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# Sections of Analytic Variety

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Abstract. We present a study of sections of a germ of analytic variety $V$ through relations of equivalences defined by geometric subgroups, introduced by J. Damon, that in some way preserve $V$ and we show that under certain conditions these equivalences relate. We also obtain a classification under $\mathcal{R}_V$-equivalence of sections of $S_k$-singularities.

## 1. Introduction

Let $(V,0)$ be the germ of an analytic subvariety of $K^n$ ($K=\mathbb{R}$ or $\mathbb{C}$). The study of equivalence of the germ $f:(K^n,0)\to(K^p,0)$ by requiring that the germs of diffeomorphisms on $K^n$ preserve $V$ has been done through groups $\mathcal{R}_V$, $\mathcal{K}_{RV}$, $\mathcal{K}_V$ and $\mathcal{A}_V$ which are geometric subgroups of the classical groups of singularity theory $\mathcal{R}$, $\mathcal{K}$, $\mathcal{A}$ (see [**29**]). This study began in [**1**] with $V$ the discriminant of a versal unfolding and has been followed by [**27**], [**31**], [**4**], [**5**], [**6**], [**11**], etc. Results for unfolding, versality, determinacy theorems and topological triviality for these subgroups were presented in [**10**], [**11**] and [**12**]. A specific study for $\mathcal{K}_V$ was given in [**8**] and [**9**], for $\mathcal{R}_V$ in [**5**]. Results about $\mathcal{R}_V$-topological triviality of families of function germs were given in [**28**] and [**29**].

In this paper we present the equivalence relations defined by these subgroups and show that under certain conditions these equivalences are related. For groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ we give algebraic and geometric conditions necessary and sufficient for finite determinacy and we obtain a classification of sections of $S_k$-singularities (defined in [**22**]) given by submersions under $\mathcal{R}_V$-equivalence.

## 2. Equivalence of Sections

Let $\mathcal{O}_n$ be the ring of analytic function germs $f:(K^n,0)\to K$, with $K=\mathbb{R}$ or $\mathbb{C}$ and let $m_n$ be its maximal ideal.

A germ of a subset $(V,0)\subset(K^n,0)$ is the germ of analytic variety if there exist analytic function germs $f_1,...,f_r$ such that $V=\{x\in K^n : f_1(x)=...=f_r(x)=0\}$. We denote by $I(V)$ the ideal in $\mathcal{O}_n$ consisting of germs vanishing on $V$, that is, $I(V)=\{\phi\in\mathcal{O}_n : \phi(x)=0\,\forall\, x\in V\}$.

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Definition 2.1. A section of $V$ is a set $C$ obtained by the intersection of $V$ with the germ of a variety $(X,0) \subset (K^n,0)$.

All germs which we consider will be analytic. Classically the three principal notions of equivalence for map germs are $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{K}$-equivalence. We consider the space of analytic germs $f:(K^n,0)\to(K^p,0)$, which we denote by $\mathcal{O}_{n,p}$. Let $\mathcal{D}_s$ denote the group of germs of diffeomorphisms $\varphi:(K^s,0)\to(K^s,0)$. The group $\mathcal{R}=\mathcal{D}_n$ acts on $\mathcal{O}_{n,p}$ by $\varphi.f=f\circ\varphi^{-1}$. The group $\mathcal{A}=\mathcal{D}_n\times\mathcal{D}_p$ acts on $\mathcal{O}_{n,p}$ by $(\varphi,\phi).f=\phi\circ f\circ\varphi^{-1}$. The group $\mathcal{K}$, contact equivalence, consists of $H\in\mathcal{D}_{n+p}$ such that there is a $\varphi\in\mathcal{D}_n$ so that $H\circ i=i\circ\varphi$ and $\pi\circ H=\varphi\circ\pi$ where $i(x)=(x,0)$ is the inclusion $i:K^n\to K^{n+p}$ and $\pi(x,y)=x$ is the projection $\pi:K^{n+p}\to K^p$, therefore $H(x,y)=(\varphi(x),\theta(x,y))$ with $\theta(x,0)=0$. Then $\mathcal{K}$ acts on $\mathcal{O}_{n,p}$ by $(\varphi(x),H.f(x))=H(x,f(x))$ that is $\text{graph}(H.f)=H(\text{graph}(f))$. Germs are $\mathcal{R}$, $\mathcal{A}$ or $\mathcal{K}$-equivalent if they lie in common orbits of the group actions.

We can study sections of $V$ in two ways: $X$ is given by a parametrization $g:(K^m,0)\to(K^n,0)$ or as an inverse image, $X=l^{-1}(0)$, $l:(K^n,0)\to(K^p,0)$. We describe below these different approaches.

In the first case, i e, taking the section of $V$ as an image of a map germ $g:(K^m,0)\to(K^n,0)$, $m<n$, the important equivalence relations are:

Definition 2.2. Two germs $g$ and $g':(K^m,0)\to(K^n,0)$ are:

**(i)** $\mathcal{A}_V$-equivalent if there exist germs $\varphi\in\mathcal{D}_m$ and $\phi\in\mathcal{D}_n$ with $g\circ\varphi=\phi\circ g'$ and $\phi(V)=V$.

**(ii)** $\mathcal{K}_V$-equivalent if there exists $H=(\varphi,\theta)\in\mathcal{K}$ such that $H(x,g(x))=(\varphi(x),g'(\varphi(x)))$ and $H(K^m\times V)\subseteq K^m\times V$. That is,

$$\mathcal{K}_V=\{H=(\varphi,\theta)\in\mathcal{K}\ :\ \theta(K^m\times V)\subseteq V\}.$$

This subgroup of $\mathcal{K}$ can be seen as the semi-direct product of $\mathcal{R}$ and $\mathcal{C}_V$ where

$$\mathcal{C}_V=\{H=(id,\theta)\in\mathcal{K}:\ \theta(K^n\times V)\subseteq V\}$$

and $id$ is the germ at 0 of the identity mapping on $K^m$.

The $\mathcal{K}$-equivalence is the one which shows the equivalence of the germs of varieties $g^{-1}(0)$, then the $\mathcal{K}_V$-equivalence shows the equivalence of the germs of varieties $g^{-1}(V)$. The group $\mathcal{K}_V$ is a geometric subgroup of $\mathcal{K}$ which satisfies the basic theorems of theory of singularities, see [**8**]. Relations between $\mathcal{A}$ and $\mathcal{K}_V$ are given in [**9**].

When $X=l^{-1}(0)$ for a germ $l:(K^n,0)\to(K^p,0)$, we consider the following relations of equivalence:

Definition 2.3. Two germs $l$, $l':(K^n,0)\to(K^p,0)$ are:

**(i)** $\mathcal{R}_V$-equivalent if there exists $\phi\in\mathcal{D}_n$ with $\phi(V)=V$ and $l\circ\phi=l'$. That is,

$$\mathcal{R}_V=\{\phi\in\mathcal{R}:\ \phi(V)=V\}$$

**(ii)** $\mathcal{K}_{RV}$-equivalent if there exists $H=(\varphi,\theta)\in\mathcal{K}$ with $\varphi(V)=V$ and $H(x,l(x))=(\phi(x),l'(\phi(x)))$. That is,

$$\mathcal{K}_{RV}=\{H=(\varphi,\theta)\in\mathcal{K}\ :\ \varphi(V)=V\}$$

When $p=1$, the group $\mathcal{R}_V$ has been studied by some authors, such as [**5**], [**13**] and [**26**].

The group $\mathcal{K}_{RV}$ can be seen as semi-direct of product $\mathcal{R}_V$ and $\mathcal{C}$. Some results on this group were obtained by R. Atique in [**2**].

Definition 2.4. Let $f : (K^n, 0) \to (K^p, 0)$ be an analytic map germ, we denote by $I_f$ the ideal generated by components $f_1, ..., f_p$ of $f$. A germ $l : (K^p, 0) \to (K^s, 0)$ is a reduced defining equation for $f$ if the ideal of the functions which vanish on $(f(K^n), 0)$ is generated by $I_l$, i.e., $I(f(K^n)) = \langle l_1, ..., l_s \rangle$ as an $\mathcal{O}_p$-module.

