REVERSIBILITY AND BRANCHING OF PERIODIC ORBITS

ANA CRISTINA MEREU[1] AND MARCO A. TEIXEIRA[2]

ABSTRACT. We study the dynamics near an equilibrium point of a 2-parameter family of a reversible system in $\mathbb{R}^6$. In particular, we exhibit conditions for the existence of periodic orbits near the equilibrium of systems having the form $x^{(vi)} + \lambda_1 x^{(iv)} + \lambda_2 x'' + x = f(x, x', x'', x''', x^{(iv)}, x^{(v)})$. The techniques used are Belitskii normal form combined with Lyapunov-Schmidt reduction.

1. INTRODUCTION

In this paper we will consider reversible systems of the form

$$\dot{X} = F(X, \lambda_1, \lambda_2)$$

with $X \in \mathbb{R}^{2n}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $F : \mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2n}$ a C^k $(k \geq 2)$ parameter-dependent vector field such that

$$R(F(X, \lambda_1, \lambda_2)) = -F(RX, \lambda_1, \lambda_2)$$

for some linear involution $R \neq I$ with $R^2 = I$ where I denotes the identity map. Moreover assume that $\dim Fix(R) = n$, where $Fix(R) = \{X \in \mathbb{R}^{2n}; RX = X\}$.

Our study is partially motivated by some models in Partial Differential Equations as described in [1, 2] but also it forms part of a program addressing the systematic classification of singularities of reversible systems in higher dimension. It is worth to say that many dynamical systems that arise in the context of applications possess robust structural properties, such as for instance symmetries, anti-symmetries or Hamiltonian structure. For a historical overview and further references about systems with time-reversal symmetries, see [8].

The Lyapunov center theorem for conservative systems asserts that in such systems fixed points with simple purely imaginary eigenvalues are surrounded by families of periodic orbits provided that a certain non-resonance condition is fulfilled. Analogous results have been extended to reversible system by Devaney [3] were establish. Interesting questions arise when the non-resonance condition is violated.

In [6, 7], results on the existence of families of periodic orbits of reversible systems in $\mathbb{R}^6$ around an equilibrium that presents a $0 : p : q$-resonances with $p, q \in \mathbb{Z}$. In [9] the non-semisimple $1 : 1 : 1$, $1 : 1 : 2$, $2 : 2 : 1$ resonant cases, were investigated.

We begin by considering the 2-parameter family of systems X_{λ_1,λ_2} represented by the equation

$$x^{(vi)} + \lambda_1 x^{(iv)} + \lambda_2 x'' + x = 0, \tag{1}$$

1991 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. Periodic orbits, reversible systems, resonance, normal form, Lyapunov Center Theorem.

* The authors are partially supported by the grant CNPq 473587/2010-6.

where λ_1 and λ_2 are real parameters. We study the existence of periodic orbits near an equilibrium of reversible perturbations of (1). System X_{λ_1,λ_2} can also be represented as

$$\begin{aligned} \dot{x}_1 &= y_1\\ \dot{y}_1 &= x_2\\ \dot{x}_2 &= y_2\\ \dot{y}_2 &= x_3\\ \dot{x}_3 &= y_3\\ \dot{y}_3 &= -\lambda_1 x_3 - \lambda_2 x_2 - x_1. \end{aligned} \tag{2}$$

Let us now describe the set up in more detail. The starting point is the family

$$Y_{\lambda_1,\lambda_2} = X_{\lambda_1,\lambda_2} + F,$$

where $F : \mathbb{R}^6 \to \mathbb{R}^6$ of class C^k, $(k \geq 2)$. So we deal with Y_{λ_1,λ_2} with $F = (f_1, f_2, f_3, f_4, f_5, f_6)$, that is, Y_{λ_1,λ_2} has the form

$$\begin{aligned} \dot{x}_1 &= y_1 + f_1(x_1, y_1, x_2, y_2, x_3, y_3)\\ \dot{y}_1 &= x_2 + f_2(x_1, y_1, x_2, y_2, x_3, y_3)\\ \dot{x}_2 &= y_2 + f_3(x_1, y_1, x_2, y_2, x_3, y_3)\\ \dot{y}_2 &= x_3 + f_4(x_1, y_1, x_2, y_2, x_3, y_3)\\ \dot{x}_3 &= y_3 + f_5(x_1, y_1, x_2, y_2, x_3, y_3)\\ \dot{y}_3 &= -\lambda_1 x_3 - \lambda_2 x_2 - x_1 + f_6(x_1, y_1, x_2, y_2, x_3, y_3), \end{aligned} \tag{3}$$

with $f_i(X) = o(\|X\|^2)$ and $i = 1, ..., 6$.

We also analyze Y_{λ_1,λ_2} with $F = (0, 0, 0, 0, 0, f)$. So we have

$$x^{(vi)} + \lambda_1 x^{(iv)} + \lambda_2 x'' + x + f(x) = 0, \tag{4}$$

and

$$\begin{aligned} \dot{x}_1 &= y_1\\ \dot{y}_1 &= x_2\\ \dot{x}_2 &= y_2\\ \dot{y}_2 &= x_3\\ \dot{x}_3 &= y_3\\ \dot{y}_3 &= -\lambda_1 x_3 - \lambda_2 x_2 - x_1 + f(x_1), \end{aligned} \tag{5}$$

with $f(x_1) = o(x_1^2)$.

The correspondent characteristic polynomial of the linearized differential equation at the origin is:

$$P(y) = y^6 + \lambda_1 y^4 + \lambda_2 y^2 + 1.$$

We are interested in the cases where all of the eigenvalues are imaginary pure. More specifically, we consider parametric values in such a way that the eigenvalues are in $\alpha : 1 : l$ resonance, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $l \in \mathbb{Z}$. That is, if we denote the eigenvalues by $\pm i\alpha_j$, $j = 1, 2, 3$ then

$$l\ \alpha_2 - \alpha_3 = 0, \qquad l \in \mathbb{Z}, \qquad \text{and}$$

$$\alpha_1 \neq k_2\alpha_2 + k_3\alpha_3, \qquad \forall\ k_2, k_3 \in \mathbb{Z}.$$

Example 1. *Consider* $\lambda_1 = \lambda_2 = \dfrac{7}{2}$*. The eigenvalues are* $\pm i$, $\pm\dfrac{i}{\sqrt{2}}$ *and* $\pm i\sqrt{2}$ *and they are in* $\sqrt{2} : 1 : 2$ *resonance.*

Our main goal in this paper is to exhibit conditions for the existence of families of periodic solutions of Y_{λ_1,λ_2}. Most of our technical analysis are based on a combined use of normal form theory and Lyapunov-Schmidt Reduction. The system is first subjected to the normalizations procedure and the Belitiskii normal form plays a crucial role in our context.

In Section 2 we discuss the Belitskii Normal Form. The Lyapunov-Schmidt Reduction is sumaryzed in Section 3. In the next sections we exhibit conditions for the existence of families of periodic orbits of Y_{λ_1,λ_2} when the non-resonance condition is violated. In Section 4 we study Y_{λ_1,λ_2} in presence of $\alpha : 1 : 2$, $\alpha : 1 : 3$, $\alpha : 1 : p$ resonance, $\alpha \in \mathbb{R}/\mathbb{Q}$ and $p \in \mathbb{N}$, $p > 3$, respectively. In Section 5 we study Y_{λ_1,λ_2} when the non-resonance condition is preserved finding a similar result to the Lyapunov center theorem. In each resonance case described above:

- the Belitiskii normal form of (3) is computed;
- we present conditions on the coefficients in the normal form of the vector field to guarantee the existence of families of periodic orbits;
- we compute the Belitiskii normal form of (5) and conditions for existence of families of periodic orbits are exhibited;
- a particular example is presented where we exhibit explicitly the coefficients of the normal form and we establish the number of families of periodic orbits.

2. Belitskii Normal Form

In this section we briefly discuss the Belitskii Normal Form.

Consider the C^∞ vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{6}$$

reversible with respect to the involution R with $f(0) = 0$ and $Df(0) = A$.

For $f \in C^\infty(\mathbb{R}^n)$, let $T_m f$ represent its Taylor polynomial of degree m and $\tilde{T}_m f$ represent the homogeneous part of degree m in $T_m f$, that is,

$$T_m f(x) = \sum_{l=0}^{m} \frac{1}{l!} D^l f(0) x^{(l)} \qquad \text{and} \qquad \tilde{T}_m f(x) = \frac{1}{m!} D^m f(0) x^{(m)}, \qquad \forall x \in \mathbb{R}^n$$

where $x^{(l)}$ denotes the l-upla $(x, ..., x)$.

We say that the vector field (6) is in *Belitskii Normal Form* up to order m, $m \geq 2$, with respect to A, if

$$A^* \tilde{T}_l f(x) = D\tilde{T}_l f(x) A^* x, \qquad l \in \{2, ..., m\},$$

that is, $\tilde{T}_l f(x)$, $l = 2, ..., m$, commute with A^*, where A^* is the adjoint matrix of A.

Consider a diffeomorphism

$$\begin{array}{llll} \Phi : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & y & \mapsto & \Phi(y) = y + h(y) \end{array}$$

with $h(y) = o(|y^2|)$. Let $\Phi^* f$ be the pull-back of f under Φ. That is, $(\Phi^* f)(y) = D\Phi(y)^{-1} f(\Phi(y))$.

The proof of the next theorem is in [9].

Theorem 1. *Let $f \in C^\infty(\mathbb{R}^n)$. For each $m \geq 2$, there exist a neighborhood $\mathcal{U}_m$ of the origin in $\mathbb{R}^n$ and a mapping $\Phi \in C^\infty(\mathcal{U}_m)$ such that $\Phi^* f(y) = Ay + g(y)$, where $Dg(0) = 0$ and*

$$A^* \tilde{T}_l g(y) = D\tilde{T}_l g(y) A^* y, \qquad l \in \{2, ..., m\}.$$

If f is reversible with respect to the involution R then we can choose its normal form such that the reversibility is preserved. See [5].

Theorem 2. *The mapping Φ of Theorem 1 may be chosen such that $\Phi R(y) = R(\Phi(y))$. Thus, if $g(y) = \Phi^* f(y)$, we have $DR(y)g(y) = -g(Ry)$.*

3. Lyapunov-Schmidt reduction

We consider the family of R-reversible ODEs

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^{2n}, \lambda \in \mathbb{R} \tag{7}$$

satisfying $f(Rx, \lambda) = -Rf(x, \lambda)$ for all $\lambda \in \mathbb{R}$ where R is a linear involution in $\mathbb{R}^{2n}$. We assume that $f(0, \lambda) = 0$ for all λ close to 0. Let $A_\lambda := D_1 f(0, \lambda)$.

By $C^0_{2\pi}$ we denote the Banach space of continuous 2π-periodic functions $x : \mathbb{R} \to \mathbb{R}^{2n}$, $n \geq 2$ and by $C^1_{2\pi}$ the corresponding C^1-subspace. We define an inner product on $C^0_{2\pi}$ by

$$(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} \langle x_1(t), x_2(t) \rangle dt$$

where $\langle ., . \rangle$ denotes the canonical scalar product in $\mathbb{R}^{2n}$.

