

Interfaces in some elliptic nonlinear boundary value problems on Riemannian manifolds: necessary condition and location.

Arnaldo S. do Nascimento
Universidade Federal de São Carlos, D.M.
13565-905 S. Carlos, S.P., Brasil
arnaldon@dm.ufscar.br
Alexandre C. Gonçalves¹
Universidade de São Paulo, DCM-FFCLRP
14040-901, Ribeirão Preto, S.P., Brasil
acasa@ffclrp.usp.br

Corresponding author phone#:55-16-3602-3781 Fax# 55-16-3602-0447

¹corresponding author

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Universidade Federal de São Carlos, D.M.
13565-905 S. Carlos, S.P., Brasil
arnaldon@dm.ufscar.br
Alexandre C. Gonçalves¹
Universidade de São Paulo, DCM-FFCLRP
14040-901, Ribeirão Preto, S.P., Brasil
acasa@ffclrp.usp.br

Abstract

We show that the equal-area condition is a necessary hypothesis for the existence of layered solutions to a nonlinear boundary value problem depending on a positive real parameter on a Riemannian manifold with or without boundary. As an application we give examples where the results can be used in order to determine in advance the location of possible limiting interface as the parameter goes to zero.

Keywords: elliptic equation; diffusivity; equal-area condition; internal layer.

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1. Introduction

In all the works where the prime concern is to show existence of a family of solutions to a scalar or system of elliptic equations with no-flux boundary condition which develops internal transition layer, as a certain parameter varies, one most certainly will find in the set of hypotheses the so-called equal-area condition also sometimes labeled bulk force balance law. When the problem is variational it may appear in a disguised form by requiring that the potential in the energy functional is of type double-well with equal depth.

In order to put our work into perspective we rather set out by letting \mathcal{M} be a smooth compact n -dimensional Riemannian manifold with smooth boundary $\partial\mathcal{M}$ (one possibly allows $\partial\mathcal{M} = \emptyset$). Then consider the following singularly perturbed elliptic boundary value problem

$$\begin{cases} \varepsilon \operatorname{div}(a\nabla v) + f(x, v) = 0 & \text{on } \mathcal{M} \\ \varepsilon a \frac{\partial v}{\partial \hat{\eta}} = g(x, v) & \text{on } \partial\mathcal{M} \end{cases} \quad (1.1)$$

¹corresponding author

where div and ∇ stand for the divergent and gradient (or Levi-Civita covariant derivative) on \mathcal{M} , respectively. Recall that those operators depend on the Riemannian metric on \mathcal{M} and have expressions in local coordinates similar, but not identical, to the standard divergent and gradient operators on flat Euclidean space. Still, $\varepsilon > 0$ is a small parameter, $\hat{\eta}$ the exterior unit vector co-normal to $\partial\mathcal{M}$, $f \in C^1(\mathcal{M} \times \mathbb{R})$, $g \in C^1(\partial\mathcal{M} \times \mathbb{R})$ and $a \in C^1(\overline{\mathcal{M}})$ is a positive function.

Roughly speaking we will say that a family $\{v_\varepsilon\}$ of solutions to (1.1) develops internal transition layer as $\varepsilon \rightarrow 0$ with limiting interface \mathcal{S} if \mathcal{S} induces a partition of \mathcal{M} in two open sets, \mathcal{M}_α and \mathcal{M}_β say, and this family of solutions exhibits a sharp spatial transition between two different states, say, α and β , across \mathcal{S} for small values of ε . Therefore except for a thin set, the so called transition layer region, the solutions are approximately constant assuming values close to α (to β) on compact sets of \mathcal{M}_α (respectively \mathcal{M}_β) for ε small. A similar situation can occur on $\partial\mathcal{M}$ with values $\bar{\alpha}$ and $\bar{\beta}$, say. Occasionally these solutions will be referred to as layered solutions.

Typically it is found in the study of stationary solutions of some non-linear time evolution diffusion processes in a heterogeneous medium whose diffusivity is given by $a(\cdot)$ with a sink or source and a nonlinear relation between the flux through the boundary and the concentration.

When $\mathcal{M} = \Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $a \equiv 1$ and g does not depend on the spacial variable x , a functional whose critical points are solutions to (1.1) (up to a scaling in the parameter ε) was considered in [1] and its Γ -limit calculated. Still under these hypotheses and based on [1] the authors in [8] found families of solutions to (1.1) (actually local minimizers of this functional) which develops internal transition layer or boundary layer, as $\varepsilon \rightarrow 0$, with $\bar{\alpha} \leq \alpha < \beta \leq \bar{\beta}$. More recent examples of nonlinear Neumann boundary condition where the parabolic counterpart of (1.1) is considered can be found in [2], [11], [3], [4], [5] and [22], for instance; it may appear, for instance, in combustion problems when reaction takes place only on the boundary of a container due to the presence of a solid catalyzer.

When $\mathcal{M} = \Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $g \equiv 0$, i.e., zero Neumann boundary condition, there is a vast literature devoted to the study of (1.1) in the context of layered solutions as it appears in many different types of applications: selection-migration model in population genetics, nonlinear heat diffusion, etc.. For $N = 2$, $a \equiv 1$, $f(x, v) = v - v^3$ and $g \equiv 0$, i.e., the Allen-Cahn equation, the authors in [6] and [17], for instance, show existence of a family of solutions exhibiting transition layers. Roughly under the same hypotheses but with a variable diffusivity function a a family of stable layered stationary solutions to the parabolic counterpart of (1.1) was found in [12] using Γ -convergence techniques; as opposed to [6] the limiting-interface depends on a but not on the geometry of $\partial\Omega$. The same kind of problem for the Allen-Cahn equation on compact manifolds have been considered in [21] and [7], for instance.

For the Cahn-Hilliard problem which is the functional, under a integral constraint, whose critical points are the solutions to the Alenn-Cahn equation, a complete picture regarding the location of the limiting interface and the exact number of local and global minimizers over rectangles, parallelepiped and circular cylinders has been given in [13].

The point here is to note that in all of these cases whenever showing existence of a family of solutions to (1.1) which develops internal transition layer, as $\epsilon \rightarrow 0$, it is tacitly required, based on physical or geometric ground, that

$$\int_{\alpha}^{\beta} f(x, \xi) d\xi = 0, \forall x \in \mathcal{S} \quad (1.2)$$

with α and β as above and $\mathcal{S} \subset \mathcal{M}$ is the sub-manifold candidate to the limiting interface.

