m = \zeta_m\varphi_i$,

$$V_k^m = \langle\zeta_m\varphi_1, \cdots, \zeta_m\varphi_k\rangle$$

and $W_k^m = (V_k^m)^\perp$. For each $m \in \mathbb{N}$, take a positive cut-off function $\eta \in \mathcal{C}_c^\infty(B_{1/m})$ and define

$$\varphi_{k+1}^m = \eta\varphi_{k+1}.$$

It follows from definitions above that

$$\text{supp} u \cap \text{supp}\varphi_{k+1}^m = \emptyset \tag{19}$$

whenever $u \in V_k^m$. We use the following lemma from [14].

Lemma 8. *As $m \to \infty$ we have*

$$\varphi_i^m \to \varphi_i \quad \textit{in } H_0^1, \quad \textit{and} \quad \max_{\left\{u\in V_k^m;\ \int u^2=1\right\}} \|u\|^2 \le \lambda_k + c_k m^{2-N}.$$

An easy consequence of this result is the following decompostion of H_0^1,

Corollary 1. *For m large enough*

$$V_k^m \oplus W_k = H_0^1.$$

Lemma 9. *There exist $\alpha > 0$ and $\rho > 0$ such that*

$$I_\lambda(u) \geq \alpha$$

whenever $u \in W_k$ and $\|u\| = \rho$.

Proof. Note that, if $u \in W_k$ we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int |\nabla u|^2 + \frac{\lambda}{q}\int |u|^q - \frac{a}{2}\int |u|^2 - \frac{b}{p}\int (u^+)^p \\ &\geq \left(\frac{1}{2} - \frac{a}{2\lambda_{k+1}}\right)\|u\|^2 - \frac{b}{q}|u|_p^p \\ &\geq \|u\|^2(A - B\|u\|^{p-2}), \end{aligned}$$

with $A, B > 0$. Then it suffices to take $\rho < (A/B)^{\frac{1}{p-2}}$. □

Lemma 10. *Given $\lambda_0 > 0$, there exist $m_0 \in \mathbb{N}$ and $R > \rho$ such that*

$$I_\lambda(u) \leq \frac{\lambda}{q}\int |u|^q,$$

whenever $u \in \partial Q_m$, where $Q_m = (B_R \cap V_k^m) \oplus [0, R\varphi_{k+1}^m]$, $m \geq m_0$ and $\lambda \leq \lambda_0$. Henceforth ∂ means the boundary relative to subspace V_k^m.

Proof. Let m large enough and $a_k < a$ be such that

$$\lambda_k + c_k m^{2-N} \leq a_k < a. \tag{20}$$

Initially, let $u \in V_k^m$, by Lemma 8 and (20) we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int |\nabla u|^2 + \frac{\lambda}{q}\int |u|^q - \frac{a}{2}\int u^2 - \frac{b}{p}\int (u^+)^p \\ &\leq \left(\frac{1}{2} - \frac{a}{2a_k}\right)\int |\nabla u|^2 + \frac{\lambda}{q}\int |u|^q - \frac{b}{p}\int (u^+)^p \\ &\leq \frac{\lambda}{q}\int |u|^q. \end{aligned} \tag{21}$$

Also,

$$\begin{aligned} I_\lambda(r\varphi_{k+1}^m) &= \frac{r^2}{2}\int |\nabla \varphi_{k+1}^m|^2 + \frac{\lambda r^q}{q}\int |\varphi_{k+1}^m|^q - \frac{ar^2}{2}\int (\varphi_{k+1}^m)^2 - \frac{r^p b^p}{p}\int ({\varphi_{k+1}^m}^+)^p \\ &\leq \frac{r^2}{2}\int |\nabla \varphi_{k+1}^m|^2 + \frac{\lambda_0 r^q}{q}\int |\varphi_{k+1}^m|^q - \frac{r^p b^p}{p}\int ({\varphi_{k+1}^m}^+)^p. \end{aligned} \tag{22}$$

Since $\varphi_{k+1}^m \to \varphi_{k+1}$ in H_0^1 as $m \to \infty$ and φ_{k+1} changes of signal, there exist $m_0 \in \mathbb{N}$ and $R > 0$ such that

$$I_\lambda(R\varphi_{k+1}^m) \leq 0 \quad \text{for all } m \geq m_0. \tag{23}$$

Then, by (19), (21) and (23) we obtain

$$I_\lambda(u) \le \frac{\lambda}{q}\int |u|^q, \tag{24}$$

whenever $u \in V_k^m \cup (V_k^m \oplus R\varphi_{k+1}^m)$. By (22), there exists $\beta > 0$ satisfying

$$I_\lambda(r\varphi_{k+1}^m) \le \beta \tag{25}$$

for all $m \ge m_0$ and $r \ge 0$. Since $a > \lambda_k$, we may take $R > 0$ such that

$$I_\lambda(u) \le \left(\frac{1}{2} - \frac{a}{2\lambda_k}\right)\|u\|^2 + \frac{\lambda}{q}\int |u| \le -\beta + \frac{\lambda}{q}\int |u|^q. \tag{26}$$

