# Norm resolvent convergence of Dirichlet Laplacian in unbounded thin waveguides*

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**Abstract**

The proof of norm resolvent convergence of Dirichlet Laplacian to effective operators, as the diameter of spatial waveguides vanishes, is extended to the unbounded case.

**Keywords:** unbounded waveguides; Dirichlet Laplacian; spectral approximation; norm resolvent convergence; dimensional reduction.

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# 1 Introduction

Let $x \in I$ denote the arc-length parameter of the curve $r(x)$ in $\mathbb{R}^3$, with $I$ denoting either $\mathbb{R}$ or $[0, L]$ $(L < \infty)$, and let $k(x)$ and $\tau(x)$ be its curvature and torsion at the point $r(x)$, respectively. Pick $S \neq \emptyset$ an open, bounded and connected subset of $\mathbb{R}^2$. We build a tube (waveguide) $\Omega$ in $\mathbb{R}^3$ by properly moving the region $S$ along $r(x)$; at each point $r(x)$ the cross-section region $S$ may present a (continuously differentiable) rotation angle $\alpha(x)$ (see details in Section 2). We consider the Laplacian inside $\Omega$ and suppose the Dirichlet condition at the boundary $\partial\Omega$, and study its behavior as the tube is squeezed to $r(x)$. Initially we demand that the functions $k, (\tau + \alpha') \in \mathrm{L}^\infty(I)$.

The self-adjoint operator associated with this problem with "thin cross-section" is

$$\psi \mapsto -\Delta_\varepsilon \psi, \qquad \psi \in \operatorname{dom} \Delta_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon) \cap \mathcal{H}^2(\Omega_\varepsilon),$$

where $\Omega_\varepsilon$ is the waveguide generated by the cross-section $\varepsilon S$, and $\Delta_\varepsilon$ denotes the Laplacian in $\Omega_\varepsilon$ ($\mathcal{H}^2$ and $\mathcal{H}_0^1$ denote the usual Sobolev spaces). Note that, as $\varepsilon \to 0$, such sequence of tubes $\Omega_\varepsilon$ approaches the curve $r$. It is expected that there is an effective operator, which should be identified with an one-dimensional operator in $\mathrm{L}^2(I)$, describing such singular limit $\varepsilon \to 0$. However, as $\varepsilon \to 0$, the region $\Omega_\varepsilon$ becomes narrower and the whole spectrum of $-\Delta_\varepsilon$ diverges. Let $\lambda_0$ be the first (i.e., the lowest) eigenvalue of the Dirichlet Laplacian restricted to $S$ and $u_0$ the (positive) associated normalized eigenfunction, that is,

$$-\Delta u_0 = \lambda_0 u_0, \quad u_0 \in \mathcal{H}_0^1(S), \quad u_0 \geq 0, \quad \int_S |u_0|^2 dy = 1. \tag{1}$$

For future use, set $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$C(S) := \int_S (\nabla_y u_0 \cdot Ry)^2 \, dy, \tag{2}$$

which is a constant that depends only on the tube cross-section $S$.

Consider the sequence of operators

$$F_\varepsilon \psi = -\Delta_\varepsilon \psi - \frac{\lambda_0}{\varepsilon^2} \psi, \qquad \operatorname{dom} F_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon) \cap \mathcal{H}^2(\Omega_\varepsilon). \tag{3}$$

The subtraction of the term $\lambda_0/\varepsilon^2$ is intended to renormalize the divergence of the transverse oscillations (see a discussion in Section 1 of [dOV1], which also includes some physical motivations).

In [BMT] the case of finite curves (i.e., $I = [0, L]$), so that the corresponding tubes are bounded, was studied. Therein it was shown that the asymptotic behavior of the eigenvalues $\lambda_j^\varepsilon$ of (3) (with $j \in \mathbb{N}$), as $\varepsilon \to 0$, are given by the eigenvalues of the effective one-dimensional self-adjoint operator

$$\tilde{T}w := -w'' + C(S)(\tau+\alpha')^2 w - \frac{k(x)^2}{4} w, \quad \operatorname{dom}\tilde{T} = \mathcal{H}_0^1(0,L) \cap \mathcal{H}^2(0,L). \quad (4)$$

More precisely, if $\mu_j$ are the eigenvalues of $\tilde{T}$, it was shown that, for each $j \in \mathbb{N}$,

$$\lambda_j^\varepsilon \to \mu_j, \quad \varepsilon \to 0. \quad (5)$$

Note that the operator (4) depends on geometric features of the tube, with nontrivial corrections to the one-dimensional Laplacian operator $w(x) \mapsto -w''(x)$.

The case of unbounded tubes (more precisely, $I = \mathbb{R}$) was studied in [deO2] through the variational technique of strong and weak $\Gamma$-convergences, and it was found a strong resolvent convergence of $F_\varepsilon$ to

$$Tw := -w'' + C(S)(\tau + \alpha')^2 w - \frac{k(x)^2}{4} w, \quad \operatorname{dom} T = \mathcal{H}^2(\mathbb{R}), \quad (6)$$

as $\varepsilon \to 0$. As a byproduct of [deO2], the eigenvalue convergence (5) (in case of bounded tubes) was justified as a result of a proof of the operator norm resolvent convergence of $F_\varepsilon$ to the effective operator $\tilde{T}$. The proof of norm resolvent convergence also in the case of unbounded tubes $I = \mathbb{R}$, thus extending some results of [BMT, deO2], was speculated in Remark 4.8 in [BMT] and proposed as an open problem at the end of [Kr]. To the best of the current authors knowledge, this is still an open problem whose proof is the main goal of this work; see Theorem 1 ahead.

In any event, it is worth mentioning that in [dOV1] such norm resolvent convergence of $F_\varepsilon$, in the case $I = \mathbb{R}$, was proven, but under the severe restriction of null curvature, i.e., $k(x) = 0$, for all $x \in \mathbb{R}$ (and consequently, $\tau(x) = 0$, for all $x \in \mathbb{R}$). In this case, as expected, the effective operator was found to be $(\hat{T}w)(x) = -w''(x) + \alpha'(x)^2 w(x)$, $\operatorname{dom}\hat{T} = \mathcal{H}^2(\mathbb{R})$.

The above touched upon known results were obtained through the study of the sequence of quadratic forms associated with the operators $F_\varepsilon$, i.e., of the sequence

$$\tilde{f}_\varepsilon(\psi) = \int_{\Omega_\varepsilon} \left( |\nabla\psi|^2 - \frac{\lambda_0}{\varepsilon^2}|\psi|^2 \right) dxdy, \quad \operatorname{dom}\tilde{f}_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon).$$

The symbol $\nabla = (\partial_x, \nabla_y)$, with $\nabla_y = (\partial_{y_1}, \partial_{y_2})$, denotes the gradient in the coordinates $(x, y) = (x, y_1, y_2)$ in $\mathbb{R}^3$. Usually we denote $\partial_x\psi$ simply by $\psi'$.

