BOURGIN-YANG VERSION OF THE BORSUK-ULAM THEOREM FOR $\mathbb{Z}_{p^k}$-EQUIVARIANT MAPS.

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ABSTRACT. Let $G = \mathbb{Z}_{p^k}$ be a cyclic group of prime power order. We give an estimate of the dimension of the inverse image of $\{0\}$ for a $\mathbb{Z}_{p^k}$-equivariant mapping from a sphere $S(V)$ of an orthogonal representation V into another orthogonal representation W of G in terms of V and W. It extends the Bourgin-Yang version of the Borsuk-Ulam theorem onto this class of groups. As a consequence, we also estimate the size of the Z_{p^k}-coincidences set of a continuous map from $S(V)$ into a real vector space W'.

1. INTRODUCTION

In 1954 and 1955 C. T. Yang [17, 18] and (independently) D. G. Bourgin [4] proved a theorem on $\mathbb{Z}_2$-equivariant mapping f from the unit sphere $S(\mathbb{R}^n)$ in $\mathbb{R}^n$ into $\mathbb{R}^m$, where the Euclidean spaces are considered as representations of $\mathbb{Z}_2$ with the antipodal action. They showed that for the set $Z_f := f^{-1}(0)$ we have the estimate

$$\dim Z_f \geq n - m - 1,$$

where dim is the covering dimension. Consequently, it generalized the classical Borsuk-Ulam theorem. Munkholm in [11] and [12] extended the Borsuk-Ulam theorem for the case of continuous maps $f : S^{2n-1} \to \mathbb{R}^m$ and free actions of a cyclic group $G = \mathbb{Z}_{p^k}$, (p prime, $k \geq 1$) on S^{2n-1}, giving an estimate to the covering dimension of the set $A(f) = \{x \in S^{2n-1} \mid f(x) = f(gx), \text{ for all } g \in G\}$.

In [6], Dold extended the Bourgin-Yang problem to a fibre-wise setting, giving an estimate for the set $Z_f = f^{-1}(0)$, where $\pi : E \to B$ and $\pi' : E' \to B$ are vector bundles and $f : S(E) \subset E \to E'$ is a $\mathbb{Z}_2$-map, which preserve fibres ($\pi' \circ f = \pi$). In [8] and [13] this problem was considered for the case of the cyclic group $G = \mathbb{Z}_p$ (p prime), and in [10] for bundles $E \to B$ whose fibre has the same cohomology (mod p) of a product of spheres. In all these cases, if B is a single point, Bourgin-Yang versions of the Borsuk-Ulam theorem are obtained for $G = \mathbb{Z}_p$, with p prime.

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Here we study the Bourgin-Yang problem for the case that G is a cyclic group of a prime power order, $G = \mathbb{Z}_{p^k}$, $k \geq 1$.

Let V, W be two orthogonal representations of G such that $V^G = W^G = \{0\}$ for the sets of fixed points of G. Let $f : S(V) \to W$ be a G-equivariant mapping. By Z_f, we denote the set $Z_f := \{v \in S(V) \,|\, f(v) = 0\}$.

For $G = \mathbb{Z}_{p^k}$, with p odd, every nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure ([16]), thus V and W admit it too. Put $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$, and correspondingly $d(W) = \dim_{\mathbb{C}} W = \frac{1}{2}\dim_{\mathbb{R}} W$, are integral numerical invariants of V, and respectively of W. If $G = \mathbb{Z}_{2^k}$ and V,W are orthogonal representations of G, then we put $d(V) = \dim_{\mathbb{R}} V$, and respectively $d(W) = \dim_{\mathbb{R}} W$.

Our main results say:

Theorem 1.1. *Let V, W be two orthogonal representations of the cyclic group $\mathbb{Z}_{p^k}$ and $f : S(V) \to W$ an equivariant map.*

Then, the covering dimension $\dim(Z_f) = \dim(Z_f/G) \geq \phi(V, W)$, where ϕ is a function depending on $d(V)$, $d(W)$ and the orders of the orbits of actions on $S(V)$ and $S(W)$, which we describe later (cf. Theorems 3.6, 3.9).

In particular, if $d(W) < d(V)/p^{k-1}$, then $\phi(V, W) \geq 0$, which means that there is no G-equivariant map from $S(V)$ into $S(W)$.