Suppose α_0 is a nonzero real number. We are interested in finding small periodic solutions of (7) with period near $\frac{2\pi}{\alpha_0}$.

Now consider a map $F : C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C^0_{2\pi}$ defined by

$$F(x, \lambda, \sigma)(t) = (1 + \sigma)\alpha_0 \dot{x}(t) - f(x(t), \lambda).$$

If $(x_0, \lambda_0, \sigma_0) \in C^1_{2\pi} \times \mathbb{R} \times \mathbb{R}$ is such that

$$F(x_0, \lambda_0, \sigma_0) = 0 \tag{8}$$

then $\tilde{x}(t) := x_0((1+\sigma_0)\alpha_0 t)$ is a $\frac{2\pi}{(1+\sigma_0)\alpha_0}$-periodic solution of (7). Thus the problem of finding all small periodic solutions of (7) with period near $\frac{2\pi}{\alpha_0}$ is reduced to determine the zeros of F with σ and λ near zero.

Observe that $(0, 0, 0)$ is a solution of (8).

Let $L := D_1 F(0, 0, 0) : C^1_{2\pi} \to C^0_{2\pi}$ be given by $Lx(t) = \dot{x}(t) - \frac{1}{\alpha_0} A_0 x(t)$.

Define $L^* : C^1_{2\pi} \to C^0_{2\pi}$ by

$$L^* x(t) = -\dot{x}(t) - \frac{1}{\alpha_0} A_0^* x(t),$$

where A_0^* is the adjoint of A_0.

We assume that A_0 has only purely imaginary eigenvalues.

Let $\{e_1, e_2, ..., e_{2n}\}$ be the canonic bases of $\mathbb{R}^{2n}$ and V_0 be the sum of generalized eigenspace of A_0 with eigenvalues integer multiples of $i\alpha_0$. Let

$$\mathcal{N} = \{q, q(t) = exp(tS_0/\alpha_0)v_0; v_0 \in V_0^{\mathbb{R}}\} \subset C^1_{2\pi}$$

where S_0 is the semisimple part of A_0 and $V_0^{\mathbb{R}}$ is the space of real vectors in V_0.

We try now to put the solutions of (8) in 1-1-correspondence with the solutions of an appropriate equation in $\mathcal{N}$. Define

$$X_1 = \{x \in C^1_{2\pi}; (x, \mathcal{N}) = 0\}, \qquad Y_1 = \{x \in C^0_{2\pi}; (x, \mathcal{N}) = 0\}$$

as the orthogonal complements of $\mathcal{N}$ in $C^1_{2\pi}$ and $C^0_{2\pi}$, respectively.

The proof of the next lemma is straightforward and will be omitted.

Lemma 3. *$L\mathcal{N} \subset \mathcal{N}$ and $L^*\mathcal{N} \subset N$.*

The proof of next result can be found in [4].

Lemma 4. *(Fredholm Alternative): Let $A(t)$ be a matrix in C^0_T and g be in C_T. Then the equation*

$$\dot{x} = A(t)x + g(t)$$

has a solution in C_T if and only if

$$\int_0^T \langle y(t), g(t)\rangle dt = 0$$

for all solution y of the adjoint equation

$$\dot{y} = -A^*(t)y$$

such that $y \in C_T$.

Lemma 5. *The mapping $L : X_1 \to Y_1$ is bijective.*

Proof. Immediate from Lemma 4 and from the fact $L\mathcal{N} \subset \mathcal{N}$. □

Now define the projection

$$P : C^0_{2\pi} \to C^0_{2\pi}$$

by

$$P(.) = \sum_{i=1}^{m} (., q_i)q_i,$$

where $\{q_i\}_{i=1}^m$ is a basis of $\mathcal{N}$. We have $Im(P) = \mathcal{N}$ and $Ker(P) = Y_1$. Hence,

$$C^1_{2\pi} = X_1 \oplus \mathcal{N} \qquad C^0_{2\pi} = Y_1 \oplus \mathcal{N}.$$

Now we consider

$$F(x, \lambda, \sigma) = F(q + x_1, \lambda, \sigma) =: \hat{F}(q, x_1, \lambda, \sigma),$$

$q \in \mathcal{N}$, $x_1 \in X_1$.

Thus

$$\hat{F}(q, x_1, \lambda, \sigma) = 0 \Leftrightarrow \begin{cases} (I - P)\hat{F}(q, x_1, \lambda, \sigma) = 0 \\ P\hat{F}(q, x_1, \lambda, \sigma) = 0. \end{cases}$$

From Lemma (5) and the Implicit Function Theorem we can solve the first equation as $x_1 = x_1^*(q, \sigma, \lambda)$. Then, we need to solve $P\hat{F}(q, x_1^*(q, \sigma, \lambda), \lambda, \sigma) = 0$. But

$$P\hat{F}(q, x_1^*(q, \sigma, \lambda), \lambda, \sigma) = 0 \Leftrightarrow (F, q_i) = 0 \Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} \langle F, q_i\rangle dt = 0 \Leftrightarrow$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{tS_0^*}{\alpha_0}} F(x^*(v_0, \lambda, \sigma), \lambda, \sigma)dt = 0$$

and since $S_0^* = -S_0$ we have

$$P\hat{F}(q, x_1^*(q, \sigma, \lambda), \lambda, \sigma) = 0 \Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{-tS_0}{\alpha_0}} F(x^*(v_0, \lambda, \sigma), \lambda, \sigma)dt = 0.$$

Notice that (v_0, λ, σ) is a solution of (8) provided that

$$B(v_0, \lambda, \sigma) = 0, \tag{9}$$

with $B : \mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2n}$ defined by

$$B(v_0, \lambda, \sigma) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{-tS_0}{\alpha_0}} F(x^*(v_0, \lambda, \sigma), \lambda, \sigma) dt,$$

and

$$x^*(v_0, \lambda, \sigma) = e^{\frac{tS_0}{\alpha_0}} v_0 + x_1^*(e^{\frac{tS_0}{\alpha_0}} v_0, \lambda, \sigma).$$

The proof of the next result can be found in [10].

Lemma 6. *The mapping B has the following properties:*

(1) $RB(v_0, \lambda, \sigma) = -B(Rv_0, \lambda, \sigma)$,
(2) $\varphi_\beta(B(v_0, \lambda, \sigma)) = B(\varphi_\beta(v_0), \lambda, \sigma)$,

where $\varphi_\beta v_0 = exp(-\beta/\alpha_0) v_0$.

The condition (1) says that the mapping B inherits the anti-symmetric properties of X whereas (2) says that B is rotationally equivariant.

Assume that (7) is in Belitskii normal form up to order m. Thus we consider the vector field $f(x, \lambda) = A_\lambda x + \tilde{f}(x, \lambda) + r(x, \lambda)$ with $r(x, \lambda) = o(|x|^{m+1})$. The proof of next result is in [10].

Theorem 7. *The following relations hold:*

(1) $x^*(v_0, \lambda, \sigma)(t) = e^{\frac{tS_0}{\alpha_0}} v_0 + o(|v_0|^{m+1})$;
(2) $B(v_0, \lambda, \sigma) = (1 + \sigma) S v_0 - A_\lambda v_0 - \tilde{f}(v_0, \lambda) + \mathcal{O}(\|v_0\|^{m+1})$.

Recall that the periodic solution of (9) is R-symmetric if and only if it intersects $Fix(R)$ in exactly two points. In conclusion, we obtain all small symmetric periodic solutions of (9) by solving the equation:

$$G(v_0, \lambda, \sigma) = 0,$$

with $G(v_0, \lambda, \sigma) = B(v_0, \lambda, \sigma)\big|_{v_0 \in Fix(R)}$.

4. Case $\alpha : 1 : p$ resonance

In this section, we shall use the Lyapunov-Schmidt reduction to find symmetric periodic solutions with period near 2π and $\dfrac{2\pi}{p}$ for reversible perturbations of the equation (1) when their eigenvalues are in $\alpha : 1 : p$ resonance, $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $p \in \mathbb{N}$, $p > 1$.

We denote system (3) by $\dot{x} = X(x)$, $x \in \mathbb{R}^6$. We can suppose, without loss of generality, that the matrix $A = DX(0)$ has the form up to third order:

$$A = \begin{pmatrix} 0 & -\alpha & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p \\ 0 & 0 & 0 & 0 & p & 0 \end{pmatrix}.$$

Moreover, we shall suppose that X is R-reversible where R is the linear involution in $\mathbb{R}^6$ given by

$$R(x_1,y_1,x_2,y_2,x_3,y_3)=(x_1,-y_1,x_2,-y_2,x_3,-y_3).$$

Denote by $\mathfrak{X}_R(\mathbb{R}^6)$ the space of C^∞ R-reversible vector fields in $\mathbb{R}^6$. Let J^kX be the k-jet of $X\in\mathfrak{X}_R(\mathbb{R}^6)$ at 0.

We obtain the following result:

Theorem 8. *Assume $X\in\mathfrak{X}_R(\mathbb{R}^6)$ such that $X(0)=0$ with $A=DX(0)$ and R satisfying the above conditions. Then X is conjugated, in a neighborhood of the origin, to the following normal form up to third order:*

a) *if $p=2$*

(10)

$$\begin{aligned}
\dot x_1 =&\ -\alpha y_1 - y_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_1 =&\ \alpha x_1 + x_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_2 =&\ -y_2 + b_1(x_3y_2-x_2y_3) - y_2(b_2(x_1^2+y_1^2)+b_3(x_2^2+y_2^2)+b_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_2 =&\ x_2 + b_1(x_2x_3+y_2y_3) + x_2(b_2(x_1^2+y_1^2)+b_3(x_2^2+y_2^2)+b_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_3 =&\ -2y_3 - 2c_1x_2y_2 - y_3(c_2(x_1^2+y_1^2)+c_3(x_2^2+y_2^2)+c_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_3 =&\ 2x_3 + c_1(x_2^2-y_2^2) + x_3(c_2(x_1^2+y_1^2)+c_3(x_2^2+y_2^2)+c_4(x_3^2+y_3^2))+o(\|x\|^4),
\end{aligned}$$

b) *if $p=3$*

(11)

$$\begin{aligned}
\dot x_1 =&\ -\alpha y_1 - y_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_1 =&\ \alpha x_1 + x_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_2 =&\ -y_2 + 2b_1x_2x_3y_2 - b_1x_2^2y_3 + b_1y_2^2y_3 - y_2(b_2(x_1^2+y_1^2)+b_3(x_2^2+y_2^2)+b_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_2 =&\ x_2 + 2b_1x_2y_2y_3 + b_1x_2^2x_3 - b_1y_2^2x_3 + x_2(b_2(x_1^2+y_1^2)+b_3(x_2^2+y_2^2)+b_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_3 =&\ -3y_3 - 3c_1x_2^2y_2 + c_1y_2^3 - y_3(c_2(x_1^2+y_1^2)+c_3(x_2^2+y_2^2)+c_4(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_3 =&\ 3x_3 - 3c_1x_2y_2^2 + c_1x_2^3 + +x_3(c_2(x_1^2+y_1^2)+c_3(x_2^2+y_2^2)+c_4(x_3^2+y_3^2))+o(\|x\|^4),
\end{aligned}$$

c) *if $p>3$*

$$\begin{aligned}
\dot x_1 =&\ -\alpha y_1 - y_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_1 =&\ \alpha x_1 + x_1(a_1(x_1^2+y_1^2)+a_2(x_2^2+y_2^2)+a_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_2 =&\ -y_2 - y_2(b_1(x_1^2+y_1^2)+b_2(x_2^2+y_2^2)+b_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_2 =&\ x_2 + x_2(b_1(x_1^2+y_1^2)+b_2(x_2^2+y_2^2)+b_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot x_3 =&\ -py_3 - y_3(c_1(x_1^2+y_1^2)+c_2(x_2^2+y_2^2)+c_3(x_3^2+y_3^2))+o(\|x\|^4)\\
\dot y_3 =&\ px_3 + x_3(c_1(x_1^2+y_1^2)+c_2(x_2^2+y_2^2)+c_3(x_3^2+y_3^2))+o(\|x\|^4),
\end{aligned}\tag{12}$$

where $x=(x_1,y_1,x_2,y_2,x_3,y_3)$.

