Spectral packing dimensions through power-law subordinacy

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Abstract. We offer a method of classification of spectral measures of discrete one-dimensional Schrödinger operators with respect to packing measures, which can be seen as dual to results for Hausdorff measures in subordinacy theory. We apply this method to classes of sparse operators, and give an example whose spectral measure has different Hausdorff and packing dimensions, and others for which such dimensions coincide. Some dynamical motivations are also mentioned.

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1. Introduction

Results on the Hausdorff and packing dimensions of the spectral measures of Schrödinger operators (in separable Hilbert spaces) are as interesting as difficult to obtain. The question becomes even more compelling if we remember that such dimensions are directly related to dynamical exponents of the moments of the position observable; let us recall such relations. The action

$$(H_\phi\psi)_n = a_n\psi_{n+1} + a_{n-1}\psi_{n-1} + b_n\psi_n \tag{1}$$

represents the discrete ("tight-binding") Schrödinger operator defined on $l^2(\mathbb{Z}_+^0, \mathbb{C})$ ($\mathbb{Z}_+^0 = \{n \in \mathbb{Z} : n \geq 0\}$), along with a phase boundary condition

$$\psi_{-1}\cos\phi - \psi_0\sin\phi = 0\ , \tag{2}$$

$\phi \in [0, \pi)$; (a_n) and (b_n) are sequences of real numbers, with $a_n \neq 0$ for all $n \in \mathbb{Z}_+^0$, and such that $\sum_{n=0}^{\infty} |a_n|^{-1} = \infty$ (which is sufficient to ensure that this operator is essentially self-adjoint; see [24] for a proof). The sequence (b_n) represents the *potential*.

Packing (or Tricot) measures were initially proposed by Tricot [25] and act in many aspects as dual to Hausdorff measures (see [8] for a vast discussion and precise results). Despite the growing interest in geometric measure theory of packing measures and correlated areas [18], our focus is on some of its implications in spectral theory.

Guarneri and Schulz-Baldes [12] have related the packing dimension of spectral measures of Schrödinger operators to their transport properties (more precisely, to the upper growth exponents of moments). This analysis is particularly important to operators with non-exactly scaling spectral measures, i.e., according to our nomenclature, with spectral measures whose Hausdorff and packing dimensions differ (see Section 2).

Let δ_j be the vector that takes 1 at $n = j$ and zero otherwise. For $p > 0$, the *upper* $\beta^+(p)$ *and lower* $\beta^-(p)$ *dynamical exponents* are defined as

$$\beta^+(p) := \limsup_{T\to\infty} \frac{\ln\langle X^p\rangle(T)}{p\ln T}, \qquad \beta^-(p) := \liminf_{T\to\infty} \frac{\ln\langle X^p\rangle(T)}{p\ln T}\ , \tag{3}$$

where $\langle X^p\rangle(T) := (2/T)\int_0^\infty \sum_n |n|^p e^{-2t/T}|\langle e^{-itH}\delta_0, \delta_n\rangle|^2 dt$ represents the average moment of order p associated with the initial state δ_0. The dynamics is called *ballistic* if $\beta^-(p) = 1$ for all $p > 0$, and *quasi-ballistic* if $\beta^+(p) = 1$ for all $p > 0$. If τ is the spectral measure of the Schrödinger operator (1), denote its packing and Hausdorff dimensions by $\dim_P(\tau)$ and $\dim_H(\tau)$, respectively (see Definition 7); the important relations we recall are given by the general inequalities [12]

$$\beta^-(p) \geq \dim_H(\tau), \qquad \beta^+(p) \geq \dim_P(\tau). \tag{4}$$

Hence, if one obtains $\dim_P(\tau) = 1$, then the quasi-ballistic dynamics follows at once.

First of all, we note that any countable set has zero packing and Hausdorff dimensions, and so the dimensions of any purely point spectral measures are zero; see the relevant definitions in Section 2.

Among the significant current questions related to such spectral dimensions and dynamics of Schrödinger operators, we mention:

(a) Is it possible to exhibit an example of Schrödinger operator whose Hausdorff and packing dimensions of its spectral measure differ from each other?
(b) By contrast with (a), is there some nontrivial Schrödinger operator for which the Hausdorff dimension of its spectral measure is equal to its packing dimension?

In this work we will have something to say about these questions. With respect to item (a), as far as we know, there is only one concrete example [6] of such operators published in the literature, and it is with some sparse potentials; more precisely, given $\alpha \in (0, 1)$, its spectral measure is one-packing dimensional and has Hausdorff dimension α. We offer in Section 4 a different proof of this result.

The point raised in item (b) will also be exemplified with sparse operators, in Section 4, such that both Hausdorff and packing dimensions of their spectral measures are equal to α; for the best of our knowledge, these are the first explicit examples of such kind of operators.

Such set of results is obtained through a practical way to calculate these dimensions. This will be the starting point of our discussion and, as another contribution, we will describe how to use subordinacy theory to obtain information on packing dimensions of spectral measures; this technique works when it is possible to control the asymptotic behavior of the solutions to the Schrödinger eigenvalue equation

$$(H_\phi \psi)_n = E\psi_n \,. \tag{5}$$

We, therefore, recall the necessary material from subordinacy theory.

By generalizing results by Gilbert and Pearson [11] (see [14] for and adaptation of [11] to discrete operators), Jitomirskaya and Last proposed [13] a way to classify the spectrum of discrete one-dimensional Schrödinger operators with respect to Hausdorff measures and dimensions. The key aspect was to relate the celebrated results obtained by Rogers and Taylor in [20], and refined in [21, 22], to partial information of the asymptotic behavior of solutions to (5).

The result in [13] that connects properties such as Hausdorff singularity and continuity of the spectral measure of H_ϕ with the local scaling behavior of the (generalized) eigenfunctions of (5) is the following:

Theorem 1 (Theorem 1.2 in [13]**).** *Let H_ϕ be as in* (1) *and τ denote the spectral measure of H_ϕ associated with the cyclic vector δ_0. For $E \in \mathbb{R}$ and $\alpha \in (0,1)$, $\beta = \alpha/(2-\alpha)$,*

$$(\overline{D}^\alpha \tau)(E) := \limsup_{\varepsilon \to 0} \frac{\tau((E-\varepsilon, E+\varepsilon))}{(2\varepsilon)^\alpha} = \infty \tag{6}$$

holds if, and only if,

$$\liminf_{l\to\infty} \frac{\|u_1\|_l}{\|u_2\|_l^\beta} = 0 \,.$$

In Theorem 1, $\|\cdot\|_l$ denotes the $l^2(\mathbb{Z}_+^0, \mathbb{C})$ norm truncated at $l \in \mathbb{R}$ ($[l]$ is the integral part of l), that is,

$$\|u\|_l^2 := \sum_{n=0}^{[l]} |u_n|^2 + (l - [l]) \left|u_{[l]+1}\right|^2 \;;$$

u_1 and u_2 are the solutions to (5) which satisfy the orthogonal initial conditions

$$\begin{array}{ll} u_{1,-1} = \sin\phi\,, & u_{1,0} = \cos\phi\,, \\ u_{2,-1} = \cos\phi\,, & u_{2,0} = -\sin\phi \end{array} \;; \tag{7}$$

$(\overline{D}^\alpha \tau)(E)$ is known as the *upper α-derivative* of τ at E.