As a consequence, we also give an estimate to the covering dimension of the set $A(f)$ of $\mathbb{Z}_{p^k}$-coincidences of a continuous map $f : S(V) \to W'$, where W' is a real vector space.

The paper is organized as follows. In Section 2, we recall the definition of a length index in the equivariant K-theory, presenting its properties and an estimate for the length index of $S(V)$, given by Bartsch in [3]. In Section 3, we estimate the length index of Z_f and as a consequence, we obtain a Bourgin-Yang version of the Borsuk-Ulam theorem for $\mathbb{Z}_{p^k}$, p prime, $k \geq 1$. The case $p = 2$ is discussed separately. Finally, in Section 4 we provide an estimate the size of the Z_{p^k}-coincidences set of a continuous map from $S(V)$ into a real vector space W'.

2. A length index in the equivariant K-theory

In this section, we briefly recall the notion of an equivariant index based on the cohomology length in a given cohomology theory. It was introduced and described in details by Thomas Bartsch in [3, Chapter 4]. He presented a very general version of the mentioned index, considering an equivariant map between two pairs of G-spaces and defining an index for this triple. We consider the case when these two pairs are equal and the map is equal to the identity. Moreover, we study this notion taking as an equivariant cohomology theory the equivariant K-theory, denoted by $K_G^*(X)$. This is a theory generated by K_G-theory, i.e., the equivariant K-theory of G-vector bundles, extended to the next gradation by use of the equivariant Bott periodicity (see[1], [14] and [15]).

Let us fix a set $\mathcal{A}$ of G-spaces. Usually, it is a family of orbits, which obviously is finite, if G is finite.

Definition 2.1. *The $(\mathcal{A}, K_G^*)$ – cup length of a pair (X, X') of G-spaces is the smallest r such that there exist A_1, A_2, ... , $A_r \in \mathcal{A}$ and G-maps $\beta_i : A_i \to X$, $1 \leq i \leq r$ with the* *property that for all* $\gamma \in K^*_G(X, X')$ *and for all* $\omega_i \in \ker \beta^*_i$ *we have*

$$\omega_1 \cup \omega_2 \cup \ldots \cup \omega_r \cup \gamma = 0 \in K^*_G(X, X').$$

If there is not such r*, we say that the* $(\mathcal{A}, K^*_G)$ *– cup length of* (X, X') *is* ∞*.* $r = 0$ *means that* $K^*_G(X, X') = 0$*. Moreover, the* $(\mathcal{A}, K^*_G)$ *– cup length of* X *is by definition the cup length of the pair* $(X, \emptyset)$*.*

Instead of $(\mathcal{A}, K^*_G)$ – *cup length*, one can consider also the notion of $(\mathcal{A}, K^*_G, R)$ – *length index* defined in a little bit different manner (cf. [3]). Once more we leave out a general situation of [3], presenting only the case necessary for our next considerations.

Recall that for the equivariant cohomology theory K^*_G and a G-pair (X, X'), the cohomology $K^*_G(X, X')$ is a module over the coefficient ring $K^*_G(\mathrm{pt})$, via the natural G-map $p_X : X \to \mathrm{pt}$. We write

$$\omega \cdot \gamma = p^*_X(\omega) \cup \gamma, \text{ and } \omega_1 \cdot \omega_2 = \omega_1 \cup \omega_2,$$

for $\gamma \in K^*_G(X, X')$ and ω_1, $\omega_2 \in K^*_G(\mathrm{pt})$.

Taking $R := K_G(\mathrm{pt}) = R(G) \subset K^*_G(\mathrm{pt})$, we obtain the following adjustment of [3, Definition 4.1].

Definition 2.2. *The* $(\mathcal{A}, K^*_G, R)$ *– length index of a pair* (X, X') *of* G*-spaces is the smallest* r *such that there exist* $A_1, A_2, \ldots, A_r \in \mathcal{A}$ *with the following property:*

For all $\gamma \in K^*_G(X, X')$ *and all* $\omega_i \in R \cap \ker(K^*_G(\mathrm{pt}) \to K^*_G(A_i)) = \ker(K_G(\mathrm{pt}) \to K_G(A_i))$, $i = 1, 2, \ldots, r$*, the product*

$$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_r \cdot \gamma = 0 \in K^*_G(X, X').$$

A comparison of this to the numerical invariants is given in the following statement (cf. [3, p. 59]).