Proof. Consider the vector field in complex coordinates (z_1,z_2,z_3) where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ and $z_3=x_3+iy_3$. In these coordinates,

$$A=\begin{pmatrix}\alpha i & 0 & 0 & 0 & 0 & 0\\ 0 & -\alpha i & 0 & 0 & 0 & 0\\ 0 & 0 & i & 0 & 0 & 0\\ 0 & 0 & 0 & -i & 0 & 0\\ 0 & 0 & 0 & 0 & pi & 0\\ 0 & 0 & 0 & 0 & 0 & -pi\end{pmatrix}.$$

Writing $\dot z = Az+h(z)+o(|z|^4)$ with $h(z)=(h_1,h_2,h_3,h_4,h_5,h_6)$ and $z=(z_1,\bar{z_1},z_2,\bar{z_2},z_3,\bar{z_3})$, the condition

$$A^*h(z)=Dh(z)A^*z,$$

implies

$$\begin{aligned} Dh_1(z) = &\quad \alpha i h_1, \\ Dh_2(z) = &\quad -\alpha i h_2, \\ Dh_3(z) = &\quad i h_3, \\ Dh_4(z) = &\quad -i h_4, \\ Dh_5(z) = &\quad p i h_5, \\ Dh_6(z) = &\quad -p i h_6, \end{aligned}$$

where

$$D := \alpha i z_1 \frac{\partial}{\partial z_1} - \alpha i \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + i z_2 \frac{\partial}{\partial z_2} - i \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} + p i z_3 \frac{\partial}{\partial z_3} - p i \bar{z}_3 \frac{\partial}{\partial \bar{z}_3}.$$

Let us check under which conditions $u = z_1^{k_1} \bar{z}_1{}^{k_2} z_2^{k_3} \bar{z}_2{}^{k_4} z_3^{h_5} \bar{z}_3{}^{k_6}$ is into the normal form up to order 3. Thus we have $|k| = \sum_{j=1}^{6} k_j = 2, 3$ with $k = (k_1, k_2, k_3, k_4, k_5, k_6)$.

Firstly, let us consider h_1. Since $Dh_1 = \alpha i h_1$ we have

$$\alpha i u = Du \Rightarrow \alpha i u = (\alpha i k_1 - \alpha i k_2 + i k_3 - i k_4 + p i k_5 - p i k_6) u \Rightarrow$$

$$k_1 - k_2 = 1 \qquad \text{and} \qquad k_3 - k_4 + p(k_5 - k_6) = 0.$$

If an integer $|k|$ satisfies the above conditions then $|k| \neq 2$. Thus h_1 doesn't possess monomial of order 2.

The elements k that satisfy the above conditions with $|k| = 3$ are

$$(2, 1, 0, 0, 0, 0), (1, 0, 1, 1, 0, 0), (1, 0, 0, 0, 1, 1),$$

that represent

$$z_1^2 \bar{z}_1, z_1 z_2 \bar{z}_2 \text{ and } z_1 z_3 \bar{z}_3,$$

respectively. Thus,

$$h_1 = \tilde{a}_1 z_1^2 \bar{z}_1 + \tilde{a}_2 z_1 z_2 \bar{z}_2 + \tilde{a}_3 z_1 z_3 \bar{z}_3.$$

Similarly, for h_2 we obtain

$$h_2 = \tilde{b}_1 \bar{z}_1{}^2 z_1 + \tilde{b}_2 \bar{z}_1 \bar{z}_2 z_2 + \tilde{b}_3 \bar{z}_1 \bar{z}_3 z_3.$$

We know that $\overline{h_1} = h_2$. So $\overline{\tilde{a}_j} = \tilde{b}_j$, $j = 1, 2, 3$.

From the R reversibility, where $R(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) = (\bar{z}_1, z_1, \bar{z}_2, z_2, \bar{z}_3, z_3)$, we have $\tilde{b}_j = -\tilde{a}_j$, $j = 1, 2, 3$. Thus $\tilde{a}_j = i a_j$, $a_j \in \mathbb{R}$, and,

$$h_1 = i(a_1 z_1^2 \bar{z}_1 + a_2 z_1 z_2 \bar{z}_2 + a_3 z_1 z_3 \bar{z}_3),$$

$$h_2 = -i(a_1 \bar{z}_1{}^2 z_1 + a_2 \bar{z}_1 \bar{z}_2 z_2 + a_3 \bar{z}_1 \bar{z}_3 z_3).$$

For h_3, from the equality $Dh_3 = i h_3$, we obtain

$$i u = (\alpha i k_1 - \alpha i k_2 + i k_3 - i k_4 + p i k_5 - p i k_6) u \Rightarrow$$

$$k_1 - k_2 = 0 \qquad \text{and} \qquad k_3 - k_4 + p(k_5 - k_6) = 1.$$

If $p = 2$ then the element k that satisfies the above conditions with $|k| = 2$ is $(0, 0, 0, 1, 1, 0)$ that represents $\bar{z}_2 z_3$.

In $p > 2$ there is no monomial of the second order satisfying the above conditions with $|k| = 2$.

The elements of third order are:

a) $(1, 1, 1, 0, 0), (0, 0, 1, 0, 1, 1), (0, 0, 2, 1, 0, 0)$, if $p = 2$,
b) $(0, 0, 1, 0, 1, 1), (0, 0, 0, 2, 1, 0), (1, 1, 1, 0, 0, 0), (0, 0, 2, 1, 0, 0)$, if $p = 3$,

c) $(0,0,1,0,1,1),(0,0,2,1,0,0),(1,1,1,0,0,0)$, if $p>3$.

that represent

a) $z_2^2\bar{z}_2, z_2z_3\bar{z}_3$and $z_1\bar{z}_1z_2$, respectively, if $p=2$.
b) $z_2z_3\bar{z}_3, \bar{z}_2{}^2z_3, z_1\bar{z}_1z_2$ and $z_2^2\bar{z}_2$, respectively, if $p=3$.
c) $z_2z_3\bar{z}_3, z_2^2\bar{z}_2$ and $z_1\bar{z}_1z_2$, respectively, if $p>3$.

Thus

a) $h_3=\tilde{c}_1\bar{z}_2z_3+\tilde{c}_2z_2^2\bar{z}_2+\tilde{c}_3z_2z_3\bar{z}_3+\tilde{c}_4z_1\bar{z}_1z_2$, if $p=2$;
b) $h_3=\tilde{c}_1z_2z_3\bar{z}_3+\tilde{c}_2\bar{z}_2{}^2z_3+\tilde{c}_3z_1\bar{z}_1z_2+\tilde{c}_4z_2^2\bar{z}_2$, if $p=3$;
c) $h_3=\tilde{c}_1z_2z_3\bar{z}_3+\tilde{c}_2z_2^2\bar{z}_2+\tilde{c}_3z_1\bar{z}_1z_2$, if $p>3$.

Similarly, we found

a) $h_4=\tilde{d}_1z_2\bar{z}_3+\tilde{d}_2\bar{z}_2{}^2z_2+\tilde{d}_3\bar{z}_2\bar{z}_3z_3+\tilde{c}_4\bar{z}_1z_1\bar{z}_2$, if $p=2$;
b) $h_4=\tilde{d}_1\bar{z}_2z_3\bar{z}_3+\tilde{d}_2z_2^2\bar{z}_3+\tilde{d}_3z_1\bar{z}_1\bar{z}_2+\tilde{d}_4z_2\bar{z}_2{}^2$, if $p=3$;
c) $h_4=\tilde{d}_1\bar{z}_2z_3\bar{z}_3+\tilde{d}_2z_2\bar{z}_2{}^2+\tilde{d}_3z_1\bar{z}_1\bar{z}_2$, if $p>3$.

Since $\overline{h_3}=h_4$ and from the R-reversibility, we have $\tilde{c_j}=ic_j$, $\tilde{d_j}=-ic_j$, $c_j\in\mathbb{R}$, $j=1,\ldots,4$.

Similar calculations can be done for h_5 and h_6.

So we obtain:

a) the normal form up to third order if $p=2$

$$\begin{aligned}\dot{z}_1 = &\ \alpha iz_1+iz_1(a_1|z_1|^2+a_2|z_2|^2+a_3|z_3|^2)+o(\|z\|^4)\\ \dot{z}_2 = &\ iz_2+i(b_1\bar{z}_2z_3+b_2z_2^2\bar{z}_2+b_3z_2z_3\bar{z}_3+b_4z_1\bar{z}_1z_2)+o(\|z\|^4)\\ \dot{z}_3 = &\ 2iz_3+i(c_1z_2^2+c_2z_3^2\bar{z}_3+c_3z_2\bar{z}_2z_3+c_4z_1\bar{z}_1z_3)+o(\|z\|^4),\end{aligned}\tag{13}$$

b) the normal form up to third order if $p=3$

$$\begin{aligned}\dot{z}_1 = &\ \alpha iz_1+iz_1(a_1|z_1|^2+a_2|z_2|^2+a_3|z_3|^2)+o(\|z\|^4)\\ \dot{z}_2 = &\ iz_2+i(b_1z_1\bar{z}_1z_2+b_2z_2^2\bar{z}_2+b_3z_2z_3\bar{z}_3+b_4\bar{z}_2{}^2z_3)+o(\|z\|^4)\\ \dot{z}_3 = &\ 3iz_3+i(c_1z_1\bar{z}_1z_3+c_2z_2\bar{z}_2z_3+c_3z_3^2\bar{z}_3+c_4z_2^3)+o(\|z\|^4),\end{aligned}\tag{14}$$

c) the normal form up to third order if $p>3$

$$\begin{aligned}\dot{z}_1 = &\ i\alpha z_1+iz_1(a_1|z_1|^2+a_2|z_2|^2+a_3|z_3|^2)+o(\|z\|^4)\\ \dot{z}_2 = &\ iz_2+iz_2(b_1|z_1|^2+b_2|z_2|^2+b_3|z_3|^2)+o(\|z\|^4)\\ \dot{z}_3 = &\ ipz_3+iz_3(c_1|z_1|^2+c_2|z_2|^2+c_3|z_3|^2)+o(\|z\|^4),\end{aligned}\tag{15}$$

with $z=(z_1,z_2,z_3)$. In coordinates $(x_1,y_1,x_2,y_2,x_3,y_3)$ normal forms (13), (14) and (15) are (10), (11) and (12), respectively.