Observe that $F_\varepsilon$ is initially considered in the Hilbert space $\mathrm{L}^2(\Omega_\varepsilon)$. In [BMT, deO2, dOV1] the authors perform a traditional change of variables (this change will be used in Section 2) and pass to work with $\mathrm{L}^2(I \times S, \beta_\varepsilon(x,y)dxdy)$ where $(x, y) = (x, y) \in I \times S$,

$$\beta_\varepsilon(x,y) := 1 - \varepsilon k(x)\langle z_{\alpha(x)}, y\rangle, \quad z_{\alpha(x)} := (\cos\alpha(x), -\sin\alpha(x)). \quad (7)$$

The measure $\beta_\varepsilon(x,y)dxdy$ comes from the Riemannian metric obtained from the change of variables that transforms the region $\Omega_\varepsilon$ into the simpler one $I \times S$. Observe that this change results in a region that doesn't depend on the parameter $\varepsilon > 0$; the price to be paid is the measure $\beta_\varepsilon(x,y)dxdy$ instead of $dxdy$, and a rather "complicated" expression for the actions of operators and quadratic forms associated with the problem.

Since $\beta_\varepsilon(x,y) \to 1$ uniformly as $\varepsilon \to 0$, the Hilbert space $\mathrm{L}^2(I\times S, \beta_\varepsilon dxdy)$ is algebraically equivalent to $\mathrm{L}^2(I \times S)$ and this identification is used in [BMT, deO2]. On the other hand, in [dOV1] the present authors have shown that this identification is not convenient to prove the norm resolvent convergence of the sequence $F_\varepsilon$ (unless $k(x) = 0$, for all $x \in I$, and then $\beta_\varepsilon(x,y) = 1$, for all $(x,y) \in I \times S$).

In this work, besides the usual change of variables mentioned above, an important step will be the simple, although crucial, additional change of variables

$$\begin{array}{ccc} \mathrm{L}^2(I \times S) & \to & \mathrm{L}^2(I \times S, \beta_\varepsilon dxdy) \\ \psi & \mapsto & \beta_\varepsilon^{-1/2}\,\psi \end{array} .$$

Thus, we pass to work in the Hilbert space $\mathrm{L}^2(I \times S)$ and with a new expression for the quadratic forms $\tilde{f}_\varepsilon(\psi)$ (different from those in [BMT, deO2, dOV2]). This procedure, with the differentiability hypotheses (11), will allow us to use the same method employed in [dOV1] to prove the norm resolvent convergence for unbounded waveguides also in the case of nonzero curvature and torsion. This method will be briefly recalled in Section 3 and it has also been used in other works [FS1, dOV2, dOV1].

Now, we state our main result in this work, which justifies $T$ (see (6)) as an effective operator for such thin unbounded waveguides. See the discussion related to equation (15) about how the dimensional reduction is implemented.

**Theorem 1.** *Assume that conditions* (11) *(see Section 2) hold true and let* $\mathcal{L} := \{w(x)u_0(y) : w \in \mathrm{L}^2(\mathbb{R})\}$*. Then, the sequence of operators* $F_\varepsilon$ *converges in the norm resolvent sense to the operator* $T \oplus 0$*, as* $\varepsilon \to 0$*, where* 0 *is the null operator on the subspace* $\mathcal{L}^\perp$*.*

In Section 2 we describe the waveguides and study the quadratic forms $\tilde{f}_\varepsilon(\psi)$; then we perform the two change of variables mentioned above. In Section 3 we present the method of reduction of dimension and prove our main result. Section 4 is of pure technical character and devoted to the proof of an auxiliary result, that is, Theorem 3.

## 2 Quadratic forms

In this section we go into details of the region where the Laplacian operator is considered and the associated quadratic forms.

We suppose that $r : \mathbb{R} \to \mathbb{R}^3$ is a simple $C^3$ curve in $\mathbb{R}^3$ parametrized by its arc-length parameter $x$. The curvature of $r$ at the position $x$ is defined by $k(x) := |r''(x)|$. We choose the usual orthonormal triad of vector fields $\{T(x), N(x), B(x)\}$, the so-called Frenet frame, given the tangent, normal and binormal vectors, respectively, moving along the curve and defined by

$$T = r', \qquad N = k^{-1}T', \qquad B = T \times N. \tag{8}$$

To justify the construction (8), it is assumed that $k > 0$, but if $r$ has a piece of a straight line (i.e., $k = 0$ identically in this piece), usually one can choose a constant Frenet frame instead. It is possible to combine constant Frenet frames with the Frenet frame (8) to include other types of curves, for instance, curves with $k(x) > 0$ only on a compact interval of values of $x$ (and so obtaining a global $C^2$ Frenet frame; see [Kl], Theorem 1.3.6).

In each situation above we assume that a global Frenet frame exists and that the Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{9}$$

where $\tau(x)$ is the torsion of $r(x)$, actually defined by (9).

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a bounded $C^1$ function so that $\alpha(0) = 0$, and $S$ an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$\vec{\mathbf{x}}(x, y) = r(x) + \varepsilon\, y_1 N_\alpha(x) + \varepsilon\, y_2 B_\alpha(x)$$

and consider the domain

$$\Omega_\varepsilon = \left\{ \vec{\mathbf{x}}(x, y) \in \mathbb{R}^3 : x \in \mathbb{R}, y = (y_1, y_2) \in S \right\},$$

where

$$\begin{aligned} N_\alpha(x) &:= \cos\alpha(x) N(x) + \sin\alpha(x) B(x), \\ B_\alpha(x) &:= -\sin\alpha(x) N(x) + \cos\alpha(x) B(x). \end{aligned}$$

Hence, this tube $\Omega_\varepsilon$ is obtained by putting the region $\varepsilon S$ along the curve $r(x)$, which is simultaneously rotated by an angle $\alpha(x)$ with respect to the cross section at the position $x = 0$.

In order to study the Laplacian $-\Delta_\varepsilon$ in $\Omega_\varepsilon$, and with Dirichlet condition at the boundary $\partial\Omega_\varepsilon$, we initially consider the corresponding family of quadratic forms

$$b_\varepsilon(\psi) := \int_{\Omega_\varepsilon} |\nabla\psi|^2 dx dy, \qquad \mathrm{dom}\, b_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon). \tag{10}$$

We are interested in the limit of the sequence $b_\varepsilon(\psi)$ as $\varepsilon \to 0$. For this singular limit, customary "regularizations" will be employed.