Proposition 2.3. *For any system* $\mathcal{A}$ *and every pair of* G*-spaces* (X, X') *we have*

$$(\mathcal{A}, K^*_G, R) - \textit{length index of } (X, X') \leq (\mathcal{A}, K^*_G) - \textit{cup length of } (X, X').$$

The $(\mathcal{A}, K^*_G, R)$ – *length index* has many properties which are important from the point of view of applications to study critical points of G-invariant functions and functionals (see [3]). Next, we shall use only a part of them.

After [3], for given two powers $1 \leq m \leq n \leq p^{k-1}$ of p we set

$$\mathcal{A}_{m,n} := \{G/H \mid H \subset G;\ m \leq |H| \leq n\}, \tag{1}$$

where $|H|$ is the cardinality of H. Next we put

$$l_n(X, X') = (\mathcal{A}_{m,n}, K^*_G, R) - \textit{length index of } (X, X'). \tag{2}$$

Remark 2.4. By [3, Observation 5.5], the index l_n does not depend on m. It says that if for $\mathcal{A}' \subseteq \mathcal{A}$ is such that for each $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ and a G-map $A \to A'$ then

$$(\mathcal{A}, K^*_G, R) - \text{length index} = (\mathcal{A}', K^*_G, R) - \text{length index}.$$

The following theorem is fundamental for our version of the Bourgin–Yang theorem for $G = \mathbb{Z}_{p^k}$ (cf. [2], [3]). We shall write $\mathcal{A}_X$ for a set of all the G-orbits of X (up to a homeomorphism, thus up to an isomorphism of finite G-sets).

Theorem 2.5. [3, Theorem 5.8] *Let V be an orthogonal representation of $G = \mathbb{Z}_{p^k}$ with $V^G = \{0\}$ and $d = d(V) = \frac{1}{2}\dim_{\mathbb{R}} V$. Fix m, n two powers of p as above. Then*

$$l_n(S(V)) \geq \begin{cases} 1 + \left\lceil \frac{(d-1)m}{n} \right\rceil & \text{if } \mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}, \\ \infty & \text{if } \mathcal{A}_{S(V)} \not\subseteq \mathcal{A}_{1,n}\,, \end{cases}$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x. Moreover, if $\mathcal{A}_{S(V)} \subset \mathcal{A}_{n,n}$, then

$$l_n(S(V)) = d.$$

3. Bourgin-Yang theorem for $\mathbb{Z}_{p^k}$

In this section we prove our main theorem. To do it, first we need to discuss a relation between the $l_n = (\mathcal{A}_{m,n}, K_G^*, R)$ – *length index* of a G-set X and its dimension.

If X is a compact G-space, where G is a compact Lie group, in [15, Proposition 5.3, p. 147], G. Segal showed that there is the Atiyah-Hirzebruch spectral sequence for equivariant K-theory

$$E_2^{s,t} = H^s(X/G; \mathcal{K}_G^t) \Rightarrow K_G^*(X),$$

where $\mathcal{K}_G^t$ is the sheaf on X/G associated to the presheaf $V \mapsto K_G^t(\pi^{-1}\overline{V})$ ($\pi : X \to X/G$ is the projection) with the stalk $\mathcal{K}_G^t$ at an orbit $Gx = G/G_x$ equal to $R(G_x)$, if t is even, and $\mathcal{K}_G^t = 0$, if t is odd.

Moreover, there exists an invariant filtration of X such that $K_G^*(X)$ is the associated module of the limit of this spectral sequence with respect to this filtration.

If X is a G-CW-complex, which is filtered by its skeletons $\{X^s\}$, it is customary to define a filtration of $K_G^*(X)$ by setting $K_{G,s}^*(X) = \ker(K_G^*(X) \to K_G^*(X^{s-1}))$. It corresponds to a filtering of X by the G-subspaces $\pi^{-1}(Y^s)$, when the orbit space $Y = X/G$ is a CW-complex and $\{Y^s\}$ its skeletons ($\pi : X \to Y$ is the projection).