Regarding (1.1) our main result assures that not only (1.2) is actually a necessary hypothesis for existence of a family of such solutions but also a similar condition must hold on the boundary:

$$\int_{\bar{\alpha}}^{\bar{\beta}} g(x, \xi) d\xi = 0, \forall x \in \partial\mathcal{M} \cap \partial\mathcal{S}, \quad (1.3)$$

where $\bar{\alpha}$ and $\bar{\beta}$ play the same role for g on $\partial\mathcal{M}$ as α and β do for f on \mathcal{M} .

In general knowing a priori the location of the limiting interface (as $\epsilon \rightarrow 0$) of a family of solutions to (1.1) is not an easy task. For the Allen-Cahn equation in flat euclidian domains, i.e., $a \equiv 1$, $f(x, v) = v - v^3$ and $g \equiv 0$, it is known (under suitable hypotheses) that interfaces evolve, as $\epsilon \rightarrow 0$, by mean curvature motion and thus a candidate, γ say, for limiting interface must have zero curvature and intersect the boundary of the domain orthogonally (the reader is referred to [17], for instance, for a more detailed account and references on this matter). These information have been used in order to construct a family of layered solutions whose limiting interface is γ (see [6], [16], for instance).

As an application of our main result, in the last section we give some specific examples for which (1.2) and/or (1.3) can be used to determine in advance the location of any limiting interface of (1.1), as $\epsilon \rightarrow 0$. All examples are spatially inhomogeneous and our criterion, namely (1.2), fails for the Alenn-Cahn equation for in this case the meaningful parameter, namely the curvature, is hidden.

For instance, we conclude that any family $\{v_{\epsilon}\}$ of layered solutions to

$$\begin{cases} \Delta v_{\epsilon} = 0 & \text{on } \mathcal{M} \\ \epsilon \frac{\partial v_{\epsilon}}{\partial \hat{\eta}} = v_{\epsilon}(1 - v_{\epsilon})(a(x) - v_{\epsilon}) & \text{on } \partial\mathcal{M} \end{cases} \quad (1.4)$$

where $0 \leq v_{\epsilon} \leq 1$, must have as limiting interface, as $\epsilon \rightarrow 0$, the level set $\{x \in \partial\mathcal{M} : a(x) = 1/2\}$.

Also any family $\{v_{\epsilon}\}$ of layered solutions to

$$\begin{cases} \epsilon \Delta v_{\epsilon} + b(x)f(v_{\epsilon}) = 0 & \text{on } \mathcal{M} \\ \frac{\partial v_{\epsilon}}{\partial \hat{\eta}} = 0 & \text{on } \partial\mathcal{M} \end{cases} \quad (1.5)$$

where $0 \leq v_{\epsilon} \leq 1$ and f is a smooth positive function on $[0, 1]$ satisfying $f(0) = f(1) = 0$ must have a limiting interface, as $\epsilon \rightarrow 0$, the nodal set $\{x \in \mathcal{M} : b(x) = 0\}$.

2. Notation, hypotheses and the meaning of internal layer

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a smooth (meaning at least C^2) compact connected Riemannian manifold of dimension n whose volume element will be denoted by $dV^{\mathcal{M}}$ and \mathcal{S} be a closed $(n-1)$ -dimensional sub-manifold of \mathcal{M} , not necessarily connected, whose volume element will be denoted by $dV^{\mathcal{S}}$. The tangent bundle of \mathcal{S} is a sub-bundle $T\mathcal{S} \subset T\mathcal{M}|_{\mathcal{S}}$ of the pullback of the tangent bundle of \mathcal{M} . It defines a complement $N\mathcal{S}$, the normal bundle of \mathcal{S} , such that $T\mathcal{M}|_{\mathcal{S}} = T\mathcal{S} \oplus N\mathcal{S}$.

Throughout this work we will assume the following topological hypotheses:

- (H1) The border of \mathcal{S} is a $(n-2)$ -dimensional differential submanifold of $\partial\mathcal{M}$, if not empty. Further, $\partial\mathcal{S} = \mathcal{S} \cap \partial\mathcal{M}$.
- (H2) The normal bundle of \mathcal{S} can be oriented in such a way that if U is a connected component of $\mathcal{M} \setminus \mathcal{S}$ then U is either “inside” \mathcal{S} or “outside” \mathcal{S} .

Remark 2.1. Hypothesis (H1) impose regularity on the layers and the obvious necessity that the border of \mathcal{S} cannot happen in the interior of \mathcal{M} . A stronger hypothesis would be requiring \mathcal{S} to be *transversal* to $\partial\mathcal{M}$ (see [14]). For the special case of Allen-Cahn equation in a flat euclidian domain Ω the limiting interface, in view of the zero Neumann boundary condition, intersects $\partial\Omega$ orthogonally. In the general case where the normal derivative along the boundary is not spatially homogeneous it is possible that the contact angle be different from $\pi/2$. Nevertheless transversality is not needed for our proofs, but only the usual regularity on the layers so that classical integration can be performed.

Hypothesis (H2) says that each component of \mathcal{S} does actually separate two distinct phases on \mathcal{M} . The advantage of this set up is that it discards the need for orientability hypotheses on the manifolds \mathcal{M}, \mathcal{S} .

Indeed $\mathcal{S} \setminus \partial\mathcal{S}$ induces a non-trivial partition in $\widehat{\mathcal{M}} \stackrel{def}{=} \mathcal{M} \setminus \partial\mathcal{M}$, i.e., $\widehat{\mathcal{M}} = \mathcal{M}_\alpha \cup (\mathcal{S} \setminus \partial\mathcal{S}) \cup \mathcal{M}_\beta$ where \mathcal{M}_α and \mathcal{M}_β are both open disjoint subsets with positive measure $dV^{\mathcal{M}}$. Notice that each component of \mathcal{M}_α borders only components of \mathcal{M}_β , and the other way around, thanks to (H2). Similarly $\partial\mathcal{M} = \mathcal{N}_\alpha \cup \partial\mathcal{S} \cup \mathcal{N}_\beta$, where \mathcal{N}_α and \mathcal{N}_β are both open disjoint subsets with positive induced measure $dV^{\mathcal{S}}$, respecting the orientation of the normal bundle of $\partial\mathcal{S}$ within $T\partial\mathcal{M}$.

