Relative Borsuk-Ulam Theorems for Spaces with a Free $\mathbb{Z}_2$-action

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Dedicated to Professor Carlos Biasi

Abstract

Let (X, A) be a pair of topological spaces, $T : X \to X$ a free involution and A a $T-$invariant subset of X. In this context, a question that naturally arises is whether or not all continuous maps $f : X \to \mathbb{R}^k$ have a T -coincidence point, that is, a point $x \in X$ with $f(x) = f(T(x))$. In this paper, we obtain results of this nature under cohomological conditions on the spaces A and X.

2010 *Mathematics Subject Classification*: Primary 55M20; Secondary 55M35.

Key words and phrases: Borsuk-Ulam theorems, multi-valued map, involutions.

The authors were supported in part by FAPESP of Brazil Grant number 08/57607-6. The first author was supported in part by CNPq of Brazil Grant number 308390/2008-3. The second author was supported in part by FAPESP of Brazil Grant number 2012/03316-6. The third author was supported in part by CNPq of Brazil Grant number 304480/2008-8

1 Introduction

One formulation of the Borsuk-Ulam Theorem [1] is that there is no map from S^m to S^n equivariant with respect to the antipodal map, when $m > n$. In [6], it was proved that if X and Y are Hausdorff, pathwise connected and paracompact spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$ such that for some natural $n \geq 1$, $\check{H}^r(X; \mathbb{Z}_2) = 0$ for $1 \leq r \leq n$ and $\check{H}^{n+1}(Y/S; \mathbb{Z}_2) = 0$, where Y/S is the orbit space of Y by S, then there is no equivariant map $f : (X, T) \to (Y, S)$. The first aim of this paper is to generalize this result for the following relative case:

Theorem 1.1. *Let X, Y be a Hausdorff, connected and paracompact spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$. Let A be a non-empty connected and $T-$invariant subset of X. Suppose that for some $n \geq 1$, $\check{H}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-1$, $i^* : \check{H}^n(X, \mathbb{Z}_2) \to \check{H}^n(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map and $\check{H}^{n+1}(Y/S; \mathbb{Z}_2) = 0$. Then there is no equivariant map $f : (X, T) \to (Y, S)$.*

The following theorem is an important consequence of Theorem 1.1.

Theorem 1.2. *Let X be a Hausdorff, connected and paracompact space with a free involution $T : X \to X$. Let A be a non-empty connected and $T-$invariant subset of X. Suppose that for some $n \geq 1$, $\check{H}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$ and $i^* : \check{H}^{n-1}(X, \mathbb{Z}_2) \to \check{H}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map. Then, if $\varphi : X \multimap \mathbb{R}^k$ is an acyclic multi-valued map and $n \geq k$, there exists $x \in X$ such that $\varphi(x) \cap \varphi(T(x)) \neq \emptyset$.*

In particular case that $\varphi = f$ is a single valued map, it follows:

Corollary 1.3. *Let X be a Hausdorff, connected and paracompact space with a free involution $T : X \to X$. Let A be a non-empty connected and $T-$invariante subset of X. Suppose that for some $n \geq 1$, $\check{H}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$ and $i^* : \check{H}^{n-1}(X, \mathbb{Z}_2) \to \check{H}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map. Then, if $f : X \to \mathbb{R}^k$ is a map with $n \geq k$, there exists $x \in X$ such that $f(x) = f(T(x))$.*

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we recall definitions, fix notations, state results needed on multi-valued map and prove Theorem 1.2. In Section 4, we show an interesting example for which Theorem 1.2 can be applied and we finished the paper with some applications.

Throughout the paper, we assume that all spaces under consideration are Hausdorff spaces. $\check{H}^*(\ ,\mathbb{Z}_2)$ denotes the Čech cohomology with coefficients in $\mathbb{Z}_2$.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We need first to prove the following lemma.