The general case is discussed in [15, §5] by use of the nerve of a G-stable closed finite covering of X. For each finite covering $\mathcal{U} = \{U_j\}_{j\in\mathcal{S}}$ of a compact G-space X by G-stable closed sets it is associated a compact G-space $\mathcal{W}_{\mathcal{U}}$, with a G-map $w : \mathcal{W}_{\mathcal{U}} \to X$ and a filtration by G-subspaces $\mathcal{W}_{\mathcal{U}}^0 \subset \mathcal{W}_{\mathcal{U}}^1 \subset \cdots \subset \mathcal{W}_{\mathcal{U}}^j \subset \cdots \subset \mathcal{W}_{\mathcal{U}}$, so that the following conditions are satisfied:

(i) $w^* : K_G^*(X) \to K_G^*(\mathcal{W}_{\mathcal{U}})$ is an isomorphism, and
(ii) when $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a G-map $\mathcal{W}_{\mathcal{U}} \to \mathcal{W}_{\mathcal{V}}$ defined up to G-homotopy, respecting the filtrations and the projections onto X.

Definition 3.1. *We say that an element of $K_G^*(X)$ is in $K_{G,s}^*(X)$ if, for some finite covering $\mathcal{U}$, it is in the kernel of $w^* : K_G^*(X) \to K_G^*(\mathcal{W}_{\mathcal{U}}^{s-1})$.*

For the filtration of $K_G^*(X)$ defined above

$$K_G^*(X) = K_{G,0}^*(X) \supset K_{G,1}^*(X) \supset \cdots \supset K_{G,s}^*(X) \supset \cdots\,, \tag{3}$$

$K_G^*(X)$ is a filtered ring in the sense that

$$K_{G,s}^*(X) \cdot K_{G,s'}^*(X) \subset K_{G,s+s'}^*(X)\,,$$

thus $K_{G,s}^*(X)$ is an ideal in $K_G^*(X)$ (see [15, p. 145-146]).

Moreover, we have the following

Proposition 3.2. (i) *An element of* $K_G^*(X)$ *is in* $K_{G,1}^*(X)$ *if, and only if, its restriction to each orbit is zero, i.e.,*

$$K_{G,1}^*(X) = \ker(K_G^*(X) \to \prod_{x\in X} K_G^*(G/G_x)) = \bigcap_{x\in X} \ker(K_G(X) \to K_G(G/G_x)) \,.$$

(ii) *If the subgroups of* G *are totally ordered and* H *is the largest isotropic subgroup on* X*, then*

$$K_{G,1}^*(X) = \ker(K_G^*(X) \to K_G^*(G/H)) \,.$$

Proof: (i) Proposition 5.1(i), p. 146 in [15].

(ii) By (i), an element ξ belongs to $K_{G,1}^*(X)$ if, and only if, ξ is in $\ker(K_G^*(X) \to K_G^*(G/G_x))$, for all $x \in X$. Then, $K_{G,1}^*(X) \subset \ker(K_G^*(X) \to K_G^*(G/H))$. Conversely, since the subgroups of G are totally ordered and H is the largest isotropic subgroup on X, there exist natural G-maps $G/G_x \to G/H$, for all $x \in X$. Then,

$$\ker(K_G^*(X) \to K_G^*(G/H)) \subset \ker(K_G^*(X) \to K_G^*(G/G_x)), \text{ for all } x \in X,$$

and consequently, $\ker(K_G^*(X) \to K_G^*(G/H)) \subset K_{G,1}^*(X)$. □

Lemma 3.3. *If* X *is a compact* G*-space such that* $\dim X/G \leq 2r-1$*, then*

(i) $K_{G,2i}(X) = K_{G,2i}^0(X) = K_{G,2i-1}^0(X) = K_{G,2i-1}(X)$*, for all* $i = 1, 2, 3, \dots,$

(ii) $(K_{G,2}(X))^r = (K_{G,2}^0(X))^r = 0$.