Note that if $n = 2$ then $\partial\mathcal{S}$ is constituted by an even number of points of $\partial\mathcal{M}$.

Let $\alpha, \beta, \bar{\alpha}$ and $\bar{\beta}$ be real numbers satisfying $\alpha < \beta$ and $\bar{\alpha} < \bar{\beta}$. Define a.e. functions $v_0 : \widehat{\mathcal{M}} \mapsto \mathbb{R}$ and $\bar{v}_0 : \partial\mathcal{M} \mapsto \mathbb{R}$ by

$$v_0 \stackrel{def}{=} \alpha \chi_{\mathcal{M}_\alpha} + \beta \chi_{\mathcal{M}_\beta} \quad (2.1)$$

and

$$\bar{v}_0 \stackrel{def}{=} \bar{\alpha} \chi_{\mathcal{N}_\alpha} + \bar{\beta} \chi_{\mathcal{N}_\beta}, \quad (2.2)$$

where χ_A stands for the characteristic function of a set A .

Definition 2.2. We will say that a family $\{v_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ of solutions to (1.1) in $C^1(\mathcal{M}) \cap C^2(\mathcal{M} \setminus \partial\mathcal{M})$ develops internal transition layers, as $\varepsilon \rightarrow 0$, with interface \mathcal{S} and $\partial\mathcal{S}$ if it holds that

$$v_{\varepsilon|\widehat{\mathcal{M}}} \xrightarrow{\varepsilon \rightarrow 0} v_{0|\widehat{\mathcal{M}}} \text{ in } L^1(\widehat{\mathcal{M}}) \quad (2.3)$$

$$v_{\varepsilon|\partial\mathcal{M}} \xrightarrow{\varepsilon \rightarrow 0} \bar{v}_{0|\partial\mathcal{M}} \text{ in } L^1(\partial\mathcal{M}) \quad (2.4)$$

where v_0 and \bar{v}_0 are given by (2.1) and (2.2), respectively.

If \mathcal{M} has no boundary then $\partial\mathcal{S} = \emptyset$ and (2.4) is void.

Remark 2.3. Instead of working in the topology of $L^1(\mathcal{M})$ we could have required the convergence in (2.3) and (2.4) to take place on compact sets of $\mathcal{M} \setminus \mathcal{S}$.

Remark 2.4. Many interesting cases of \mathcal{M} may require a multiple component \mathcal{S} , a simple example being the 2-dimensional torus with an interface containing a non-simply-connected closed curve.

3. The equal-area as a necessary condition

Theorem 3.1. Let $\{v_\varepsilon\}_{0 < \varepsilon < \varepsilon_0}$ be a family of solutions to (1.1) which develops a internal transition layer with interface \mathcal{S} and $\partial\mathcal{S}$ in the sense of Definition 2.2. Then it holds

$$\int_{\alpha}^{\beta} f(x, \xi) d\xi = 0, \forall x \in \mathcal{S} \quad (3.1)$$

and

$$\int_{\bar{\alpha}}^{\bar{\beta}} g(x, \xi) d\xi = 0, \forall x \in \partial\mathcal{S}. \quad (3.2)$$

Moreover

$$f(x, v_0) = 0, \forall x \in \widehat{\mathcal{M}} \setminus \mathcal{S} \text{ and } g(x, \bar{v}_0) = 0, \forall x \in \partial\mathcal{M} \setminus \partial\mathcal{S}. \quad (3.3)$$

If \mathcal{M} has no boundary then (3.2) is void.

Proof Let W be any C^1 vector field on $\overline{\mathcal{M}}$ and denote by Wv the directional derivative of a function $v : \mathcal{M} \rightarrow \mathbb{R}$ with respect to W , i.e.,

$$Wv = dv(W) = \langle W, \nabla v \rangle . \quad (3.4)$$

For simplicity of notation we delete the index ε of v_ε in the next computations. Multiplying the first equation of (1.1) by (Wv) we get

$$\varepsilon(Wv) \operatorname{div}(a\nabla v) + (Wv)f(x, v) = 0 . \quad (3.5)$$

Integrating on \mathcal{M} and using the boundary condition yield

$$\begin{aligned}
\int_{\mathcal{M}} (Wv)f(x,v)dV^{\mathcal{M}} &= -\varepsilon \int_{\mathcal{M}} (Wv) \operatorname{div}(a\nabla v) dV^{\mathcal{M}} \\
&= -\varepsilon \int_{\mathcal{M}} \{ \operatorname{div}[(Wv)a\nabla v] - \langle \nabla(Wv), a\nabla v \rangle \} dV^{\mathcal{M}} \\
&= -\varepsilon \int_{\partial\mathcal{M}} (Wv)a \langle \nabla v, \hat{\eta} \rangle dV^{\partial\mathcal{M}} + \varepsilon \int_{\mathcal{M}} \langle \nabla(Wv), a\nabla v \rangle dV^{\mathcal{M}} \\
&= -\int_{\partial\mathcal{M}} (Wv)g(x,v) dV^{\partial\mathcal{M}} + \varepsilon \int_{\mathcal{M}} \langle \nabla(Wv), a\nabla v \rangle dV^{\mathcal{M}} \tag{3.6}
\end{aligned}$$

where $dV^{\partial\mathcal{M}}$ denotes the induced metric in $\partial\mathcal{M}$ and $\hat{\eta}$ is the unitary outward vector field on $\partial\mathcal{M}$. Note that

$$\begin{aligned}
\nabla(Wv) &= \nabla \langle W, \nabla v \rangle \\
&= [\langle \nabla W, \nabla v \rangle + \langle W, H_v \rangle]^* = (\nabla W)^* \nabla v + H_v(W), \tag{3.7}
\end{aligned}$$

where H_v is the hessian of v and $*$ is the metric dual of a tensor.