Thus Theorem 8 is proved. □

Definition 1. *We say that the R-reversible vector field*

$$\dot{X}=AX+F(X),\qquad X\in\mathbb{R}^{2n},$$

satisfies the normal condition on $FixR$ if

$$\langle A,NF\rangle\big|_{FixR}=0,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^{2n}$ and

$$N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Define now the following set
$\mathcal{U}^0 = \{X \in \mathfrak{X}_R(\mathbb{R}^6);\ J^2X$ is expressed by (10) with $b_1c_1 > 0\}$.
$\mathcal{U}^1 = \{X \in \mathfrak{X}_R(\mathbb{R}^6);\ J^3X$ is expressed by (10) with $c_4 \neq 0\}$.
$\mathcal{V}^0 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (11) with $c_4 \neq 0\}$,
$\mathcal{V}^1 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (11) with $b_3 \neq 0\}$,
$\mathcal{V}^2 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (11) with $\Phi \geq 0,\ \dfrac{\Gamma - b_1c_1\sqrt{\Phi}}{\Lambda} \neq 0\}$
$\mathcal{V}_0^i = \{X \in \mathcal{V}^i; X$satisfy the normal condition on $FixR\},\ i = 0, 1, 2$.
$\mathcal{W}^0 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (12) with $c_3 \neq 0\}$,
$\mathcal{W}^1 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (12) with $b_2 \neq 0\}$,
$\mathcal{W}^2 = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$ is expressed by (12) with $b_2(p(b_3+b_2)-(c_2+c_3)) \neq 0\}$
and
$\mathcal{W}_0^i = \{X \in \mathcal{W}^i; X$ satisfies the normal condition on $FixR\}$, $i = 0, 1, 2$.
where

$$\begin{aligned} \Phi &= 9b_1^2 + 36b_3b_4 - 12b_4c_3 + 12b_3c_4 - 4c_3c_4 \\ \Gamma &= 36b_3^2b_4 + 3b_1^2c_3 - 18b_3b_4c_3 + 2b_4c_3^2 + 6b_3^2c_4 - 2b_3c_3c_4 \\ \Lambda &= 36b_3^2b_4^2 + 6b_1^2b_4c_3 - 12b_3b_4^2c_3 + b_4^2c_3^2 + 12b_3^2b_4c_4 + b_1^2c_3c_4 - 2b_3b_4c_3c_4 + b_3^2c_4^2. \end{aligned}$$

Considering the vector field in the Belitskii normal form, we will use the Lyapunov-Schmidt reduction to study the existence of symmetric periodic solutions and to prove the following:

Theorem 9. *The following statements hold:*

i) *Each* $X \in \mathcal{U}^0 \cup \mathcal{V}_0^0 \cup \mathcal{V}_0^1 \cup \mathcal{W}_0^0 \cup \mathcal{W}_0^1$ *possesses a one-parameter family of R-symmetric periodic orbits terminating at the equilibrium with period converging to* 2π*;*
ii) *Each* $X \in \mathcal{U}^1$ *possesses a one-parameter family of R-symmetric periodic orbits terminating at the equilibrium with period converging to* π*.*
iii) *Each* $X \in \mathcal{V}^0$ *possesses a one-parameter family of symmetric periodic solutions converging to the origin with period converging to* $\dfrac{2\pi}{3}$*.*
iv) *Each* $X \in \mathcal{V}_0^2$ *possesses two one-parameter families of symmetric periodic orbits converging to the origin with period converging to* 2π*.*
v) *Each* $X \in \mathcal{W}_0^2$ *possesses two one-parameter families of symmetric periodic orbits converging to the origin with period converging to* 2π*.*
vi) *Each* $X \in \mathcal{W}^0$ *possesses a one-parameter family of symmetric periodic orbits converging to the origin with period converging to* $\dfrac{2\pi}{p}$*.*

Proof. The equation of reduction $B : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ for the solutions with period near 2π is given by

$$B^p(u_4,\sigma) = (1+\sigma)S_4^p u_4 - A_4^p u_4 - \tilde{h}_p(u_4) + o(\|u_4\|^i)$$

where

$$S_4^p = A_4^p = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & p & 0 \end{pmatrix}, \qquad u_4 = \begin{pmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix}, \qquad \tilde{h}_p = \begin{pmatrix} h_3^p \\ h_4^p \\ h_5^p \\ h_6^p \end{pmatrix},$$

with

$$\begin{aligned} h_3^2 =& \quad b_1(x_3y_2 - x_2y_3) \\ h_4^2 =& \quad b_1(x_2x_3 + y_2y_3) \\ h_5^2 =& \quad -2c_1x_2y_2 \\ h_6^2 =& \quad c_1(x_2^2 - y_2^2). \end{aligned}$$

$$\begin{aligned} h_3^3 =& \quad 2b_1x_2x_3y_2 - b_1x_2^2y_3 + b_1y_2^2y_3 - y_2(b_3(x_2^2+y_2^2) + b_4(x_3^2+y_3^2)) \\ h_4^3 =& \quad 2b_1x_2y_2y_3 + b_1x_2^2x_3 - b_1y_2^2x_3 + x_2(b_3(x_2^2+y_2^2) + b_4(x_3^2+y_3^2)) \\ h_5^3 =& \quad -3c_1x_2^2y_2 + c_1y_2^3 - y_3(c_3(x_2^2+y_2^2) + c_4(x_3^2+y_3^2)) \\ h_6^3 =& \quad -3c_1x_2y_2^2 + c_1x_2^3 + +x_3(c_3(x_2^2+y_2^2) + c_4(x_3^2+y_3^2)). \end{aligned}$$

$$\begin{aligned} h_3^p =& \quad -y_2(b_2(x_2^2+y_2^2) + b_3(x_3^2+y_3^2)) \\ h_4^p =& \quad x_2(b_2(x_2^2+y_2^2) + b_3(x_3^2+y_3^2)) \\ h_5^p =& \quad -y_3(c_2(x_2^2+y_2^2) + c_3(x_3^2+y_3^2)) \\ h_6^p =& \quad x_3(c_2(x_2^2+y_2^2) + c_3(x_3^2+y_3^2)). \end{aligned}$$

We obtain:

$$(16) \qquad \begin{cases} -\sigma y_2 - b_1(x_3y_2 - x_2y_3) + o(\|x\|^3) = 0 & \text{(a);} \\ \sigma x_2 - b_1(x_2x_3 + y_2y_3) + o(\|x\|^3) = 0 & \text{(b);} \\ -2\sigma y_3 + 2c_1x_2y_2 + o(\|x\|^3) = 0 & \text{(c);} \\ 2\sigma x_3 - c_1(x_2^2 - y_2^2) + o(\|x\|^3) = 0 & \text{(d),} \end{cases}$$

(17)

$$\begin{cases} -\sigma y_2 - b_1(2x_2x_3y_2 - x_2^2y_3 + y_2^2y_3) + y_2(b_3(x_2^2+y_2^2) + b_4(x_3^2+y_3^2)) + \tilde{\varphi}_1(x_2,y_2,x_3,y_3) = 0, \\ \sigma x_2 - b_1(2x_2y_2y_3 + x_2^2x_3 - y_2^2x_3) - x_2(b_3(x_2^2+y_2^2) + b_4(x_3^2+y_3^2)) + \tilde{\varphi}_2(x_2,y_2,x_3,y_3) = 0, \\ -3\sigma y_3 + c_1(3x_2^2y_2 - y_2^3) + y_3(c_3(x_2^2+y_2^2) + c_4(x_3^2+y_3^2)) + \tilde{\varphi}_3(x_2,y_2,x_3,y_3) = 0, \\ 3\sigma x_3 + c_1(3x_2y_2^2 - x_2^3) - x_3(c_3(x_2^2+y_2^2) + c_4(x_3^2+y_3^2)) + \tilde{\varphi}_4(x_2,y_2,x_3,y_3) = 0, \end{cases}$$

$$(18) \qquad \begin{cases} -\sigma y_2 + y_2(b_2(x_2^2+y_2^2) + b_3(x_3^2+y_3^2)) + \tilde{\varphi}_1(x_2,y_2,x_3,y_3) = 0, \\ \sigma x_2 - x_2(b_2(x_2^2+y_2^2) + b_3(x_3^2+y_3^2)) + \tilde{\varphi}_2(x_2,y_2,x_3,y_3) = 0, \\ -p\sigma y_3 + y_3(c_2(x_2^2+y_2^2) + c_3(x_3^2+y_3^2)) + \tilde{\varphi}_3(x_2,y_2,x_3,y_3) = 0, \\ p\sigma x_3 - x_3(c_2(x_2^2+y_2^2) + c_3(x_3^2+y_3^2)) + \tilde{\varphi}_4(x_2,y_2,x_3,y_3) = 0, \end{cases}$$

from $B_4^2(u,\sigma) = 0$, $B_4^3 = (u,\sigma) = 0$ and $B_4^p(u,\sigma) = 0$, $p > 3$, respectively, where $\tilde{\varphi}_j(x_2,y_2,x_3,y_3) = o(\|x\|^4)$, $x = (x_2,y_2,x_3,y_3)$, $j = 1,...,4$.

First, we will resolve system (16).

From equations (c) and (d) we obtain

$$(19) \qquad y_3 = \frac{2c_1x_2y_2 + o(\|x\|^3)}{2\sigma}, \qquad x_3 = \frac{c_1(x_2^2 - y_2^2) + o(\|x\|^3)}{2\sigma},$$

respectively. We substitute them in (a) and (b), and we have

$$(20) \qquad \begin{cases} -2\sigma^2y_2 + b_1c_1x_2^2y_2 + b_1c_1y_2^3 + \tilde{\varphi}_1(x_2,y_2) = 0 \\ 2\sigma^2x_2 - b_1c_1x_2y_2^2 - b_1c_1x_2^3 + \tilde{\varphi}_2(x_2,y_2) = 0, \end{cases}$$

where $\tilde{\varphi}_i(x_2, y_2) = o(\|(x_2, y_2)\|^3)$, $i = 1, 2$. From R-reversibility of B we get

$$R(\tilde{\varphi}_1(x_2, y_2), \tilde{\varphi}_2(x_2, y_2)) = -(\tilde{\varphi}_1 R(x_2, y_2), \tilde{\varphi}_2 R(x_2, y_2)) \Rightarrow$$
$$(\tilde{\varphi}_1(x_2, y_2), -\tilde{\varphi}_2(x_2, y_2)) = (-\tilde{\varphi}_1(x_2, -y_2), -\tilde{\varphi}_2(x_2, -y_2)) \Rightarrow$$
$$\tilde{\varphi}_1(x_2, y_2) = -\tilde{\varphi}_1(x_2, -y_2).$$

Thus $\tilde{\varphi}_1(x_2, y_2) = y_2 \Theta(x_2, y_2)$, where $\Theta(x_2, y_2) = \Theta(x_2, -y_2)$.