Recall that $\lambda_0$ is the lowest eigenvalue of the negative Laplacian with Dirichlet boundary conditions in the cross section region $S$, and $u_0 \geq 0$ the corresponding eigenfunction of this restricted problem. This eigenfunction $u_0$ is directly related to transverse oscillations in $\Omega_\varepsilon$. Due to this fact, we will remove the diverging energy $\lambda_0/\varepsilon^2$ from the quadratic forms (this strategy was also used in [BMT, deO2, dOV1, dOV2]; see a simple example and a discussion in the introductory section of [dOV1]). We also add an specific constant $c > 0$ in the quadratic forms; its precise values will be chosen later ahead and this fact will be convenient later on. Thus, we pass to study the sequence

$$\begin{aligned}\hat{h}_\varepsilon(\psi) &:= b_\varepsilon(\psi) - \frac{\lambda_0}{\varepsilon^2}\|\psi\|^2 + c\|\psi\|^2 \\ &= \int_{\Omega_\varepsilon} \left( |\nabla\psi|^2 - \frac{\lambda_0}{\varepsilon^2}|\psi|^2 + c|\psi|^2 \right) dxdy,\end{aligned}$$

$\mathrm{dom}\,\hat{h}_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon)$.

It is convenient to perform a "traditional" change of variables so that the integration region in the definition of $\hat{h}_\varepsilon(\psi)$, and consequently the domains, become fixed, that is, independent of $\varepsilon > 0$. Although this change is usual and can be found in [BMT, deO2, dOV1], we are going to repeat some details in this work because without them the expressions can be seem artificial.

Consider the mapping

$$\begin{aligned} f_\varepsilon : \quad \mathbb{R} \times S &\to \Omega_\varepsilon \\ (x, y_1, y_2) &\mapsto r(x) + \varepsilon y_1 N_\alpha(x) + \varepsilon y_2 B_\alpha(x), \end{aligned}$$

and suppose the boundedness $\|k\|_\infty < \infty$. This condition is to guarantee that $f_\varepsilon$ will be a diffeomorphism for small $\varepsilon$. With this change of variables we work with a fixed region $\mathbb{R} \times S$ for all $\varepsilon > 0$; more precisely, in the new variables the domain of the quadratic form (11) turns out to be $\mathcal{H}_0^1(\mathbb{R} \times S)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon$ which is induced by $f_\varepsilon$, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j \rangle = G_{ji}, \qquad 1 \leq i, j \leq 3,$$

where

$$e_1 = \frac{\partial f_\varepsilon}{\partial x}, \qquad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \qquad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$J = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \beta_\varepsilon & -\varepsilon(\tau + \alpha')\langle z_\alpha^\perp, y\rangle & \varepsilon(\tau + \alpha')\langle z_\alpha, y\rangle \\ 0 & \varepsilon\cos\alpha & \varepsilon\sin\alpha \\ 0 & -\varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix},$$

where

$$\beta_\varepsilon(x,y) = 1 - \varepsilon k(x)\langle z_\alpha, y\rangle, \quad z_\alpha = (\cos\alpha, -\sin\alpha), \quad z_\alpha^\perp = (\sin\alpha, \cos\alpha).$$

The inverse matrix of $J$ is given by

$$J^{-1} = \begin{pmatrix} \frac{1}{\beta_\varepsilon} & \frac{(\tau+\alpha')y_2}{\beta_\varepsilon} & -\frac{(\tau+\alpha')y_1}{\beta_\varepsilon} \\ 0 & \frac{\cos\alpha}{\varepsilon} & \frac{-\sin\alpha}{\varepsilon} \\ 0 & \frac{\sin\alpha}{\varepsilon} & \frac{\cos\alpha}{\varepsilon} \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2 \beta_\varepsilon(x,y)$. Since $k$ is a bounded function, for $\varepsilon$ small enough $\beta_\varepsilon$ does not vanish in $\mathbb{R}\times S$. Thus, $\beta_\varepsilon > 0$ and $f_\varepsilon$ is a local diffeomorphism. By requiring that $f_\varepsilon$ is injective (that is, the tube is not self-intersecting), a global diffeomorphism is obtained.

Introducing the notation

$$||\psi||_G^2 := \int_{\mathbb{R}\times S} |\psi(x,y)|^2 \beta_\varepsilon(x,y)\, dxdy,$$

and the unitary transformation

$$\begin{array}{rcl} U_\varepsilon : \mathrm{L}^2(\Omega_\varepsilon) & \to & \mathrm{L}^2(\mathbb{R}\times S, \beta_\varepsilon(x,y)dxdy) \\ \phi & \mapsto & \varepsilon\, \phi \circ f_\varepsilon \end{array}$$

we obtain a sequence of quadratic forms

$$\tilde h_\varepsilon(U_\varepsilon\psi) := \left|\left|J^{-1}\nabla\,(U_\varepsilon\psi)\right|\right|_G^2 - \frac{\lambda_0}{\varepsilon^2}\|U_\varepsilon\psi\|_G^2 + c\,\|U_\varepsilon\psi\|_G^2, \qquad \operatorname{dom}\tilde h_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S).$$

However, we still denote $U_\varepsilon\psi$ by $\psi$. Thus,

$$\tilde h_\varepsilon(\psi) = \left|\left|J^{-1}\nabla\psi\right|\right|_G^2 - \frac{\lambda_0}{\varepsilon^2}\|\psi\|_G^2 + c\,\|\psi\|_G^2, \quad \operatorname{dom}\tilde h_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S).$$

After the norms are written out, we obtain

$$\begin{aligned} \tilde h_\varepsilon(\psi) &= \int_{\mathbb{R}\times S}\Big(\frac{1}{\beta_\varepsilon^2(x,y)}\left|\psi' + (\nabla_y\psi\cdot Ry)\,(\tau+\alpha')(x)\right|^2 \\ &+ \frac{|\nabla_y\psi|^2}{\varepsilon^2} - \frac{\lambda_0}{\varepsilon^2}|\psi|^2 + c|\psi|^2\Big)\beta_\varepsilon(x,y)\, dxdy\,, \end{aligned}$$

$\operatorname{dom}\tilde h_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S)$, where $R$ is the rotation matrix that has appeared in the Introduction.

We list some remarks.

1. $\operatorname{dom}\tilde h_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S)$ is a subspace of $\mathrm{L}^2(\mathbb{R}\times S, \beta_\varepsilon(x,y)dxdy)$. The measure $\beta_\varepsilon(x,y)dxdy$ comes from of the Riemannian metric obtained from the change of variables.

2. Since $\beta_\varepsilon(x,y) \to 1$ uniformly as $\varepsilon \to 0$, the spaces $\mathrm{L}^2(\mathbb{R}\times S, \beta_\varepsilon(x,y)dxdy)$ and $\mathrm{L}^2(\mathbb{R}\times S)$ are algebraically equivalents and this identification was used in [BMT, deO2].