Therefore the second term in (3.6) can be written as

$$\begin{aligned}
\varepsilon \int_{\mathcal{M}} \langle \nabla(Wv), a\nabla v \rangle dV^{\mathcal{M}} &= \varepsilon \int_{\mathcal{M}} \langle (\nabla W)^* \nabla v + H_v(W), a\nabla v \rangle dV^{\mathcal{M}} \\
&= \varepsilon \int_{\mathcal{M}} a \langle (\nabla W) \nabla v, \nabla v \rangle dV^{\mathcal{M}} + \varepsilon \int_{\mathcal{M}} a \langle H_v(\nabla v), W \rangle dV^{\mathcal{M}} \tag{3.8}
\end{aligned}$$

using standard properties of tensor algebra. Substituting (3.8) in (3.6) yields the equality

$$\begin{aligned}
&\int_{\mathcal{M}} (Wv)f(x,v) dV^{\mathcal{M}} + \int_{\partial\mathcal{M}} (Wv)g(x,v) dV^{\partial\mathcal{M}} \\
&= \varepsilon \int_{\mathcal{M}} a \langle (\nabla W) \nabla v, \nabla v \rangle dV^{\mathcal{M}} + \varepsilon \int_{\mathcal{M}} a \langle H_v(\nabla v), W \rangle dV^{\mathcal{M}} \tag{3.9}
\end{aligned}$$

Noting that

$$\nabla(|\nabla v|^2) = (2 \langle H_v, \nabla v \rangle)^* = 2H_v(\nabla v)$$

the last integral in the second term in (3.9) by its turn can be written as

$$\begin{aligned}
\varepsilon \int_{\mathcal{M}} a \langle H_v(\nabla v), W \rangle dV^{\mathcal{M}} &= \varepsilon \int_{\mathcal{M}} \frac{1}{2} \langle \nabla(|\nabla v|^2), aW \rangle dV^{\mathcal{M}} \\
&= \frac{\varepsilon}{2} \int_{\mathcal{M}} \{ \operatorname{div}(|\nabla v|^2 aW) - |\nabla v|^2 \operatorname{div}(aW) \} dV^{\mathcal{M}} \\
&= \frac{\varepsilon}{2} \int_{\partial\mathcal{M}} |\nabla v|^2 a \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} - \frac{\varepsilon}{2} \int_{\mathcal{M}} |\nabla v|^2 \operatorname{div}(aW) dV^{\mathcal{M}}. \tag{3.10}
\end{aligned}$$

thus yielding

$$\begin{aligned} & \int_{\mathcal{M}} (Wv)f(x, v) dV^{\mathcal{M}} + \int_{\partial\mathcal{M}} (Wv)g(x, v) dV^{\partial\mathcal{M}} \\ &= \varepsilon \int_{\mathcal{M}} a \langle (\nabla W)\nabla v, \nabla v \rangle dV^{\mathcal{M}} \\ &+ \frac{\varepsilon}{2} \int_{\partial\mathcal{M}} |\nabla v|^2 a \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} - \frac{\varepsilon}{2} \int_{\mathcal{M}} |\nabla v|^2 \operatorname{div}(aW) dV^{\mathcal{M}}. \end{aligned} \quad (3.11)$$

We now focus our attention on the first (second) integral of the first term of (3.11). Let $F(x, \xi)$ ($G(x, \xi)$) be any regular function with $\frac{\partial F}{\partial \xi}(x, \xi) = f(x, \xi)$ ($\frac{\partial G}{\partial \xi}(x, \xi) = g(x, \xi)$), for all $(x, \xi) \in \mathcal{M} \times \mathbb{R}$ ($(x, \xi) \in \partial\mathcal{M} \times \mathbb{R}$), and set for simplicity on notation

$$\tilde{F}(x) \stackrel{def}{=} F(x, v_\varepsilon(x)) \text{ and } \tilde{G}(x) \stackrel{def}{=} G(x, v_\varepsilon(x)).$$

By denoting the differentials of F respect to the first and second variables as $\partial_1 F$ and $\partial_2 F$ we have $\partial_2 F = f(x, \xi)d\xi$ and $\nabla \tilde{F} = (\partial_1 F)^* + (\partial_2 F \circ dv)^* = (\partial_1 F)^* + f\nabla v$, with a similar expression for \tilde{G} . Hence

$$\begin{aligned} & \int_{\mathcal{M}} (Wv)f(x, v) dV^{\mathcal{M}} = \int_{\mathcal{M}} \left\{ \langle W, \nabla \tilde{F} \rangle - \langle W, (\partial_1 F)^* \rangle \right\} dV^{\mathcal{M}} \\ &= \int_{\mathcal{M}} \left\{ \operatorname{div}(\tilde{F}W) - \tilde{F}\operatorname{div}W - \langle W, (\partial_1 F)^* \rangle \right\} dV^{\mathcal{M}} \\ &= \int_{\partial\mathcal{M}} \tilde{F} \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} - \int_{\mathcal{M}} \left\{ \tilde{F}\operatorname{div}W + \langle W, (\partial_1 F)^* \rangle \right\} dV^{\mathcal{M}} \end{aligned} \quad (3.12)$$

For the second integral of the first term of (3.11) we must pay attention that W may not be tangential to $\partial\mathcal{M}$. There is an orthogonal splitting $W(x) = W^T(x) + W^N(x)$ for all $x \in \partial\mathcal{M}$, with $W^T(x) \in T\partial\mathcal{M}$. The divergence law will work only for the tangential component, thus

$$\begin{aligned} & \int_{\partial\mathcal{M}} (Wv)g(x, v) dV^{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} (W^T v + W^N v)g(x, v) dV^{\partial\mathcal{M}} \\ &= - \int_{\partial\mathcal{M}} \left\{ \tilde{G}\operatorname{div}W + \langle W, (\partial_1 G)^* \rangle \right\} dV^{\partial\mathcal{M}} \\ &\quad + \int_{\partial\mathcal{M}} (W^N v)g(x, v) dV^{\partial\mathcal{M}} \end{aligned} \quad (3.13)$$

where we used the fact that the boundary of $\partial\mathcal{M}$ is empty.