Theorem 1 is a direct consequence of the following

Theorem 2 (Theorem 1.1 in [13]**).** *Let H_ϕ be as in* (1) *and let $E \in \mathbb{R}$, $\varepsilon > 0$ be given. Then,*

$$\frac{5-\sqrt{24}}{|m(E+i\varepsilon)|} < \frac{\|u_1\|_{l(\varepsilon)}}{\|u_2\|_{l(\varepsilon)}} < \frac{5+\sqrt{24}}{|m(E+i\varepsilon)|} \,. \tag{8}$$

The function $m(z)$ is the so-called Weyl-Titchmarsh coefficient (see [5] for details), which coincides with the Borel-Stieltjes transform of the spectral measure τ:

$$m(z) = \int_{-\infty}^{\infty} \frac{d\tau(E)}{E - z} ;$$

given $\varepsilon > 0$, the length $l(\varepsilon) \in (0, \infty)$ is defined by the equality

$$\|u_1\|_{l(\varepsilon)} \|u_2\|_{l(\varepsilon)} = \frac{1}{2\varepsilon} , \tag{9}$$

where u_1 and u_2 are the solutions to (5) that satisfy the initial conditions (7). We observe that since we are in the limit-point case, there is just one (normalized) solution in $l^2(\mathbb{Z}_+^0; \mathbb{C})$.

Namely, Theorems 1 and 2 are the results that allow a generalization of the subordinacy theory [11, 14] to Hausdorff measures. A solution ψ to (5) is called α-Hausdorff subordinate if

$$\liminf_{l \to \infty} \frac{\|\psi\|_l}{\|\Phi\|_l^{\beta}} = 0$$

holds for any linearly independent solution Φ to (5), with $\beta = \alpha/(2 - \alpha)$. In particular, the α-Hausdorff continuous part of τ (see Definition 5) is supported on the set of energies E for which (5) does not have α-subordinate solutions (this is the set of energies, up to a set of null α-Hausdorff and τ measures, where τ has a finite upper α-derivative). The α-Hausdorff singular part of τ (see Definition 5) is supported on the set of energies for which the solutions that obey the boundary condition (2) are α-subordinate.

Analyzing the previous results, it is natural to ask if there is a dual to Theorem 1, i.e., if there is a connection between the asymptotic behavior of the eigenfunctions u_1, u_2, and the so-called *lower α-derivative* of τ at E,

$$(\underline{D}^{\alpha}\tau)(E) := \liminf_{\varepsilon \to 0} \frac{\tau((E - \varepsilon, E + \varepsilon))}{(2\varepsilon)^{\alpha}} . \tag{10}$$

In point of fact, our first goal in this work is to give a precise answer to this question in Theorem 14 ahead. There, the *packing* measures will play a role instead of Hausdorff measures. This result, as we will see in Section 3, allows an extension of the subordinacy theory to packing measures.

The paper is organized as follows. In Section 2, we recall a series of definitions and results regarding Hausdorff and packing measures and dimensions, with few contributions. In Section 3, we extend the analysis developed in [13] to the packing-dimensional setting. Finally, in Section 4 we present direct applications of the results developed in Section 3 to some classes of sparse operators introduced in [13, 27].

2. Hausdorff and packing measures and dimensions

We present in this section some definitions and concepts regarding Hausdorff and packing measures. Most of the material exposed here is based on [10, 12, 15]. Let μ be a Borel measure on $\mathbb{R}$; recall that a Borel set $S \subseteq \mathbb{R}$ has σ-finite measure μ if $S = \cup_{j=1}^{\infty} S_j$ and $\mu(S_j) < \infty$ for each j, and that μ is supported on a Borel set S (or that S supports μ) if $\mu(\mathbb{R} \setminus S) = 0$.

Definition 3. Given a set $S \subset \mathbb{R}$ and $\alpha \in [0,1]$, consider the number

$$Q_{\alpha,\delta}(S) = \inf \left\{ \sum_{k=1}^{\infty} |b_k|^{\alpha} : |b_k| < \delta, \, \forall k; \; S \subset \bigcup_{k=1}^{\infty} b_k \right\} , \tag{11}$$

with the infimum taken over all covers by intervals b_k of size at most δ. The limit

$$h^{\alpha}(S) = \lim_{\delta \downarrow 0} Q_{\alpha,\delta}(S) , \tag{12}$$

is called the α-*dimensional Hausdorff measure* of S.

The counting measure (which assigns to each set S the number of points in it), at $\alpha = 0$, and the Lebesgue measure, at $\alpha = 1$, are particular cases. $h^{\alpha}(S)$ is an outer measure on subsets of $\mathbb{R}$ (see [19]), and for $\beta < \alpha < \gamma$,

$$\delta^{\alpha-\gamma} Q_{\gamma,\delta}(S) \le Q_{\alpha,\delta}(S) \le \delta^{\alpha-\beta} Q_{\beta,\delta}(S)$$

holds for any $\delta > 0$ and $S \subset \mathbb{R}$. So, if $h^{\alpha}(S) < \infty$, then $h^{\gamma}(S) = 0$ for $\gamma > \alpha$; if $h^{\alpha}(S) > 0$, then $h^{\beta}(S) = \infty$ for $\beta < \alpha$. Hence, for every set S, there is a unique α_S such that $h^{\alpha}(S) = 0$ if $\alpha > \alpha_S$ and $h^{\alpha}(S) = \infty$ if $\alpha_S < \alpha$. The number α_S is called the *Hausdorff dimension* of the set S, usually denoted by $\dim_H(S)$.

