COINCIDENCE THEOREMS FOR MAPS OF FREE $\mathbb{Z}_p$-SPACES

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Abstract. Let us consider $\mathbb{Z}_p$, p a prime number, acting freely on Hausdorff paracompact topological space X and let Y be a k-dimensional metrizable space (or k-dimensional CW-complex). In this paper, by using the genus of X; $\mathrm{gen}\,(X, \mathbb{Z}_p)$, we prove a $\mathbb{Z}_p$-coincidence theorem for maps $f : X \to Y$. Such theorem generalizes the main theorem proved by Aarts, Fokkink and Vermeer in [1].

Key words: $\mathbb{Z}_p$-coincidence point, free $\mathbb{Z}_p$-action, genus of $\mathbb{Z}_p$-space.

1. Introduction

The classic Borsuk-Ulam theorem says that every map of S^n into the euclidean k-dimensional space $\mathbb{R}^k$ has an antipodal coincidence if $n \geq k$. This result can be generalized in many ways: S^n and $\mathbb{R}^k$ can be replaced by more general spaces X and Y, and the antipodal action $\mathbb{Z}_2$ on S^n can be replaced by actions of others groups. In one of these generalizations Aarts, Fokkink and Vermeer [1, Theorem 1] proved that if $i : X \to X$ is a fixed-point free involution of a normal space X with color number $n+2$ and k is a natural number then for every k-dimensional cone CW-complex Y and every continuous map $\varphi : X \to Y$ there is an $\mathbb{Z}_2$-coincidence, whenever $n \geq 2k$; and this result is the best possible. Let us observe that for $X = S^n$ the result was obtained independently by Shchepin in [8].

In this paper, requiring that X is a Hausdorff paracompact space, we generalized the Aarts, Fokkink and Vermeer's result for free $\mathbb{Z}_p$-actions, p prime. Specifically, we prove

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2000 Mathematics Subject Classification. Primary 55M10, 55M20; Secondary 55M35.

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Theorem 1.1. *Let X be a Hausdorff paracompact space equipped with a free $\mathbb{Z}_p$-action generated by $\alpha : X \to X$ such that* $\text{gen}(X, \mathbb{Z}_p) \geq n+1$ *and let k be a natural number. Then the following holds.*

(a) *If $n > p\,k$, then for every k-dimensional metrizable space Y and every continuous map $f : X \to Y$ there is a $\mathbb{Z}_p$-coincidence point, i.e., there is $x \in X$ such that $f(x) = f(\alpha^i(x)),\ \forall\ i \in \{1, 2, \ldots, p-1\}$.*

(b) *If $n = p\,k$, then for every k-dimensional cone* CW*-complex Y and for every continuous map $f : X \to Y$ there is a $\mathbb{Z}_p$-coincidence point.*

(c) *If $n < p\,k$ and* $\text{gen}\,(X, \mathbb{Z}_p) = n+1$*, then there exists a k-dimensional cone* CW*-complex Y and a continuous map $f : X \to Y$ such that f has no $\mathbb{Z}_p$-coincidence points.*

In the case $n = pk$, we exhibit an interesting example showing that the result does not hold for the larger class of CW-complexes of dimension k.

Example 1.2. Consider $Y = \Delta^{ps+p-2}_{s-1}$ (the $(s-1)$-skeleton of the $(ps+p-2)$-simplex) and $Y^* = \Pi^p_{i=1} Y^i - \Delta$, where $Y^i = Y, \forall i$ and Δ is the diagonal. We have that $\mathbb{Z}_p$ acts freely on Y^* and Y^* is Hausdorff, paracompact space. Moreover, it follows from[12] and [2] that $\text{gen}\,(Y^*, \mathbb{Z}_p) = p(s-1)+1$.

Define $\pi : Y^* \to Y$ by $\pi(y_1, \ldots, y_p) = y_1,\ \forall\ (y_1, \ldots, y_p) \in Y^*$ and clearly π has no $\mathbb{Z}_p$-coincidence points. From this, we conclude that the theorem does not hold in the case $n = p\,k$ when Y is any CW-complex.