Before we pass (3.11) to the limit as $\varepsilon \rightarrow 0$ we rather first check which terms will vanish. From now on we recover the sub-index ε in the notation wherever it applies. We claim that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\mathcal{M}} a |\nabla v_\varepsilon|^2 dV^{\mathcal{M}} = 0. \quad (3.14)$$

First one realizes that (1.1) implies

$$\begin{aligned} & \int_{\partial\mathcal{M}} v_\varepsilon g(x, v_\varepsilon) dV^{\partial\mathcal{M}} + \int_{\mathcal{M}} v_\varepsilon f(x, v_\varepsilon) dV^{\mathcal{M}} \\ &= \varepsilon \int_{\mathcal{M}} a |\nabla v_\varepsilon|^2 dV^{\mathcal{M}} \end{aligned} \quad (3.15)$$

Supposing for the time being that (3.3) holds and using our hypotheses, we pass (3.15) to the limit as $\varepsilon \rightarrow 0$ to obtain (3.14).

Thus regarding the first integral on the right hand-side of (3.11) we obtain

$$\begin{aligned} & \varepsilon \left| \int_{\mathcal{M}} a \langle (\nabla W) \nabla v_\varepsilon, \nabla v_\varepsilon \rangle dV^{\mathcal{M}} \right| \\ & \leq \varepsilon \int_{\mathcal{M}} a |\nabla W| |\nabla v_\varepsilon|^2 dV^{\mathcal{M}} \leq \|\nabla W\|_\infty \varepsilon \int_{\mathcal{M}} a |\nabla v_\varepsilon|^2 dV^{\mathcal{M}}. \end{aligned} \quad (3.16)$$

Hence it follows from (3.14) and (3.16) that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\mathcal{M}} a \langle (\nabla W) \nabla v_\varepsilon, \nabla v_\varepsilon \rangle dV^{\mathcal{M}} = 0. \quad (3.17)$$

The C^1 regularity of a and W implies

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \left| \int_{\mathcal{M}} |\nabla v_\varepsilon|^2 \operatorname{div}(aW) dV^{\mathcal{M}} \right| \\ & \leq \lim_{\varepsilon \rightarrow 0} \|\operatorname{div}(aW)\|_\infty \frac{\varepsilon}{2} \int_{\mathcal{M}} |\nabla v_\varepsilon|^2 dV^{\mathcal{M}} = 0. \end{aligned} \quad (3.18)$$

Taking into account (3.17) and (3.18) we obtain from (3.11)

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} (W v_\varepsilon) f(x, v_\varepsilon) dV^{\mathcal{M}} + \lim_{\varepsilon \rightarrow 0} \int_{\partial \mathcal{M}} (W v_\varepsilon) g(x, v_\varepsilon) dV^{\partial \mathcal{M}} \\ & = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \int_{\partial \mathcal{M}} |\nabla v_\varepsilon|^2 a \langle W, \hat{\eta} \rangle dV^{\partial \mathcal{M}}. \end{aligned} \quad (3.19)$$

Our next goal is to compute the first term of the left hand-side of (3.19) and this send us back to (3.12). Using our hypotheses and the dominated convergence theorem we compute the limit as $\varepsilon \rightarrow 0$ of the last integral in (3.12) as

$$\begin{aligned} & \int_{\mathcal{M}} \{(\operatorname{div} W) F(x, v_0) + \langle W, (\partial_1 F)^*(x, v_0) \rangle\} dV^{\mathcal{M}} \\ & = \int_{\mathcal{M}_\alpha} \{(\operatorname{div} W) F(x, \alpha) + \langle W, (\partial_1 F)^*(x, \alpha) \rangle\} dV^{\mathcal{M}} \\ & \quad + \int_{\mathcal{M}_\beta} \{(\operatorname{div} W) F(x, \beta) + \langle W, (\partial_1 F)^*(x, \beta) \rangle\} dV^{\mathcal{M}}. \end{aligned} \quad (3.20)$$

Let $\hat{\eta}_\alpha$ ($\hat{\eta}_\beta$) be the unitary outward vector on \mathcal{M}_α (\mathcal{M}_β). Clearly $\hat{\eta}_\alpha(x) = \hat{\eta}(x)$ if $x \in \mathcal{N}_\alpha = \partial \mathcal{M}_\alpha - \mathcal{S}$. Analyzing the integral on the domain \mathcal{M}_α :

$$\begin{aligned} & \int_{\mathcal{M}_\alpha} \{(\operatorname{div} W) F(x, \alpha) + \langle W, (\partial_1 F)^*(x, \alpha) \rangle\} dV^{\mathcal{M}} \\ & = \int_{\mathcal{M}_\alpha} \{\operatorname{div}(F(x, \alpha) W) - \langle (\partial_1 F)^*(x, \alpha), W \rangle + \langle W, (\partial_1 F)^*(x, \alpha) \rangle\} dV^{\mathcal{M}} \\ & = \int_{\mathcal{S}} F(x, \alpha) \langle W, \hat{\eta}_\alpha \rangle dV^{\mathcal{S}} + \int_{\mathcal{N}_\alpha} F(x, \alpha) \langle W, \hat{\eta} \rangle dV^{\partial \mathcal{M}}. \end{aligned} \quad (3.21)$$

In a similar fashion we obtain for the integral on M_β

$$\begin{aligned} & \int_{\mathcal{M}_\beta} \{(\operatorname{div}W)F(x, \beta) + \langle W, (\partial_1 F)^*(x, \beta) \rangle\} dV^{\mathcal{M}} \\ &= \int_{\mathcal{S}} F(x, \beta) \langle W, \hat{\eta}_\beta \rangle dV^{\mathcal{S}} + \int_{\mathcal{N}_\beta} F(x, \beta) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}}. \end{aligned} \quad (3.22)$$

Observing that $\partial M_\alpha \setminus \partial\mathcal{M}$ e $\partial M_\beta \setminus \partial\mathcal{M}$ are topologically identical but have normal opposite orientations, so that $\hat{\eta}_\alpha = -\hat{\eta}_\beta$, we obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} \{F(x, v_\varepsilon) \operatorname{div}W + \langle W, (\partial_1 F)^* \rangle\} dV^{\mathcal{M}} \\ &= \int_{\mathcal{S}} [F(x, \beta) - F(x, \alpha)] \langle W, \hat{\eta}_\beta \rangle dV^{\mathcal{S}} \\ &+ \int_{\mathcal{N}_\alpha} F(x, \alpha) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} + \int_{\mathcal{N}_\beta} F(x, \beta) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}}. \end{aligned} \quad (3.23)$$