3. In [BMT] the authors show the following inequality. For each $\psi \in \mathcal{H}_0^1(\mathbb{R}\times S)$,

$$\int_S \beta_\varepsilon \left( \frac{|\nabla_y \psi|^2}{\varepsilon^2} - \frac{\lambda_0}{\varepsilon^2}|\psi|^2 \right) dy \geq \gamma_\varepsilon(x) \int_S |\psi|^2 dy, \quad \text{q.t.p.[x]},$$

where $\gamma_\varepsilon(x)$ is a function that converges uniformly to $-k^2(x)/4$ as $\varepsilon \to 0$. This fact is also used in [deO2] and it is the responsible for the appearance of the curvature in the action of the effective operator $T$ in Theorem 1.

By means of a counterexample, in [dOV1] it is argued that the above setup is not convenient to get a norm convergence of operators through quadratic forms. Thus, in this work we consider an additional change of variables that will permit us to get such norm convergence in the case of unbounded waveguides as well. Specifically, we consider the isometry

$$\begin{array}{rccc} V_\varepsilon: & \mathrm{L}^2(\mathbb{R}\times S) & \to & \mathrm{L}^2(\mathbb{R}\times S, \beta_\varepsilon(x,y)) \\ & \psi & \mapsto & \beta_\varepsilon^{-1/2}\psi \end{array},$$

and for simplicity of notation we will denote $V_\varepsilon\psi$ by $\psi$. In the new variables, the above quadratic form reads

$$\begin{aligned} \dot{h}_\varepsilon(\psi) = \int_{\mathbb{R}\times S} \frac{1}{\beta_\varepsilon^2} \Big| \frac{\partial \psi}{\partial x} + \psi\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) + (\nabla_y \psi \cdot Ry)\ (\tau + \alpha')(x) \\ + \psi\, \beta_\varepsilon^{1/2} \left( \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry \right) (\tau+\alpha')(x) \Big|^2 dxdy \\ + \int_{\mathbb{R}\times S} \frac{\beta_\varepsilon}{\varepsilon^2} \left( \left| \nabla_y \left(\psi/\beta_\varepsilon^{1/2}\right)\right|^2 - \lambda_0 \left(\psi/\beta_\varepsilon^{1/2}\right)^2 \right) dxdy + \int_{I\times S} c\,\psi^2 dxdy, \end{aligned}$$

and $\operatorname{dom} \dot{h}_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S)$.

Although natural, the above change of variables results in a more complicated action of the quadratic forms; we will show, however, that suitable control of their terms will be possible and useful.

Now we are going to conveniently rewrite the second integral in the definition of $\dot{h}_\varepsilon$ above. This step is very important because in this way the curvature will appear explicitly in the quadratic form without making use of a variational argument [BMT]. First note that

$$\int_S \frac{\beta_\varepsilon}{\varepsilon^2} \left[ \frac{\partial}{\partial y_1} \left(\psi/\beta_\varepsilon^{1/2}\right) \right]^2 dy$$

$$= \int_S \left[ \frac{1}{\varepsilon^2} \left( \frac{\partial \psi}{\partial y_1} \right)^2 + \frac{\beta_\varepsilon}{\varepsilon^2} \left[ \frac{\partial}{\partial y_1} \left( \frac{1}{\beta_\varepsilon^{1/2}} \right) \right]^2 |\psi|^2 + 2 \frac{\beta_\varepsilon^{1/2}}{\varepsilon^2} \frac{\partial}{\partial y_1} \left( \frac{1}{\beta_\varepsilon^{1/2}} \right) \frac{\partial \psi}{\partial y_1} \psi \right] dy$$

$$= \int_S \left[ \frac{1}{\varepsilon^2} \left( \frac{\partial \psi}{\partial y_1} \right)^2 - \frac{k^2(x)}{4} \frac{\cos^2 \alpha(x)}{\beta_\varepsilon^2} |\psi|^2 \right] dy$$

because an integration by parts shows that

$$\int_S \frac{\beta_\varepsilon}{\varepsilon^2} \left[ \frac{\partial}{\partial y_1} \left( \frac{1}{\beta_\varepsilon^{1/2}} \right) \right]^2 |\psi|^2 dy$$

$$= \int_S \left[ -\frac{k^2(x)}{4} \frac{\cos^2 \alpha(x)}{\beta_\varepsilon^2} |\psi|^2 - 2 \frac{\beta_\varepsilon^{1/2}}{\varepsilon^2} \frac{\partial}{\partial y_1} \left( \frac{1}{\beta_\varepsilon^{1/2}} \right) \frac{\partial \psi}{\partial y_1} \psi \right] dy.$$

Similarly

$$\int_S \frac{\beta_\varepsilon}{\varepsilon^2} \left[ \frac{\partial}{\partial y_2} \left( \psi / \beta_\varepsilon^{1/2} \right) \right]^2 dy = \int_S \left[ \frac{1}{\varepsilon^2} \left( \frac{\partial \psi}{\partial y_2} \right)^2 - \frac{k^2(x)}{4} \frac{\sin^2 \alpha(x)}{\beta_\varepsilon^2} |\psi|^2 \right] dy.$$

Therefore,

$$\int_{\mathbb{R} \times S} \frac{\beta_\varepsilon}{\varepsilon^2} \left( \left| \nabla_y \left( \psi / \beta_\varepsilon^{1/2} \right) \right|^2 - \lambda_0 \left| \psi / \beta_\varepsilon^{1/2} \right|^2 \right) dx dy$$

$$= \int_{\mathbb{R} \times S} \left[ \frac{1}{\varepsilon^2} (|\nabla_y \psi|^2 - \lambda_0 |\psi|^2) - \frac{k^2(x)}{4 \beta_\varepsilon^2} |\psi|^2 \right] dx dy.$$

Finally, the quadratic form $\dot{h}_\varepsilon$ can be rewritten as

$$\dot{h}_\varepsilon(\psi) = \int_{\mathbb{R} \times S} \frac{1}{\beta_\varepsilon^2} \left| \frac{\partial \psi}{\partial x} + \psi \, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1 / \beta_\varepsilon^{1/2} \right) + (\nabla_y \psi \cdot R y) \, (\tau + \alpha')(x) \right.$$

$$\left. + \psi \, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1 / \beta_\varepsilon^{1/2} \right) \cdot R y \right) \, (\tau + \alpha')(x) \right|^2 dx dy$$

$$+ \int_{\mathbb{R} \times S} \left( \frac{|\nabla_y \psi|^2}{\varepsilon^2} - \lambda_0 \frac{|\psi|^2}{\varepsilon^2} \right) dx dy - \int_{\mathbb{R} \times S} \frac{k^2(x)}{4 \beta_\varepsilon^2} |\psi|^2 dx dy + \int_{\mathbb{R} \times S} c |\psi|^2 dx dy,$$

and with $\operatorname{dom} \dot{h}_\varepsilon = \mathcal{H}_0^1(\mathbb{R} \times S)$.