Let us recall the definition of packing measure. A δ-packing of an arbitrary set $S \subset \mathbb{R}$ is a countable disjoint collection $(B(x_k, r_k))_{k \in \mathbb{N}}$ of closed intervals centered at $x_k \in S$ with radius $r_k \le \delta/2$. The (α, δ)-premeasure $P_{\delta}^{\alpha}(S)$ is defined by

$$P_{\delta}^{\alpha}(S) = \sup \left\{ \sum_{k=1}^{\infty} (2r_k)^{\alpha} : (B(x_k, r_k))_k \text{ is a } \delta\text{-packing of } S \right\} ,$$

the supremum taken over all δ-packings of S.

Definition 4. The α-*packing measure* $P^{\alpha}(S)$ of S is constructed by a procedure in two steps: first, take the decreasing limit

$$\overline{P}^{\alpha}(S) = \lim_{\delta \to 0} P_{\delta}^{\alpha}(S) ,$$

and then

$$P^{\alpha}(S) = \inf \left\{ \sum_{k=1}^{\infty} \overline{P}^{\alpha}(S_k) : S \subset \bigcup_{k=1}^{\infty} S_k, \; S_k \text{ disjoint Borel sets} \right\} .$$

It follows, by Definition 4, that $P^{\alpha}(S)$ is an outer measure on $\mathbb{R}$. The so-called *packing dimension* of the set S, denoted by $\dim_P(S)$, is defined as the infimum of all α such that $P^{\alpha}(S) = 0$, which coincides with the supremum of all α with $P^{\alpha}(S) = \infty$.

It is possible to show (see [10]) that the Hausdorff and packing dimensions are related by the inequality $\dim_H(S) \le \dim_P(S)$. In general, this inequality is strict; we will see some examples ahead. When the Hausdorff and packing dimensions of S coincide, one says that S has *exact dimension* $\dim(S)$; hence, in this case, $\dim(S) = \dim_H(S) = \dim_P(S)$.

The above notions of Hausdorff (packing) measures and dimensions lead to a series of notions of continuity and singularity of Borel measures with respect to them (see [15] for details).

Definition 5. Let μ be a Borel measure on $\mathbb{R}$ and $\alpha \in [0,1]$.

1. μ is called α-Hausdorff (α-packing) continuous if $\mu(S) = 0$ for every Borel set S with $h^\alpha(S) = 0$ (resp. $P^\alpha(S) = 0$).
2. μ is called α-Hausdorff (α-packing) singular if it is supported on some Borel set S with $h^\alpha(S) = 0$ (resp. $P^\alpha(S) = 0$).

Remark 6. Definition 5 is only a partial characterization of continuity and singularity with respect to Hausdorff (packing) measures. This concept was introduced, regarding Hausdorff measures, in [20] for any continuous additive set function F in Euclidean space (see Section 3 of Chapter 3 of [19] for an application to monotonic continuous functions defined on the interval $[0,1]$). It was Last [15] who established the connection between Rogers and Taylor's work and the spectral theory of Schrödinger operators. The definition is somewhat new for packing measures and produces results equivalent to those obtained by Cutler in [7].

Definition 7. A Borel measure μ on $\mathbb{R}$ is said to have *exact Hausdorff (packing)* dimension α, for some $\alpha \in (0,1)$ and denoted by $\dim_H(\mu)$ (resp. $\dim_P(\mu)$), if two requirements hold:

1. For every set S with $\dim_H(S) < \alpha$ (resp. $\dim_P(S) < \alpha$), one has $\mu(S) = 0$.
2. There is a Borel set S_0 of Hausdorff (resp. packing) dimension α which supports μ.

Definition 8. A Borel measure μ on $\mathbb{R}$ is called:

1. *zero-Hausdorff (packing) dimensional* if it is supported on a set with $\dim_H(S) = 0$ (resp. $\dim_P(S) = 0$).
2. *one-Hausdorff (packing) dimensional* if $\mu(S) = 0$ for any set S with $\dim_H(S) < 1$ (resp. $\dim_P(S) < 1$).

Remark 9. According to Definition 7, a Borel measure μ on $\mathbb{R}$ is of exact Hausdorff (packing) dimension α if, for every $\varepsilon > 0$, it is simultaneously $(\alpha - \varepsilon)$-Hausdorff (resp. packing) continuous and $(\alpha + \varepsilon)$-Hausdorff (resp. packing) singular.

By combining the above definitions, a Borel measure μ on $\mathbb{R}$ has exact dimension α if: (1) for every $0 \le \beta < \alpha$ and every Borel set S with $\dim(S) = \beta$, one has $\mu(S) = 0$, and (2) there is a Borel set S_0 of exact dimension α which supports μ.

Remark 10. It turns out that μ has exact dimension α if it is simultaneously $(\alpha - \varepsilon)$-Hausdorff continuous (which implies $(\alpha - \varepsilon)$-packing continuity; see

Remark 18) and $(\alpha+\varepsilon)$-packing singular (which implies $(\alpha+\varepsilon)$-Hausdorff singularity; see Remark 22), for all $\varepsilon > 0$.

The following result, motivated by [19] and Theorem 2 in [12], relates the continuity with respect to Hausdorff (packing) measures with the local scaling behavior of the measure.

Theorem 11. *Let μ be a Borel measure on $\mathbb{R}$ and denote*

$$T_\infty^\alpha := \{E : (\overline{D}^\alpha\mu)(E) = \infty\}, \qquad U_\infty^\alpha := \{E : (\underline{D}^\alpha\mu)(E) = \infty\},$$

where $(\overline{D}^\alpha\mu)(E)$ and $(\underline{D}^\alpha\mu)(E)$ are, respectively, defined in (6), (10). *Then T_∞^α and U_∞^α are Borel sets, and*

1. $h^\alpha(T_\infty^\alpha) = 0$.
2. $P^\alpha(U_\infty^\alpha) = 0$.
3. $\mu(S \cap (\mathbb{R}\setminus T_\infty^\alpha)) = 0$ *for any S with $h^\alpha(S) = 0$.*
4. $\mu(S \cap (\mathbb{R}\setminus U_\infty^\alpha)) = 0$ *for any S with $P^\alpha(S) = 0$.*

Proof. We only prove the items regarding packing measures; items *(1)* and *(3)* are well known and proven in Chapter 3 of [19] (Section 3).