Therefore from (3.12)

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} (Wv_\varepsilon) f(x, v_\varepsilon) dV^{\mathcal{M}} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}} F(x, v_\varepsilon) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} - \int_{\mathcal{S}} [F(x, \beta) - F(x, \alpha)] \langle W, \hat{\eta}_\beta \rangle dV^{\mathcal{S}} \\ &- \int_{\mathcal{N}_\alpha} F(x, \alpha) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} - \int_{\mathcal{N}_\beta} F(x, \beta) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} \\ &= - \int_{\mathcal{S}} [F(x, \beta) - F(x, \alpha)] \langle W, \hat{\eta}_\beta \rangle dV^{\mathcal{S}}. \end{aligned} \quad (3.24)$$

Next we turn our attention to the second term of the left hand-side of (3.19) and this send us back to (3.13). A computation similar to the one carried out in (3.23) yields

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}} (Wv_\varepsilon) g(x, v_\varepsilon) dV^{\partial\mathcal{M}} \\ &= - \int_{\partial\mathcal{S}} [G(x, \bar{\beta}) - G(x, \bar{\alpha})] \langle W^T, \hat{\eta}_{\bar{\beta}} \rangle dV^{\partial\mathcal{S}} \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}} (W^N v_\varepsilon) g(x, v_\varepsilon) dV^{\partial\mathcal{M}}, \end{aligned} \quad (3.25)$$

where $\hat{\eta}_{\bar{\beta}}$ is the normal vector to $\partial\mathcal{S}$ into $\partial\mathcal{M}$ going outward of \mathcal{N}_β . Notice that $W^N v_\varepsilon = \langle W, \hat{\eta} \rangle \frac{\partial v_\varepsilon}{\partial \hat{\eta}}$. Taking into account (3.24) and (3.25) we can write (3.19) as

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}} \left(g(x, v_\varepsilon) \frac{\partial v_\varepsilon}{\partial \hat{\eta}} - \frac{\varepsilon}{2} |\nabla v_\varepsilon|^2 a \right) \langle W, \hat{\eta} \rangle dV^{\partial\mathcal{M}} \\ &= \int_{\mathcal{S}} [F(x, \beta) - F(x, \alpha)] \langle W, \hat{\eta}_\beta \rangle dV^{\mathcal{S}} \\ &+ \int_{\partial\mathcal{S}} [G(x, \bar{\beta}) - G(x, \bar{\alpha})] \langle W^T, \hat{\eta}_{\bar{\beta}} \rangle dV^{\partial\mathcal{S}}. \end{aligned} \quad (3.26)$$

Recall that W is an arbitrary C^1 vector field on \mathcal{M} . Let ρ be a smooth positive function on \mathcal{S} and such that $\rho = 0$ in $\partial\mathcal{S}$. We first set

- $W(x) = [F(x, \beta) - F(x, \alpha)]\rho\hat{\eta}_\beta$, $\forall x \in \mathcal{S} \setminus \partial\mathcal{S}$, $W = 0$ on $\partial\mathcal{M}$ and let it be smooth on $\mathcal{M} \setminus \mathcal{S}$.

This choice is possible by virtue of (H2) and turns (3.26) into

$$\int_{\mathcal{S}} [F(x, \beta) - F(x, \alpha)]^2 \rho dV^{\mathcal{S}} = 0 \quad (3.27)$$

which establishes (3.1).

Next we devise another C^1 vector field W with the following properties:

- $W(x) = [G(x, \bar{\beta}) - G(x, \bar{\alpha})]\hat{\eta}_{\bar{\beta}}$ for all $x \in \partial\mathcal{S}$, $W(x) \in T\partial\mathcal{M}$ for all $x \in \partial\mathcal{M}$ and W arbitrary in $\mathcal{M} \setminus \partial\mathcal{M}$.

This choice for W when substituted into (3.26) and using that $F(x, \beta) = F(x, \alpha)$ everywhere in \mathcal{S} yields

$$\int_{\partial\mathcal{S}} [G(x, \bar{\beta}) - G(x, \bar{\alpha})]^2 dV^{\partial\mathcal{S}} = 0, \quad (3.28)$$

which establishes (3.2).

It remains to verify (3.3). To that goal suppose $\exists x_0 \in \widehat{\mathcal{M}} \setminus \mathcal{S}$ such that $f(x_0, \alpha) > 0$, say. By continuity, $\exists \delta > 0$ and a geodesic ball $B_\delta(x_0) \subset \widehat{\mathcal{M}} \setminus \mathcal{S}$ so that $f(x, \alpha) > 0$ for $x \in B_\delta(x_0)$.

Integrating equation (1.1) on this ball yields

$$\varepsilon \int_{\partial B_\delta(x_0)} a \langle \nabla v_\varepsilon, \hat{\eta}_B \rangle dV^{\partial B} = - \int_{B_\delta(x_0)} f(x, v_\varepsilon) dV^{\mathcal{M}},$$

where $\hat{\eta}_B$ stands for the unitary outward normal vector at $\partial B_\delta(x_0)$.

It is known that $\varepsilon |\nabla v_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$ pointwise in \mathcal{M} . The proof, based in a blow-up technique, is standard for Euclidean spaces, but that being of a local nature, holds to our case as well.

Therefore, up to a sub-sequence, we pass to the limit to obtain

$$\int_{B_\delta(x_0)} f(x, \alpha) dV^{\mathcal{M}} = 0,$$

which contradicts our hypothesis that $f(\cdot, \alpha)$ is positive on $B_\delta(x_0)$.