In [**18**], it's shown that $f$ and $g : (K^n, 0) \to (K^p, 0)$ are $\mathcal{K}$-equivalent if only if there exists $h \in \mathcal{D}_n$ taking $I_f$ to $I_g$, that is $h^*(I_f) = I_g$, where $h^* : \mathcal{O}_n \to \mathcal{O}_n$ is the ring isomorphism induced by $h$. It follows that if $f$ and $g$ are $\mathcal{K}$-equivalent then there is diffeomorphism $h : (K^n, 0) \to (K^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$. We remark that the converse is not true, for example, consider $f(x) = x$ and $g(x) = x^2$. However, we can show the following.

Lemma 2.5. *Let $f$, $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $I_f$ and $I_g$ radical ideals. If there exists $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$, then $f$ and $g$ are $\mathcal{K}$-equivalent.*

Proof. It's enough to show that $h^*(I_g) = I_f$. Let $l \in I_g$ then $l(z) = 0 \, \forall \, z \in g^{-1}(0)$. Taking $x \in f^{-1}(0)$, by hypothesis it follows that $h(x) \in g^{-1}(0)$. Hence,

$$h^*(l)(x) = (l \circ h)(x) = l(h(x)) = 0 \Rightarrow l \circ h \in I(V(I_f)) = \sqrt{I_f} = I_f$$

where the penultimate equality follows from the Hilbert's Nullstellensatz Theorem for germs of analytic functions (see [**21**]), therefore $h^*(I_g) \subseteq I_f$. As $h^*$ is an isomorphism, we can show similarly that $(h^*)^{-1}(I_f) \subseteq I_g$, thus $I_f \subseteq h^*(I_g)$. Therefore $h^*(I_g) = I_f$. □

The following result connects $\mathcal{A}_V$ classification of section to $\mathcal{K}_{RV}$ classification of reduced defining equation of sections.

Theorem 2.6. *Given $g$, $g^{'} : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, let $l$, $l^{'} : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be defining equation for $g$, $g'$ respectively.*
(a) *If $g$, $g'$ are $\mathcal{A}_V$- equivalent then $l$, $l'$ are $\mathcal{K}_{RV}$-equivalent.*
(b) *If $g$, $g'$ are $\mathcal{A}$-finitely determined with $n > 2$ and $l$, $l'$ are $\mathcal{K}_{RV}$-equivalent then $g$, $g'$ are $\mathcal{A}_V$-equivalent.*

Proof. (a) If $g$, $g'$ are $\mathcal{A}_V$-equivalent then there exists $\phi \in \mathcal{D}_m$ and $\varphi \in \mathcal{D}_n$ such that $g \circ \phi = \varphi \circ g'$ and $\varphi(V) = V$. Hence, $\varphi(\mathrm{Im}(g')) = \mathrm{Im}(g)$,

$$(l \circ \varphi)^{-1}(0) = \varphi^{-1}(l^{-1}(0)) \supseteq \varphi^{-1}(\mathrm{Im}(g)) = \mathrm{Im}(g'), \text{then}$$

$$I((l \circ \varphi)^{-1}(0)) \subseteq I_{l'} \Rightarrow (l \circ \varphi)^{-1}(0) \supseteq {l'}^{-1}(0)$$

we have

$$(l' \circ \varphi^{-1})^{-1}(0) = \varphi({l'}^{-1}(0)) \supseteq \varphi(\mathrm{Im}(g')) = \mathrm{Im}(g), \text{ then}$$

$$I((l' \circ \varphi^{-1})^{-1}(0)) \subseteq I_l \Rightarrow \varphi({l'}^{-1}(0)) \supseteq l^{-1}(0) \Rightarrow {l'}^{-1}(0) \supseteq (l \circ \varphi)^{-1}(0)$$

Therefore

$$(l \circ \varphi)^{-1}(0) = {l'}^{-1}(0)$$

as $I_l$ and $I_{l'}$ are radical ideals it follows from Lemma 2.5 that $l$, $l'$ are $\mathcal{K}_{RV}$-equivalent, since $\varphi(V) = V$.

(b) If $l$, $l'$ are $\mathcal{K}_{RV}$ equivalent then there exists $h \in \mathcal{D}_n$, $h(V) = V$ such that $(l \circ h)^{-1}(0) = {l'}^{-1}(0)$. Thus, $(h^{-1} \circ g)(\mathbb{C}^m) = g'(\mathbb{C}^m)$, and by uniqueness of the normalisation (see [**16**]) it follows that there exists $\alpha \in \mathcal{D}_m$ such that $h^{-1} \circ g = g' \circ \alpha$, that is, $g$, $g'$ are $\mathcal{A}_V$-equivalent. □

With the hypothesis added that $g$, $g'$ are germs of immersion we can connect the $\mathcal{A}_V$-equivalence to the $\mathcal{K}_V$-equivalence.

THEOREM 2.7. *Let $g$, $g' : (K^m,0) \to (K^n,0)$ be germs of immersion. Then $g$ and $g'$ are $\mathcal{A}_V$-equivalent if only if they are $\mathcal{K}_V$-equivalent.*

PROOF. We can suppose that $g(x) = (x, \varphi(x))$ and $g'(x) = (\phi_1(x), \phi_2(x))$ (changing the coordinate only in the source). Supposing $g$, $g'$ are $\mathcal{K}_V$-equivalent then there exists $H \in \mathcal{D}_{m+n}$ given by $H(x,y) = (h(x), \theta(x,y))$ with $\theta(x,0) = 0$, $\theta(x,V) \subseteq V$ and

$$H(x, x, \varphi(x)) = (h(x), \theta(x, x, \varphi(x))) = (h(x), \phi_1(h(x)), \phi_2(h(x)))$$

Let $\tilde{\theta} : (K^n,0) \to (K^n,0)$ with $\tilde{\theta}(y) = \theta(y_1, ..., y_m, y)$. Then

$$\tilde{\theta} \circ g(x) = \tilde{\theta}(x, \phi(x)) = \theta(x, x, \phi(x)) = (\phi_1(h(x)), \phi_2(h(x))) = g' \circ h(x)$$

and $\tilde{\theta}(V) \subseteq V$, $\tilde{\theta} \in \mathcal{D}_n$, since $H \in \mathcal{D}_n$ . Therefore $g$, $g'$ are $\mathcal{A}_V$-equivalent.

Since $\mathcal{A}_V$ is a subgroup of $\mathcal{K}_V$, it is immediate that if $g$ and $g'$ are $\mathcal{A}_V$-equivalent then they are $\mathcal{K}_V$-equivalent. □

The following example shows that if $g$ or $g'$ is not an immersion then Proposition 2.7 can be false.

EXAMPLE 2.8. Define $g, g' : (K,0) \to (K^2,0)$ by $g(t) = (t^2,0)$, $g'(t) = (t^2,t^3)$ and $V = \{(0,y), y \in K\}$. Taking $H(t,x,y) = (t,x,y-tx)$, $H(t,V) \subseteq K \times V$ and $H(t,t^2,t^3) = (t,t^2,0)$. Therefore $g$, $g'$ are $\mathcal{K}_V$-equivalent, but $g$, $g'$ are not $\mathcal{A}_V$-equivalent, since $g$, $g'$ are not $\mathcal{A}$-equivalent, since $g'$ is 3-$\mathcal{A}$-determined and $g$ is not $\mathcal{A}$-finitely determined.

LEMMA 2.9. *If $I(V) = \langle \phi_i \rangle_{i=1,\dots,r}$ as an $\mathcal{O}_n$-module then $I(K^m \times V) = \langle \phi_i \rangle_{i=1,\dots,r}$ as an $\mathcal{O}_{m+n}$-module.*

PROOF. $P \in I(K^m \times V)$ if and only if $P(x,y) = 0$, $\forall\,(x,y) \in K^m \times V$.