(2) Let $U_n(\delta) = \{E \in S : \inf_{\varepsilon<\delta}\mu([E-\varepsilon, E+\varepsilon])\,\varepsilon^{-\alpha} \geq n\}$, where S represents a Borel subset of $\mathbb{R}$. Then, for any $E \in U_n(\delta)$, $\mu([E-\varepsilon, E+\varepsilon]) \geq n\varepsilon^\alpha$ for every $\varepsilon < \delta$. This shows that $U_n(\delta)$ is open relative to S, being therefore a Borel set.

Hence, if $(B(E_k, r_k))_k$ is a δ-packing of $U_n(\delta)$, we have

$$\sum_{k=1}^\infty (2r_k)^\alpha \leq 2^\alpha \sum_{k=1}^\infty \mu([E_k - r_k, E_k + r_k])/n \leq 2^\alpha\mu(S)/n\ ,$$

since the elements of the δ-packing are pairwise disjoint. Thus, $P^\alpha(U_n(\delta)) \leq P_\delta^\alpha(U_n(\delta)) \leq 2^\alpha\mu(S)/n$.

Since $\bigcup_{i=1}^\infty U_n(1/i) = \{E \in S : (\underline{D}^\alpha\mu)(E) \geq n\}$, we obtain from the σ-additivity of P^α that

$$P^\alpha\left(\{E \in S : (\underline{D}^\alpha\mu)(E) \geq n\}\right) \leq \sup_i P^\alpha\left(U_n(1/i)\right) \leq 2^\alpha\mu(S)/n. \qquad (13)$$

To observe that U_∞^α is a Borel set, it is enough to note that $U_\infty^\alpha \bigcap S = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty U_n(1/i)$ is a Borel set. Also, by (13), we have $P^\alpha(U_\infty^\alpha \bigcap S) \leq P^\alpha(\{E \in S : \underline{D}^\alpha(E) \geq n\}) \leq 2^\alpha\mu(S)/n$, for each positive integer n; thus $P^\alpha(U_\infty^\alpha \bigcap S) = 0$. This concludes the proof of *(2)*, since S is an arbitrary Borel subset of $\mathbb{R}$.

(4) To prove this item, we must use the fact that any Borel measure μ on $\mathbb{R}$ possesses the centered Vitali covering property: every centered Vitali covering of S contains a countable set of disjoint balls B_k such that $\mu\left(S \setminus \bigcup_{k=1}^\infty B_k\right) = 0$ (a centered Vitali covering of S is a set of closed balls containing, for any $E \in S$ and $\delta > 0$, some closed ball $B(E, r)$ with $r \leq \delta$).

Let then $L_m(n) \subset L(n) = \{E \in S : (\underline{D}^\alpha\mu)(E) < n\}$. For any $E \in L_m(n)$ and $\delta > 0$, there exists $r \leq \delta$ such that $\mu([E-r, E+r]) \leq nr^\alpha$. Thus, the set of closed balls $B(E, r) = \{E \in L_m(n) : \mu([E-r, E+r]) \leq$

$nr^\alpha, r \leq \delta\}$ is a centered Vitali covering of $L_m(n)$. Let $(B(E_k, r_k))_k$ be the associated δ-packing given by the Vitali covering property; therefore, $\mu\left(L_m(n) \setminus \bigcup_{k=1}^\infty B(E_k, r_k)\right) = 0$. Then, $\mu(L_m(n)) \leq \sum_{k=1}^\infty \mu(B(E_k, r_k)) \leq n\sum_{k=1}^\infty r_k^\alpha$. As this holds for any $\delta > 0$ and the decomposition $L(n) = \bigcup_{m=1}^\infty L_m(n)$ can be chosen disjoint, we have $\mu(L_m(n)) \leq 2^{-\alpha} n \overline{P}^\alpha(L_m(n))$ and consequently

$$\mu(L(n)) \leq 2^{-\alpha} n P^\alpha(L(n)) \,. \tag{14}$$

It is a direct consequence of (14) and $\bigcup_{n=1}^\infty L(n) = S \cap (\mathbb{R} \setminus U_\infty^\alpha)$ that $\mu(S \cap (\mathbb{R} \setminus U_\infty^\alpha)) = 0$ for any S with $P^\alpha(S) = 0$; this concludes the proof. □

Remark 12.
1. Theorem 11 establishes that it is sufficient to know $(\overline{D}^\alpha \mu)(E)$ (resp. $(\underline{D}^\alpha \mu)(E)$), for every $\alpha \in [0,1]$ and a.e. E w.r.t. μ, to completely determine the μ decomposition (in the sense of Definition 5) with respect to Hausdorff (resp. packing) measures.
2. It is worth emphasizing again that the results regarding packing measures are essentially due to Cutler [7]. In fact, Cutler demonstrated that the knowledge of the local upper dimension
$$\sigma^+(E) := \limsup_{\varepsilon \to 0} \frac{\ln[\mu((E-\varepsilon, E+\varepsilon))]}{\ln \varepsilon} \,, \tag{15}$$
for a.e. E w.r.t. μ, is sufficient to determine the μ decomposition with respect to packing dimensions. In particular, μ is of exact packing dimension α if, and only if, it is supported on a set of exact packing dimension α (see [7]), which, according to Theorem 11 and Definition 7, implies that $\sigma^+(E) = \alpha$ almost everywhere with respect to it. We have opted to present our results with respect to Hausdorff (packing) measures due to the local scaling behavior of the eigenfunctions of (5) (see Theorems 1 and 14).

Let us decompose μ according to the sets T_∞^α and U_∞^α, i.e., as

$$\mu = \chi_{T_\infty^\alpha}\mu + (1 - \chi_{T_\infty^\alpha})\mu, \quad \text{and} \quad \mu = \chi_{U_\infty^\alpha}\mu + (1 - \chi_{U_\infty^\alpha})\mu,$$

respectively, where χ_A denotes the characteristic function of the set A. Theorem 11 implies the following

Corollary 13. *Let μ be a finite Borel measure on $\mathbb{R}$. Then, for every $\alpha \in [0,1]$, μ has the unique decompositions*

$$\mu = \mu_{\alpha Hs} + \mu_{\alpha Hc} \,, \qquad \mu = \mu_{\alpha Ps} + \mu_{\alpha Pc} \,,$$

with respect to the sets T_∞^α and U_∞^α, respectively, where $\mu_{\alpha Hs} := \mu(T_\infty^\alpha \cap \cdot)$ is α-Hausdorff singular, $\mu_{\alpha Ps} := \mu(U_\infty^\alpha \cap \cdot)$ is α-packing singular, and $\mu_{\alpha Hc} := \mu((\mathbb{R} \setminus T_\infty^\alpha) \cap \cdot)$ is α-Hausdorff continuous, $\mu_{\alpha Pc} := \mu((\mathbb{R} \setminus U_\infty^\alpha) \cap \cdot)$ is α-packing continuous.