Remark 1.3. In the case that Y is a cone CW-complex, Theorem 1.1 is the best possible. Note that, if we consider X a Hausdorff paracompact free $\mathbb{Z}_2$-space with color number $n+2$, by Theorem 2.3, we have that $\text{gen}(X, \mathbb{Z}_2) = n+1$ and in this way, Theorem 1.1 generalizes main result of [1].

Remark 1.4. Let us consider $G = \mathbb{Z}_p$ and X satisfying the assumptions of [5, Theorem 1]. We have that $H^{m+1}(\mathbb{Z}_p, \mathbb{Z}) \neq 0$, for all m odd, and since $H_i(X, \mathbb{Z}) = 0$, for $0 < i < m$, by Proposition 2.9, $\text{gen}\,(X, \mathbb{Z}_p) \geq m+1$. Therefore, it follows from Theorem 1.1(i) that, whenever $n > pk$, for every continuous map $f : X \to Y$, with Y CW-complex k-dimensional, there is a $\mathbb{Z}_p$-coincidence point. Then, in the case $G = \mathbb{Z}_p$ and $n > pk$, Theorem 1.1 includes the result proved by Gonçalves, Jaworowski, Pergher and Volovikov in [5].

2. Preliminaries

Aarts, Brouwer, Fokkink and Vermeer, in [2], defined the genus, $\text{gen}\,(X, G)$, in the sense of Švarc, as follows.

Let G be a finite group which acts freely on a space X Hausdorff paracompact. Let G^* denote $G \backslash \{e\}$. We say that an open subset U of X is a *color* if $U \cap g \cdot U = \emptyset$ for all $g \in G^*$ and we shall say that a cover $\mathcal{U}$ of X by colors is a *coloring*. If (X, G) admits a finite coloring, then the *color number* $\text{col}\,(X, G)$ is the minimal cardinality of a coloring. If U is a color, then the set $G \cdot U = \bigcup_{g \in G} g \cdot U$ is called a set of the first kind and $G \cdot U$ is said to be *generated* by the color U. As G is a group, the collection $\{g \cdot U |\ g \in G\}$ is pairwise disjoint. The space X together with the group action is usually called a G-space.

Definition 2.1. Suppose that X is a G-space and let U be a color. We say that a set $G \cdot U$ is a *set of the first kind*. The *genus*, $\text{gen}\,(X, G)$, is defined as the minimal cardinality of a covering of X by sets of the first kind.

It follows from the definition that the genus in non-decreasing under equivariant maps.

Proposition 2.2. *Let X and Y be free G-spaces Hausdorff paracompacts and let $F : X \to Y$ be G-equivariant map. Then,* $\text{gen}\,(X, G) \leq \text{gen}\,(Y, G)$.

Hartskamp [6] and Bogatyi [3, Theorem 5] proved independently the following result:

Theorem 2.3. *Suppose that X is a Hausdorff paracompact G-space. The following statements are equivalent.*
(i) $\text{gen}\,(X, G) = n + 1$;
(ii) $\text{col}\,(X, G) = n + |G|$.

Other papers in connection with Theorem 2.3 are the papers of Steinlein [10, 11].

Krasnosel'skiĭ in [7], proved the following theorem:

Theorem 2.4. $\text{gen}\,(S^n, \mathbb{Z}_p) = n + 1$.

For two simplicial spaces X and Y, recall that the join $X * Y$ is the simplicial space realized by all simplices $[x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_m]$ for simplices $[x_1, x_2, \ldots, x_k]$ and $[y_1, y_2, \ldots, y_m]$ in X and Y, respectively. If X and Y are G-spaces, then so is $X * Y$. The k-fold join $G * G * \ldots * G$ is the simplicial space realized by all $[g_1, g_2, \ldots, g_k]$, with $g_i \in G$, for $i = 1, \ldots, k$. The G-action on the join is induced by $g_i \mapsto gg_i$ on the vertices.