By Hadamard's Lemma (see [**18**]), $P(x,y) - P(0,y) = \sum_{i=1}^{m} q_i(x,y)x_i$ where

$$q_i(x,y) = \int_0^1 \frac{\partial P}{\partial x_i}(tx_1, ..., tx_m, y_1, ..., y_n)dt$$

since $P|_{K^m \times \{y\}} \equiv 0$, we have that $q_i|_{K^m \times \{y\}} \equiv 0\ \forall\, y \in V$.

Therefore, $q_i(K^m \times V) \equiv 0$, i.e., $q_i \in I(K^m \times V)$. From the hypothesis, $P(0,y) = \sum_{i=1}^{r} h_i(y)\phi_i(y)$, with $h_i \in \mathcal{O}_n$, then

$$P(x,y) = \sum_{i=1}^{r} h_i(y)\phi_i(y) + \sum_{i=1}^{m} q_i(x,y)x_i,$$

therefore $I(K^m \times V) \subseteq I(V)_{\mathcal{O}_n} + I(K^m \times V)\,\langle x_1, ..., x_m \rangle$, hence

$$I(K^m \times V) \subseteq I(V)_{\mathcal{O}_{m+n}} + I(K^m \times V)\,\langle x_1, ..., x_m, y_1, ..., y_n \rangle\,.$$

By Nakayama Lemma (see [**18**]), we have

$$I(K^m \times V) \subseteq I(V)_{\mathcal{O}_{m+n}} = \langle \phi_i \rangle_{i=1,\dots,r} \text{ as an } \mathcal{O}_{m+n}\text{-module.}$$

As the other inclusion is immediate, the result follows. □

THEOREM 2.10. *Two germs $g$, $g' : (K^m,0) \to (K^n,0)$ are $\mathcal{K}_V$-equivalent if only if there exists $h \in \mathcal{D}_m$ such that $g' \circ h$, $g$ are $\mathcal{C}_V$-equivalent.*

PROOF. If $g$, $g'$ are $\mathcal{K}_V$-equivalent then there exist germs of diffeomorphism $H$ and $h$ with $H(x,y) = (h(x), \theta(x,y))$, $\theta(x,0) = 0$, $\theta(x,V) \subseteq V$ $\forall x \in K^m$ and $H \circ (id, g) = (h, g' \circ h)$. Let $H' = (h^{-1}, \pi_2)$, where $\pi_2(x,y) = y$ thus, $H' \circ H(id,g) = (id, \theta(id,g)) = (id, g' \circ h)$, that is, $g$ and $g' \circ h$ are $\mathcal{C}_V$-equivalent.

If $g$, $g' \circ h$ are $\mathcal{C}_V$-equivalent then there exist germs of diffeomorphism $H$, with $H(x,y) = (x, \theta(x,y))$, $\theta(x,V) \subseteq V$ $\forall x \in K^m$ and $H(id,g) = (id, g' \circ h)$. Let $H'(x,y) = (h(x), y)$, thus $H' \circ H(x,y) = (h(x), \theta(x,y))$. Therefore $H' \circ H(id,g) = (h, \theta(id,g)) = (h, g' \circ h)$, that is, $g$ and $g'$ are $\mathcal{K}_V$-equivalent. □

THEOREM 2.11. *Let $g$, $g' : (K^m, 0) \to (K^n, 0)$ and $I(V)$ be the ideal generated by $\phi_1, ..., \phi_r$ as an $\mathcal{O}_n$-module, then:*
(a) *If $g$ and $g'$ are $\mathcal{C}_V$-equivalent then $\langle \phi_i \circ g \rangle_{i=1,\dots,r} = \langle \phi_i \circ g' \rangle_{i=1,\dots,r}$.*
(b) *If $g$ and $g'$ are $\mathcal{K}_V$-equivalent then $\langle \phi_i \circ g \rangle_{i=1,\dots,r}$ and $\langle \phi_i \circ g' \rangle_{i=1,\dots,r}$ are isomorphic. Therefore the germs $F = (\phi_1 \circ g, ..., \phi_r \circ g)$ and $G = (\phi_1 \circ g', ..., \phi_r \circ g')$ are $\mathcal{K}$-equivalent.*

PROOF. $(a)$ It is enough to show that $\langle \phi_i \circ g' \rangle \subseteq \langle \phi_i \circ g \rangle$, since the other inclusion is shown analogously. We show that each $\phi_i \circ g'$ can be written $\phi_i \circ g' = \sum_{j=1}^{r} a_{ij}(\phi_j \circ g)$ with $a_{ij} \in \mathcal{O}_{m+n}$, as $g$ and $g'$ are $\mathcal{C}_V$-equivalent, there exists a germ of diffeomorphism $H$ such that $H(x,y) = (x, \theta(x,y))$, $\theta(x,0) = 0$, $\theta(x,V) \subseteq V$ and $H(x, g(x)) = (x, \theta(x, g(x))) = (x, g'(x))$ as $\theta(x,V) \subseteq V \Rightarrow \phi_i(\theta(x,V)) \equiv 0$ $\Rightarrow \phi_i \circ \theta \in I(K^m \times V)$.

By Lemma 2.9, we can write each $\phi_i \circ \theta$ as

$$\phi_i \circ \theta(x,y) = \sum_{j=1}^{r} a_{ij}(x,y)\phi_j(y) \text{ with } a_{ij} \in \mathcal{O}_{m+n}$$

therefore

$$\phi_i \circ \theta(x, g(x)) = \sum_{j=1}^{r} a_{ij}(x, g(x))\phi_j(g(x)),$$

then $\phi_i \circ \theta(x, g(x)) = \phi_i \circ g'(x)$. It follows that $\phi_i \circ g' = \sum_{j=1}^{r} a_{ij}(\phi_j \circ g)$.

$(b)$ If $g$ and $g'$ are $\mathcal{K}_V$-equivalent then there exists a diffeomorphism $h$ of $K^m$ such that $g' \circ h$ and $g$ are $\mathcal{C}_V$-equivalent, by part $(a)$, $\langle \phi_i \circ g \rangle = \langle \phi_i \circ g' \circ h \rangle$. Thus the isomorphism $h^* : \mathcal{O}_m \to \mathcal{O}_m$ takes $\langle \phi_i \circ g' \rangle$ to $\langle \phi_i \circ g \rangle$. Therefore these ideals are isomorphic. □

The following example shows that the condition $\langle \phi_i \circ g' \rangle$ isomorphic $\langle \phi_i \circ g \rangle$ does not imply that $g$ and $g'$ are $\mathcal{K}_V$-equivalent.

EXAMPLE 2.12. Let $V = \{(x,0), x \in K\} \subseteq K^2$, $I(V) = \langle y \rangle_{\mathcal{O}_2} = \langle \phi \rangle_{\mathcal{O}_2}$. We take $g$, $g' : (K,0) \to (K^2,0)$ with $g(t) = (t, t^2)$, $g'(t) = (0, t^2)$, so $(\phi \circ g)(t) = t^2 = (\phi \circ g')(t)$, therefore $\langle \phi \circ g \rangle = \langle \phi \circ g' \rangle$. But $g$ and $g'$ are not $\mathcal{K}_V$-equivalent, since they are not $\mathcal{K}$-equivalent.

However, the following result is true.

THEOREM 2.13. ([**14**]) Let $V = \phi^{-1}(0)$ where $\phi : (K^n, 0) \to (K, 0)$ be a submersion, let $g$, $g' : (K^m, 0) \to (K^n, 0)$ be embedding and $h : (K^n, 0) \to (K^n, 0)$ be a diffeomorphism such that $h^*(\langle \phi \circ g' \rangle) = \langle \phi \circ g \rangle$, then $g$ and $g'$ are $\mathcal{K}_V$-equivalent.