Thus, the standard spectral decomposition with respect to Lebesgue measure can be extended to include decompositions with respect to packing measures in the exact way obtained previously in [20] for Hausdorff measures.

We could, in fact, refine our results by decomposing μ with respect to the sets U_∞^α, $U_+^\alpha := \{E \in \mathbb{R} : 0 < (\underline{D}^\alpha\mu)(E) < \infty\}$ and $U_0^\alpha := \{E \in \mathbb{R} : (\underline{D}^\alpha\mu)(E) = 0\}$, similarly to [20].

We underline that Rogers and Taylor presented in [21] an example of additive set function F, defined on the interval $[0, 1]$, whose spectrum does not have exact dimension. Namely, F was used by Rogers and Taylor to show the impossibility of the use of the lower α-derivative to decompose set functions (which Borel measures are a subclass) in α-Hausdorff singular and α-Hausdorff continuous parts. What was shown is that F, despite being 1/2-Hausdorff singular (i.e., F is supported on a set of Hausdorff dimension less than 1/2), it has lower derivative

$$(\underline{D}^{1/2}F)(x) = \liminf_{\varepsilon\to 0} \frac{F((x-\varepsilon, x+\varepsilon))}{(2\varepsilon)^{1/2}} = 0$$

for every point x in the spectrum of F. This, according to Theorem 11, shows that F is 1/2-packing continuous (i.e., does not give weight to sets of packing dimension less than 1/2).

3. Packing decomposition of spectral measures

We obtain in this section the results which permit the decomposition of spectral measures of Schrödinger operators (1) with respect to packing measures. The material exposed is in numerous aspects dual to [13]. The next result is central to our analysis.

Theorem 14. *Let H_ϕ be the operator* (1), *τ its spectral measure associated with the cyclic vector δ_0, $E \in \mathbb{R}$, $\alpha \in (0, 1)$ and $\beta = \alpha/(2-\alpha)$. Then $E \in U_\infty^\alpha$, that is, $(\underline{D}^\alpha\tau)(E) = \infty$, if, and only if,*

$$\limsup_{l\to\infty} \frac{\|u_1\|_l}{\|u_2\|_l^\beta} = 0\ .$$

Proof. For any two functions $f(\varepsilon)$ and $g(\varepsilon)$, write $f \sim g$ if there exist positive constants C_1, C_2 such that $C_1 f < g < C_2 f$ for every $\varepsilon > 0$. From (8) and (9) one gets

$$\varepsilon^{1-\alpha}|m(E+i\varepsilon)| \sim \|u_2\|_{l(\varepsilon)}^{\alpha-1}\|u_1\|_{l(\varepsilon)}^{\alpha-1}\frac{\|u_2\|_{l(\varepsilon)}}{\|u_1\|_{l(\varepsilon)}} = \left(\frac{\|u_2\|_{l(\varepsilon)}^\beta}{\|u_1\|_{l(\varepsilon)}}\right)^{2-\alpha},$$

and so

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{1-\alpha}|m(E+i\varepsilon)| = \infty \qquad \Longleftrightarrow \qquad \limsup_{l\to\infty} \frac{\|u_1\|_{l(\varepsilon)}}{\|u_2\|_{l(\varepsilon)}^\beta} = 0\ .$$

In order to proceed, we need a dual to Theorem 3.1 in [9]; see also [26], which contains an insight of such result.

Theorem 15. *Given the Borel measure μ, fix $E_0 \in \mathbb{R}$ and $\alpha \in [0,1)$. Thus, the quantities $(\underline{D}^\alpha\mu)(E_0)$ and $\underline{R}_\mu^{(1-\alpha)}(E_0)$ are either all infinite or all in $[0,\infty)$, where we define*

$$\underline{R}_\mu^{(1-\alpha)}(E_0) = \liminf_{\varepsilon\downarrow 0} \varepsilon^{(1-\alpha)}\left|\int \frac{d\mu(y)}{y - E_0 - i\varepsilon}\right| .$$

Proof. The proof of Theorem 15 traces the same steps of the proof of Theorem 3.1 in [9], where we must replace limit superiors with limit inferiors in the estimates. □

By Theorem 15,

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{1-\alpha}|m(E+i\varepsilon)| = \infty \qquad \Longleftrightarrow \qquad (\underline{D}^\alpha\tau)(E) = \infty ,$$

and this concludes the proof of Theorem 14. □

Remark 16.
1. Theorem 14 is the dual to Theorem 1 and, as we will see, produces the decomposition of the spectral measure with respect to packing measures (instead of Hausdorff measures). In fact, what Theorem 14 shows is that the α-packing singular part of τ is supported on the set of energies where u_1 is an *α-packing subordinate* solution to (5), in the sense that $\limsup_{l\to\infty} \|u_1\|_l/\|u_2\|_l^\beta = 0$.
2. It can be proven that the bounds stated in Lemmas 3.2 and 3.3 in [9] are also valid for the α-derivative $(D^\alpha\mu)(E)$.

In the sequel, we need two basic known facts. The first establishes the following bound for the solution u_1:

$$\liminf_{l\to\infty} \frac{\|u_1\|_l}{l^{1/2}\ln l} < \infty \tag{16}$$

(inequality (16) is a direct consequence of Theorem 3.10 in [16]; this theorem provides, in fact, the bound $\lim_{l\to\infty} \frac{\|u_1\|_l}{l^{1/2}\ln l} < \infty$).

The second is the constancy of the Wronskian (recall that the Wronskian of two functions $\varphi,\psi : \mathbb{Z}_+^0 \to \mathbb{C}$ is given by $W[\varphi,\psi](n) = a_n(\varphi_n\overline{\psi}_{n+1} - \varphi_{n+1}\overline{\psi}_n)$). It follows by Green's identity (see [24]) that

$$\sum_{n=0}^{N}\left(\overline{\psi}_n(H_\phi\varphi)_n - \overline{(H_\phi\psi)}_n\phi_n)\right) = W[\varphi,\psi](N) - W[\varphi,\psi](-1) = 0,$$

i.e., the Wronskian of the solutions $\{\varphi,\psi\}$ to (5) is constant; in particular,

$$W[u_1,\overline{u_2}](n) = a_n\left(u_{1,n+1}u_{2,n} - u_{2,n+1}u_{1,n}\right) = 1$$

for all $n \in \mathbb{Z}_+^0$, which results in

$$\|u_1(E)\|_l\|u_2(E)\|_l \geq \frac{1}{2\sqrt{2}}\left(\sum_{n=0}^{[l]-1}|a_n|^{-1} + (l-[l])\left|a_{[l]}\right|^{-1}\right) . \tag{17}$$

We are now ready to present important consequences of Theorem 14 to packing measures and dimension, which connect the properties of spectral measures with the scaling behavior of the eigenfunctions of (5). They are counterparts of some of the results in [13] on Hausdorff measures. Recall that, for $\alpha \in [0, 1]$ one has $\beta = \alpha/(2 - \alpha)$.