Proof: Consider the Atiyah-Hirzebruch spectral sequence in the equivariant K-theory

$$E_2^{s,t} = H^s(X/G; \mathcal{K}_G^t) \Rightarrow K_G^*(X),$$

as above. We have that $E_2^{s,t} = 0$, if $s < 0$ or if t is odd and this implies that

$$E_2^{s,t} = E_\infty^{s,t} = 0 \text{ for } s < 0 \text{ or } t \text{ odd.}$$

It follows from the definition of the filtration of $K_G(X) = K_G^0(X)$,

$$K_G^0(X) = K_{G,0}^0(X) \supset K_{G,1}^0(X) \supset \cdots \supset K_{G,s}^0(X) \supset \cdots \,,$$

that $K_{G,s}^0(X) = 0$, for all $s > \bar{s}$, where $\bar{s} = \dim X/G$. Since by assumption $\bar{s} = \dim X/G \leq 2r-1$, in particular, we have that $K_{G,2r}^0(X) = 0$. Consequently, since $(K_{G,2}^0(X))^r \subset K_{G,2r}^0(X)$ we conclude that $(K_{G,2}^0(X))^r = 0$.

Moreover, for all $s > \bar{s}$ and for all t we have that $0 = H^s(X/G; \mathcal{K}_G^t) \cong E_2^{s,t} \cong E_\infty^{s,t}$ and using the definition of the infinite term $E_\infty^{s,t} \cong K_{G,s}^{s+t}(X)/K_{G,s+1}^{s+t}(X)$ we have the following exact sequences:

$$0 \longrightarrow K^0_{G,1}(X) \longrightarrow K^0_G(X) \longrightarrow E^{0,0}_\infty \longrightarrow 0$$
$$0 \longrightarrow K^0_{G,2}(X) \longrightarrow K^0_{G,1}(X) \longrightarrow E^{1,-1}_\infty \longrightarrow 0$$
$$\cdots\cdots\cdots$$
$$0 \longrightarrow K^0_{G,2i}(X) \longrightarrow K^0_{G,2i-1}(X) \longrightarrow E^{2i-1,-2i+1}_\infty \longrightarrow 0$$
$$0 \longrightarrow K^0_{G,2i+1}(X) \longrightarrow K^0_{G,2i}(X) \longrightarrow E^{2i,-2i}_\infty \longrightarrow 0$$
$$\cdots\cdots\cdots$$
$$0 \longrightarrow K^0_{G,\bar{s}}(X) \longrightarrow K^0_{G,\bar{s}-1}(X) \longrightarrow E^{\bar{s}-1,-\bar{s}+1}_\infty \longrightarrow 0$$
$$0 \longrightarrow K^0_{G,\bar{s}}(X) \longrightarrow E^{\bar{s},-\bar{s}}_\infty \longrightarrow 0.$$

Since $E^{2i-1,-2i+1}_\infty = 0$, it follows that $K^0_{G,2i}(X) = K^0_{G,2i-1}(X)$, for all $i = 1, 2, 3, \ldots$. □

Remark 3.4. From [5, Theorem 1.1] it follows that for an action of a finite group G on a paracompact Hausdorff space X we have $\dim X = \dim X/G$. Consequently, in this case, to estimate $\dim X$ it is enough to estimate $\dim X/G$.

Theorem 3.5. *Let X be a compact G-space, with $G = \mathbb{Z}_{p^k}$, and suppose that $\mathcal{A}_X \subset \mathcal{A}_{m,n}$. If $l_n(X) \geq r + 1$, then* $\dim X = \dim X/G \geq 2r$.

Proof: First, we observe that to compute $l_n = (\mathcal{A}_{m,n}, K^*_G(X), R)$- length index with

$$\mathcal{A}_{m,n} = \{G/H \mid H \subset G;\ m \leq |H| \leq n\}$$

it is no loss of generality to assume that $\mathcal{A}_{m,n}$ consists of just one element G/H, because the subgroups of G are totally ordered (see Remark 2.4, [3, p. 76], and (4) below). Take $H \in \mathcal{A}_{m,n}$ such that $|H| = n$. Once more, if $K \subset H$ then there is a G-map $\phi : G/K \to G/H$, and consequently

$$\ker(K_G(\mathrm{pt}) \to K_G(G/H)) \subset \ker(K_G(\mathrm{pt}) \to K_G(G/K)). \tag{4}$$