By replacing the global multiplicative factors $\beta_\varepsilon$ by 1 in the first, third and last integral in the expression of $\dot{h}_\varepsilon$, we arrive now at the quadratic form

$$h_\varepsilon(\psi) = \int_{\mathbb{R} \times S} \left| \frac{\partial \psi}{\partial x} + \psi \, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1 / \beta_\varepsilon^{1/2} \right) + (\nabla_y \psi \cdot R y) \, (\tau + \alpha')(x) \right.$$

$$\left. + \psi \, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1 / \beta_\varepsilon^{1/2} \right) \cdot R y \right) \, (\tau + \alpha')(x) \right|^2 dx dy$$

$$+\int_{\mathbb{R}\times S}\left(\frac{|\nabla_y\psi|^2}{\varepsilon^2}-\lambda_0\frac{|\psi|^2}{\varepsilon^2}\right)dxdy-\int_{\mathbb{R}\times S}\frac{k^2(x)}{4}|\psi|^2dxdy+\int_{\mathbb{R}\times S}c|\psi|^2dxdy,$$

and with $\operatorname{dom} h_\varepsilon=\mathcal{H}_0^1(\mathbb{R}\times S)$.

If one chooses $c$ so that $\|k^2/4\|_\infty<c$, the quadratic forms $\dot{h}_\varepsilon$ and $h_\varepsilon$ become positive. Let $\dot{H}_\varepsilon$ and $\mathcal{H}_\varepsilon$ denote the self-adjoint operators associated with $\dot{h}_\varepsilon$ and $h_\varepsilon$, respectively.

Until now we only suppose that $k,(\tau+\alpha')\in \mathrm{L}^\infty(\mathbb{R})$. For the next results we are going to add the following hypotheses:

$$k'(x),k''(x),\alpha'(x),\alpha''(x),\tau'(x)\in \mathrm{L}^\infty(\mathbb{R}). \tag{11}$$

**Theorem 2.** *Suppose that conditions in (11) are satisfied. Then, there exists a number $C>0$ so that*

$$\|\dot{H}_\varepsilon^{-1}-H_\varepsilon^{-1}\|\leq C\varepsilon,$$

*for $\varepsilon>0$ small enough.*

The main point of this theorem is that $\beta_\varepsilon(x,y)\to 1$ uniformly as $\varepsilon\to 0$. Its proof is quite similar to proof of Theorem 3.1 in [dOV2] and will not be presented here.

# 3 Reduction of dimension and main results

Recall that $u_0(y)$ is the positive and normalized eigenfunction corresponding to the first eigenvalue $\lambda_0$ of the Laplacian in $\mathcal{H}_0^1(S)$. Let $\mathcal{L}$ be the subspace of $\mathrm{L}^2(\mathbb{R}\times S)$ generated by all functions of the form $w(x)u_0(y)$, $w\in \mathrm{L}^2(\mathbb{R})$. By considering the orthogonal decomposition

$$\mathrm{L}^2(\mathbb{R}\times S)=\mathcal{L}\oplus\mathcal{L}^\perp, \tag{12}$$

for $\psi\in \mathrm{L}^2(\mathbb{R}\times S)$, one can write

$$\psi(x,y)=w(x)u_0(y)+\eta(x,y), \tag{13}$$

with $w\in \mathrm{L}^2(\mathbb{R})$ and $\eta\in\mathcal{L}^\perp$. Note that $wu_0\in\mathcal{H}_0^1(\mathbb{R}\times S)$ if $w\in\mathcal{H}^1(\mathbb{R})$. Correspondingly, for $\psi\in\mathcal{H}_0^1(\mathbb{R}\times S)$, write $\psi=wu_0+\eta$ with $w\in\mathcal{H}^1(\mathbb{R})$ and $\eta\in\mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}^\perp$.

Now we are going to study the quadratic form $h_\varepsilon$ restricted to the subspace $\mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}$. Under the conditions (11), for $w\in\mathcal{H}^1(\mathbb{R})$ some long calculations (see Appendix A) show that

$$h_\varepsilon(wu_0)=\int_{\mathbb{R}\times S}\left[\left|\frac{\partial w}{\partial x}\right|^2+\left(C(S)(\tau+\alpha')^2(x)-\frac{k^2(x)}{4}+g_\varepsilon(x)+c\right)|w|^2\right]dx,$$

with $C(S)$ given by (2) and $g_\varepsilon(x)\to 0$ uniformly as $\varepsilon\to 0$.

Due to the uniform convergence of $g_\varepsilon(x)$ and the definition of the constant $c > 0$, there exists a number $d > 0$ so that

$$b_\varepsilon(wu_0) \geq d \int_{\mathbb{R}} |w|^2 dx, \quad \forall w \in \mathcal{H}^1(\mathbb{R}), \tag{14}$$

for $\varepsilon > 0$ small enough. This inequality will be important ahead.

Taking into account the isometry

$$\mathcal{L} \ni wu_0 \mapsto w \in \mathrm{L}^2(\mathbb{R}), \qquad \int_{\mathbb{R}\times S} |w(x)u_0(y)|^2 \, dxdy = \int_{\mathbb{R}} |w(x)|^2 \, dx, \tag{15}$$

we may think of $h_\varepsilon$ restricted to $\mathrm{L}^2(\mathbb{R})$, that is, $h_\varepsilon(w) := h_\varepsilon(wu_0)$. The self-adjoint operator associated with $h_\varepsilon$ in $\mathrm{L}^2(\mathbb{R})$ is

$$(T_\varepsilon^c w)(x) := -w''(x) + \left( C(S)(\tau + \alpha')^2(x) - \frac{k^2(x)}{4} + g_\varepsilon(x) + c \right) w(x), \tag{16}$$

$\operatorname{dom} T_\varepsilon^c = \mathcal{H}^2(\mathbb{R})$, and the above choice of $c$ implies that zero is not a spectral point of $T_\varepsilon^c$.

In the previous section we made a series of change of variables in the operator $-\Delta_\varepsilon - \lambda_0/\varepsilon^2 \mathbf{1} + c\mathbf{1}$ to get $H_\varepsilon$; more specifically,

$$V_\varepsilon^{-1} U_\varepsilon \left( -\Delta_\varepsilon + \lambda_0/\varepsilon^2 \mathbf{1} + c\mathbf{1} \right) U_\varepsilon^{-1} V_\varepsilon = H_\varepsilon;$$

however, for simplicity of notation, we shall write $-\Delta_\varepsilon - \lambda_0/\varepsilon^2 \mathbf{1} + c\mathbf{1}$ for the operator associated with the quadratic form $h_\varepsilon$.