Definition 2.5. The k-fold join $G * G * \cdots * G$ with the standard action is a G-space, denoted by S^k_G.

From [2], it follows the result:

Theorem 2.6. *Let X be a free G-space Hausdorff paracompact such that* $\text{gen}\,(X, G) \leq k$. *Then, there exists a G-equivariant map $F : X \to S^k_G$.*

In [9], Švarc obtained the following theorem:

Theorem 2.7. *Suppose that X is a Hausdorff paracompact G-space of* $\dim X = n$. *Then* $\text{gen}\,(X, G) \leq n + 1$.

In [2], Aarts, Brouwer, Fokkink and Vermeer proved that

Theorem 2.8. *Let G be a finite group.*
(i) $\text{gen}\,(S^k_G, G) = k$.
(ii) *Suppose that X is a free G-space paracompact Hausdorff $(k-2)$-connected. Then, there exists a G-equivariant map $F : S^k_G \to X$ and as a consequence,* $\text{gen}\,(X, G) \geq k$.

Volovikov, in [12], proved the following proposition:

Proposition 2.9. *Suppose that $G = \mathbb{Z}_p^n$ acts on X without fixed points. If $\tilde{H}^i(X) = 0$ for $i \leq N-1$, then* $\text{gen}\,(X, G) \geq N + 1$.

3. Proof of Theorem 1.1

Proof. **(Case (a) $n > pk$).** Let $\alpha : X \to X$ be a map that generates a free $\mathbb{Z}_p$-action on X with $\text{gen}\,(X, \mathbb{Z}_p) \geq n + 1$ and let $f : X \to Y$ be a continuous map.

Suppose, by contradiction, that for each $x \in X$, exists i, $j \in \{1, 2, \ldots, p\}$, $i \neq j$, satisfying $f(\alpha^i(x)) \neq f(\alpha^j(x))$, where $\alpha^p = id$.

Consider $Y^* = \Pi_{i=1}^p Y^i - \Delta$, where

$$\Delta = \{(y_1, y_2, \ldots, y_p) \in \Pi_{i=1}^p Y^i, \mid y_1 = y_2 = \ldots = y_p\}.$$

We have that Y^* is a metrizable space with $\dim Y^* \leq p\,k$.

Define $\sigma : Y^* \to Y^*$ by

$$\sigma(y_1, y_2, \ldots, y_p) = (y_p, y_1, \ldots, y_{p-1}),\ \forall\ (y_1, y_2, \ldots, y_p) \in Y^*,$$

and $\phi : X \to Y^*$ by

$$\phi(x) = (f(\alpha^{p-1}(x)), f(\alpha^{p-2}(x)), \ldots, f(\alpha(x)), f(x)),\ \forall\ x \in X.$$

Note that, σ generates a free $\mathbb{Z}_p$-action on Y^* and ϕ is a continuous map well-defined. Moreover, ϕ is a $\mathbb{Z}_p$-equivariant map.

Since the genus is non-decreasing under equivariant maps, we have that $\text{gen}\,(X, \mathbb{Z}_p) \leq \text{gen}\,(Y^*, \mathbb{Z}_p)$.

Now, since $\dim Y^* \leq p\,k$, it follows from Theorem 2.7 that

$$\text{gen}\,(Y^*, \mathbb{Z}_p) \leq p\,k + 1. \tag{3.1}$$

Therefore, $\text{gen}\,(X, \mathbb{Z}_p) \leq \text{gen}\,(Y^*, \mathbb{Z}_p) \leq p\,k + 1 < n + 1$, which contradicts $\text{gen}\,(X, \mathbb{Z}_p) \geq n + 1$.

(Case (b) $n = pk$**).** The strategy used to show the case $n = pk$ for a cone CW-complex Y, is the following: we shall show that the upper bound of equation (3.1) can be reduced by one, ie, we shall prove that $\text{gen}\,(Y^*, \mathbb{Z}_p) \leq p\,k$.