Lemma 2.1. *(cf.,[6]) Let X, Y be Hausdorff and paracompact spaces, equipped with free involutions $T : X \to X$ and $S : Y \to Y$. Let $e \in \check{H}^1(X/T,\mathbb{Z}_2)$ and $u \in \check{H}^1(Y/T,\mathbb{Z}_2)$ be the Euler classes of the $\mathbb{Z}_2$-principal bundles $X \to X/T$ and $Y \to Y/T$, respectively. If $e^{n+1} \neq 0$ and $u^{n+1} = 0$, then there is no equivariant map $f : (X,T) \to (Y,S)$.*

Proof. First consider $B\mathbb{Z}_2$ the classifying space for $\mathbb{Z}_2$, and denote by $\alpha \in \check{H}^1(B\mathbb{Z}_2,\mathbb{Z}_2)$ the Euler class of the universal principal $\mathbb{Z}_2$-bundle over $B\mathbb{Z}_2$. Since X is a Hausdorff paracompact space, one can take a classifying map $h : X/T \to B\mathbb{Z}_2$ for the principal $\mathbb{Z}_2$-bundle $X \to X/T$, and from $h^* : \check{H}^1(B\mathbb{Z}_2,\mathbb{Z}_2) \to \check{H}^1(X/T,\mathbb{Z}_2)$ one gets the Euler class $e = h^*(\alpha) \in \check{H}^1(X/T,\mathbb{Z}_2)$ of $X \to X/T$. Now suppose $f : (X,T) \to (Y,S)$ an equivariant map, and let $g : Y/S \to B\mathbb{Z}_2$ be a classifying map for $Y \to Y/S$. Then $g \circ \bar{f}$ also is a classifying map for $X \to X/T$ and, therefore, it is homotopic to the previous classifying map $h : X/T \to B\mathbb{Z}_2$, where $\bar{f} : X/T \to Y/S$ is the map induced by f. Since $u = g^*(\alpha)$, we have that $e = h^*(\alpha) = (g\circ\bar{f})^*(\alpha) = \bar{f}^*\circ g^*(\alpha) = \bar{f}^*(u)$, and thus $\bar{f}^*(u^{n+1}) = e^{n+1} \neq 0$, which contradicts the fact that $u^{n+1} = 0$. □

Proof of Theorem 1.1. Let $\hat{e} \in \check{H}^1(A/T,\mathbb{Z}_2)$ be the Euler class of the principal $\mathbb{Z}_2$-bundle $A \to A/T$ and let $\bar{i} : A/T \hookrightarrow X/T$ be the map induced by inclusion $i : A \hookrightarrow X$. We have that

$$\bar{i}^*(e) = \hat{e} \in \check{H}^1(A/T,\mathbb{Z}_2),$$

where $e \in \check{H}^1(X/T,\mathbb{Z}_2)$ is the Euler class for $X \to X/T$.

Now, let us consider the following diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \check{H}^0(X/T) & \xrightarrow{p^*} & \check{H}^0(X) & \xrightarrow{\tau} & \check{H}^0(X/T,) & \xrightarrow{\smile e} & \check{H}^1(X/T) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \bar{i}^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \bar{i}^*} & & \downarrow{\scriptstyle \bar{i}^*} & \\ 0 \longrightarrow & \check{H}^0(A/T) & \xrightarrow{p^*} & \check{H}^0(A) & \xrightarrow{\tau} & \check{H}^0(A/T) & \xrightarrow{\smile \hat{e}} & \check{H}^1(A/T) & \longrightarrow \cdots \end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \check{H}^r(X/T) & \xrightarrow{p^*} & \check{H}^r(X) & \xrightarrow{\tau} & \check{H}^r(X/T) & \xrightarrow{\smile e} & \check{H}^{r+1}(X/T) & \longrightarrow \cdots \\
& \bar{i}^* \downarrow & & i^* \downarrow & & \bar{i}^* \downarrow & & \bar{i}^* \downarrow & \\
\cdots \longrightarrow & \check{H}^r(A/T) & \xrightarrow{p^*} & \check{H}^r(A) & \xrightarrow{\tau} & \check{H}^r(A/T) & \xrightarrow{\smile \hat{e}} & \check{H}^{r+1}(A/T) & \longrightarrow \cdots
\end{array}$$

where the rows are Gysin exact sequences (see for example [3, Theorem 17.9.2]) and each square commutes by the naturality, p is the quotient map and τ is the transfer homomorphism. Since A is connected, $p^* : \check{H}^0(A/T, \mathbb{Z}_2) \to \check{H}^0(A, \mathbb{Z}_2)$ is an isomorphism, hence $\cup\hat{e} : \check{H}^0(A/T, \mathbb{Z}_2) \to \check{H}^1(A/T, \mathbb{Z}_2)$ is injective and thus $\hat{e} = 1 \cup \hat{e} \in \check{H}^1(A/T, \mathbb{Z}_2)$ is a nonzero class. The fact that $\check{H}^r(A, \mathbb{Z}_2) = 0$, for $1 \leq r \leq n-1$ implies that $\cup\hat{e} : \check{H}^r(A/T, \mathbb{Z}_2) \to \check{H}^{r+1}(A/T, \mathbb{Z}_2)$ is an isomorphism for $1 \leq r \leq n-2$ and injective for $r = n-1$, hence $\hat{e}^n \in \check{H}^n(A/T, \mathbb{Z}_2)$ is nonzero. Since $\bar{i}^*(e^n) = \hat{e}^n \neq 0$, we see that $e^n \in \check{H}^n(X/T, \mathbb{Z}_2)$ is nonzero.