**Theorem 3.** *By assuming conditions* (11)*, there exists a number $D > 0$ so that, for $\varepsilon > 0$ small enough,*

$$\left\| \left( -\Delta_\varepsilon - \frac{\lambda_0}{\varepsilon^2} \mathbf{1} + c\mathbf{1} \right)^{-1} - (T_\varepsilon^c)^{-1} \oplus 0 \right\| \leq D\,\varepsilon,$$

*where* 0 *is the null operator on the subspace $\mathcal{L}^\perp$.*

Now define the operator $\operatorname{dom} T^c = \mathcal{H}^2(\mathbb{R})$, with action

$$(T^c w)(x) := -w''(x) + \left( C(S)(\tau + \alpha')^2(x) - \frac{k^2(x)}{4} + c \right) w(x).$$

**Proposition 1.** *$T_\varepsilon^c$ converge to $T^c$ in the norm resolvent sense as $\varepsilon \to 0$.*

*Proof.* We take $\mu < 0$. Thus, $\mu \in \rho(T_\varepsilon^c) \cap \rho(T^c)$ because $T_\varepsilon^c$ and $T^c$ are positive operators. By the third resolvent identity [deO1],

$$(T_\varepsilon^c - \mu\mathbf{1})^{-1} - (T^c - \mu\mathbf{1})^{-1} = (T_\varepsilon^c - \mu\mathbf{1})^{-1}(T^c - T_\varepsilon^c)(T^c - \mu\mathbf{1})^{-1}.$$

Since $\|(T^c_\varepsilon - \mu\mathbf{1})^{-1}\| \le 1$, and $\|(T^c - \mu\mathbf{1})^{-1}\| \le 1$, we have

$$\begin{aligned}\|(T^c_\varepsilon - \mu\mathbf{1})^{-1} - (T^c - \mu\mathbf{1})^{-1}\| &\le \|(T^c_\varepsilon - \mu\mathbf{1})^{-1}\|\|T^c - T^c_\varepsilon\|\|(T^c - \mu\mathbf{1})^{-1}\| \\ &\le \|T^c - T^c_\varepsilon\| \\ &= \|g_\varepsilon\|_\infty.\end{aligned}$$

Since $g_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, the result follows. □

*Proof.* (Theorem 1) Put $Q_\varepsilon := \big(-\Delta_\varepsilon - \lambda_0/\varepsilon^2\mathbf{1} + c\mathbf{1}\big) = F_\varepsilon + c\mathbf{1}$, and note that

$$\begin{aligned}\|Q_\varepsilon^{-1} - (T^c)^{-1} \oplus 0\| &\le \|Q_\varepsilon^{-1} - (T^c_\varepsilon)^{-1} \oplus 0\| + \|(T^c_\varepsilon)^{-1} \oplus 0 - (T^c)^{-1} \oplus 0\| \\ &= \|Q_\varepsilon^{-1} - (T^c_\varepsilon)^{-1} \oplus 0\| + \|(T^c_\varepsilon)^{-1} - (T^c)^{-1}\|.\end{aligned}$$

By Theorem 3 and Proposition 1, one gets the norm resolvent convergence of $Q_\varepsilon$ to $T^c$, and consequently the proof of Theorem 1. □

# 4 Proof of Theorem 3

We begin with some observations. Take $\eta \in \mathcal{H}^1_0(\mathbb{R} \times S) \cap \mathcal{L}^\perp$. Thus,

$$\int_S u_0(y)\, \eta(x,y)\, dy = 0, \quad \text{q.t.p.[x]}. \tag{17}$$

An integration by parts shows that

$$\int_S \nabla_y u_0(y)\, \nabla_y \eta(x,y)\, dy = 0, \quad \text{q.t.p. [x]}. \tag{18}$$

Let $\lambda_1$ be the second eigenvalue of the Dirichlet Laplacian in $\mathcal{H}^1_0(S)$. Recall that $\lambda_0$ is a simple eigenvalue; thus $\lambda_1 > \lambda_0$. Since $\eta \in \mathcal{H}^1_0(\mathbb{R}\times S) \cap \mathcal{L}$,

$$\int_S |\nabla_y \eta|^2 dy \ge \lambda_1 \int_S \eta^2 dy,$$

and consequently

$$\int_S \big(|\nabla_y \eta|^2 - \lambda_0 |\eta|^2\big)\, dy \ge (\lambda_1 - \lambda_0) \int_S \eta^2 dy. \tag{19}$$

Now recall the decompositions (12) and (13). Each $\psi \in \mathcal{H}^1_0(\mathbb{R} \times S) \cap \mathrm{L}^2(\mathbb{R} \times S)$ can be written as

$$\psi(x,y) = w(x) u_0(y) + \eta(x,y), \quad w \in \mathcal{H}^1(\mathbb{R}), \quad \eta \in \mathcal{H}^1_0(\mathbb{R} \times S) \cap \mathcal{L}.$$

Denote by $h_\varepsilon(\psi_1, \psi_2)$ the sesquilinear form associated with the quadratic form $h_\varepsilon(\psi)$. Thus, for $\psi \in \mathcal{H}_0^1(\mathbb{R} \times S)$,

$$h_\varepsilon(\psi) = h_\varepsilon(w) + h_\varepsilon(\eta) + 2\, h_\varepsilon(wu_0, \eta).$$

We are going to check that there are $c_0 > 0$ and functions $0 \le q(\varepsilon)$, $0 \le p(\varepsilon)$ and $c(\varepsilon)$ so that $h_\varepsilon(w)$, $h_\varepsilon(\eta)$ and $h_\varepsilon(wu_0, \eta)$ satisfy the following conditions:

$$h_\varepsilon(w) \ge c(\varepsilon) \|w\|^2_{\mathrm{L}^2(\mathbb{R})}, \quad \forall w \in \mathcal{H}^1(\mathbb{R}), \quad c(\varepsilon) \ge c_0 > 0; \tag{20}$$

$$h_\varepsilon(\eta) \ge p(\varepsilon) \|\eta\|^2_{\mathrm{L}^2(\mathbb{R}\times S)}, \quad \forall \eta \in \mathcal{H}_0^1(\mathbb{R} \times S) \cap \mathcal{L}^\perp; \tag{21}$$

$$|h_\varepsilon(wu_0, \eta)|^2 \le q(\varepsilon)^2\, h_\varepsilon(w)\, h_\varepsilon(\eta), \quad \forall \psi \in \mathcal{H}_0^1(\mathbb{R} \times S); \tag{22}$$

and with

$$p(\varepsilon) \to \infty,\ c(\varepsilon) = O(p(\varepsilon)),\ q(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{23}$$

Thus, Theorem 2 in [dOV1] (which is in fact just an abstract reformulation of Proposition 3.1 in [FS2]) guarantees that, for $\varepsilon > 0$ small enough,

$$\left\| H_\varepsilon^{-1} - (T_\varepsilon^c)^{-1} \oplus 0 \right\| \le p(\varepsilon)^{-1} + D_1\, q(\varepsilon)\, c(\varepsilon)^{-1}, \tag{24}$$

for some number $D_1 > 0$.

As previously observed (see inequality (14)), there exists $d > 0$ so that

$$h_\varepsilon(wu_0) \ge d \int_{\mathbb{R}} |w|^2 dx, \quad \forall w \in \mathcal{H}^1(\mathbb{R}),$$

for all $\varepsilon > 0$ small enough. Thus, by defining $c(\varepsilon) := d$ it follows that condition (20) is satisfied.