Since $s_\phi B(u, \sigma) = B(s_\phi u, \sigma)$, where $u = (x_1, y_1, x_2, y_2, x_3, y_3)$ and $s_\phi u = \exp(-\phi S_4)u$ we have that if $\phi = \dfrac{\pi}{2}$ and $B = (f_1, f_2, f_3, f_4, f_5, f_6)$ then

$$\begin{pmatrix} f_1(u)\cos\left(\alpha\frac{\pi}{2}\right) + f_2(u)\sin\left(\alpha\frac{\pi}{2}\right) \\ -f_1(u)\sin\left(\alpha\frac{\pi}{2}\right) + f_2(u)\cos\left(\alpha\frac{\pi}{2}\right) \\ f_4(u) \\ -f_3(u) \\ -f_5(u) \\ -f_6(u) \end{pmatrix} = \begin{pmatrix} f_1(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \\ f_2(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \\ f_3(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \\ f_4(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \\ f_5(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \\ f_6(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) \end{pmatrix},$$

where $H_1(x_1, y_1) = \cos\left(\alpha\dfrac{\pi}{2}\right)x_1 + \sin\left(\alpha\dfrac{\pi}{2}\right)y_1$ and $H_2 = -\sin\left(\alpha\dfrac{\pi}{2}\right)x_1 + \cos\left(\alpha\dfrac{\pi}{2}\right)y_1$. From the equality

$$f_3(H_1(x_1, y_1), H_2(x_1, y_1), y_2, -x_2, -x_3, -y_3) = f_4(u),$$

we get

$$\tilde{\varphi}_2(x_2, y_2) = \tilde{\varphi}_1(y_2, -x_2) = -x_2\Theta(y_2, -x_2) = -x_2\Theta(y_2, x_2) = -x_2\Theta_1(x_2, y_2),$$

where Θ, $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ are given above and $\Theta_1(x_2, y_2) = \Theta(y_2, x_2)$. Thus system (20) can be written as

$$\begin{aligned} y_2(-2\sigma^2 + b_1c_1x_2^2 + b_1c_1y_2^2 + \Theta(x_2, y_2)) &= 0 \\ -x_2(-2\sigma^2 + b_1c_1y_2^2 + b_1c_1x_2^2 + \Theta_1(x_2, y_2)) &= 0. \end{aligned}$$

We assume $y_2 = 0$ to obtain R-symmetric solutions. Thus

$$-x_2(-2\sigma^2 + b_1c_1x_2^2 + \Theta_1(x_2, 0)) = 0,$$

and the non trivial solutions for system (16) are

$$x_2 \approx \pm\sqrt{\frac{2\sigma^2}{b_1c_1}}, \qquad x_3 \approx \frac{\sigma}{b_1} \qquad y_2 = y_3 = 0$$

if $b_1c_1 > 0$.

In $\mathcal{U}^0$ system (16) possesses two non trivial solutions which converge to the equilibrium when σ tends to zero. This two solutions correspond to a single orbit because B is s_ϕ-equivariant. Thus X possesses a one-parameter family of symmetric periodic orbits terminating at the equilibrium with period converging to 2π.

Now we will resolve system (17).

From the R-reversibility of B, it holds

$$\begin{aligned} \tilde{\varphi}_1(x_2, y_2, x_3, y_3) &= -\tilde{\varphi}_1(x_2, -y_2, x_3, -y_3) \\ \tilde{\varphi}_3(x_2, y_2, x_3, y_3) &= -\tilde{\varphi}_3(x_2, -y_2, x_3, -y_3). \end{aligned}$$

So

$$\tilde{\varphi}_1(x_2, y_2, x_3, y_3) = y_2\Theta_1(x_2, y_2, x_3, y_3) + y_3\Theta_2(x_2, y_2, x_3, y_3)$$

and

$$\tilde{\varphi}_3(x_2, y_2, x_3, y_3) = y_2\Theta_5(x_2, y_2, x_3, y_3) + y_3\Theta_6(x_2, y_2, x_3, y_3).$$

Since $s_\phi B(u,\sigma) = B(s_\phi u, \sigma)$ if $\phi = \dfrac{\pi}{2}$ we obtain

$$\tilde{\varphi_2}(x_2,y_2,x_3,y_3) = \tilde{\varphi_1}(y_2,-x_2,-y_3,x_3) = -x_2\Theta_1(y_2,-x_2,-y_3,x_3)+x_3\Theta_2(y_2,-x_2,-y_3,x_3) \Rightarrow$$

$$\tilde{\varphi_2}(x_2,y_2,x_3,y_3) = x_2\Theta_3(x_2,y_2,x_3,y_3) + x_3\Theta_4(x_2,y_2,x_3,y_3).$$

Similarly

$$\tilde{\varphi_4}(x_2,y_2,x_3,y_3) = x_2\Theta_7(x_2,y_2,x_3,y_3) + x_3\Theta_8(x_2,y_2,x_3,y_3).$$

So system (17) on $Fix\ R$ can be written as:

$$(21) \qquad \begin{cases} \sigma x_2 - b_1x_2^2x_3 - b_3x_2^3 - b_4x_2x_3^2 + x_2\Theta_3(x_2,x_3) + x_3\Theta_4(x_2,x_3) = 0, \\ 3\sigma x_3 - c_1x_2^3 - c_3x_2^2x_3 - c_4x_3^3 + x_2\Theta_7(x_2,x_3) + x_3\Theta_8(x_2,x_3) = 0. \end{cases}$$

Imposing the normal condition $\langle A_4^3, NF\rangle\big|_{FixR} = 0$, we obtain

$$x_2f_4(x_2,0,x_3,0) + 3x_3f_2(x_2,0,x_3,0) = 0,$$

where $F = (f_1,f_2,f_3,f_4)$ and A_4^3 is given above.

Thus in points $(x_2,0,0,0)$ we have $x_2f_4(x_2,0,0,0) = 0$. This implies that $f_4(x_2,0,x_3,0) = x_3\tilde{f}_4(x_2,x_3)$. Similarly, $f_2(x_2,0,x_3,0) = x_2\tilde{f}_2(x_2,x_3)$. So system (21) becomes

$$\begin{cases} x_2(\sigma - b_1x_2x_3 - b_3x_2^2 - b_4x_3^2 + \tilde{\Theta}_3(x_2,x_3)) = 0 & (G_1), \\ x_3(3\sigma - c_3x_2^2 - c_4x_3^2 + \tilde{\Theta}_8(x_2,x_3)) = 0 & (G_2). \end{cases}$$

From G_1 we have $x_2 = 0$ or $\sigma - b_1x_2x_3 - b_3x_2^2 - b_4x_3^2 + \tilde{\Theta}_3(x_2,x_3) = 0$.

If $x_2 = 0$ then $x_3(3\sigma - c_4x_3^2 + \Theta_8(x_3)) = 0$. Thus if $c_4 \neq 0$ then all non trivial solutions of system (4) are

$$x_2 = 0 \qquad x_3 \approx \pm\sqrt{\frac{3\sigma}{c_4}}.$$

If $\sigma - b_1x_2x_3 - b_3x_2^2 - b_4x_3^2 + \tilde{\Theta}_3(x_2,x_3) = 0$ then from $G_2 = 0$ we obtain:

$$x_3(3b_1x_2x_3 + 3b_3x_2^2 + 3b_4x_3^2 - c_3x_2^2 - c_4x_3^2 + \tilde{\Theta}(x_2,x_3)) = 0.$$

We have $x_3 = 0$ or $3b_1x_2x_3 + 3b_3x_2^2 + 3b_4x_3^2 - c_3x_2^2 - c_4x_3^2 + \tilde{\Theta}(x_2,x_3) = 0$.

If $x_3 = 0$ we have the non trivial solution

$$x_2 \approx \pm\sqrt{\frac{\sigma}{b_3}}, \qquad x_3 = 0.$$

If $3b_1x_2x_3 + 3b_3x_2^2 + 3b_4x_3^2 - c_3x_2^2 - c_4x_3^2 + \tilde{\Theta}(x_2,x_3) = 0$, we find 4 non trivial solutions:

$$x_2 \approx \pm\left(\frac{3b_1 \pm \sqrt{\Phi}}{2\sqrt{2}(3b_3 - c_3)}\right)\sqrt{\frac{\sigma(\Gamma - b_1c_3\sqrt{\Phi})}{\Lambda}} \qquad x_3 \approx = \pm\sqrt{\frac{\sigma(\Gamma - b_1c_3\sqrt{\Phi})}{2\Lambda}},$$

In $\mathcal{V}_0^0 \cup \mathcal{V}_0^1$ system (17) possesses 2 non trivial solutions converging to the origin when $\sigma \to 0$. Thus, there exists a one-parameter family of symmetric periodic orbits converging to the equilibrium with period converging to 2π.

In $\mathcal{V}_0^2$ system (17) possesses 4 non trivial solutions converging to the origin when $\sigma \to 0$. Thus there exist two one-parameter families of symmetric periodic orbits converging to the equilibrium with period converging to 2π.

Finally, we will resolve system (18).

Imposing the normal condition, $\langle A_4^p, NF\rangle\big|_{FixR} = 0$, with A_4^p given above and $F = (f_1, f_2, f_3, f_4)$ we obtain

$$x_2 f_4(x_2, 0, x_3, 0) + p x_3 f_2(x_2, 0, x_3, 0) = 0.$$

Thus in points $(x_2, 0, 0, 0)$ we have $x_2 f_4(x_2, 0, 0, 0) = 0$. So $f_4(x_2, 0, x_3, 0) = x_3 \tilde{f}_4(x_2, x_3)$. Similarly, $f_2(x_2, 0, x_3, 0) = x_2 \tilde{f}_2(x_2, x_3)$. Thus system (18) restricted to $FixR$ becomes

$$\begin{cases} x_2(\sigma - b_2 x_2^2 - b_3 x_3^2 + \Theta_1(||(x_2, x_3)||^3) = 0 & (G_1), \\ x_3(p\sigma - c_2 x_2^2 - c_3 x_3^2 + \Theta_2(||(x_2, x_3)||^3) = 0 & (G_2). \end{cases}$$

From G_1 we obtain $x_2 = 0$ or $\sigma = b_2 x_2^2 + b_3 x_3^2 + \Theta_1(||(x_2, x_3)||^3$.

If $x_2 = 0$ then $x_3(p\sigma - c_3 x_3^2 + \Theta_2(||(x_2, x_3)||^3)) = 0$. Thus the non trivial solution is

$$x_2 = 0 \qquad x_3 \approx \pm\sqrt{\frac{p\sigma}{c_3}},$$

if $c_3 \neq 0$.

If $\sigma = b_2 x_2^2 + b_3 x_3^2 + \Theta_1(||(x_2, x_3)||^3$ then $x_3(p b_2 x_2^2 + p b_3 x_3^2 - c_2 x_2^2 - c_3 x_3^2 + \bar{\Theta}(||(x_2, x_3)||^3)) = 0$. We have $x_3 = 0$ or $p b_2 x_2^2 + p b_3 x_3^2 - c_2 x_2^2 - c_3 x_3^2 + \bar{\Theta}(||(x_2, x_3)||^3) = 0$. Case $x_3 = 0$ then $\sigma = b_2 x_2^2 + \Theta_1(||(x_2, x_3)||^3$ and so the non trivial solution is:

$$x_2 \approx \pm\sqrt{\frac{\sigma}{b_2}},$$

if $b_2 \neq 0$.