A similar argument can be used to prove that $g(x, \bar{v}_0) = 0$, $\forall x \in \partial\mathcal{M} \setminus \partial\mathcal{S}$. We sketch the proof in this case. Suppose that $\exists x_0 \in \partial\mathcal{M} \setminus \partial\mathcal{S}$ satisfying, for definiteness $g(x_0, \bar{\alpha}) > 0$, say. By continuity of g we can take a geodesic ball $B_\delta(x_0) \subset \mathcal{M} \setminus \mathcal{S}$ so that $g(x, \bar{\alpha}) > 0 \forall x \in B_\delta(x_0) \cap \partial\mathcal{M}$. Likewise we obtain

$$\begin{aligned} & \int_{B_\delta(x_0) \cap \partial\mathcal{M}} \varepsilon a \langle \nabla v_\varepsilon, \hat{\eta} \rangle dV^{\partial\mathcal{M}} + \int_{\partial B_\delta(x_0) \setminus \partial\mathcal{M}} \varepsilon a \langle \nabla v_\varepsilon, \hat{\eta}_B \rangle dV^{\partial B} \\ & + \int_{B_\delta(x_0)} f(x, v_\varepsilon) dV^{\mathcal{M}} = 0. \end{aligned} \quad (3.29)$$

Arguing as before we get a contradiction from the fact that, as $\varepsilon \rightarrow 0$, the second and third integrals in (3.29) go to zero whereas the first one converges to

$$\int_{B_\delta(x_0) \cap \partial\mathcal{M}} g(x, \bar{\alpha}) dV^{\partial\mathcal{M}} .$$

This establishes the proof. ■

Remark 3.2. Equation (3.26) also gives us a clue on the asymptotic behavior of $\{v_\varepsilon\}$, as $\varepsilon \rightarrow 0$, on the boundary which may have some interest of its own. Since the right hand-side of (3.26) vanishes, the boundary condition implies

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{M}} \left(\frac{g^2(x, v_\varepsilon)}{a\varepsilon} - \frac{\varepsilon}{2} a |\nabla v_\varepsilon|^2 \right) w(x) dV^{\partial\mathcal{M}} = 0 , \quad (3.30)$$

where $w(x)$ is any C^1 function defined on $\partial\mathcal{M}$.

Remark 3.3. In Definition 2.2 we supposed that the limiting interface \mathcal{S} intersects the boundary of \mathcal{M} and this intersection is $\partial\mathcal{S}$. The same result holds when $\mathcal{S} \cap \partial\mathcal{M} = \emptyset$. The proof is somehow easier and thus omitted.

Remark 3.4. The real numbers α , β , $\bar{\alpha}$ and $\bar{\beta}$ in (2.1) and (2.2) could have been considered as functions in $C^1(\widehat{\mathcal{M}})$ and $C^1(\partial\mathcal{M})$, respectively, instead of constants at the expense of some more computations.

Remark 3.5. The ε -scalings for the equations in \mathcal{M} and on the boundary could be different, i.e., we could have considered $\lambda_\varepsilon a \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = g(x, v_\varepsilon)$ on $\partial\mathcal{M}$ as long as $\lambda_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. A case where this really occurs will be presented in the applications.

4. Applications

In all applications below \mathcal{M} will denote a Riemannian manifold with or without boundary having dimension $n \geq 2$.

4.1 An example in population genetics.

The following problem often appears in the literature when studying the stationary solutions of a selection-migration model in population genetics (see, [15])

$$\begin{cases} \varepsilon \Delta v_\varepsilon + b(x)f(v_\varepsilon) = 0 & \text{on } \mathcal{M} \\ \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = 0 & \text{on } \partial\mathcal{M} \end{cases} \quad (4.1)$$

where ε is a positive parameter, $0 \leq v_\varepsilon \leq 1$ is the gene frequency, $b \in C(\mathcal{M})$ is the local relative selective advantage (disadvantage where $b(x) < 0$) of the gene at position $x \in \mathcal{M}$ and f is essentially a smooth positive function on $[0, 1]$ satisfying $f(0) = f(1) = 0$. Thus given a family of

solutions $\{v_\varepsilon\}$ to (4.1), according to Theorem 3.1, a necessary condition for a family of solutions $\{v_\varepsilon\}$ to develop internal transition layers with interface \mathcal{S} , as $\varepsilon \rightarrow 0$, is that

$$b(x) \int_0^1 f(\zeta) d\zeta = 0, \quad \forall x \in \mathcal{S}.$$

Since $\int_0^1 f > 0$, b must change sign on \mathcal{M} and the location of a limiting internal interface \mathcal{S} , whenever it exists, is known in advance; it is contained in the nodal set

$$Z_{\mathcal{M}}(b) := \{x \in \mathcal{M} : b(x) = 0\}$$

and in case $Z(b)$ is connected they must coincide. Indeed in [15], when \mathcal{M} is a bounded domain in \mathbb{R}^N , existence of a family $\{u_\varepsilon\}$ of solutions to (4.1) which are asymptotically stable stationary solutions to the corresponding parabolic problem has been proved; moreover u_ε develops internal transition layer, as $\varepsilon \rightarrow 0$, with limiting interface given by $Z_{\mathcal{M}}(b)$.

The same conclusion is true if \mathcal{M} has no boundary. Also a similar result holds if $f \in C^1(\mathbb{R})$ has only a finite number of zeros, $\alpha_1 < \alpha_2 < \dots < \alpha_k$ satisfying

$$\int_{\alpha_i}^{\alpha_j} f(\xi) d\xi \neq 0 \quad (i, j \in \{1, \dots, k\}, i \neq j), \quad (4.2)$$

i.e., no pair of zeros of f satisfies an equal-area condition.

4.2 Harmonic functions with nonlinear boundary condition.

Let us consider the following nonlinear boundary value problem

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{on } \mathcal{M} \\ \varepsilon \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = v_\varepsilon(1 - v_\varepsilon)(a(x) - v_\varepsilon) & \text{on } \partial\mathcal{M} \end{cases} \quad (4.3)$$

where Δ stands for the Laplace-Beltrami operator, $a \in C^1(\partial\mathcal{M})$ and $0 \leq v_\varepsilon \leq 1$. Set

$$I = \{x \in \partial\mathcal{M} : a(x) = 1/2\}.$$

Then Theorem 3.1 assures us that, given a family of solutions $\{v_\varepsilon\}$ to (4.3) a necessary condition for its concentration around 0 and 1 on $\partial\mathcal{M}$ is that

$$\int_0^1 g(x, \zeta) d\zeta = \int_0^1 [\zeta(1 - \zeta)(a(x) - \zeta)] d\zeta = \frac{1}{6}(a(x) - \frac{1}{2}) = 0.$$

In other words, we must have $x \in I$. In this case the location of the interface $\partial\mathcal{S}$ in $\partial\mathcal{M}$, whenever it exists, is known a priori and $\partial\mathcal{S} \subset I$. If I is connected then $\partial\mathcal{S} \equiv I$.