Corollary 17. *Define, for every $l > 0$,*

$$R_l := \frac{1}{2\sqrt{2}} \left(\sum_{n=0}^{[l]-1} |a_n|^{-1} + (l - [l]) \left| a_{[l]} \right|^{-1} \right) .$$

Suppose that, for some $\alpha \in (0, 1]$ and every E in some Borel subset A of $\mathbb{R}$, every solution Φ to (5) satisfies

$$\liminf_{l \to \infty} \frac{\|\Phi\|_l^2}{R_l^{2-\alpha}} < \infty . \tag{18}$$

Then, the restriction $\tau(A \cap \cdot)$ is α-packing continuous.

Remark 18. Corollary 17 shows that α-packing continuity is less restrictive than α-Hausdorff continuity, since the existence of just one bounded subsequence of $\|\Phi\|_l^2 R_l^{\alpha-2}$ is sufficient to assure α-packing continuity, while we must, according to Corollary 4.4 in [13], demand the boundedness of the whole sequence $\|\Phi\|_l^2 R_l^{\alpha-2}$ to assure α-Hausdorff continuity. This result is consistent with the fact that $(\mathbb{R} \setminus T_\infty^\alpha) \subseteq (\mathbb{R} \setminus U_\infty^\alpha)$.

Proof. Let $E \in A$. We obtain from (17) the inequality $\|u_1\|_l \|u_2\|_l \geq R_l$, and since there exists, by hypothesis, a subsequence $(l_j)_j$ converging to infinity such that $\|u_2\|_{l_j}^2 < C R_{l_j}^{2-\alpha}$ (C is a constant), we have $\|u_1\|_{l_j} > C^{-1/2} R_{l_j}^{\alpha/2}$. Thus,

$$\frac{\|u_1\|_{l_j}}{\|u_2\|_{l_j}^{\beta}} > C^{-(1+\beta)/2} R_{l_j}^{\alpha/2 - \beta(2-\alpha)/2} = C^{-(1+\beta)/2} > 0 .$$

It follows that $\limsup_{l \to \infty} \|u_1\|_l \|u_2\|_l^{-\beta} > 0$, and $\tau(A \cap \cdot)$ is α-packing continuous by Theorem 14. □

Keeping in mind applications of the previous results to one-dimensional spectral theory, it is then convenient to rewrite Corollary 17 in terms of the transfer matrices $T_n(E) = A_n(E) A_{n-1}(E) \cdots A_0(E)$, with

$$A_n(E) = \begin{pmatrix} \dfrac{E - b_n}{a_n} & -\dfrac{a_{n-1}}{a_n} \\ 1 & 0 \end{pmatrix} ;$$

in fact, given an arbitrary pair of initial conditions $(u_0, u_{-1})^t$ (satisfying (7), for instance), it is possible to obtain any solution to (5) by the successive application of $T_n(E)$, $n \in \mathbb{Z}_+^0$, that is,

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = T_n(E) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} .$$

Thus,

$$T_n(E) = \begin{pmatrix} u_1(n+1) & u_2(n+1) \\ u_1(n) & u_2(n) \end{pmatrix} . \tag{19}$$

Corollary 19. *Suppose that, for some $\alpha \in (0,1]$ and every E in the Borel subset $A \subset \mathbb{R}$,*

$$\liminf_{l\to\infty} \frac{1}{R_l^{2-\alpha}} \sum_{n=0}^{l} \|T_n(E)\|^2 < \infty , \tag{20}$$

where $\|\cdot\|$ is any matricial norm. Then, the restriction $\mu(A\cap\cdot)$ is α-packing continuous.

Proof. By (19) and the equivalence of norms, there exists some positive constant C such that

$$C\left(|u_{1,n+1}|^2 + |u_{1,n}|^2 + |u_{2,n+1}|^2 + |u_{2,n}|^2\right) \le \|T_n(E)\|^2 ,$$

which implies

$$C'\left(\|u_1\|_{l+1}^2 + \|u_2\|_{l+1}^2\right) \le \sum_{n=0}^{l} \|T_n(E)\|^2 . \tag{21}$$

We see, from (20) and (21), that the hypothesis (18) in Corollary 17 is fulfilled. This concludes the proof of the corollary. □

Remark 20. Corollary 19 is actually a dual to Corollary 3.7 in Carvalho *et al.* [3], a result about Hausdorff dimension first proposed in the context of Schrödinger sparse operators.

Corollary 21. *Define R_l as in Corollary 17. Suppose that*

$$\limsup_{l\to\infty} \frac{\|u_1\|_l^2}{R_l^\alpha} = 0 \tag{22}$$

is satisfied for every E in some Borel subset A of $\mathbb{R}$. Then, the restriction $\tau(A\cap\cdot)$ is α-packing singular.

Proof. Let $E \in A$. Again, from (17), $\|u_2\|_l^\beta \ge (R_l/\|u_1\|_l)^\beta$. This result implies

$$\limsup_{l\to\infty} \frac{\|u_1\|_l}{\|u_2\|_l^\beta} \le \limsup_{l\to\infty} \frac{\|u_1\|_l^{1+\beta}}{R_l^\beta} = \limsup_{l\to\infty} \left(\frac{\|u_1\|_l^2}{R_l^\alpha}\right)^{1/(2-\alpha)} = 0 .$$

It follows, by Theorem 14, that $\tau(A\cap\cdot)$ is α-packing singular. □

Remark 22. Note that the result of Corollary 21 is more restrictive than its dual (Corollary 4.5 in [13]), since it demands that $\|\psi\|_l^2/l^\delta > c > 0$ for all l large enough. This shows that α-packing singularity is more restrictive than α-Hausdorff singularity; that is, if the restriction $\tau(A\cap\cdot)$ is α-packing singular, then it is necessarily α-Hausdorff singular (in agreement with $U_\infty^\alpha \subseteq T_\infty^\alpha$).