THEOREM 2.14. *Let $(V, 0) \subset (K^n, 0)$. Two germs $l$, $l' : (K^n, 0) \to (K^r, 0)$ are $\mathcal{K}_{RV}$-equivalent if and only if there exists germ of diffeomorphism $\phi : (K^n, 0) \to (K^n, 0)$ with $\phi(V) = V$ such that $l' \circ \phi$ and $l$ are $\mathcal{C}$-equivalent.*

PROOF. If $l$ and $l'$ are $\mathcal{K}_{RV}$-equivalent, then there exist germs of diffeomorphism $H$ and $\phi$ with $H(x, y) = (\phi(x), \theta(x, y))$, $\theta(x, 0) = 0$, $\phi(V) = V$ and $H \circ (id, l) = (\phi, l' \circ \phi)$.

Let $H' = (\phi^{-1}, \pi_2)$, where $\pi_2(x, y) = y$. Then $H' \circ H(id, l) = (id, \theta(id, l)) = (id, l' \circ \phi)$, that is, $l$ and $l' \circ \phi$ are $\mathcal{C}$-equivalent and $\phi(V) = V$.

Conversely, if $l$ and $l' \circ \phi$ are $\mathcal{C}$-equivalent and $\phi(V) = V$, then there exists germ of diffeomorphism $H$ with $H(x, y) = (x, \theta(x, y)), H(id, l) = (id, l' \circ \phi)$.

Let $H'(x, y) = (\phi(x), y)$. We have that $H' \circ H(x, y) = (\phi(x), \theta(x, y))$. Therefore $H' \circ H(id, l) = (\phi, \theta(id, l)) = (\phi, l' \circ \phi)$, then $l$, $l'$ are $\mathcal{K}_{RV}$-equivalent. □

In this work, given an analytic variety germ $(V, 0) \subset (K^n, 0)$, we will always define a section of $V$ through an equation $h = 0$, with $h : (K^n, 0) \to (K^p, 0)$. Hence, our main interest is action of the groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ on the set $\mathcal{O}(n, p)$ of the germs of analytic maps $h : (K^n, 0) \to (K^p, 0)$.

As an application of the equivalence studied, we will see in the following proposition that the $\mathcal{A}$ classification of bigerms reduces to $\mathcal{A}_V$ or $\mathcal{K}_{RV}$ classification of sections. This result extends for any dimension of the result of R. Atique [**2**].

A bigerm $G : (K^m, S) \to (K^n, 0)$, $S = \{x_1, x_2\}$ and $x_i \in K^m$, $i = 1, 2$ is the equivalence classes of the maps that coincide in the neighborhood of $S$. Without loss of generality we can consider $x_1 = x_2 = 0$ and the germ $G$ as

$$G : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g : (K^m, 0) \to (K^n, 0) \end{cases}$$

(The equivalence of bigerm is defined by different diffeomorphisms in source and a unique diffeomorphism in the target, see [**2**]).

THEOREM 2.15. *Consider the bigerms*

$$G : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g : (K^m, 0) \to (K^n, 0) \end{cases} \qquad G' : \begin{cases} f : (K^m, 0) \to (K^n, 0) \\ g' : (K^m, 0) \to (K^n, 0) \end{cases}$$

*with $n > 2$ and $f$, $g$, $g^{'}$ $\mathcal{A}$-finitely determined. Let $l$ and $l^{'}$ be the defining equation for $g(K^m)$ and $g^{'}(K^m)$ respectively and let $V$ be the image of $f$. The following statements are equivalent*
(i) *$G$, $G'$ are $\mathcal{A}$-equivalent,*
(ii) *$g$, $g'$ are $\mathcal{A}_V$-equivalent,*
(iii) *$l$, $l^{'}$ are $\mathcal{K}_{RV}$-equivalent.*

PROOF. If $G$, $G'$ are $\mathcal{A}$-equivalent then there exist diffeomorphisms $\phi_f$, $\phi_g$ : $(K^m, 0) \to (K^m, 0)$ and $\varphi : (K^n, 0) \to (K^n, 0)$ such that $f \circ \phi_f = \varphi \circ f$ and $g \circ \phi_g = \varphi \circ g'$.

Hence $\varphi(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f)$. It follows that $\varphi(V) \subseteq V$, then $g$, $g'$ are $\mathcal{A}_V$-equivalent.

Assuming that $g$, $g^{'}$ are $\mathcal{A}_V$- equivalent, there exist diffeomorphisms $\phi_g$ : $(K^m, 0) \to (K^m, 0)$ and $\varphi : (K^n, 0) \to (K^n, 0)$ such that $g \circ \phi_g = \varphi \circ g'$ and $\varphi(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f)$. Therefore $(\varphi \circ f)(K^n)) \subseteq f(K^n)$, by uniqueness of normalisation (see [**16**], [**17**]), there exists $\phi$ such that $f \circ \phi = \varphi \circ f$, hence $G$, $G'$ are $\mathcal{A}$-equivalent. It follows from Proposition 2.6 that (ii) is equivalent to (iii). □

## 3. Finite Determinacy

Results on finite determinacy of groups ($\mathcal{G}_V = \mathcal{R}_V$ or $\mathcal{K}_{RV}$) have been obtained by some authors. The group $\mathcal{R}_V$ was studied in [**5**], [**13**] and [**26**]. There are not many results about group $\mathcal{K}_{RV}$ but there are some results in [**2**].

Our interest is in groups of diffeomorphisms that preserve an analytic variety $V$. To get these diffeomorphisms the basic technique is the integration of vector fields tangent to $V$. The main references for studying these vector fields are [**5**], [**8**] and [**30**].

Definition 3.1. ([**5**]) Let $(V,0) \subset (K^n,0)$ be the germ of an analytic variety and let $\xi$ be a germ of analytic vector field on $(K^n,0)$. Then $\xi$ is said to be *logarithmic* for $(V,0)$ if, when considered as a derivation $\xi : \mathcal{O}_n \to \mathcal{O}_n$, $g \mapsto \xi g$, we have $\xi g \in I(V)$ for all $g \in I(V)$, where $\xi g = dg(\xi)$. The $\mathcal{O}_n$- module of such vector fields is denoted by $\Theta_V$ or $Derlog(V)$, that is,

$$\Theta_V = \{\xi \in \mathcal{O}_n^n \ : \ \xi g \in I(V), \forall\, g \in I(V)\}$$

The main properties of $\Theta_V$ are given by the following proposition.

Theorem 3.2. ([**5**]) (i) *$\Theta_V$ is the set of the vector field germs at $0 \in K^n$ which are tangent to the set of regular points of $V$.*
(ii) *$\Theta_V$ is finitely generated as an $\mathcal{O}_n$-module.*
(iii) *If $\xi \in \Theta_V$ and $\xi(0) = 0$, then the flow $\phi_t$ generated by $\xi$ preserves $(V,0)$. Thus $\phi_t \in \mathcal{R}_V$ for all $t$.*

Definition 3.3. A germ $f : (K^n,0) \to (K^p,0)$ is $k$-$\mathcal{G}_V$-finitely determined if all germs $g$, such that $j^k g(0) = j^k f(0)$ is $\mathcal{G}_V$-equivalent to $f$. If $f$ is $k$-$\mathcal{G}_V$-finitely determined for some $k$, we say that $f$ is $\mathcal{G}_V$-finitely determined.

Definition 3.4. Let $f : (K^n,0) \to (K^p,0)$. We denote by $\Theta_V^0$ the subset of $\Theta_V$ constituted by vector fields that vanish at origin. The tangent spaces and the extended tangent spaces of the groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$ are respectively:

$$T\mathcal{R}_V(f) = df(\Theta_V^0), \qquad T\mathcal{R}_{V,e}(f) = df(\Theta_V)$$
$$T\mathcal{K}_{RV}(f) = df(\Theta_V^0) + f^*(m_p)\mathcal{O}_n^p, \quad T\mathcal{K}_{RV,e}(f) = df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p$$

The next proposition gives a necessary condition to $\mathcal{G}_V$-finite determinacy.