Let $x \in X$ and $G_x \subset H \in \mathcal{A}_{m,n}$ with biggest isotropy subgroup. If $\omega \in \ker(K_G(\mathrm{pt}) \to K_G(G/G_x))$, then $p^*_X(\omega) \in \ker(\beta^* : K_G(X) \to K_G(G/G_x))$, for any G-map $\beta : G/G_x \to X$, in particular for the inclusion $Gx \subset X$ (see [3, p. 59]). Consequently, if $\omega \in \ker(K_G(\mathrm{pt}) \to K_G(G/H))$, then

$$p^*_X(\omega) \in K^0_{G,1}(X) = K^0_{G,2}(X)\,,$$

by (4), Proposition 3.2 (ii), and Lemma 3.3 (i). Therefore, for every $\omega_i \in \ker(K_G(\mathrm{pt}) \to K_G(G/H))$, $i = 1, 2, \ldots, r$, the product

$$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_r = p^*_X(\omega_1) \cup p^*_X(\omega_2) \cup \ldots \cup p^*_X(\omega_r) \in (K^0_{G,2}(X))^r \subset K_G(X). \tag{5}$$

If $\dim X/G \leq 2r - 1$, by Lemma 3.3(ii), $(K^0_{G,2}(X))^r = 0$ and it follows from (5) and from definition of the length index that $l_n(X) \leq r$. □

We are in position to prove our main theorem.

Theorem 3.6. *Let V, W be two complex orthogonal representations of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \xrightarrow{G} W$ be an equivariant* *map and* $Z_f = f^{-1}(0)$. *Suppose* $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ *and* $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. *Then*

$$l_n(Z_f) \ \geq 1 + \Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W).$$

Consequently,

$$\dim(Z_f) \ \geq \ 2\Big(\Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W)\Big) := \phi(V,W).$$

In particular, if $d(W) < d(V)/p^{k-1}$, *then* $\phi(V,W) \geq 0$, *which means that there is no* G-*equivariant map from* $S(V)$ *into* $S(W)$.

Proof: Denote by $\mathcal{U} = S(V) \setminus Z_f$, which is an open and invariant set. From the continuity of the equivariant K-theory, it follows that there exists an open invariant set $\mathcal{V} \subset S(V)$ such that

$$Z_f \subset \mathcal{V} \text{ and } \ K^*_G(\mathcal{V}) = K^*_G(Z_f).$$

It yields $l_n(Z_f) = l_n(\mathcal{V})$. Moreover,

$$K^*_G(W \setminus \{0\}) = K^*_G(S(W)),$$

by the equivariant deformation argument and then $l_n(W \setminus \{0\}) = l_n(S(W))$. Since f maps equivariantly $\mathcal{U}$ into $W \setminus \{0\}$ we have

$$l_n(\mathcal{U}) \leq l_n(W \setminus \{0\}) = l_n(S(W)),$$

by the corresponding monotonicity property of the length index (cf. [3, Theorem 4.6]). Obviously, $\mathcal{U} \cup \mathcal{V} = S(V)$. It follows by the additivity property of the length index (cf. [3, Theorem 4.6]) that

$$l_n(S(V)) \leq l_n(\mathcal{V}) + l_n(\mathcal{U}),$$

which gives

$$l_n(Z_f) \geq l_n(S(V)) - l_n(S(W)).$$

By Theorem 3.5, we have $l_n(S(W)) \leq d(W)$. Further, by assumption $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and it follows from Theorem 2.5 that

$$l_n(Z_f) \geq 1 + \Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W).$$

Consequently, from Theorem 3.5

$$\dim(Z_f) \ \geq \ 2\Big(\Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W)\Big) := \phi(V,W).$$

□

Now, let us consider V, W real orthogonal representations of the group $G = \mathbb{Z}_{2^k}$ and let $f : S(V) \to W$ be an equivariant map. Since Bartsch's computation works for complex orthogonal representations we consider

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}, \ \text{ and } \ W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}.$$

Note that $V_{\mathbb{C}}$ is a complex space of complex dimension $d(V_{\mathbb{C}}) = \dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{R}}(V) = d(V)$, and analogously $d(W_{\mathbb{C}}) = \dim_{\mathbb{C}}(W_{\mathbb{C}}) = \dim_{\mathbb{R}}(W) = d(W)$.