Observe that

$$h_\varepsilon(\eta) \ge \frac{1}{\varepsilon^2} \int_{\mathbb{R}\times S} \left( |\nabla_y \eta|^2 - \lambda_0 |\eta|^2 \right) dxdy.$$

Thus, by inequality (19),

$$h_\varepsilon(\eta) \ge \frac{\lambda_1 - \lambda_0}{\varepsilon^2} \int_{\mathbb{R}\times S} |\eta|^2 dxdy. \tag{25}$$

We take $p(\varepsilon) := (\lambda_1 - \lambda_0)/\varepsilon^2$ and then the condition (21) is satisfied.

Now we need to analyze $h_\varepsilon(wu_0, \eta)$. Some calculations (see Appendix B) show that there exists $D_2 > 0$ so that

$$|h_\varepsilon(wu_0, \eta)| \le D_2\, \varepsilon\, h_\varepsilon(wu_0)^{1/2} h_\varepsilon(\eta)^{1/2}. \tag{26}$$

By taking $q(\varepsilon) := D_2\, \varepsilon$, it is found that conditions (22) and (23) are satisfied.

Now it is immediate to conclude (24), i.e., there exists $D > 0$ so that

$$\|H_\varepsilon^{-1} - (T_\varepsilon^c)^{-1} \oplus 0\| \le D\, \varepsilon,$$

for $\varepsilon > 0$ small enough. This completes the proof of Theorem 3.

# A Restriction to a subspace

In this appendix we restrict the quadratic form $h_\varepsilon(\psi)$ to the subspace $\mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}$.

First note that

$$\int_{\mathbb{R}\times S}|w|^2\left(\frac{|\nabla_y u_0|^2}{\varepsilon^2}-\lambda_0\frac{|u_0|^2}{\varepsilon^2}\right)dxdy=0,$$

because $u_0$ is an eigenfunction of the Dirichlet Laplacian in $\mathcal{H}_0^1(S)$. Using that $\int_S|u_0|^2dy=1$, it is follows that

$$\int_{\mathbb{R}\times S}\left(-\frac{k^2(x)}{4}+c\right)|wu_0|^2dxdy=\int_{\mathbb{R}}\left(-\frac{k^2(x)}{4}+c\right)|w|^2dx.$$

Now we are going to study the first integral that appears in the definition of $h_\varepsilon(\psi)$ restrict to the subspace $\mathcal{H}_0^1(\mathbb{R}\times S)\cap\mathcal{L}$. Some calculations results in

$$\begin{aligned}
&\int_{\mathbb{R}\times S}\left(\frac{\partial w}{\partial x}u_0+wu_0\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)+w\left(\nabla_y u_0\cdot Ry\right)(\tau+\alpha')(x)\right.\\
+&\left.\quad wu_0\beta_\varepsilon^{1/2}\left(\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\right)(\tau+\alpha')(x)\right)^2dxdy\\
=&\int_{\mathbb{R}\times S}\left(\frac{\partial w}{\partial x}\right)^2u_0^2dxdy+\int_{\mathbb{R}\times S}w^2u_0^2\beta_\varepsilon\left(\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right)^2dxdy\\
+&\int_{\mathbb{R}\times S}w^2\left(\nabla_y u_0\cdot Ry\right)^2(\tau+\alpha')^2(x)dxdy\\
+&\int_{\mathbb{R}\times S}w^2u_0^2\beta_\varepsilon\left(\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\right)^2(\tau+\alpha')^2(x)dxdy\\
+&\,2\int_{\mathbb{R}\times S}\frac{\partial w}{\partial x}wu_0^2\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)dxdy\\
+&\,2\int_{\mathbb{R}\times S}\frac{\partial w}{\partial x}wu_0\left(\nabla_y u_0\cdot Ry\right)(\tau+\alpha')(x)dxdy\\
+&\,2\int_{\mathbb{R}\times S}\frac{\partial w}{\partial x}wu_0^2\beta_\varepsilon^{1/2}\left(\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\right)(\tau+\alpha')(x)dxdy\\
+&\,2\int_{\mathbb{R}\times S}w^2\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)u_0\left(\nabla_y u_0\cdot Ry\right)(\tau+\alpha')(x)dxdy\\
+&\,2\int_{\mathbb{R}\times S}w^2u_0^2\beta_\varepsilon\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\left(\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\right)(\tau+\alpha')(x)dxdy\\
+&\,2\int_{\mathbb{R}\times S}w^2\beta_\varepsilon^{1/2}u_0\left(\nabla_y u_0\cdot Ry\right)\left(\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\right)(\tau+\alpha')^2(x)dxdy
\end{aligned}$$

We perform an integration by parts in the integrals that contain the derivative of $w$ with respect to variable $x$. For example,

$$2\int_{\mathbb{R}\times S}\frac{\partial w}{\partial x}wu_0^2\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)dxdy=-\frac{1}{2}\int_{\mathbb{R}\times S}\frac{\partial(w^2)}{\partial x}u_0^2\left(1/\beta_\varepsilon\right)\frac{\partial\beta_\varepsilon}{\partial x}dxdy$$

$$= \frac{1}{2}\int_{\mathbb{R}\times S} w^2 u_0^2 \left( \frac{\partial(1/\beta_\varepsilon)}{\partial x}\frac{\partial \beta_\varepsilon}{\partial x} + (1/\beta_\varepsilon)\frac{\partial^2 \beta_\varepsilon}{\partial x^2} \right) dxdy$$

Similarly,

$$2\int_{\times S} \frac{\partial w}{\partial x} w u_0^2 \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x)\, dxdy$$

$$= -\int_{\mathbb{R}\times S} w^2 u_0^2 \frac{\partial}{\partial x}\left[ \beta_\varepsilon^{1/2} \nabla_y \left( 1/\beta_\varepsilon^{1/2} \cdot Ry \right) (\tau + \alpha')(x) \right] dxdy.$$