Now, we will show that $e^{n+1} \in \check{H}^{n+1}(X/T, \mathbb{Z}_2)$ is nonzero. Suppose $e^{n+1} = e^n \cup e = 0$, then $e^n \in \ker(\cup e) = \operatorname{im}(\tau)$ and there exists a nonzero class $a \in \check{H}^n(X, \mathbb{Z}_2)$ such that $\tau(a) = e^n$. Therefore, $\bar{i}^* \circ \tau(a) = \bar{i}^*(e^n) = \hat{e}^n \neq 0$.

On the other hand, since $i^* : \check{H}^n(X, \mathbb{Z}_2) \to \check{H}^n(A, \mathbb{Z}_2)$ is the null homomorphism and each square commutes in the diagram, we have that $\bar{i}^* \circ \tau(a) = \tau \circ i^*(a) = 0$. Thus $e^{n+1} \neq 0$.

Finally, since $\check{H}^{n+1}(Y/S, \mathbb{Z}_2) = 0$, we have that $u^{n+1} \in \check{H}^{n+1}(Y/S, \mathbb{Z}_2)$ is zero. By Lemma 2.1 there is no equivariant map $f : (X, T) \to (Y, S)$. □

3 Results on multi-valued map and Proof of Theorem 1.2

Let X and Y be two spaces and assume that for each point $x \in X$ a non-empty closed subset $\varphi(x)$ of Y is given; in this case, we say that φ is a multi-valued map from X into Y and we write $\varphi : X \multimap Y$. More precisely, a multi-valued map $\varphi : X \multimap Y$ can be defined as a subset φ of $X \times Y$ for which the following condition is satisfied: for every $x \in X$ the set $\varphi_x = \{y \in Y \mid (x, y) \in \varphi\}$ is a non-empty closed subset of Y.

A multi-valued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) if for every open subset U of Y the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is an open subset of X.

A compact space X is acyclic (with respect to the functor $\check{H}^*(\ , \mathbb{Z}_2)$) if $\check{H}^0(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\check{H}^q(X, \mathbb{Z}_2) = 0$ for all $q > 0$. In words, X has the cohomology of a point.

An u.s.c. multi-valued map $\varphi : X \multimap Y$ is called acyclic if for every

$x \in X$ the set $\varphi(x)$ is an acyclic subset of Y.

Let $\varphi : X \multimap Y$ be an u.s.c. multi-valued map and consider

$$\Gamma_\varphi = \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$$

the graph of φ. There are two projections associated with φ. They are $p : \Gamma_\varphi \to X$ and $q : \Gamma_\varphi \to Y$ given by $p(x,y) = x$ and $q(x,y) = y$.

Below, we listed some basic properties of the u.s.c. multi-valued mappings.

Lemma 3.1. *Let X be a connected space and $\varphi : X \multimap Y$ an u.s.c. multi-valued map with connected values. Then $\varphi(X) = \bigcup_{x\in X} \varphi(x)$ is a connected space.*

A continuous function $p : X \to Y$ is said perfect if it is closed, surjective and $p^{-1}(y)$ is compact, for each $y \in Y$.

Lemma 3.2 ([2], Theorem 5.3)**.** *Let $p : X \to Y$ be a perfect function. If Y is paracompact, so also is X.*

Lemma 3.3. *Let $\varphi : X \multimap Y$ an u.s.c. multi-valued map with compact values. Then, the projection $p : \Gamma_\varphi \to X$ is a perfect function. In particular, if X is paracompact, so also is Γ_φ.*

Proof. See [4], Proposition 32.3. □

In [7], we have:

Theorem 3.4. *Let X, Y be Hausdorff paracompact spaces and $p : X \to Y$ a continuous, closed onto map such that $p^{-1}(y)$ is acyclic for every $y \in Y$. Then, the induced homomorphism*

$$p^* : \check{H}^*(Y, \mathbb{Z}_2) \xrightarrow{\sim} \check{H}^*(X, \mathbb{Z}_2)$$

is an isomorphism.