Theorem 3.5. *If $f : (K^n,0) \to (K^p,0)$ is $\mathcal{G}_V$-finitely determined, then*

$$T\mathcal{G}_V(f) \supset m_n^l \mathcal{O}_n^p,$$

*for some positive integer $l$.*

For groups $\mathcal{R}_V$ and $\mathcal{K}_{RV}$, the essential part of proof, as in groups $\mathcal{K}$, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{R}$, is to note that $J^k\mathcal{G}_V$, the set of $k$-jets of elements $\mathcal{G}_V$, is a Lie group acting smoothly on $J^k(n,p)$. R. Pellikaan in [**25**] shows that $J^k\mathcal{R}_V$ is a Lie group and as $\mathcal{K}_{RV} = \mathcal{R}_V.\mathcal{C}$ (semi-direct product) it follows that $J^k\mathcal{K}_{RV}$ is Lie group.

The theorems on infinitesimal criterion for $\mathcal{G}_V$-finite determinacy are known, see for instance [**2**], [**6**], [**5**].

Theorem 3.6. *If $T\mathcal{G}_V(f) \supseteq m_n^k\mathcal{O}_n^p$ then $f$ is $k$-$\mathcal{G}_V$-determined, $\mathcal{G}_V = \mathcal{R}_V$, $\mathcal{K}_{RV}$.*

The following corollary is useful for the classification of germs, since it reduces the estimate of tangent space in the finite dimension case.

COROLLARY 3.7. *If* $T\mathcal{G}_V(f) + m_n^{k+1}\mathcal{O}_n^p \supseteq m_n^k\mathcal{O}_n^p$ *then* $f$ *is* $k$-$\mathcal{G}_V$*-determined.*

PROOF. This follows Nakayama Lemma, since $T\mathcal{G}_V(f)$ is an $\mathcal{O}_n$-module, for $\mathcal{G}_V = \mathcal{R}_V,\ \mathcal{K}_{RV}$. □

DEFINITION 3.8. (a) We denote by $\Theta_V^1$ the subset of $\Theta_V$ constituted by vector fields whose 1-jets are null. If $f : (K^n, 0) \to (K^p, 0)$ we denote

$$\begin{array}{l} T\mathcal{R}_V^1(f) = df(\Theta_V^1) \\ T\mathcal{K}_{RV}^1(f) = df(\Theta_V^1) + f^*(m_p)m_n\mathcal{O}_n^p. \end{array}$$

(b) We denote by $H^k(n,p)$ the vector space of all the maps $K^n \to K^p$, where the components are homogeneous polynomials of degree $k$.
(c) The standard $k-$jet space $m_n\mathcal{O}_n/m_n^{k+1}\mathcal{O}_n$ is denoted $J^k(n,1)$.

An efficient method for the classification of singularities finitely determined is the *complete transversal method* described by Bruce, Kirk and du Plessis in [**3**].

THEOREM 3.9. ([**3**]) Let $f : (K^n, 0) \to (K^p, 0)$ be an analytic germ and let $T$ be a vector subspace of the set $H^{k+1}(n,p)$, such that $J^{k+1}(T\mathcal{G}_V^1 f) + T \supset H^{k+1}(n,p)$. Then every $(k+1)$-jet $g$ with $j^k g = j^k f$ lies in the same $\mathcal{G}_V^1$-orbit of some $(k+1)$-jet of the form $j^{k+1}f + t$, for some $t \in T$.

COROLLARY 3.10. *If* $T\mathcal{G}_V^1 f \supseteq m_n^{k+1}\mathcal{O}_n^p$ *then* $f$ *is* $k$-$\mathcal{G}_V$*-determined.*

LEMMA 3.11. *Let* $f : (K^n, 0) \to (K^p, 0)$ *be an analytic germ and let* $\{\xi_1, ..., \xi_r\}$ *be generators of* $\Theta_V$*. Then* $f$ *is* $\mathcal{K}_{RV}$*-finitely determined if and only if* $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$ *for some* $k$*, where* $Jf(\Theta_V)$ *is the ideal in* $\mathcal{O}_n$ *generated by all the* $(p \times p)$*-minors of the matrix* $(\nabla f_i(\xi_j))_{i=1,\dots,p}^{j=1,\dots,r}$ *( if* $p > r$*, then* $Jf(\Theta_V) = 0$*).*

PROOF. The proof is an adaptation of the result correspondent for the group $\mathcal{K}$ ([**15**], Lemma 2.12).

If $f$ is $\mathcal{K}_{RV}$-finitely determined, then by Proposition 3.5, there exists $l$ such that $df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p \supset m_n^l\mathcal{O}_n^p$. Let $u \in {m_n^l}^p = m_n^k$, $u = u_1.\dots.u_p$ where $u_i \in m_n^l$. Since

$$(0, \dots, 0, u_i, 0, \dots, 0) \in df(\Theta_V) + f^*(m_p)\mathcal{O}_n^p$$

there exists $\eta_i \in \Theta_V$ such that

$$df(\eta_i) = (0, \dots, 0, u_i, 0, \dots, 0) \mathrm{mod} f^*(m_p)\mathcal{O}_n^p$$

with $\eta_i = a_1^i\xi_1 + \dots + a_r^i\xi_r$ and $a_j^i \in \mathcal{O}_n$.

Hence,

$$df \begin{pmatrix} a_1^1\xi_1^1 + \dots + a_r^1\xi_r^1 & \dots & a_1^p\xi_1^1 + \dots + a_r^p\xi_r^1 \\ a_1^1\xi_1^2 + \dots + a_r^1\xi_r^2 & \dots & a_1^p\xi_1^2 + \dots + a_r^p\xi_r^2 \\ \vdots & \vdots & \vdots \\ a_1^1\xi_1^n + \dots + a_r^1\xi_r^n & \dots & a_1^p\xi_1^n + \dots + a_r^p\xi_r^n \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

$\mathrm{mod} f^*(m_p)\mathcal{O}_n^p$. Thus,

$$df \begin{pmatrix} \xi_1^1 & \xi_2^1 & \dots & \xi_r^1 \\ \xi_1^2 & \xi_2^2 & \dots & \xi_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^n & \xi_2^n & \dots & \xi_r^n \end{pmatrix} \cdot \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^p \\ a_2^1 & a_2^2 & \dots & a_2^p \\ \vdots & \vdots & \vdots & \vdots \\ a_r^1 & a_r^2 & \dots & a_r^p \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

$\text{mod} f^*(m_p)\mathcal{O}_n^p$, otherwise,

$$\begin{pmatrix} \nabla f_1(\xi_1) & \dots & \nabla f_1(\xi_r) \\ \nabla f_2(\xi_1) & \dots & \nabla f_2(\xi_r) \\ \vdots & \vdots & \vdots \\ \nabla f_p(\xi_1) & \dots & \nabla f_p(\xi_r) \end{pmatrix} \cdot \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^p \\ a_2^1 & a_2^2 & \dots & a_2^p \\ \vdots & \vdots & \vdots & \vdots \\ a_r^1 & a_r^2 & \dots & a_r^p \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_p \end{pmatrix}$$

$\text{mod} f^*(m_p)\mathcal{O}_n^p$. Taking the determinant of each side of the equation, it follows that $u \in Jf(\Theta_V) + f^*(m_p)$.