Note that the sixth integral is null because after an integration by parts one gets

$$\int_S u_0 \left( \nabla_y u_0 \cdot Ry \right) dy = 0.$$

Thus,

$$b_\varepsilon(wu_0) = \int_{\mathbb{R}} \left[ \left| \frac{\partial w}{\partial x} \right|^2 + \left( C(S)(\tau + \alpha')^2(x) - \frac{k^2(x)}{4} + g_\varepsilon(x) + c \right) |w|^2 \right] dx,$$

where

$$\begin{aligned} g_\varepsilon(x) \;=\; & \int_S u_0^2\, \beta_\varepsilon \left( \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) \right)^2 dy \\ + & \int_S u_0^2\, \beta_\varepsilon \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right)^2 (\tau + \alpha')^2(x)\, dy \\ + & \frac{1}{2}\int_S u_0^2 \left( \frac{\partial(1/\beta_\varepsilon)}{\partial x}\frac{\partial \beta_\varepsilon}{\partial x} + (1/\beta_\varepsilon)\frac{\partial^2 \beta_\varepsilon}{\partial x^2} \right) dy \\ - & \int_S u_0^2 \frac{\partial}{\partial x}\left[ \beta_\varepsilon^{1/2} \nabla_y \left( 1/\beta_\varepsilon^{1/2} \cdot Ry \right) (\tau + \alpha')(x) \right] dy \\ + & 2\int_S \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) u_0 \left( \nabla_y u_0 \cdot Ry \right) (\tau + \alpha')(x)\, dy \\ + & 2\int_S u_0^2\, \beta_\varepsilon \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \cdot Ry \right) \right) (\tau + \alpha')(x)\, dy \\ + & 2\int_S \beta_\varepsilon^{1/2} u_0 \left( \nabla_y u_0 \cdot Ry \right) \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')^2(x)\, dy. \end{aligned}$$

Recalling the definition of $\beta_\varepsilon(x, y)$, taking into account that $k, \tau + \alpha' \in \mathrm{L}^\infty(\mathbb{R})$ and the conditions in (11), we can see that $g_\varepsilon(x)$ converges to zero uniformly as $\varepsilon \to 0$.

# B Proof of inequality (26)

After some calculations one has

$$b_\varepsilon(wu_0, \eta) = \int_{\mathbb{R}\times S} dxdy \Big[ \frac{\partial w}{\partial x} u_0 + w\, u_0\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) + w \left( \nabla_y u_0 \cdot Ry \right) (\tau + \alpha')(x)$$

$$+ w\, u_0\, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \Big]$$

$$\times \Big[ \frac{\partial \eta}{\partial x} + \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) + (\nabla_y \eta \cdot Ry)\, (\tau + \alpha')(x)$$

$$+ \eta\, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \Big]$$

$$+ \int_{\mathbb{R}\times S} \frac{w^2}{\varepsilon^2} \left( \nabla_y u_0 \nabla_y \eta - \lambda_0 u_0 \eta \right) \, dxdy + \int_{\mathbb{R}\times S} \left( -\frac{k^2(x)}{4} + c \right) (w u_0 \eta)\, dxdy.$$

The two last integrals are null due to conditions (17) and (18). Thus

$$b_\varepsilon(wu_0, \eta) = \sum_i^4 b_\varepsilon^i(wu_0, \eta),$$

where

$$\begin{aligned}
b_\varepsilon^1(wu_0, \eta) &= \int_{\mathbb{R}\times S} \frac{\partial \eta}{\partial x} \left[ \frac{\partial w}{\partial x} u_0 + w\, u_0\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) + w\, (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x) \right. \\
&+ \left. w u_0 \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \right] dxdy, \\
b_\varepsilon^2(wu_0, \eta) &= \int_{\mathbb{R}\times S} \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) \left[ \frac{\partial w}{\partial x} u_0 + w u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) \right. \\
&+ w\, (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x) \\
&+ \left. w\, u_0\, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \right] dxdy, \\
b_\varepsilon^3(wu_0, \eta) &= \int_{\mathbb{R}\times S} (\nabla_y \eta \cdot Ry)\, (\tau + \alpha')(x) \left[ \frac{\partial w}{\partial x} u_0 + w u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) \right. \\
&+ w\, (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x) \\
&+ \left. w\, u_0\, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \right] dxdy, \\
b_\varepsilon^4(wu_0, \eta) &= \int_{\mathbb{R}\times S} \eta\, \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \left[ \frac{\partial w}{\partial x} u_0 \right. \\
&+ w\, u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left( 1/\beta_\varepsilon^{1/2} \right) + w\, (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x) \\
&+ \left. w\, u_0 \beta_\varepsilon^{1/2} \left( \nabla_y \left( 1/\beta_\varepsilon^{1/2} \right) \cdot Ry \right) (\tau + \alpha')(x) \right] dxdy.
\end{aligned}$$

Note that

$$b_\varepsilon^1(wu_0, \eta) = \int_{\mathbb{R}\times S} \frac{\partial \eta}{\partial x}\, w\, (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x)\, dxdy$$

$$= \; -\int_{\mathbb{R}\times S} \eta \left[ \frac{\partial w}{\partial x} (\nabla_y u_0 \cdot Ry)\, (\tau + \alpha')(x) + w\, (\nabla_y u_0 \cdot Ry) \frac{\partial}{\partial x} (\tau + \alpha')(x) \right] dxdy.$$

By conditions (11), there exist numbers $E_1, E_2 > 0$ so that

$$\begin{aligned} |b_\varepsilon^1(wu_0,\eta)| &\leq E_1 \left( \int_{\mathbb{R}\times S} \eta^2 \, dxdy \right)^{1/2} \left( \int_{\mathbb{R}} \left( \frac{\partial w}{\partial x} \right)^2 dx \right)^{1/2} \\ &+ E_2 \left( \int_{\mathbb{R}\times S} \eta^2 \, dxdy \right)^{1/2} \left( \int_{\mathbb{R}} w^2 dx \right)^{1/2}. \end{aligned}$$

Since

$$b_\varepsilon(wu_0) \geq \int_{\mathbb{R}} |w'|^2 dx,$$

and due to (14), there exists $E_3 > 0$ so that

$$|b_\varepsilon^1(wu_0,\eta)| \leq E_3 \left( \int_{\mathbb{R}\times S} \eta^2 dxdy \right)^{1/2} (b_\varepsilon(wu_0))^{1/2},$$

for $\varepsilon > 0$ small enough.

This same procedure can be applied to $b_\varepsilon^3(wu_0,\eta)$ and so there exists $E_4 > 0$ so that

$$|b_\varepsilon^3(wu_0,\eta)| \leq E_4 \left( \int_{\mathbb{R}\times S} \eta^2 dxdy \right)^{1/2} (b_\varepsilon(wu_0))^{1/2},$$

for $\varepsilon > 0$ small enough.

Now we are going to estimate $b_\varepsilon^2(wu_0,\eta)$ and $b_\varepsilon^4(wu_0,\eta)$. Observe that

$$|b_\varepsilon^2(wu_0,\eta)| \leq E_5 \left( \int_{\mathbb{R}\times S} \eta^2 \, dxdy \right)^{1/2} b_\varepsilon(wu_0)^{1/2},$$

and

$$|b_\varepsilon^4(wu_0,\eta)| \leq E_5 \left( \int_{\mathbb{R}\times S} \eta^2 \, dxdy \right)^{1/2} b_\varepsilon(wu_0)^{1/2},$$

for some number $E_5 > 0$. By virtue of (14), we conclude that there exists $D_2 > 0$ so that

$$|b_\varepsilon(wu_0,\eta)| \leq D_2 \, \varepsilon \, (b_\varepsilon(\eta))^{1/2} \, (b_\varepsilon(wu_0))^{1/2}.$$

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