For this, let Y be a k-dimensional CW-complex, which is a cone CW-complex, ie, $Y = CA = \dfrac{A \times [0,1]}{\sim}$, where A is a CW-complex of dimension $k-1$ and $\sim$ is the following equivalence relation: $(a, 1) \sim (a', 1)$, $\forall\, a, a' \in A$.

In this sense, we obtain coordinates for Y. A point in Y is represented by class $[a,\, u]$, with $a \in A$ and $u \in [0, 1]$.

We take $Y^* = \Pi_{i=1}^{p} Y^i - \Delta$. Define $\sigma :\, Y^* \to Y^*$ by

$$\sigma(y_1, y_2, \ldots, y_p) = (y_p, y_1, \ldots, y_{p-1}),\ \forall\ (y_1, y_2, \ldots, y_p) \in Y^*,$$

or using coordinates, by

$$\sigma([a_1, u_1], \ldots, [a_p, u_p]) = ([a_p, u_p], [a_1, u_1], \ldots, [a_{p-1}, u_{p-1}]), \forall\ a_1, \ldots, a_p \in A \\ \text{e}\ \forall\ u_1, \ldots, u_p \in [0, 1].$$

Note that, σ generates a free $\mathbb{Z}_p$-action on Y^*.

Lemma 3.1. $\text{gen}\,(Y^*,\, \mathbb{Z}_p) \leq p\,k$.

Proof. Let $Z = [0, 1] \times \cdots \times [0, 1] \setminus \{(1, \ldots, 1)\}$ and let $s :\, Z \to Z$ be given by

$$s(u_1, \ldots, u_p) = (u_p, u_1, \ldots, u_{p-1}),\ \forall\ (u_1, \ldots, u_p) \in Z.$$

The projection $\pi :\, Y^* \to Z$ defined by

$$\pi([a_1, u_1], \ldots, [a_p, u_p]) = (u_1, \ldots, u_p),\ \forall\ ([a_1, u_1], \ldots, [a_p, u_p]) \in Y^*,$$

is well-defined, is continuous and $s \circ \pi = \pi \circ \sigma$.

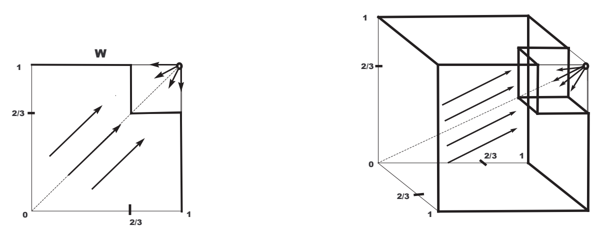
Let us consider the following subsets of Z,

$$\begin{aligned}
W_1 &= \{2/3\} \times [2/3,1] \times [2/3,1] \times \cdots \times [2/3,1] \\
W_2 &= [2/3,1] \times \{2/3\} \times [2/3,1] \times \cdots \times [2/3,1] \\
&\vdots \\
W_p &= [2/3,1] \times [2/3,1] \times \cdots \times [2/3,1] \times \{2/3\} \\
\\
W_1^1 &= \{1\} \times [0,2/3] \times [0,1] \times [0,1] \times \cdots \times [0,1] \\
W_2^1 &= \{1\} \times [2/3,1] \times [0,2/3] \times [0,1] \times \cdots \times [0,1] \\
&\vdots \\
W_{p-1}^1 &= \{1\} \times [2/3,1] \times \cdots \times [2/3,1] \times [0,2/3] \\
&\vdots \\
W_1^p &= [0,2/3] \times [0,1] \times [0,1] \times \cdots \times [0,1] \times \{1\} \\
W_2^p &= [2/3,1] \times [0,2/3] \times [0,1] \times \cdots \times [0,1] \times \{1\} \\
&\vdots \\
W_{p-1}^p &= [2/3,1] \times [2/3,1] \times \cdots \times [2/3,1] \times [0,2/3] \times \{1\},
\end{aligned}$$

and we define

$$W = (\cup_{i=1}^{p} W_i) \cup (\cup_{j=1}^{p-1} W_j^1) \cup \ldots \cup (\cup_{j=1}^{p-1} W_j^p).$$

We have that W is the union of $p^2 = p + p\,(p-1)$ closed subsets of Z (Figure 1 illustrates the cases $p = 2$ and $p = 3$):



(a) case $p = 2$ (b) case $p = 3$

FIGURE 1.