Assuming now that $Jf(\Theta_V) + f^*(m_p) \supseteq m_n^k$. It is enough to show that

$$Jf(\Theta_V)\mathcal{O}_n^p \subseteq df(\Theta_V).$$

Let $M = (\nabla f_i(\xi_{j_s}))_{i=1,\dots,p}^{s=1,\dots,p}$ be a $(p \times p)$ sub-matrix of the matrix $(\nabla f_i(\xi_j))_{i=1,\dots,p}^{j=1,\dots,r}$. Let $\{e_1, ..., e_p\}$ be the standard base of $K^p$, then $(\det M)e_k \in Jf(\Theta_V)\mathcal{O}_n^p$ and

$$(\det M)e_k = \begin{pmatrix} \nabla f_1(\xi_1) & \dots & \nabla f_1(\xi_r) \\ \nabla f_2(\xi_1) & \dots & \nabla f_2(\xi_r) \\ \vdots & \vdots & \vdots \\ \nabla f_p(\xi_1) & \dots & \nabla f_p(\xi_r) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ cof(\nabla f_k(\xi_{j_1})) \\ \vdots \\ \text{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ 0 \end{pmatrix}$$

where $\text{cof}(\nabla f_k(\xi_{j_s}))$ is the cofactor of $\nabla f_i(\xi_{j_s})$ in the matrix $M$, then

$$(\det M)e_k = df \begin{pmatrix} \xi_1^1 & \xi_2^1 & \dots & \xi_r^1 \\ \xi_1^2 & \xi_2^2 & \dots & \xi_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^n & \xi_2^n & \dots & \xi_r^n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ cof(\nabla f_k(\xi_{j_1})) \\ \vdots \\ \text{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ 0 \end{pmatrix},$$

Hence

$$(\det M)e_k = df \begin{pmatrix} \xi_{j_1}^1 \text{cof}(\nabla f_k(\xi_{j_1})) + \dots + \xi_{j_p}^1 \text{cof}(\nabla f_k(\xi_{j_p})) \\ \xi_{j_1}^2 \text{cof}(\nabla f_k(\xi_{j_1})) + \dots + \xi_{j_p}^2 \text{cof}(\nabla f_k(\xi_{j_p})) \\ \vdots \\ \xi_{j_1}^n \text{cof}(\nabla f_k(\xi_{j_1})) + \dots + \xi_{j_p}^n \text{cof}(\nabla f_k(\xi_{j_p})) \end{pmatrix}$$

$$= df(\text{cof}(\nabla f_k(\xi_{j_1}))\xi_{j_1} + \text{cof}(\nabla f_k(\xi_{j_2}))\xi_{j_2} + \dots + \text{cof}(\nabla f_k(\xi_{j_p}))\xi_{j_p})$$

therefore, $(\det M)e_k \in df(\Theta_V)$ and $Jf(\Theta_V)\mathcal{O}_n^p \subseteq df(\Theta_V)$. □

The following proposition is the geometric criterion for the $\mathcal{K}_{RV}$-finitely determinacy.

THEOREM 3.12. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *and let* $(V, 0) \subseteq (\mathbb{C}^n, 0)$. *Then* $f$ *is* $\mathcal{K}_{RV}$*-finitely determined if and only if* $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$.

Proof. By Lemma 3.11, $f$ is $\mathcal{K}_{RV}$-finitely determined if and only if $Jf(\Theta_V)+f^*(m_p) \supseteq m_n^k$ for some $k$. Then, $V(Jf(\Theta_V)+f^*(m_p)) \subseteq V(m_n^k)$. Hence,

$$V(Jf(\Theta_V)) \cap V(f^*(m_p)) \subseteq \{0\} \text{ and } V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}.$$

Going in the other direction, assuming that $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$ we get

$$I(V(Jf(\Theta_V)+f^*(m_p))) \supseteq I(\{0\}).$$

Thus, by *Hilbert's Nullstellensatz Theorem* for analytic function germs (see [**21**]),

$$\sqrt{Jf(\Theta_V)+f^*(m_p)} \supseteq m_n,$$

hence $Jf(\Theta_V)+f^*(m_p) \supseteq m_n^k$ for some $k$. □

The following theorem is the geometric criterion for the $\mathcal{R}_V$-finite determinacy and its proof is a consequence of the proof of Proposition 3.12.

Theorem 3.13. *Let $(V,0) \subseteq (\mathbb{C}^n,0)$ be an analytic variety germ and let $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be an analytic function germ. Let $V(f) = \{x \in \mathbb{C}^n : \xi f(x) = 0$ for all $\xi \in \Theta_V\}$. Then $f$ is $\mathcal{R}_V$-finitely determined if and only if $V(f) = \{0\}$ or $\emptyset$.*

In the real case, the necessary condition remains true, that is, if $f$ is $\mathcal{G}_V$-finitely determined then: $V(Jf(\Theta_V)) \cap f^{-1}(0) \subseteq \{0\}$ for $\mathcal{G}_V = \mathcal{K}_{RV}$ and the set $\{x \in \mathbb{R}^n : \xi f(x) = 0 \text{ for all } \xi \in \Theta_V\}$ is $\{0\}$ or $\emptyset$ for $\mathcal{G}_V = \mathcal{R}_V$.

Theorem 3.14. *Let $(V,0) \subseteq (K^n,0)$ be an analytic variety germ and let $f : (K^n,0) \to (K,0)$. If $f$ is $\mathcal{R}_V$-finitely determined then $f^{-1}(0)$ is transverse to $V$ away from 0.*

Proof. By hypothesis there exists a positive integer $l$ such that $df(\Theta_V) \supseteq m_n^l$, thus we have that $V(df(\Theta_V)) \subseteq \{0\}$. However, $V(df(\Theta_V))$ describes the set of points where $f^{-1}(0)$ is not transverse to $V$, thus $f^{-1}(0)$ is transverse to $V$ away from 0. □

In the following example we show that the converse of Theorem 3.14 is not true.

Example 3.15. Let $V \subseteq (\mathbb{C}^2,0)$ be defined by $\varphi(x,y) = x^3 - y^2 = 0$. We have that $\Theta_V$ is generated by $\alpha_1 = (2x,3y), \alpha_2 = (2y,3x^2)$. Let $f(x,y) = y^2$, $f^{-1}(0) = x$-axis. It is transverse to $V$ away from 0 but $f$ is not $\mathcal{R}_V$-finitely determined.

Bruce and Roberts, in [**5**], introduced a generalization of Milnor number of a function germ $f$ related to a variety $V$ by $\mu_{BR}(f,V) = \dim \frac{\mathcal{O}_n}{df(\theta_V)}$. We will call it the Bruce-Roberts number of $f$ with respect to $V$. Like the Milnor number of $f$, this number shows some geometric properties of $f$ and $V$. For instance, if one considers the group $\mathcal{R}_V$, then $f$ is finitely determined with respect to the action of $\mathcal{R}_V$ on $\mathcal{O}_n$ if and only if $\mu_{BR}(f,V)$ is finite and, in this case, the codimension of the orbit of $f$ under this action is equal to $\mu_{BR}(f,V)$. We refer to [**5**] for more details.

## 4. Section of the Singularities $S_k$

The goal of this section is to classify germs of submersion $h : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ by action of $\mathcal{R}_V$ where $V$ is the image of $S_k$-singularities classified by Mond in [**22**], $S_k = \{(x, y^2, y^3 + x^{k+1}y)\}$, $k \geq 1$, in this case the sections are space curves $C_k = S_k \cap h^{-1}(0)$.

We have that $\varphi_k(x, y, z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$ is a reduced defining equation for the images of $S_k$ (obtained using Gröbner basis methods) and the germ of the singular set of $S_k$ is a smooth curve given by $\Sigma_k = \{(x, -x^{k+1}, 0)\}$.

Given a finite function germ $f : (C, 0) \to (\mathbb{C}, 0)$ on a complex analytic reduced space curve $(C, 0)$, in [**24**] is shown that

$$\mu(f|_C) = \mu(C) + \deg(f) - 1,$$

where $\mu(C)$ is the Milnor number of the curve, as defined by Buchweitz and Greuel [**7**], $\deg(f)$ is the degree of $f$ (that is, the number of inverse images of a generic value), and $\mu(f|_C)$ is the Milnor number of $f$, introduced by Goryunov [**19**] for curves in $(\mathbb{C}^3, 0)$ and by Mond and D. van Straten [**23**] for the general case of curves in $(\mathbb{C}^n, 0)$. As $\Sigma_k$ is a smooth curve it follows that $\mu(h|_{\Sigma_k}) = \deg(h|_{\Sigma_k}) - 1$.