Thus, by (19), (25) and (26) we get

$$I_\lambda(u + r\varphi_{k+1}^m) = I_\lambda(u) + I_\lambda(r\varphi_{k+1}^m) \le \frac{\lambda}{q}\int |u|^q \tag{27}$$

for all $m \ge m_0$ and $u \in \partial(B_R \cap V_k^m)$. Therefore, by (24) and (27) we conclude the proof. □

Conclusion of Theorem 1: subcritical case

Let α given by the Lemma 9. Take λ enough small in order that

$$\frac{\lambda}{q}\int |u|^q \le \mu < \alpha$$

for all $u \in \partial Q_m$. Then by Lemma 10 we have

$$I_\lambda(u) \le \mu < \alpha$$

whenever $u \in \partial Q_m$ and $m \ge m_0$. Applying the Linking Theorem, I_λ possesses a critical point u at level c, where

$$c_\lambda = \inf_{\Gamma}\max_{u \in Q_m} I_\lambda(h(u))$$

and

$$\Gamma = \{h \in \mathcal{C}(\overline{Q_m}, H_0^1);\ h = Id \text{ on } \partial Q_m\}.$$

Finally, since $c_\lambda \ge \alpha$, $I_\lambda(u) \ge \alpha > 0$ and $c_\lambda^\pm \to 0$ as $\lambda \to 0$. Therefore, if λ is small enough $c_\lambda^\pm < \alpha \le c_\lambda$, and, consequently, u may to be neither of the critical points found above for I_λ^+ and I_λ^-. □

Conclusion of Theorem 2: critical case

For the critical case, we consider the family of functions taken from [4]

$$u_\epsilon(x) = \frac{[N(N-2)\epsilon^2]^{\frac{N-2}{4}}}{[\epsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \epsilon > 0.$$

We recall that u_ϵ satisfies $\|u_\epsilon\|^2 = \|u_\epsilon\|_{2^*}^{2^*} = S^{N/2}$ for all $\epsilon > 0$. Let $u_\epsilon^m = \eta u_\epsilon$, where η is given as above, and $Q_m^\epsilon = (B_R \cap V_k^m) \oplus [0, Ru_\epsilon^m]$. Replacing u_ϵ^m by φ_{k+1}^m in Lemma 10, we obtain

$$I_\lambda(u) \leq \frac{\lambda}{q} \int_\Omega |u|^q \quad \text{for all } u \in \partial Q_m^\epsilon$$

whenever m is large. Therefore, to conclude the proof of Theorem 2, it remains to show that

$$\sup_{u \in Q_m^\epsilon} I_\lambda(u) < \frac{b^{\frac{2-N}{2}} S^{N/2}}{N}$$

for ϵ and λ small enough.

Lemma 11. *There exist $m_0 > 0$, $\lambda_0 > 0$ and $\epsilon_0 > 0$ such that*

$$\sup_{u \in Q_m^\epsilon} I_\lambda(u) < \frac{b^{\frac{2-N}{2}} S^{N/2}}{N}$$

for all $m \geq m_0$, $\lambda < \lambda_0$ and $\epsilon < \epsilon_0$.

Proof. We write

$$I_\lambda(u) = J(u) + \frac{\lambda}{q}|u|_q^q, \quad \text{where} \quad J(u) :\equiv \frac{1}{2}\|u\|^2 - \frac{a}{2}|u|_2^2 - \frac{b}{2^*}|u^+|_{2^*}^{2^*}.$$

We note that it is sufficient to prove that there exist $m_0 > 0$ and $\epsilon_0 > 0$ such that

$$\sup_{u \in Q_m^\epsilon} J(u) < \frac{b^{\frac{2-N}{2}} S^{N/2}}{N}$$

for all $m \geq m_0$ and $\epsilon < \epsilon_0$. Let $u = v + tu_\epsilon^m \in Q_m^\epsilon$. We first observe that

$$\max_{t \geq 0} J(tu_\epsilon^m) = \frac{b^{\frac{2-N}{2}}}{N} \left(\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|_2^2}{|u_\epsilon^m|_{2^*}^2} \right)^{N/2}.$$

Fixe $m_0 > 0$ such that $\lambda_k + c_k m_0^{2-N} \leq \sigma < a$. For $m \geq m_0$, we have

$$J(v) = \frac{1}{2}\|v\|^2 - \frac{a}{2}|v|_2^2 - \frac{1}{2^*}|v^+|_{2^*}^{2^*} \leq \frac{1}{2}\|v\|^2 - \frac{a}{2}|v|_2^2 \leq \frac{\sigma}{2}|v|_2^2 - \frac{a}{2}|v|_2^2.$$

Hence

$$J(u) = J(v) + J(tu_\epsilon^m) \leq J(tu_\epsilon^m).$$

Therefore, it remains to prove that

$$\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|_2^2}{|u_\epsilon^m|_{2^*}^2} < S$$

whenever ϵ is small. But this follows from identities

$$\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|_2^2}{|u_\epsilon^m|_{2^*}^2} = \begin{cases} S - ad\epsilon^2|\ln\epsilon| + O(\epsilon^{N-2}) & \text{if } N = 4, \\ S - ad\epsilon^2 + O(\epsilon^{N-2}) & \text{if } N \geq 5, \end{cases}$$

for their details see [23, Page 52]. □

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