Case $p b_2 x_2^2 + p b_3 x_3^2 - c_2 x_2^2 - c_3 x_3^2 + \bar{\Theta}(||(x_2, x_3)||^3) = 0$, if $p b_2 - c_2 \neq 0$ this equation can be solved for x_2^2 in terms of x_3. Thus:

$$x_2^2 \approx \frac{-(p b_3 - c_3) x_3^2}{p b_2 - c_2}$$

and so $x_3^2 \approx \dfrac{\sigma}{-b_2\left(\dfrac{p b_3 - c_3}{p b_2 - c_2} + 1\right)}$. In this way non trivial solutions are given by

$$x_2 \approx \pm\sqrt{\frac{\sigma(p b_3 - c_3)}{-b_2(p(b_3 + b_2) - (c_2 + c_3))}}, \qquad x_3 \approx \pm\sqrt{\frac{\sigma(p b_2 - c_2)}{-b_2(p(b_3 + b_2) - (c_2 + c_3))}},$$

if $b_2(p(b_3 + b_2) - (c_2 + c_3)) \neq 0$.

In $\mathcal{W}_0^0 \cup \mathcal{W}_0^1$ system (18) possesses 2 non trivial solutions converging to the origin when $\sigma \to 0$. In this case we conclude that there exists a one-parameter family of symmetric period orbits converging to the equilibrium with period converging to 2π.

In $\mathcal{W}_0^2$ system (18) possesses 4 non trivial solutions converging to the origin when $\sigma \to 0$. In this case we conclude that there exist two one-parameter families of symmetric periodic orbits converging to the equilibrium with period converging to 2π.

For periodic solutions with period near $2\pi/p$, the reduction mapping $B_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is given by:

$$B_2^p(u_2, \sigma) = (1 + \sigma) S_2^p u_2 - A_2^p u_2 - \tilde{h}_p(u_2) + o(\|u_2\|^4),$$

where

$$S_2^p = A_2^p = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \qquad \tilde{h_2}_p = \begin{pmatrix} h_5 \\ h_6 \end{pmatrix}.$$

Similar to the previous case, from the R-reversibility of B and from the relation $s_\phi B(u,\sigma) = B(s_\phi u, \sigma)$ we have that $B_2^p(u_2,\sigma) = 0$ implies:

a) If $p = 2$ then $x_3(2\sigma - c_4 x_3^2 + \Theta_2(x_3)) = 0$ whose non-trivial solutions are

$$x_3 \approx \pm\sqrt{\frac{2\sigma}{c_4}},$$

provided that $\dfrac{\sigma}{c_4} > 0$.

b) If $p = 3$ then $3\sigma x_3 - c_4 x_3^3 + x_3\tilde{\Theta}_8(x_3) = 0$ whose non trivial solutions are

$$x_3 \approx \pm\sqrt{\frac{3\sigma}{c_4}},$$

if $c_4 \neq 0$.

c) If $p > 3$ then $x_3(p\sigma - c_3 x_3^2 + \tilde{\Theta}_2(x_3)) = 0$ whose non trivial solutions are

$$x_3 \approx \pm\sqrt{\frac{p\sigma}{c_3}},$$

if $c_3 \neq 0$.

Thus in $\mathcal{U}^1$ there is a one-parameter family of symmetric periodic orbits terminating at the equilibrium with period converging to π.

In $\mathcal{V}^0$ there exist a one-parameter family of symmetric periodic solutions converging to the equilibrium with period converging to $\dfrac{2\pi}{3}$.

In $\mathcal{W}^0$, there exist a one-parameter family of symmetric period orbits converging to the equilibrium with period converging to $\dfrac{2\pi}{p}$.

□

Now we consider (4) with $f(x) = ax^3 + o(x^4)$, where $a \neq 0$, $a \in \mathbb{R}$, and $o(x^4)$ denotes the terms of superior order to 3.

System (5) can be written as

$$\dot{X} = AX + F(X) \tag{22}$$

where $X = (x_1, y_2, x_2, y_2, x_3, y_3)$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -\lambda_2 & 0 & -\lambda_1 & 0 \end{pmatrix} \tag{23}$$

and

$$F(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ ax_1^3 + o(\|x_1\|^4) \end{pmatrix}. \tag{24}$$

Assume that the eigenvalues of A are in $\alpha : 1 : p$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, resonance. Thus, there is a matrix P such that $P^{-1}AP = J$ where J is the Jordan matrix of A. That is,

$$J = \begin{pmatrix} 0 & -\alpha & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p \\ 0 & 0 & 0 & 0 & p & 0 \end{pmatrix}.$$

We denote $P = \left[j_{m\times n}\right]_{6\times 6}$ and its inverse $P^{-1} = \left[l_{m\times n}\right]_{6\times 6}$.

Lemma 10. *Consider system* (5) *with* $f(x_1) = ax_1^3 + o(x_1^4)$, $a \in \mathbb{R}$. *Then there exists a mapping* $x = \Phi(y)$ *with* $\Phi \in C^\infty(\mathbb{R}^6)$, *tangent to the identity, such that* (5) *is transformed into:*

a) if $p = 2$, *the normal form* (10) *up to third order with coefficients given by*

$$\begin{aligned} a_1 &= \frac{3a}{8}(j_{12}^3 l_{16} - j_{11}^3 l_{26}), & b_1 &= 0, \\ a_2 &= \frac{3a}{8}(2j_{14}^2 j_{12} l_{16} - 2j_{13}^2 j_{11} l_{26}), & b_2 &= \frac{3a}{8}(2j_{12}^2 j_{14} l_{36} - 2j_{11}^2 j_{13} l_{46}), \\ a_3 &= \frac{3a}{8}(2j_{16}^2 j_{12} l_{16} - 2j_{15}^2 j_{11} l_{26}), & b_3 &= \frac{3a}{8}(-j_{13}^3 l_{46} + j_{14}^3 l_{36}), \\ & & b_4 &= \frac{3a}{8}(-2j_{15}^2 j_{13} l_{46} + 2j_{16}^2 j_{14} l_{36}), \end{aligned}$$

$$\begin{aligned} c_1 &= 0, \\ c_2 &= \frac{3a}{8}(2j_{12}^2 j_{16} l_{56} - 2j_{11}^2 j_{15} l_{66}), \\ c_3 &= \frac{3a}{8}(2j_{14}^2 j_{16} l_{56} - 2j_{13}^2 j_{15} l_{66}), \\ c_4 &= \frac{3a}{8}(j_{16}^3 l_{56} - j_{15}^3 l_{66}). \end{aligned}$$

b) if $p = 3$, *the normal form* (11) *up to third order with coefficients:*

$$\begin{aligned} a_1 &= -\frac{3a}{8}(j_{12}^3 l_{16} - j_{11}^3 l_{26}), & b_1 &= \frac{3a}{8}(j_{13}^2 j_{15} l_{46} + j_{14}^2 j_{16} l_{36}), \\ a_2 &= -\frac{3a}{8}(2j_{14}^2 j_{12} l_{16} - 2j_{13}^2 j_{11} l_{26}), & b_2 &= -\frac{3a}{8}(2j_{12}^2 j_{14} l_{36} - 2j_{11}^2 j_{13} l_{46}), \\ a_3 &= -\frac{3a}{8}(2j_{16}^2 j_{12} l_{16} - 2j_{15}^2 j_{11} l_{26}), & b_3 &= -\frac{3a}{8}(-j_{13}^3 l_{46} + j_{14}^3 l_{36}), \\ & & b_4 &= -\frac{3a}{8}(-2j_{15}^2 j_{13} l_{46} + 2j_{16}^2 j_{14} l_{36}), \end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{a}{8}(j_{14}^3 l_{56} + j_{13}^3 l_{66}),\\
c_2 &= -\frac{3a}{8}(2j_{12}^2 j_{16} l_{56} - 2j_{11}^2 j_{15} l_{66}),\\
c_3 &= -\frac{3a}{8}(2j_{14}^2 j_{16} l_{56} - 2j_{13}^2 j_{15} l_{66}),\\
c_4 &= -\frac{3a}{8}(j_{16}^3 l_{56} - j_{15}^3 l_{66}).
\end{aligned}$$

c) if $p > 3$ *the normal form* (12) *up to third order with coefficients:*

$$\begin{aligned}
a_1 &= -\frac{3a}{8}(j_{12}^3 l_{16} - j_{11}^3 l_{26}), & b_1 &= -\frac{3a}{8}(2j_{12}^2 j_{14} l_{36} - 2j_{11}^2 j_{13} l_{46}),\\
a_2 &= -\frac{3a}{8}(2j_{14}^2 j_{12} l_{16} - 2j_{13}^2 j_{11} l_{26}), & b_2 &= -\frac{3a}{8}(-j_{13}^3 l_{46} + j_{14}^3 l_{36}),\\
a_3 &= -\frac{3a}{8}(2j_{16}^2 j_{12} l_{16} - 2j_{15}^2 j_{11} l_{26}), & b_3 &= -\frac{3a}{8}(-2j_{15}^2 j_{13} l_{46} + 2j_{16}^2 j_{14} l_{36}),
\end{aligned}$$

$$\begin{aligned}
c_1 &= -\frac{3a}{8}(2j_{12}^2 j_{16} l_{56} - 2j_{11}^2 j_{15} l_{66}),\\
c_2 &= -\frac{3a}{8}(2j_{14}^2 j_{16} l_{56} - 2j_{13}^2 j_{15} l_{66}),\\
c_3 &= -\frac{3a}{8}(j_{16}^3 l_{56} - j_{15}^3 l_{66}).
\end{aligned}$$

Proof. The mapping $x = P\tilde{x}$ with $x = (x_1, y_1, x_2, y_2, x_3, y_3)$, $\tilde{x} = (\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3)^T$ transforms system (5) into the following system $\dot{\tilde{X}} = A\tilde{X} + F(\tilde{X})$ with same linear part as (10):

(25)

$$\begin{aligned}
\dot{\tilde{x}}_1 &= -\alpha\tilde{y}_1 + a\ l_{16}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4)\\
\dot{\tilde{y}}_1 &= \alpha\tilde{x}_1 + a\ l_{26}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4)\\
\dot{\tilde{x}}_2 &= -\tilde{y}_2 + a\ l_{36}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4)\\
\dot{\tilde{y}}_2 &= \tilde{x}_2 + a\ l_{46}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4)\\
\dot{\tilde{x}}_3 &= -2\tilde{y}_3 + +a\ l_{56}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4)\\
\dot{\tilde{y}}_3 &= 2\tilde{x}_3 + a\ l_{66}\,(j_{11}\tilde{x}_1 + j_{12}\tilde{y}_1 + j_{13}\tilde{x}_2 + j_{14}\tilde{y}_2 + j_{15}\tilde{x}_3 + j_{16}\tilde{y}_3)^3 + o(\|\tilde{x}\|^4).
\end{aligned}$$

System (25) is carried in the normal form (10) up to order 3 by a mapping $\tilde{x} = \Phi(y) = y + G(y)$. We found $\Phi : \mathbb{R}^6 \to \mathbb{R}^6$ by using the MATHEMATICA 6.0 program in the following way.