Theorem 4.1. *$\Theta_{S_k}$ is the $\mathcal{O}_3$-module generated by*

$$\eta_1 = (2x, 2(k+1)y, 3(k+1)z) \qquad \eta_2 = (0, 2z, x^{2(k+1)} + 4x^{k+1}y + 3y^2)$$
$$\eta_3 = (x^{k+1} + 3y, -2(k+1)x^k y, 0) \qquad \eta_4 = (z, 0, (k+1)(x^{2k+1}y + x^k y^2))$$

Proof. If $\varphi_k = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y$ then $S_k = \varphi_k^{-1}(0)$. We have that

$$d\varphi_k = (\varphi_{k_x}, \varphi_{k_y}, \varphi_{k_z}) = (2(k+1)x^k y^2 + 2(k+1)x^{2k+1}y, 4x^{k+1}y + 3y^2 + x^{2(k+1)}, -2z),$$

$$d\varphi_k(\eta_1) = 6(k+1)\varphi_k \text{ and } d\varphi_k(\eta_2) = d\varphi_k(\eta_3) = d\varphi_k(\eta_4) = 0,$$

so $\eta_1, \eta_2, \eta_3$ and $\eta_4$ are in $\Theta_{S_k}$.

Now if $\xi \in \Theta_{S_k}$ then $d\varphi_k(\xi) = \lambda\varphi_k$ for some $\lambda \in \mathcal{O}_3$, then

$$d\varphi_k(\xi - \frac{1}{6(k+1)}\lambda\eta_1) = d\varphi_k(\xi) - \frac{\lambda}{6(k+1)} d\varphi_k(\eta_1) = d\varphi_k(\xi) - \lambda\varphi_k = 0,$$

so we need only to check that $\eta_1, \eta_2, \eta_3$ and $\eta_4$ generate all germs of vector $\xi$ such that $d\varphi_k(\xi) = 0$. Writing $\xi = (\xi_1, \xi_2, \xi_3)$, we have $d\varphi_k(\xi) = 0$ if and only if

$$\xi_1\varphi_{k_x} + \xi_2\varphi_{k_y} - \xi_3\varphi_{k_z} = 0,$$

if and only if $(\xi_1, \xi_2, \xi_3)$ is a syzygy of $(\varphi_{k_x}, \varphi_{k_y}, \varphi_{k_z})$ which is generated by

$$(\varphi_{k_y}, -\varphi_{k_x}, 0) = (y + x^{k+1})\eta_1,\ (\varphi_{k_z}, 0, -\varphi_{k_x}) = -\frac{1}{2}\eta_4 \text{ and } (0, -\varphi_{k_z}, \varphi_{k_y}) = \eta_2,$$

so the result follows. □

We follow the standard classification techniques, namely to classify germs inductively at the jet level until a sufficient jet is obtained (and hence produces a finitely determined germ). For this we use the complete transversal method, Mather's Lemma and the results on finite determinacy. To calculate the Milnor number and the Bruce-Roberts number we use SINGULAR (see [**20**]).

THEOREM 4.2. *Let $V$ be the image of $S_k$, $k \geq 1$ and $h : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a germ of submersion $\mathcal{R}_V$-finitely determined with $\mu(h_{|\Sigma_k}) \leq k$. Then $h$ is $\mathcal{R}_V$-equivalent to one of the following germs*

| $h$ | $\mu_{BR}(h, S_k)$ | $\mu(C_k)$ | $\mu(h_{\vert\Sigma_k})$ |
|---|---|---|---|
| $x$ | 1 | 2 | 0 |
| $y + a_2x^2,\ \ a_2 \neq 0$ | 3 | 5 | 1 |
| $y + a_3x^3 + a_4x^4,\ \ a_3 \neq 0$ | 5 | 8 | 2 |
| $y + a_4x^4 + a_5x^5 + a_6x^6,\ \ a_4 \neq 0$ | 7 | 11 | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \cdots + a_{2k}x^{2k},\ \ a_{k+1} \neq 0, 1$ | $2k+1$ | $3k+2$ | $k$ |
| $y + x^{k+1} + a_lx^l + \cdots + a_{l+k-1}x^{l+k-1},\ \ l \geq k+2,\ a_l \neq 0$ | $l+k$ | $2l+k$ | $k$ |
| $y + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{l+k-1}x^{l+k-1},\ \ l \geq k+2,\ a_l \neq 0$ | $l+k$ | $l+2k+1$ | $k$ |
| $z + a_2x^2,\ \ a_2 \neq 0$ | 4 | 6 | 1 |
| $z + a_3x^3 + a_4x^4,\ \ a_3 \neq 0$ | 7 | 10 | 2 |
| $z + a_4x^4 + a_5x^5 + a_6x^6,\ \ a_4 \neq 0$ | 10 | 14 | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2},\ \ a_l \neq 0, l \leq k+1$ | $3l-2$ | $4l-2$ | $l-1$ |

In what follows we give a sketch of the proof.

With the intention of setting ideas, we start with the case $k = 1$ :

*Section of $S_1$.*

We start classifying by the 1-jets. By Mather's Lemma and change of scale the orbit in $J^1(3,1)$ of the action of $J^1\mathcal{R}_V$ reduces to: $x, y, z$ and 0.

**Case (i):** Classification of germs $h$ for which the 1-jet is $x$.

Consider the germ $h(x,y,z) = x$. Then $T\mathcal{R}_V(h) + \mathcal{M}^2 \supset \mathcal{M}^1$ and $x$ is 1-$\mathcal{R}_V$-determined.

**Case (ii):** Classification of germs $h$ for which the 1-jet is $y$.

Consider the germ $h(x,y,z) = y$, by geometric criterion, $y$ is not $\mathcal{R}_V$-finite. Then a complete transversal is given by $h = y + a_lx^l$.

**(a):** If $l = 2$, $a_2 \neq 0$ and $a_2 \neq 1$ then $h(x,y,z) = y + a_2x^2$ is 2-$\mathcal{R}_V$-determined.

When $a_2 = 1$, $h = y + x^2$ is not $\mathcal{R}_V$-finite, a complete transversal is $y + x^2 + a_lx^l$.

Let $h = y + x^2 + a_lx^l$ with $a_l \neq 0$, then $T\mathcal{R}_V(h) + \mathcal{M}^{l+2} \supseteq \mathcal{M}^{l+1}$ hence $y + x^2 + a_lx^l$ is $(l+1)$-$\mathcal{R}_V$-determined. It shows that in fact it is $(l)$-$\mathcal{R}_V$-determined for all $a_l \neq 0$.

**(b):** If $l \neq 2$ and $a_l \neq 0$ then $h = y + a_lx^l$ is $l$-$\mathcal{R}_V$-determined.

**Case (iii):** Classification of the germs for which the 1-jet is $z$.

Consider the germ $h(x,y,z) = z$ which is not $\mathcal{R}_V$-finite.

As $J^2(T\mathcal{R}^1_V(h)) + \mathbb{C}\{x^2, xy, y^2\} \supseteq H^2$, all the 2-jets which 1-jet is $z$ are $\mathcal{R}_V$-equivalent to $z + a_2x^2 + bxy + cy^2$. We have

$$J^2(T\mathcal{R}_V(h)) = \left\langle 2a_2x^2 + 3bxy + 4cy^2 + 3z, 2bxz + 4cyz + 3y^2, 6a_2xy + 3by^2, 2a_2xz + byz \right\rangle,$$

therefore $y^2 \in j^2(T\mathcal{R}_V(h))$, $\forall\, a_2,\ b,\ c$ if $a_2 \neq 0$, $xy \in j^2(T\mathcal{R}_V(h))$, $\forall\, b, c$. Hence, by Mather's Lemma, we have:

For $a_2 \neq 0,\ h = z + a_2x^2 + bxy + cy^2 \sim z + a_2x^2$ is $2-\mathcal{R}_V$-determined.
For $a_2 = 0,\ b \neq 0,\ h = z + bxy + cy^2 \sim z + bxy$ and $\mu(h|_{\Sigma_1}) = 2 > 1$.
For $a_2 = b = 0,\ j^2(h) = z$ and $\mu(h|_{\Sigma_1}) \geq 2$.