Moreover, $V_{\mathbb{C}} \simeq V \oplus V$ is a real representation of the group $G = \mathbb{Z}_{2^k}$, and has a natural structure of unitary representation of G given by $g(x \cdot 1 + y \cdot \imath) = gx \cdot 1 + gy \cdot \imath$. Analogously for $W_{\mathbb{C}}$. We know that $S(V \oplus V) = S(V) * S(V)$, as a G-space.

Now, we define an equivariant extension $\tilde{f} : S(V_{\mathbb{C}}) \to W_{\mathbb{C}}$ of the map f by the formula

$$\tilde{f}((x,t,y)) = tf(x) \cdot 1 + (1-t)f(y) \cdot \imath. \tag{6}$$

Lemma 3.7. *The map $\tilde{f}$ is G-equivariant and we have*

$$Z_{\tilde{f}} = Z_f * Z_f\,.$$

Proof: Surely $\tilde{f}$ is equivariant. Note that the vectors $tf(x) \cdot 1$ and $(1-t)f(y) \cdot \imath$ are in perpendicular subspaces (orthogonal sub-representations). Consequently, their sum is equal to 0 if, and only if, the both are equal to 0. This shows that a point $(x,t,y) \in S(V) * S(V)$ is mapped by $\tilde{f}$ onto 0 iff it belongs to $Z_f * Z_f$, which shows the lemma. □

Corollary 3.8.

$$2\, l_n(Z_f) \geq l_n(Z_{\tilde{f}}) \text{ and } \dim Z_{\tilde{f}} = 2 \dim Z_f + 1.$$

Proof: The inequality of the statement follows from Lemma 3.7 and [3, Corollary 4.10]. The equality is a direct consequence of Lemma 3.7 and of a well-known fact about the dimension of the join. □

As a consequence of the previous results we obtain the following Bourgin–Yang version of the Borsuk-Ulam theorem for real orthogonal representations of $G = \mathbb{Z}_{2^k}$, $k \geq 1$.

Theorem 3.9. *Let V, W be two real orthogonal representations of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \stackrel{G}{\to} W$ be an equivariant map and $Z_f = f^{-1}(0)$. Suppose that $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then*

$$2\, l_n(Z_f) \;\geq\; 1 + \Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W)$$

and

$$\dim(Z_f) \;\geq\; \Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W) = \phi(V,W).$$

In particular, if $d(W) < d(V)/2^{k-1}$, then $\phi(V,W) \geq 0$, which means that there is no G-equivariant map from $S(V)$ into $S(W)$.

Proof: Let us consider the equivariant extension $\tilde{f} : S(V_{\mathbb{C}}) \to W_{\mathbb{C}}$ of f as defined in (6). Applying Theorem 3.6 and using Corollary 3.8, we obtain the desired result. □

Remark 3.10. In particular, if $\mathcal{A}_{S(V)} \subset \mathcal{A}_{n,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{n,n}$ by Theorem 3.6 we have

$$\dim(Z_f) \;\geq\; 2\,(d(V) - d(W)) - 2.$$

We note that, if we consider $G = \mathbb{Z}_p$ (p prime), that is, $k = 1$, using the equivariant Borel cohomology it is possible to show that

$$\dim(Z_f) \;\geq\; 2\,(d(V) - d(W)) - 1,$$

which is the same estimate given by Izydorek and Rybicki, by considering $B = pt$ in the parametrized Borsuk-Ulam problem [8, Corollary 2.1].

Remark 3.11. We observe that Theorem 3.9 extends the Bourgin-Yang theorem for $G = \mathbb{Z}_{2^k}$, since for $k = 1$ we have

$$\dim(Z_f) \;\geq\; d(V) - d(W) - 1.$$

Remark 3.12. We note that opposite to the case $G = \mathbb{Z}_p$, for $G = \mathbb{Z}_{p^k}$ the classical formulation of the Borsuk-Ulam theorem does not hold. Bartsch in [3, Theorem 3.22] showed that for any p-group G which contains an element g of order p^2, there exists an equivariant map $f : S(V) \to S(W)$ between two complex representations V, W such that $\dim V > \dim W$.