Proof of Theorem 1.2. Let $\varphi : X \multimap \mathbb{R}^k$ be an acyclic multi-valued map. Let $\tilde{X}$ and $\tilde{A}$ be the sets defined by

$$\tilde{X} = \{(x, T(x), u, v) \in X^2 \times \mathbb{R}^{2k} \mid u \in \varphi(x),\ v \in \varphi(T(x))\}$$

$$\tilde{A} = \{(x, T(x), u, v) \in A^2 \times \mathbb{R}^{2k} \mid u \in \varphi(x),\ v \in \varphi(T(x))\}.$$

Thus, the space $\tilde{X}$ is the graph of the u.s.c. multi-valued map $\Phi : \{(x, T(x)) \mid x \in X\} \to \mathbb{R}^{2k}$ given by

$$\Phi(x, T(x)) = \varphi(x) \times \varphi(T(x))$$

and $\tilde{A}$ is the graph of Φ restrict to the set $\{(a, T(a)) \mid a \in A\}$. By Lemma 3.3, $\tilde{X}$ and $\tilde{A}$ are paracompact spaces. Moreover, since X and A are connected and $\Phi(x, T(x))$ is connected for each $x \in X$, by Lemma 3.1, $\tilde{X}$ and $\tilde{A}$ are connected spaces. The map $\tilde{T} : \tilde{X} \to \tilde{X}$ defined by

$$\tilde{T}(x, T(x), u, v) = (T(x), x, v, u)$$

is a free involution on $\tilde{X}$. Moreover, $\tilde{A}$ is $\tilde{T}$-invariant. Now, using Theorem 3.4, one can prove that $\check{H}^r(\tilde{A}, \mathbb{Z}_2) = 0$ for $1 \le r \le n-2$ and $j^* : \check{H}^{n-1}(\tilde{X}, \mathbb{Z}_2) \to \check{H}^{n-1}(\tilde{A}, \mathbb{Z}_2)$ is the null homomorphism, where $j : \tilde{A} \hookrightarrow \tilde{X}$ is the inclusion map. In fact, let $s : \tilde{X} \to X$ be defined by

$$s(x, T(x), u, v) = x$$

and $s|_{\tilde{A}} : \tilde{A} \to A$ be the restriction of s to $\tilde{A}$. Then, s and $s|_{\tilde{A}}$ are continuous, closed and onto maps. Moreover, $s^{-1}(x) = \{(x, T(x))\} \times \varphi(x) \times \varphi(T(x))$, which is acyclic for each $x \in X$. Hence, by Theorem 3.4, $s^* : \check{H}^*(X, \mathbb{Z}_2) \to \check{H}^*(\tilde{X}, \mathbb{Z}_2)$ and $(s|_{\tilde{A}})^* : \check{H}^*(A, \mathbb{Z}_2) \to \check{H}^*(\tilde{A}, \mathbb{Z}_2)$ are isomorphisms. From the isomorphism $(s|_{\tilde{A}})^*$, it follows that $\check{H}^r(\tilde{A}, \mathbb{Z}_2) = 0$ for $1 \le r \le n-2$. Note that $i \circ (s|_{\tilde{A}}) = s \circ j$, following the commutative diagram

$$\begin{array}{ccc} \check{H}^{n-1}(\tilde{X}, \mathbb{Z}_2) & \xrightarrow{j^*} & \check{H}^{n-1}(\tilde{A}, \mathbb{Z}_2) \\ {\scriptstyle s^*}\uparrow \simeq & & \simeq \uparrow {\scriptstyle (s|_{\tilde{A}})^*} \\ \check{H}^{n-1}(X, \mathbb{Z}_2) & \xrightarrow{i^*} & \check{H}^{n-1}(A, \mathbb{Z}_2) \end{array}$$

Since $i^* : \check{H}^{n-1}(X, \mathbb{Z}_2) \to \check{H}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism and s^* and $(s|_{\tilde{A}})^*$ are isomorphisms, it follows that $j^* : \check{H}^{n-1}(\tilde{X}, \mathbb{Z}_2) \to \check{H}^{n-1}(\tilde{A}, \mathbb{Z}_2)$ is the null homomorphism.