We define $G(x_1,y_1,x_2,y_2,x_3,y_3) = \begin{pmatrix} g_1(x_1,y_1,x_2,y_2,x_3,y_3) \\ g_2(x_1,y_1,x_2,y_2,x_3,y_3) \\ g_3(x_1,y_1,x_2,y_2,x_3,y_3) \\ g_4(x_1,y_1,x_2,y_2,x_3,y_3) \\ g_5(x_1,y_1,x_2,y_2,x_3,y_3) \\ g_6(x_1,y_1,x_2,y_2,x_3,y_3) \end{pmatrix}$ where $g_i = \sum_{|k|=3} a_{k_1k_2k_3k_4k_5k_6} x_1^{k_1} y_1^{k_2} x_2^{k_3} y_2^{k_4} x_3^{k_5} y_3^{k_6}$ with $|k| = \sum_{i=1}^{6} k_i$.

We impose the condition $R\Phi = \Phi R$ and then we compute $B(x_1,y_1,x_2,y_2,x_3,y_3) = J\Phi - D\Phi J$.

Thus, the system assumes the form $\dot{X} = AX + F(X) + B(X)$, with $X = (x_1,y_1,x_2,y_2,x_3,y_3)$. Identifying this vector field to (10) we obtain the coefficients of Φ and also the coefficients of the normal form (5).

The full code of this program is available in the author's website:
https://sites.google.com/site/mereuufscar/publications.

In the same way we find the coefficients if the normal form of the $p = 3$ and $p > 3$ cases. □

Example: Consider (5) with $f(x_1) = x_1^3$ and $\lambda_1 = \lambda_2 = \dfrac{31}{5}$. Thus the eigenvalues of A are $\pm i$, $\pm\dfrac{i}{\sqrt{5}}$ and $\pm i\sqrt{5}$, that are in $\sqrt{5} : 1 : 5$ resonance.

Firstly, we make a time rescaling by: $t = \sqrt{5}\tau$ transforming system (5) into:

$$\begin{aligned} \dot{x}_1 &= \sqrt{5}y_1 \\ \dot{y}_1 &= \sqrt{5}x_2 \\ \dot{x}_2 &= \sqrt{5}y_2 \\ \dot{y}_2 &= \sqrt{5}x_3 \\ \dot{x}_3 &= \sqrt{5}y_3 \\ \dot{y}_3 &= -\sqrt{5}\lambda_1 x_3 - \sqrt{5}\lambda_2 x_2 - \sqrt{5}x_1 + \sqrt{5}x_1^3. \end{aligned} \tag{26}$$

The mapping $x = P\tilde{x}$ with $x = (x_1,y_1,x_2,y_2,x_3,y_3)$, $\tilde{x} = (\tilde{x}_1,\tilde{y}_1,\tilde{x}_2,\tilde{y}_2,\tilde{x}_3,\tilde{y}_3)^T$ and

$$P = \begin{pmatrix} 0 & 1 & 0 & 25\sqrt{5} & 0 & \frac{1}{25\sqrt{5}} \\ 1 & 0 & 25 & 0 & \frac{1}{25} & 0 \\ 0 & -1 & 0 & -5\sqrt{5} & 0 & -\frac{1}{5\sqrt{5}} \\ -1 & 0 & -5 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & 0 & \sqrt{5} & 0 & \frac{1}{\sqrt{5}} \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

transforms system (26) into

$$\dot{\tilde{X}} = J\tilde{X} + F(\tilde{X}) \tag{27}$$

with same linear part as (10) by putting $\alpha = \sqrt{5}$:

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= -\sqrt{5}\tilde{y}_1 - \frac{5\sqrt{5}}{16}\left(\tilde{y}_1 + 25\sqrt{5}\tilde{y}_2 + \frac{\tilde{y}_3}{25\sqrt{5}}\right)^3 \\
\dot{\tilde{y}}_1 &= \sqrt{5}\tilde{x}_1 \\
\dot{\tilde{x}}_2 &= -\tilde{y}_2 + \frac{1}{96}\sqrt{5}\left(\tilde{y}_1 + 25\sqrt{5}\tilde{y}_2 + \frac{\tilde{y}_3}{25\sqrt{5}}\right)^3 \\
\dot{\tilde{y}}_2 &= \tilde{x}_2 \\
\dot{\tilde{x}}_3 &= -5\tilde{y}_3 + \frac{125}{96}\sqrt{5}\left(\tilde{y}_1 + 25\sqrt{5}\tilde{y}_2 + \frac{\tilde{y}_3}{25\sqrt{5}}\right)^3 \\
\dot{\tilde{y}}_3 &= 5\tilde{x}_3.
\end{aligned}
\tag{28}
$$

Thus, by Lemma 10, there exists a diffeomorphism Φ transforming $F(\tilde{X})$ into the terms of third order of the normal form (12) with coefficients given by:

$$
\begin{array}{lll}
a_1 = \dfrac{15\sqrt{5}}{128} & b_1 = -\dfrac{125}{128} & c_1 = -\dfrac{5}{128} \\
a_2 = \dfrac{46875\sqrt{5}}{64} & b_2 = -\dfrac{390625}{256} & c_2 = -\dfrac{15625}{128} \\
a_3 = \dfrac{3\sqrt{5}}{40000} & b_3 = -\dfrac{1}{3200} & c_3 = -\dfrac{1}{160000}.
\end{array}
$$

That is, Φ transforms (27) in $\dot{\tilde{X}} = J\tilde{X} + J^3\tilde{X} + R(\tilde{X})$. Since we don't know if $R(\tilde{X})$ satisfies the normal condition, we cannot apply Theorem 9.

Now we can written system (26) as $\dot{X} = \sqrt{5}AX + \sqrt{5}F(X)$ where A and F are given by (23) and (24), respectively. We consider a perturbation $\dot{X} = \sqrt{5}AX + \sqrt{5}F(X) + \tilde{h}$, where $\tilde{h} = (h_1, h_2, h_3, h_4, h_5, h_6)$ satisfies $\tilde{h}(x) = P(-R)P^{-1}(\tilde{X})$. In this way, the system becomes $\dot{\tilde{X}} = J\tilde{X} + J^3\tilde{X}$ and applying Theorem 9 we have immediately that:

i) $X \in \mathcal{W}_0^0 \cup \mathcal{W}_0^1$ and so possesses a one-parameter family of symmetric periodic solutions converging to the origin with period converging to 2π.
ii) $X \in \mathcal{W}_0^2$ and so possesses two one-parameter families of symmetric periodic solutions converging to the origin with period converging to 2π.
iii) $X \in \mathcal{W}^3$ and so possesses two one-parameter families of symmetric periodic solutions converging to the origin with period converging to $\dfrac{2\pi}{5}$.

5. Case $\alpha_1 : \alpha_2 : \alpha_3$ non resonant

For any small reversible perturbations of equation (1), the Lyapunov center theorem states the existence of three one-parameter families of periodic solutions with period near $\dfrac{2\pi}{\alpha_i}$, $i = 1, 2, 3$. In this section, we will use Lyapunov-Schmidt reduction to find a similar result. Firstly, we will find the Belitskii normal form .

Without loss of generality, we can suppose

$$A = DX(0) = \begin{pmatrix} 0 & -\alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 \end{pmatrix},$$

$\alpha_j \in \mathbb{R}$, $j = 1, 2, 3$.

Theorem 11. *Assume $X \in \mathfrak{X}_R(\mathbb{R}^6)$ such that $X(0) = 0$ with $A = DX(0)$ and R given by conditions above. Then X is formally conjugated, in a neighborhood of the origin, to the following normal form:*

$$\begin{array}{rl} \dot{x}_1 = & -\alpha_1 y_1 - y_1\varphi_1(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2) \\ \dot{y}_1 = & \alpha_1 x_1 + x_1\varphi_1(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2) \\ \dot{x}_2 = & -\alpha_2 y_2 - y_2\varphi_2(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2) \\ \dot{y}_2 = & \alpha_2 x_2 + x_2\varphi_2(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2) \\ \dot{x}_3 = & -\alpha_3 y_3 - y_3\varphi_3(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2) \\ \dot{y}_3 = & \alpha_3 x_3 + x_3\varphi_3(x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2). \end{array} \tag{29}$$

Proof. Considering the vector field in complex coordinates, we have

$$A = \begin{pmatrix} \alpha_1 i & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1 i & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 i & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 i & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 i & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 i \end{pmatrix}.$$

If $h(x) = (h_1(x), ..., h_6(x))$ the condition

$$A^* h(x) = Dh(x) A^* x,$$

with $x = (z_1, \bar{z_1}, z_2, \bar{z_2}, z_3, \bar{z_3})$, implies that

$$\begin{array}{rl} Dh_1(x) = & -\alpha_1 i h_1, \\ Dh_2(x) = & \alpha_1 i h_2, \\ Dh_3(x) = & -\alpha_2 i h_3, \\ Dh_4(x) = & \alpha_2 i h_4, \\ Dh_5(x) = & -\alpha_3 i h_5, \\ Dh_6(x) = & \alpha_3 i h_6, \end{array}$$

where

$$D := -i\alpha_1 z_1 \frac{\partial}{\partial z_1} + i\alpha_1 \bar{z_1} \frac{\partial}{\partial \bar{z_1}} - i\alpha_2 z_2 \frac{\partial}{\partial z_2} + i\alpha_2 \bar{z_2} \frac{\partial}{\partial \bar{z_2}} - i\alpha_3 z_3 \frac{\partial}{\partial z_3} + i\alpha_3 \bar{z_3} \frac{\partial}{\partial \bar{z_3}}.$$

Consider $u = z_1^{k_1} \bar{z_1}^{k_2} z_2^{k_3} \bar{z_2}^{k_4} z_3^{k_5} \bar{z_3}^{k_6}$.

From $Dh_1(x) = -i\alpha_1 h_1$ we have

$$-i\alpha_1 u = (-i\alpha_1 k_1 + i\alpha_1 k_2 - i\alpha_2 k_3 + i\alpha_2 k_4 - i\alpha_3 k_5 + i\alpha_3 k_6) u \Rightarrow$$

$$-\alpha_1 = -\alpha_1(k_1 - k_2) - \alpha_2(k_3 - k_4) - \alpha_3(k_5 - k_6).$$

As the eigenvalues are not in resonance, we have

$$k_1 - k_2 = 1 \qquad k_3 = k_4 \qquad k_5 = k_6,$$

that is,

$$u = z_1^{k_2+1}\bar{z}_1^{k_2} z_2^{k_3}\bar{z}_2^{k_3} z_3^{k_5}\bar{z}_3^{k_5}.$$

Thus

$$h_1 = z_1\varphi_1(|z_1|^2, |z_2|^2, |z_3|^2).$$

Similarly,

$$h_2 = \bar{z}_1\varphi_2(|z_1|^2, |z_2|^2, |z_3|^2).$$

If we denote $\tilde{\phi}_1$ and $\tilde{\phi}_2$ the coefficients of φ_1 and φ_2, respectively, we will have $\overline{\tilde{\varphi}_1} = \tilde{\phi}_2$, since $\overline{h_1} = h_2$. From the R reversibility we have $\tilde{\phi}_2 = -\tilde{\phi}_1$. So $\tilde{\phi}_1 = i\phi_1$, $\phi_1 \in \mathbb{R}$.