*Sections of $S_k$.*
We have that orbits in $J^1(3,1)$ of the act of $J^1\mathcal{R}_V$ are: $x, y, z$ and $0$.

**Case (i):** Classification of the germs $h$ for which the 1-jet is $x$.
Let $h(x,y,z) = x$, so $T\mathcal{R}_V(h) \supset \mathcal{M}^1$ and $x$ is 1-$\mathcal{R}_V$-determined.

**Case (ii):** Classification of the germs $h$ for which the 1-jet is $y$.
The generators of the tangent space are: $\eta_1 h = 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = -2(k+1)x^3y$ and $\eta_4 h = 0$ and a $l$-complete transversal is $y + a_l x^l$.
Let $h = y + a_l x^l$, then $\eta_1 h = 2la_lx^l + 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = la_lx^{k+l} + 3la_lx^{l-1}y - 2(k+1)x^ky$ and $\eta_4 h = la_lx^{l-1}z$.

**(a):** If $l = k+1$, $a_{k+1} \neq 0$ and $a_{k+1} \neq 1$ then $h = y + a_{k+1}x^{k+1}$ is $\mathcal{R}_V$-finitely determined.
In this case $T\mathcal{R}_V(h) \supseteq \mathcal{M}^{2k+1}$, if $a_{k+1} \neq 0$ and $a_{k+1} \neq 1$, thus $y + a_{k+1}x^{k+1}$ is $(2k+1)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $F = y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \ldots + a_{2k}x^{2k} + a_{2k+1}x^{2k+1}$, as $x^{2k+1} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \ldots + a_{2k}x^{2k}$.
For $a_{k+1} = 1$, let $h = y + x^{k+1}$, then $\eta_1 h = 2(k+1)x^{k+1} + 2(k+1)y$, $\eta_2 h = 2z$, $\eta_3 h = (k+1)x^{2k+1} + (k+1)x^ky$ and $\eta_4 h = (k+1)x^kz$. Hence by geometric criterion, $y + x^{k+1}$ is not $\mathcal{R}_V$-finite. A $l$-complete transversal for $y + x^{k+1}$ is $y + x^{k+1} + a_lx^l$, $l > (k+1)$.
Let $h = y + x^{k+1} + a_lx^l$ with $a_l \neq 0$. Then $\eta_1 h = 2(k+1)x^{k+1} + 2(k+1)y + 2a_llx^l$, $\eta_2 h = 2z$, $\eta_3 h = (k+1)x^{2k+1} + (k+1)x^ky + la_lx^{k+l} + 3la_lx^{l-1}y$ and $\eta_4 h = (k+1)x^kz + la_lx^{l-1}z$. Hence $T\mathcal{R}_V(h) + \mathcal{M}^{k+l+1} \supseteq \mathcal{M}^{k+l}$, thus $y + x^{k+1} + a_lx^l$ is $(k+l)$-$\mathcal{R}_V$-determined. Successively using complete transversal method we obtain $F = y + x^{k+1} + a_lx^l + \ldots + a_{k+l-1}x^{k+l-1} + a_{k+l}x^{k+l}$, as $x^{k+l} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + x^{k+1} + a_lx^l + \ldots + a_{k+l-1}x^{k+l-1}$.

**(b):** If $l \neq (k+1)$ and $a_l \neq 0$ then $h = y + a_lx^l$ is $\mathcal{R}_V$-finitely determined.
If $l > (k+1)$, $T\mathcal{R}_V(h) + \mathcal{M}^{k+l+1} \supseteq \mathcal{M}^{k+l}$ and $y + a_lx^l$ is $(k+l)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $F = y + a_lx^l + \ldots + a_{k+l-1}x^{k+l-1} + a_{k+l}x^{k+l}$, as $x^{k+l} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + x^{k+1} + a_lx^l + \ldots + a_{k+l-1}x^{k+l-1}$.
If $l < (k+1)$ then $T\mathcal{R}_V(h) + \mathcal{M}^{2l} \supseteq \mathcal{M}^{2l-1}$ and $y + a_lx^l$ is $(2l-1)$-$\mathcal{R}_V$-determined. Successively using the complete transversal method we obtain $y + a_lx^l + \ldots + a_{2l-2}x^{2l-2} + a_{2l-1}x^{2l-1}$, as $x^{2l-1} \in T\mathcal{R}_V(F)$, then by Mather's Lemma $F \sim y + a_lx^l + \ldots + a_{2l-2}x^{2l-2}$.

**Case (iii):** Classification of the germs $h$ for which the 1-jet is $z$.
By geometric criterion, $h = z$ is not $\mathcal{R}_V$-finite. A 2-complete transversal for $z$ is $h = z + a_2x^2 + bxy + cy^2$.

$$J^2(T\mathcal{R}_V) = \begin{array}{l}\langle 4a_2x^2 + (4+2k)bxy + 4(k+1)cy^2 + 3(k+1)z, \\ 2bxz + 4cyz + 3y^2, 6a_2xy + 3by^2, 2a_2xz + byz\rangle .\end{array}$$

We have $j^2x\eta_1 h = 3(k+1)xz$, $j^2y\eta_1 h = 3(k+1)yz$, $j^2\eta_2 h = 2bxz +$

$4cyz + 3y^2$ e $j^2\eta_3 h = 6a_2xy + 3by^2$. Then $a_2xy, y^2 \in j^2(T\mathcal{R}_V(h))$ and by Mather's Lemma:

For $a_2 \neq 0,\ h = z + a_2x^2 + bxy + cy^2 \sim z + a_2x^2$.

For $a_2 = 0,\ b \neq 0,\ h = z + bxy + cy^2 \sim z + bxy$.

For $a_2 = b = 0,\ h = z$.

If $a_2 \neq 0$, $h = z + a_2x^2$ is 2-$\mathcal{R}_V$-determined.

If $a_2 = 0$ and $b \neq 0$ taking $h = z + bxy$. We have $\mu(h|_{\Sigma_k}) = k+1 > k$.

If $b = 0$, let $j^{l-1}h = z$, $l \geq 3$, a $l$-complete transversal for $z$ is $z + a_lx^l + b_{l-1}x^{l-1}y$ if $3 \leq l \leq (2k+2)$ and $z + a_lx^l$ if $l \geq (2k+3)$.

Then for $h = z + a_lx^l + b_{l-1}x^{l-1}y$, $\eta_1 h = 2la_lx^l + 2(l+k)b_{l-1}x^{l-1}y + 3(k+1)z$, $\eta_2 h = 2b_{l-1}x^{l-1}z + x^{2k+2} + 4x^{k+1}y + 3y^2$, $\eta_3 h = la_lx^{l+k} + 3la_lx^{l-1}y + (l-2k-3)b_{l-1}x^{l+k-1}y + (3l-3)b_{l-1}x^{l-2}y^2$ and $\eta_4 h = la_lx^{l-1}z + (l-1)b_{l-1}x^{l-2}yz + (k+1)x^{2k+1}y + (k+1)x^ky^2$

thus, for $a_l \neq 0$, $x^{l-1}y \in T\mathcal{R}_V(h)$ and by Mather's Lemma $h \sim z + a_lx^l$.

Let $h = z + a_lx^l$, with $a_l \neq 0$. If $l > k+1$, $\mu(h|_{\Sigma_k} \geq k+1 > k$, then we do not need to classify. If $l \leq k+1$, $h$ is $(2l-1)$-$\mathcal{R}_V$-determined and $x^{2l-1} \in T\mathcal{R}_V(\tilde{h})$ with $\tilde{h} = z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2}$, then $h \sim z + a_lx^l + a_{l+1}x^{l+1} + \cdots + a_{2l-2}x^{2l-2}$.

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