Define a retraction $r: Z \to W$ as follows:

In the right upper corner of Z, the retraction r is the central projection to W with center of projection $(1, 1, \ldots, 1)$. In the lower part of Z, the retraction r is the projection to W parallel to the diagonal of Z $(\{(z, z, \ldots, z) \mid z \in [0, 1]\})$. Note that,

(i)

$$\begin{aligned} r(\{1\} \times [0,1) \times [0,1] \times \ldots \times [0,1]) &\subset \cup_{j=1}^{p-1} W_j^1 \\ r([0,1] \times \{1\} \times [0,1) \times \ldots \times [0,1]) &\subset \cup_{j=1}^{p-1} W_j^2 \\ &\vdots \\ r([0,1) \times [0,1] \times \ldots \times [0,1] \times \{1\}) &\subset \cup_{j=1}^{p-1} W_j^p \end{aligned}$$

(ii) Let $z \in Z$ such that z has $1 < n \leq p-1$ coordinates equal to 1.

- If $z \in W$ then $r(z) = z$, ie, $r(z)$ has all the coordinates equal to coordinates of z.
- If $z \in Z - W$ then z belongs to the top of Z and we can assume, without loss of generality, that $z = (1, \ldots, 1, x_{n+1}, \ldots, x_p)$. Thus,

$$\begin{aligned} r(z) &= (1, \ldots, 1, x_{n+1}, \ldots, x_p) + \lambda \cdot \overrightarrow{z(1, \ldots, 1)} \\ &= (1, 1, \ldots, 1, x_{n+1} + \lambda \cdot (x_{n+1} - 1), \ldots, x_p + \lambda \cdot (x_p - 1)) \cap W, \end{aligned}$$

Therefore, the coordinates in z that are equal to 1 remain equal to 1 in $r(z) \in W$.

Using the retraction r, we define a retraction $\rho: Y^* \to \pi^{-1}(W)$ by

$$\rho([a_1, u_1], \ldots, [a_p, u_p]) = ([a_1, u_1'], \ldots, [a_p, u_p']), \ \forall\ ([a_1, u_1], \ldots, [a_p, u_p]) \in Y^*,$$

where $(u_1', \ldots, u_p') = r(u_1, \ldots, u_p)$.

We have that ρ is continuous and from (i) and (ii), it follows that ρ is well-defined.

Now, we shall show that $s \circ r = r \circ s$. First, we observe that $s(W) \subseteq W$.

Let $P \in Z$ such that P belongs to the bottom of Z. We take the vector $\vec{v} = \overrightarrow{(1, 1, \ldots, 1)}$. Then, $(s \circ r)(P) = s(r(P)) = s(P + \lambda \vec{v})$, for some $\lambda \in \mathbb{R}$ such that $P + \lambda \vec{v} \in W$. Thus,

$$(s \circ r)(P) = s(P + \lambda \vec{v}) = s(P) + \lambda \vec{v} \overset{s(W) \subset W}{=} r(s(P)) = (r \circ s)(P).$$

Now, let $P \in Z$ such that P belongs to the top of Z. Let $\vec{u} = \overrightarrow{P(1, 1, \ldots, 1)}$ and $\overrightarrow{u'} = \overrightarrow{s(P)(1, 1, \ldots, 1)}$. Then, $(s \circ r)(P) = s(r(P)) = s(P + \lambda \vec{u})$, for some $\lambda \in \mathbb{R}$ and such that $P + \lambda \vec{u}$ intersects W. Then,

$$(s \circ r)(P) = s(P + \lambda \vec{u}) = s(P) + \lambda \overrightarrow{u'} \overset{s(W) \subset W}{=} r(s(P)) = (r \circ s)(P).$$

Therefore, $s\circ r = r\circ s$. Thus, $s^{p-1}\circ r = r\circ s^{p-1}$.