Actually when $\mathcal{M} \subset \mathbb{R}^2$ is a smooth bounded domain existence of such solutions have been proved in [2].

A similar result with obvious modifications holds true for the problem

$$\begin{cases} \varepsilon \Delta v_\varepsilon + v_\varepsilon(1 - v_\varepsilon)(a(x) - v_\varepsilon) = 0 & \text{on } \mathcal{M} \\ \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = 0 & \text{on } \partial\mathcal{M} \end{cases} \quad (4.4)$$

whose solutions when \mathcal{M} is a smooth bounded domain in \mathbb{R}^N ($N \geq 2$) have been proved in [10].

Another example of this type is the following problem

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{on } \mathcal{M} \\ \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = \varepsilon^{-1} b(x) f(v_\varepsilon) & \text{on } \partial \mathcal{M} \end{cases} \quad (4.5)$$

where b and f satisfy the same hypotheses as in (4.1) above except that now on $\partial \mathcal{M}$ instead of \mathcal{M} . Any family of solutions u_ε to (4.5) which develops internal transition layer, as $\varepsilon \rightarrow 0$, must have its limiting interface located at $Z_{\partial \mathcal{M}}(b) := \{x \in \partial \mathcal{M} : b(x) = 0\}$. Indeed in [11], when \mathcal{M} is a bounded domain in \mathbb{R}^N , existence of a family $\{u_\varepsilon\}$ of solutions to (4.5) which are asymptotically stable stationary solutions to the corresponding parabolic problem has been proved; moreover u_ε develops internal transition layer in $\partial \mathcal{M}$, as $\varepsilon \rightarrow 0$, with limiting interface given by $Z_{\partial \mathcal{M}}(b)$.

4.3 An example with $f \not\equiv 0$ and $g \not\equiv 0$.

Existence of layered solutions with internal interface (in the sense of Definition 2.2) for the problem

$$\begin{cases} \varepsilon \Delta v_\varepsilon + f(x, v_\varepsilon) = 0 & \text{on } \mathcal{M} \\ \varepsilon \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = \delta_\varepsilon g(x, v_\varepsilon) & \text{on } \partial \mathcal{M} \end{cases} \quad (4.6)$$

has been proved in [8], using Γ -convergence techniques, when \mathcal{M} is smooth bounded (non-convex) domain in \mathbb{R}^3 . There in addition to some technical hypotheses the equal-area conditions (3.1) and (3.2) are assumed, i.e., they are sufficient conditions. Therefore if the results of [8] holds when \mathcal{M} is a smooth Riemannian manifold (which is likely to be the case) then one realizes that conditions (3.1) and (3.2) cannot be discarded as they are, by Theorem 3.1, necessary conditions as well.

4.4 A geometric boundary condition

Let $\mathcal{M} \subset \mathbb{R}^3$ be a 2-dimensional smooth surface where the boundary $\partial \mathcal{M}$ is a smooth planar curve and consider the problem

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{on } \mathcal{M} \\ \varepsilon \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = \kappa(\cdot) g(v_\varepsilon) & \text{on } \partial \mathcal{M}, \end{cases} \quad (4.7)$$

where κ is the curvature of the border curve $\partial \mathcal{M}$ and $g \in C^1(\mathbb{R})$. Problem (4.7) appears in the study of stationary solutions of some diffusion problem, heat diffusion for instance, whose flux on the boundary is proportional to its curvature multiplied by a function of the temperature.

Suppose g has only a finite number of zeros, $a_1 < a_2 < \dots < a_k$, and that

$$\int_{a_i}^{a_j} g(\xi) d\xi \neq 0 \quad (i, j \in \{1, \dots, k\}, i \neq j). \quad (4.8)$$

Then, according to Theorem 3.1, the only possibility of a given family $\{v_\varepsilon\}$ of solutions to (4.7) to develop an internal transition layer as $\varepsilon \rightarrow 0$ (in the sense of Definition 2.2) is the existence of

points $P_1, P_2 \in \partial\mathcal{M}$ such that $\kappa(P_j) = 0$ ($j = 1, 2$). In particular this is not possible if the curve $\partial\mathcal{M}$ is strictly convex in which case we believe only formation of spikes in $\partial\mathcal{M}$ are possible.

In the absence of the curvature term in the boundary condition, problem (4.7) reads

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{on } \mathcal{M} \\ \varepsilon \frac{\partial v_\varepsilon}{\partial \hat{\eta}} = g(v_\varepsilon) & \text{on } \partial\mathcal{M}, \end{cases} \quad (4.9)$$

Then, under hypothesis (4.8), a similar argument yields that no family $\{v_\varepsilon\}$ of solutions to (4.9) develops internal transition layer in $\partial\mathcal{M}$, as $\varepsilon \rightarrow 0$. Therefore as long as concentration phenomenon is concerned, we believe that only spike solutions can occur in $\partial\mathcal{M}$. When $\mathcal{M} \subset \mathbb{R}^2$ is a square and $g(u) = u - u^3$ the authors in [3], in a computer-assisted proof, claimed have been proved existence of solutions to (4.9) which are non-constant stable stationary solutions to its parabolic counterpart.

4.4 An example of no internal limiting interface

Let us consider, as a particular case of (1.1), the problem

$$\begin{cases} -\varepsilon \Delta u + u - u^p = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \hat{\eta}} = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \quad (4.10)$$

where $\Omega \subset \mathbb{R}^N$. The authors in [19] prove the existence of a family of solutions concentrating along k -dimensional minimal sub-manifolds of $\partial\Omega$, for $N \geq 3$, $k \in \{1, \dots, N - 2\}$ and $1 < p < (N - k + 2)/(N - k - 2)$.

Regarding others concentration phenomena one might wonder whether there is a family of solutions to (4.10) developing internal transition layers in the sense of Definition 2.2, in other words, a family of solutions $u_\varepsilon > 0$ concentrating on the values 0 and 1 on sets whose measures do not depend on ε . This possibility is ruled out since according to Theorem 3.1 this would imply $\int_0^1 (\xi - \xi^p) d\xi = 0$, which is not the case.

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