4. Estimating the size of the $\mathbb{Z}_{p^k}$-coincidences set

Let V be an orthogonal representation of a group G and let W' be a real vector space. Given a continuous map $f : S(V) \to W'$, we denote by $A(f)$ the G-coincidences set of f, that is,

$$A(f) = \{v \in S(V) \,|\, f(gv) = f(v), \text{ for all } g \in G\}.$$

In this section, we estimate the size of the set $A(f)$, for $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, as follows.

Theorem 4.1. *Let V be an orthogonal representation of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f : S(V) \to W'$ be a continuous map. If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{k-1}}$, then*

$$\dim A(f) \geq 2\Big[\frac{d(V)-1}{p^{k-1}}\Big] - (p^k - 1)\dim W'.$$

Proof: Let us consider the real vector space $\overset{p^k}{\underset{i=1}{\oplus}} W'$, which is the direct sum of p^k copies of W'. We have that $\overset{p^k}{\underset{i=1}{\oplus}} W'$ admits an action of the cyclic group $G = \mathbb{Z}_{p^k}$, given by

$$g(w_1, w_2, \ldots, w_{p^k}) = (w_2, \ldots, w_{p^k}, w_1),$$

for a fixed generator $g \in G$ and for each $(w_1, \ldots, w_{p^k}) \in \overset{p^k}{\underset{i=1}{\oplus}} W'$.

Let us denote by Δ the subspace vector of $\overset{p^k}{\underset{i=1}{\oplus}} W'$ consisting of all points $(w_1, w_2, \ldots, w_{p^k})$ in $\overset{p^k}{\underset{i=1}{\oplus}} W'$ such that $w_1 = \cdots = w_{p^k}$. We have $\overset{p^k}{\underset{i=1}{\oplus}} W' = \Delta \oplus \Delta^{\perp}$, where $\Delta^{\perp}$ is the orthogonal complement of Δ. Since Δ is a $\dim W'$-dimensional G-subspace of $\overset{p^k}{\underset{i=1}{\oplus}} W'$, let us observe that $\Delta^{\perp}$ is a $(p^k-1)\dim W'$-dimensional G-subrepresentation of $\overset{p^k}{\underset{i=1}{\oplus}} W'$, for which $(\Delta^{\perp})^G = \{0\}$. Consequently, by the same argument as for Theorem 1.1, it has a complex structure compatible with the action of G.

Consider the G-equivariant map $F : S(V) \to \Delta \oplus \Delta^{\perp}$ defined by

$$F(v) = (f(v), f(gv), \ldots, f(g^{p^k-1}v)).$$

The linear orthogonal projection along the diagonal Δ defines a G-equivariant map $r : \Delta \oplus \Delta^{\perp} \to \Delta^{\perp}$. Let us denote by h the composition

$$S(V) \xrightarrow{F} \Delta \oplus \Delta^{\perp} \xrightarrow{r} \Delta^{\perp},$$

with $Z_h = h^{-1}(0) = (r \circ F)^{-1}(0) = F^{-1}(\Delta) = A(f)$. Since $h : S(V) \to \Delta^{\perp}$ is a G-equivariant map and $\mathcal{A}_{S(\Delta^{\perp})} \subset \mathcal{A}_{1,p^{k-1}}$, it follows from Theorem 3.6 that

$$\dim A(f) = \dim(Z_h) \geq 2\left\lceil \frac{d(V)-1}{p^{k-1}} \right\rceil - (p^k - 1) \dim W'.$$

□

For $G = \mathbb{Z}_{2^k}$, $k \geq 1$, using the same steps of the proof of Theorem 4.1 and applying Theorem 3.9 we have the following

Theorem 4.2. *Let V be a real orthogonal representation of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f : S(V) \to W'$ be a continuous map. If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,2^{k-1}}$, then*

$$\dim A(f) \geq \left\lceil \frac{d(V)-1}{2^{k-1}} \right\rceil - (2^k - 1) \dim W'.$$

□

Remark 4.3. In [12], for the group $G = \mathbb{Z}_{p^k}$, p an odd prime, $k \geq 1$, Munkholm studied the dimension of $A(f)$ under the assumption that the action on $S(V)$ is free by another approach. His formula for an estimate of $\dim A(f)$ from below is of a different form than this presented here, and there is not a direct comparison between them. He studied the case of free action, and then his estimate seems to be better than ours. On the other hand, our formula holds for every representation V with $S(V)^G = \emptyset$.

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