Thus

$$\begin{aligned} h_1 &= iz_1\varphi_1(|z_1|^2, |z_2|^2, |z_3|^2) \\ h_2 &= -i\bar{z}_1\varphi_1(|z_1|^2, |z_2|^2, |z_3|^2). \end{aligned}$$

In a similar way we obtain h_3, h_4, h_5 e h_6 and the normal form:

$$\begin{aligned} \dot{z}_1 &= i\alpha_1 z_1 + iz_1\varphi_1(|z_1|^2, |z_2|^2, |z_3|^2) \\ \dot{z}_2 &= i\alpha_2 z_2 + iz_2\varphi_2(|z_1|^2, |z_2|^2, |z_3|^2) \\ \dot{z}_3 &= i\alpha_3 z_3 + iz_3\varphi_3(|z_1|^2, |z_2|^2, |z_3|^2). \end{aligned} \tag{30}$$

In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ system 30 is transformed in system 29.

□

Define

$\mathcal{U}^i = \{X \in \mathfrak{X}_R(\mathbb{R}^6); J^3X$is expressed by (29) with $a_i \neq 0$
where a_i is the coefficient of monomial x_i^2 in $\varphi_i\}$, $i = 1, 2, 3$.

Using the Lyapunov-Schmidt reduction, we obtain the following result:

Theorem 12. *Each $X \in \mathcal{U}^j$ possesses a one-parameter family of symmetric periodic solutions converging to the origin with period converging to $\dfrac{2\pi}{\alpha_j}$, $j = 1, 2, 3$.*

Proof. In order to find the periodic solutions of period near $\dfrac{2\pi}{\alpha_1}$, the reduction mapping $B : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is given by:

$$B(u, \sigma) = (1 + \sigma)Su - Au - \tilde{h}(u) + o(\|u\|^4),$$

where

$$S = A = \begin{pmatrix} 0 & -\alpha_1 \\ \alpha_1 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \tilde{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

So we should find the non trivial solutions of the system

$$\begin{aligned} -\sigma\alpha_1 y_1 + y_1(a_1(x_1^2 + y_1^2)) + y_1\tilde{\varphi}_1(\|(x_1, y_1)\|^4) &= 0 \\ \sigma\alpha_1 x_1 - x_1(a_1(x_1^2 + y_1^2)) + x_1\tilde{\varphi}_2(\|(x_1, y_1)\|^4) &= 0. \end{aligned} \tag{31}$$

To obtain the solutions R-symmetries we do $y_1 = 0$ and so

$$x_1(\sigma\alpha_1 + a_1 x_1^2 + \tilde{\varphi}_2(\|(x_1)\|^4)) = 0,$$

and non trivial solutions for system (31) are given by

$$x_1 \approx \pm\sqrt{\frac{\sigma\alpha_1}{a_1}} \qquad y_1 = 0,$$

if $a_1 \neq 0$.

Similarly we obtain the results for symmetric period orbits with period near $\dfrac{2\pi}{\alpha_2}$ and $\dfrac{2\pi}{\alpha_3}$. □

Now, we consider perturbations of (1) of the form (4) with $f(x) = ax^3 + o(x^4)$, $a \in \mathbb{R}$, $a \neq 0$, where $o(x^4)$ denote the terms of superior order to 3.

System (5) can be written as $\dot{X} = AX + F(X)$, where $X = (x_1, y_2, x_2, y_2, x_3, y_3)$, A and F are given by (23) and (24), respectively.

Consider λ_1 and λ_2 such that the eigenvalues of A are $\alpha_1 : \alpha_2 : \alpha_3$ no resonance, $\alpha_i \in \mathbb{R}$. In this way, there exists a matrix P such that $P^{-1}AP = J$ where J is the Jordan matrix of A:

$$J = \begin{pmatrix} 0 & -\alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 \end{pmatrix}.$$

We denote $P = \left[j_{m\times n}\right]_{6\times 6}$ and its inverse $P^{-1} = \left[l_{m\times n}\right]_{6\times 6}$.

Lemma 13. *There exists a mapping $x = \Phi(y)$ where $\Phi \in C^\infty(\mathbb{R}^6)$, tangent to identity, such that* (5) *with $f(x_1) = ax_1^3 + o(x_1^4)$, $a \in \mathbb{R}$, is transformed into the normal form* (29) *up to third order with coefficients:*

$$\begin{aligned} a_1 &= -aj_{12}^3 l_{16}, & b_1 &= -3aj_{12}^2 j_{14} l_{36}, \\ a_2 &= -3aj_{14}^2 j_{12} l_{16}, & b_2 &= -aj_{14}^3 l_{36}, \\ a_3 &= -3a2j_{16}^2 j_{12} l_{16}, & b_3 &= -3aj_{16}^2 j_{14} l_{36}, \end{aligned}$$

$$\begin{aligned} c_1 &= -3aj_{12}^2 j_{16} l_{56}, \\ c_2 &= -3aj_{14}^2 j_{16} l_{56}, \\ c_3 &= -aj_{16}^3 l_{56}. \end{aligned}$$

We find the coefficients of the normal form using the MATHEMATICA 6.0 program in the same way as the $\alpha : 1 : 2$ case.

Example: Consider (5) with $f(x_1) = x_1^3$, $\lambda_1 = \dfrac{31}{6}$ and $\lambda_2 = \dfrac{41}{6}$. In this case the eigenvalues of A are $\pm i\sqrt{2}$, $\pm i\sqrt{3}$ and $\pm \dfrac{i}{\sqrt{6}}$.

The mapping $x = P\tilde{x}$ with $x = (x_1, y_1, x_2, y_2, x_3, y_3)$, $\tilde{x} = (\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3)^T$ and

$$P = \begin{pmatrix} 0 & \dfrac{1}{4\sqrt{2}} & 0 & \dfrac{1}{9\sqrt{2}} & 0 & 36\sqrt{6} \\ \dfrac{1}{4} & 0 & \frac{1}{9} & 0 & 36 & 0 \\ 0 & -\dfrac{1}{2\sqrt{2}} & 0 & -\dfrac{1}{3\sqrt{3}} & 0 & -6\sqrt{6} \\ -\dfrac{1}{2} & 0 & -\dfrac{1}{3} & 0 & -6 & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{3}} & 0 & \sqrt{6} \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

transforms system (26) into the following system $\dot{\tilde{X}} = AX + F(\tilde{X})$ with same linear part as (29) and $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$ and $\alpha_3 = \dfrac{1}{\sqrt{6}}$:

$$\begin{aligned} \dot{\tilde{x}}_1 &= -\sqrt{2}\tilde{y}_1 - \frac{24}{11}\left(\frac{\tilde{y}_1}{4\sqrt{2}} + \frac{\tilde{y}_2}{9\sqrt{3}} + 36\sqrt{6}\tilde{y}_3\right)^3 \\ \dot{\tilde{y}}_1 &= \sqrt{2}\tilde{x}_1 \\ \dot{\tilde{x}}_2 &= -\sqrt{3}\tilde{y}_2 + \frac{54}{17}\left(\frac{\tilde{y}_1}{4\sqrt{2}} + \frac{\tilde{y}_2}{9\sqrt{3}} + 36\sqrt{6}\tilde{y}_3\right)^3 \\ \dot{\tilde{y}}_2 &= \sqrt{3}\tilde{x}_2 \\ \dot{\tilde{x}}_3 &= -\frac{\tilde{y}_3}{\sqrt{6}} + \frac{1}{187}\left(\frac{\tilde{y}_1}{4\sqrt{2}} + \frac{\tilde{y}_2}{9\sqrt{3}} + 36\sqrt{6}\tilde{y}_3\right)^3 \\ \dot{\tilde{y}}_3 &= \frac{\tilde{x}_3}{\sqrt{6}}. \end{aligned} \tag{32}$$

Thus the coefficients of the normal form established in Lemma 13 are given by:

$$\begin{aligned} a_1 &= \frac{3\sqrt{2}}{352} & b_1 &= -\frac{3\sqrt{3}}{272} & c_1 &= -\frac{81}{748\sqrt{6}} \\ a_2 &= \frac{\sqrt{2}}{297} & b_2 &= -\frac{2\sqrt{3}}{4131} & c_2 &= -\frac{8}{561\sqrt{6}} \\ a_3 &= \frac{69984\sqrt{2}}{11} & b_3 &= -\frac{46656\sqrt{3}}{17} & c_3 &= -\frac{1679616}{187\sqrt{6}}. \end{aligned}$$

So from Theorem 12 we have immediately that $X \in \mathcal{U}^j$ and thus it possesses a one-parameter family of symmetric periodic solutions converging to the origin with period converging to $\dfrac{2\pi}{\alpha_j}$, $j = 1, 2, 3$ with $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$ and $\alpha_3 = \dfrac{1}{\sqrt{6}}$, since $a_i \neq 0$, $b_i \neq 0$, $c_i \neq 0$, $i = 1, 2, 3$.

References

[1] A.R. Champneys, *Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics*, Physica D **112** (1998), 158–186.

[2] J. V. Chaparova, L. A. Peletier and S. A. Tersian, *Existence and nonexistence of nontrivial solutions of semilinear fourth- and sixth-order differential equations*, Advances in Differential Equations **8** (2003), 1237–1258.

[3] R. L. Devaney, *Reversible diffeomorphisms and flows*, Trans. Am. Math. Soc. **218** (1976), 89–113.

[4] J. Hale, *Ordinary Differential Equations*, New York, Wilay-Interscience, 1969.

[5] G. Iooss and M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, Adv. Ser. Nonlinear Dynamics **3** (1992), World Scientific, Singapure.
[6] A. Jacquemard, M. F. S. Lima and M. Teixeira , *Degenerate resonances and branching of periodic orbits*, Annali di Matematica ed Applicata **187** (2008), 105–117.
[7] M. F. S. Lima and M. Teixeira , *Families of periodic orbits in resonant reversible systems*, Bull. Braz. Math. Soc. **40** (2009), 521–547.
[8] J.S.W. Lamb and J.A.G. Roberts *Time-reversal symmetry in dynamical systems: a survey.* Phys. D **112** (1998), 1–39
[9] C. W. Shi, *Bifurcations of Symmetric Periodic Orbits near Equilibrium in Reversible Systems,* Int. J. Bifurcation and Chaos **7** (1997), 569–584.
[10] T. Wagenknecht, *An analytical study of a two degrees of freedom Hamiltonian System associated the Reversible Hyperbolic Umbilic,* Thesis, University Ilmenau, Germany, 1999.

[1] Departamento de Física, Química e Matemática, Universidade Federal de São Carlos, 18052-780, S.P, Brazil

[2] Departamento de Matemática, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-970, Campinas, S.P, Brazil
E-mail address: anamereu@ufscar.br,teixeira@ime.unicamp.br