Let us consider $\sigma' = \sigma|_{\pi^{-1}(W)}$. We have that $\sigma'(\pi^{-1}(W))\subseteq \pi^{-1}(W)$ and σ' generates a free $\mathbb{Z}_p$-action on $\pi^{-1}(W)$.

Claim: $\rho\colon (Y^*,\sigma)\to(\pi^{-1}(W),\sigma')$ is a $\mathbb{Z}_p$-equivariant map. Indeed,

$$\begin{aligned}(\sigma'\circ\rho)([a_1,u_1],\dots,[a_p,u_p]) &= \sigma'([a_1,u_1'],\dots,[a_p,u_p'])\\ &= ([a_p,u_p'],[a_1,u_1'],\dots,[a_{p-1},u_{p-1}']),\end{aligned}$$

where $(u_1',\dots,u_p') = r(u_1,\dots,u_p)$.

$$\begin{aligned}(\rho\circ\sigma)([a_1,u_1],\dots,[a_p,u_p]) &= \rho([a_p,u_p],[a_1,u_1],\dots,[a_{p-1},u_{p-1}])\\ &= ([a_p,\tilde{u}_p],[a_1,\tilde{u}_1],\dots,[a_{p-1},\tilde{u}_{p-1}]),\end{aligned}$$

with $(\tilde{u}_p,\tilde{u}_1,\dots,\tilde{u}_{p-1}) = r(u_p,u_1,\dots,u_{p-1})$.

Now, we have that

$$\begin{aligned}s^{p-1}(\tilde{u}_p,\tilde{u}_1,\dots,\tilde{u}_{p-1}) &= s^{p-2}(\tilde{u}_{p-1},\tilde{u}_p,\tilde{u}_1,\dots,\tilde{u}_{p-2})\\ &\ \vdots\\ &= s(\tilde{u}_2,\tilde{u}_3,\dots,\tilde{u}_p,\tilde{u}_1)\\ &= (\tilde{u}_1,\dots,\tilde{u}_p).\end{aligned}$$

On the other hand

$$\begin{aligned}s^{p-1}(\tilde{u}_p,\tilde{u}_1,\dots,\tilde{u}_{p-1}) &= s^{p-1}(r(u_p,u_1,\dots,u_{p-1}))\\ &\overset{s^{p-1}\circ r = r\circ s^{p-1}}{=} r(s^{p-1}(u_p,u_1,\dots,u_{p-1}))\\ &= r(u_1,\dots,u_p).\end{aligned}$$

Then, $(\tilde{u}_1,\dots,\tilde{u}_p) = r(u_1,\dots,u_p)$ and thus, $(\tilde{u}_1,\dots,\tilde{u}_p) = (u_1',\dots,u_p')$. Therefore, $\rho\circ\sigma = \sigma'\circ\rho$ and then, ρ is a $\mathbb{Z}_p$-equivariant map.

From this, we conclude that $\text{gen}\,(Y^*,\mathbb{Z}_p)\leq \text{gen}\,(\pi^{-1}(W),\mathbb{Z}_p)$.

Note that $W = (\cup_{i=1}^{p} W_i)\cup(\cup_{j=1}^{p-1} W_j^1)\cup\ldots\cup(\cup_{j=1}^{p-1} W_j^p)$ is written as an union of p^2 closed subsets of Z. By simplicity, we rewrite $W = \cup_{i=1}^{p^2} W^i$. Let $W_1 = \{\frac{2}{3}\}\times[\frac{2}{3},1]\times\cdots\times[\frac{2}{3},1]$ and we compute $\pi^{-1}(W_1)$:

$$\begin{aligned}\pi^{-1}(W_1) &= \{([a_1,u_1],\dots,[a_p,u_p])\in Y^* \mid \pi([a_1,u_1],\dots,[a_p,u_p])\in W_1\}\\ &= \{([a_1,u_1],\dots,[a_p,u_p])\in Y^* \mid (u_1,\dots,u_p)\in W_1\}\\ &= \left(\frac{A\times\{\frac{2}{3}\}}{\sim}\times\frac{A\times[0,1]}{\sim}\times\cdots\times\frac{A\times[0,1]}{\sim}\right)\cap Y^*.\end{aligned}$$