Finally, suppose that $\varphi(x) \cap \varphi(T(x)) = \emptyset$ for all $x \in X$. Then it is well defined an equivariant map $F : (\tilde{X}, \tilde{T}) \to (S^{k-1}, a)$ by

$$F(x, T(x), u, v) = \frac{u - v}{\|u - v\|},$$

where $a : S^{k-1} \to S^{k-1}$ is the antipodal map. Since $n \ge k$, $\check{H}^n(S^{k-1}/a, \mathbb{Z}_2) = 0$, which contradicts Theorem 1.1.

Therefore, $\varphi(x) \cap \varphi(T(x)) \ne \emptyset$ for some $x \in X$. □

4 Examples and applications

Example 4.1. Let $T_n = T_1 \sharp \cdots \sharp T_1$ be the n-fold a connected sum of tori $T_1 = S^1 \times S^1$, which is embedded in $\mathbb{R}^3$ symmetrically with respect to the origin. Let $T : T_n \to T_n$ be the antipodal map given by $T(x, y, z) = (-x, -y, -z)$. If n is even, there exists a loop $A = \{(x, 0, z); x^2 + z^2 = 1\} \subset T_n$ homeomorphic to S^1, which separates T_n into two components symmetrical with respect to the origin such that $T(A) = A$. We have that $i^* : \check{H}^1(T_n, \mathbb{Z}_2) \to \check{H}^1(A, \mathbb{Z}_2)$ is the null homomorphism, so by Theorem 1.2, for any u.s.c. multi-valued map $\varphi : T_n \multimap \mathbb{R}^2$ with connected compact values, there exists $x \in T_n$ such that $\varphi(x) \cap \varphi(T(x)) \neq \emptyset$. In particular, for any continuous map $f : T_n \to \mathbb{R}^2$, there exists $x \in T_n$ such that $f(x) = f(T(x))$.

Remark 4.2. In the Example 4.1, let us note that $\check{H}^1(T_n, \mathbb{Z}_2) \neq 0$, and therefore [6, Theorem 1 or Theorem A'] cannot be applied to show this result.

4.1 Maximizing simultaneously two related functions

Let $f : X \times Y \to \mathbb{R}$ be a real map defined in a cartesian product, $X \times Y$, where Y is compact. For each $x \in X$, let $f_x : Y \to \mathbb{R}$ be the map defined by $f_x(y) = f(x, y)$, for every $y \in Y$. Thus, f define a family of maps, $\{f_x : Y \to \mathbb{R}\}_{x \in X}$. If X admits a free involution $T : X \to X$, we ask if there exists a point $x_0 \in X$ such that the maps f_{x_0} and $f_{T(x_0)}$ can be simultaneously maximized, that is, there exists a point $y_0 \in Y$ such that

$$f_{x_0}(y_0) \geq f_{x_0}(y) \text{ and } f_{T(x_0)}(y_0) \geq f_{T(x_0)}(y), \text{ for every } y \in Y.$$

This problem is related to a Borsuk-Ulam problem for a multi-valued map. Namely, let $\alpha : X \to \mathbb{R}$ be the map defined by

$$\alpha(x) = \max f_x(Y), \text{ for each } x \in X,$$

and $\varphi : X \multimap Y$ be the multi-valued defined by

$$\varphi(x) = \{y \in Y \mid f_x(y) = \alpha(x)\}.$$

Then, the maps f_{x_0} and $f_{T(x_0)}$ can be simultaneously maximized if and only if $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$.

Let $Y \subset \mathbb{R}^n$ be a convex subset. A function $f : Y \to \mathbb{R}$ is said quasiconcave if, for each $\lambda \in (0, 1)$, we have $f(\lambda y_1 + (1 - \lambda) y_2) \geq \min\{f(y_1), f(y_2)\}$, for all $y_1, y_2 \in Y$. In particular, if $f : Y \to \mathbb{R}$ is quasiconcave and Y is compact, then the set $\{y \in Y \mid f(y) = \max f(Y)\}$ is a non-empty, compact and convex subset of Y.

In view of Theorem 1.2 and Example 4.1, we can assert that:

Corollary 4.3. *If $Y \subset \mathbb{R}^2$ is a compact and convex subset of $\mathbb{R}^2$ and $f : T_{2n} \times Y \to \mathbb{R}$ is a continuous function such that $f_x : Y \to \mathbb{R}$ is quasiconcave for each $x \in T_{2n}$, then there exists $x_0 \in T_{2n}$ such that f_{x_0} and f_{-x_0} can be simultaneously maximized.*

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