4. Applications to sparse operators

We present in this section some applications of the previous results to sparse Schrödinger operators. Initially, we devote our attention to H_ϕ with $a_n = 1$, for all n, and

$$b_n = \begin{cases} v & \text{if} \quad n = x_j^\omega \in \mathcal{A}\,, \\ 0 & \text{if} \quad n \notin \mathcal{A}\,, \end{cases} \tag{23}$$

with $0 \neq v \in \mathbb{R}$, and $\mathcal{A} = (x_j^\omega)_{j\geq 1}$ is a random sequence of natural numbers of the form $x_j^\omega = x_j + \omega_j$, such that the sequence (x_j) satisfies

$$x_j - x_{j-1} = \gamma^j\,, \qquad\qquad j = 2, 3, \ldots, \tag{24}$$

with $x_1 + 1 = \gamma \geq 2$ (we might define γ as any real number greater than 1 and rewrite (24) as $x_j - x_{j-1} = [\gamma^j]$); $\omega = (\omega_1, \omega_2, \ldots)$ represents a sequence of independent random variables defined on a probability space $(\Xi, \mathcal{B}, \nu)$, where we assume that there is some $\eta > 0$ with ω_j uniformly distributed over the finite set $\{0, 1, 2, \ldots, [j^\eta]\}$, for all j.

Let us denote by H_ϕ^ω the operator defined above. This model was initially proposed by Zlatoš in [27], who, among other results, has determined the exact Hausdorff dimension of the spectral measure τ of H_ϕ^ω for a.e. phase-boundary condition ϕ (w.r.t. Lebesgue measure) and a.e. realization of ω (w.r.t. the probability measure ν).

Marchetti *et al.* presented in [17] a slightly modified version of the deterministic analogue of H_ϕ^ω, by taking $b_n = 0$ and

$$a_n = \begin{cases} p & \text{if} \quad n = x_j^\omega \in \mathcal{A}\,, \\ 0 & \text{if} \quad n \notin \mathcal{A}\,, \end{cases}$$

with $0 < p < 1$ (the cases $p = 0$ and $p = 1$ produce, respectively, operators with purely point and absolutely continuous spectra) and $\mathcal{A}$ as in (23). It is possible to show that the spectral properties of both models are equivalent. With respect to [27], in [17] it was presented a simpler and more accurate estimate of the asymptotic behavior of the eigenfunctions of (5). Carvalho *et al.* [4] extended the results of [17] to the above random analogue of H_ϕ^ω, establishing the exact asymptotic behavior (up to set of ν measure zero) of the eigenfunctions of (5); in particular, the authors demonstrated the uniform distribution, on the interval $[0, \pi)$, of the Prüfer angles of these solutions (see [4] and its references for a definition of Prüfer variables). The determination of the exact asymptotic behavior of the eigenfunctions of (5), together with Theorem 14 and its corollaries, lead us to the following

Theorem 23. *Let τ be the spectral measure of H_ϕ^ω. Given a closed interval of energies $L \subset [-2, 2]$, for a.e. $\phi \in [0, \pi)$ and a.e. $\omega \in \Xi$, the spectral measure τ restricted to L has exact local dimension*

$$\dim_\tau(E) = \max\left\{0, 1 - \frac{\ln r(E)}{\ln \gamma}\right\}\,, \tag{25}$$

with $r(E) = 1 + \dfrac{v^2}{4 - E^2}$.

Proof. The proof of the theorem involves a two-part strategy: first, we must prove, under the conditions in the statement of the theorem, that $\dim_{H,\tau}(E)$ (i.e., the local Hausdorff dimension of τ at E) is given by (25); this is exactly the content of Theorem 6.3 in [27]; second, we must repeat these steps and show that $\dim_{P,\tau}(E)$ (i.e., the local packing dimension of τ at E) is also given by (25).

In particular, by the one-dimensional version of Theorem 3.11 in [3] (see it for details), for a.e. $E \in L$, it follows that the estimates

$$\sum_{k=0}^{l} \|T_k(E)\|^2 \leq c\, l^{1+\zeta}, \tag{26}$$

with $\zeta := \ln r / \ln \beta$, for some $c > 0$, and

$$\|u_1(E)\|_l^2 \leq c'\, l^{1-\zeta}, \tag{27}$$

for some $c' > 0$ and for a.e. $E \in [-2, 2]$ w.r.t. τ, hold true.

Thus, by (26) one has

$$\liminf_{l\to\infty} \frac{1}{l^{2-\alpha}} \sum_{k=0}^{l} \|T_k(E)\|^2 < \infty\ ,$$

if $2 - \alpha = 1 + \zeta + \varepsilon$, and by (27)

$$\limsup_{l\to\infty} \frac{\|u_1(E)\|_l^2}{l^{\alpha'}} = 0\ ,$$

if $\alpha' = 1 - \zeta + \varepsilon$, for arbitrary $\varepsilon > 0$.

It follows, by Corollaries 19 and 21, that the spectral measure τ is simultaneously $(1-\zeta-\varepsilon)$-packing continuous and $(1-\zeta+\varepsilon)$-packing singular. Since ε is arbitrary, we have, by Remark 9, that the restriction $\tau((L \setminus B) \cap \cdot)$ is such that $\dim_{P,\tau}(E)$ is given by (25), where $E \in L \setminus B$, with B some set of Lebesgue null measure.

Finally, from the theory of rank one perturbations, we know that $\tau(B) = 0$ holds for a.e. ϕ; therefore, for a.e. ϕ, the restriction $\tau(L \cap \cdot)$ has (25) as its local packing dimension.

We obtain from the results above that the restriction $\tau(L \cap \cdot)$ has (25) as its exact local dimension. This concludes the proof of the theorem. □

There are other examples of sparse operators with spectral measures of exact local dimension that can be obtained as above; see [3] for an extension of [17, 27] to a strip of $\mathbb{Z}_+^2$, [2] for a Dirac operator analogue to [27], and [1] for a continuous counterpart of [2, 27].

All these examples have a common feature: thanks to the determination of the exact asymptotic behavior of the solutions to the eigenvalue equations, it is possible to obtain the α-derivative $(D^\alpha \tau)(E)$ for a.e. E (w.r.t. Lebesgue measure) in the essential spectrum of the corresponding operator.

Now we show that a distinct situation occurs with the sparse operator considered in Theorem 1.3 in [13].

Theorem 24. *Let $\alpha \in (0,1)$ and denote by H_ϕ^J the Schrödinger operator H_ϕ given by* (1)*, with $a_n = 1$ for every $n \in \mathbb{N}$ and*

$$b_n = \begin{cases} x_j^{(1-\alpha)/2\alpha} & \text{if} \quad n = x_j \in \mathcal{B}\,, \\ 0 & \text{if} \quad n \notin \mathcal{B}\,, \end{cases} \tag{28}$$

where $\mathcal{B} = (x_j)_j = \left(2^{(j^j)}\right)_j$. Then

1. *For every boundary phase $\phi \in [0,\pi)$, the essential spectrum of H_ϕ consists of the interval $[-2,2]$ (which is its essential part) along with some discrete point spectrum outside this interval.*
2. *For Lebesgue a.e. boundary phase $\phi \in [0,\pi)$, the restriction of the spectral measure τ to $(-2,2)$ has exact Hausdorff dimension α.*
3. *For every boundary phase $\phi \in [0,\pi)$, the restriction of the spectral measure τ to $(-2,2)$ is one-packing dimensional.*

Remark 25. 1. Items *(1)* and *(2)* are discussed in Theorem 1.3 in [13].
2. It is an immediate consequence of *(3)* and the second inequality in equation (4) that the dynamics of H_ϕ^J is quasi-ballistic.
3. Tcheremchantsev presented in [23] (item 2 of Corollary 4.5) an improved version of *(2)*, valid for every boundary phase $\phi \in [0,\pi)$. The result was obtained by the exact determination of the dynamical properties of H_ϕ.
4. Combes and Mantica obtained in [6] a version of item *(3)* (Theorem 2). Their proof is based on direct estimates of the lower derivative $(\underline{D}^1\tau)(E)$ for every $E \in (-2,2)$. Our proof, however, is different and relies on estimates of the asymptotic behavior of the solutions to (5).

Proof. We only need to prove *(3)*, and our strategy is based on results of Section 3; we will also make use of some estimates discussed in [13].

Consider any closed interval I of $(-2,2)$. It is sufficient to show that the restriction $\tau(I \cap \cdot)$ has exact packing dimension 1 in order to conclude the same for $\tau((-2,2) \cap \cdot)$. For every $E \in I$, $p > k \geq -1$, define

$$T(p,k;E) = A_p(E)A_{p-1}(E)\cdots A_{k+1}(E)\ .$$

Thanks to the sparsity of the potential, $T(p,k;E)$ corresponds to the product of $p-k$ free transfer matrices (the free transfer matrix is nothing but the transfer matrix of the free laplacian) for every $x_n \leq k < p < x_{n+1}$, $n \in \mathbb{Z}_+^0$. Thus, there is a constant C_I, depending only on the interval I, such that $1 \leq \|T(p,k;E)\| < C_I$ for any such p,k and $E \in I$. Now, for any $n \in \mathbb{Z}_+^0$, we have the bound $\|A_{x_n}(E)\| \leq b_{x_n} + 3$.

Combining the estimates above, (28) and the decomposition of $T_m(E)$ in the product

$$T_m(E) = T(m,x_n;E)A_{x_n}(E)T(x_n-1,x_{n-1};E)\cdots A_{a_0}(E)T(a_0-1,-1;E)\,, \tag{29}$$

where $x_n \le m < x_{n+1}$, for some $n \in \mathbb{Z}_+^0$, we see that

$$\|T_m(E)\| \le C_I^{n+1} \prod_{k=0}^{n} (b_{x_k} + 3) \le C_1^n \left(\prod_{k=0}^{n} x_k \right)^{(1-\alpha)/2\alpha} ,$$

with C_1 a constant depending on C_I and α.

Since $\prod_{k=0}^n x_k = 2^{\sum_{k=0}^n k^k}$ and $(1/n^n)\sum_{k=0}^n k^k$ converges to 1 as $n \to \infty$, we have, for sufficiently large n,

$$\prod_{k=0}^{n} x_k < x_n^{1+t_n} , \tag{30}$$

where $(t_n)_n$ is a sequence which goes to zero as $n \to \infty$. Taking into account that $C_1^n < x_n^{t'_n}$, we can deduce from (29) and (30) that

$$\|T_m(E)\| \le x_n^{(1-\alpha)/2\alpha+\eta_n} , \tag{31}$$

a result valid for $x_n \le m < x_{n+1}$, where n is sufficiently large and $(\eta_n)_n$ represents a sequence which goes to zero as $n \to \infty$. By considering the sequence $l_n = x_n + \left[x_n^{\alpha/(1-\alpha)} \right]$, clearly $x_n < l_n < x_{n+1}$, so it follows by (31) that

$$\begin{aligned} \sum_{k=0}^{l_n} \|T_k(E)\|^2 &\le x_n x_{n-1}^{(1-\alpha)/\alpha+2\eta_{n-1}} + (l_n - x_n) x_n^{(1-\alpha)/\alpha+2\eta_n} \\ &< x_n^{1+\delta'_n} + x_n^{1+2\eta_n} \le x_n^{1+t''_n} , \end{aligned} \tag{32}$$

where $(t''_n)_n$ converges to zero. Thus, by (32), one has

$$\liminf_{l\to\infty} \frac{1}{l^{2-\alpha}} \sum_{k=0}^{l} \|T_k(E)\|^2 < \infty ,$$

provided $2 - \alpha = 1 + \varepsilon$, for each $\varepsilon > 0$.

It follows, by Corollary 19, that the spectral measure τ is $(1-\varepsilon)$-packing continuous. Since $\varepsilon > 0$ is arbitrary, we have, by Definition 8, that the restriction $\tau(I \cap \cdot)$ is one-packing dimensional. □

One could wonder why Theorems 23 and 24 present results so distinct despite H_ϕ^ω and H_ϕ^J have been defined in a quite similar way. Indeed, Theorem 23 shows that the α-derivative $D_\tau^\alpha(E)$ exists (for a.e. ω w.r.t. the probability measure ν and Lebesgue a.e. E), which guarantees, according to Theorem 11, that τ has exact dimension given by (25). On the other hand, $D_\tau^\alpha(E)$ does not exist for H_ϕ^J, since there are sequences of natural numbers where the asymptotic behavior of the eigenfunctions of (5) are distinct; this, given Theorems 1, 11 and 14, results in different values of packing and Hausdorff dimensions for its spectral measure. Summing up, while the asymptotic behavior of the eigenfunctions of (5) is exactly determined for the operator H_ϕ^ω, this is not the case for H_ϕ^J, for which we only have estimates on its eigenfunctions.

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