Then,

$$\dim \pi^{-1}(W_1) \leq (k-1) + (p-1)k = p\,k - 1.$$

For each W_i, $i = 2, 3, \ldots, p^2-1, p^2$, in the analogous way, we obtain that

$$\dim \pi^{-1}(W_i) \leq p\,k - 1, \ \forall\ i = 2, 3, \ldots, p^2-1, p^2.$$

Since $\pi^{-1}(W) = \pi^{-1}\left(\bigcup_{i=1}^{p^2} W_i\right) = \bigcup_{i=1}^{p^2} \pi^{-1}(W_i)$ is an union of closed subsets with $\dim \pi^{-1}(W_i) \leq p\,k - 1$, $\forall\ i = 1, 2, \ldots, p^2$, by [4, The Sum Theorem], it follows that

$$\dim \pi^{-1}(W) \leq p\,k - 1.$$

Then, by Theorem 2.7,

$$\text{gen}\,(\pi^{-1}(W), \mathbb{Z}_p) \leq \dim \pi^{-1}(W) + 1 \leq (p\,k-1) + 1 = p\,k.$$

Therefore,

$$\text{gen}\,(Y^*, \mathbb{Z}_p) \leq \text{gen}\,(\pi^{-1}(W), \mathbb{Z}_p) \leq p\,k,$$

which completes the proof of lemma. □

Now, suppose that $f : X \to Y$ has no $\mathbb{Z}_p$-coincidence points. As in the proof of Theorem 1.1 (a), there is a $\mathbb{Z}_p$-equivariant map $\phi : X \to Y^*$. Then, it follows from Lemma 3.1, that

$$\text{gen}(X, \mathbb{Z}_p) \leq \text{gen}(Y^*, \mathbb{Z}_p) \leq p\,k,$$

which contradicts $\text{gen}(X, \mathbb{Z}_p) \geq n + 1 = pk + 1$. This completes the proof of Theorem 1.1 (b). □

(Case (c) $n < pk$ **and** $\text{gen}(X, \mathbb{Z}_p) = n + 1$**).**

Proof. In this case, we have that $\text{gen}(X, \mathbb{Z}_p) = n + 1 \leq pk$ and, it follows from Theorem 2.6 that there is a $\mathbb{Z}_p$-equivariant map $F : X \to S^{pk}_{\mathbb{Z}_p}$, where $S^{pk}_{\mathbb{Z}_p} = \mathbb{Z}_p * \mathbb{Z}_p * \cdots * \mathbb{Z}_p$ is the pk-fold join (Definition 2.5). On the other hand, it follows from [12, Corollary 6.1] that there are a $\mathbb{Z}_p$-space X', a cone CW-complex Y of dimension k and a map $\varphi : X' \to Y$ without $\mathbb{Z}_p$-coincidence points. Further, there is a $\mathbb{Z}_p$-equivariant map $E : S^{pk}_{\mathbb{Z}_p} \to X'$ and, consequently, the map $f = \varphi \circ E \circ F : X \to Y$ has no $\mathbb{Z}_p$-coincidence points. We observe that this construction shows that the hypothesis $n \geq p\,k$ in Theorem 1.1 (a) and (b) is the best condition to guarantee the existence of $\mathbb{Z}_p$-coincidence points, when we consider any Hausdorff paracompact $\mathbb{Z}_p$-space X of $\text{gen}(X, \mathbb{Z}_p) = n + 1$. This completes the proof of Theorem 1.1. □

References

[1] J.M. Aarts, R.J. Fokkink, H. Vermeer, *Coincidence theorems for involutions*, Topology Appl. 85 (1998) 3-8.

[2] J.M. Aarts, G.A. Brouwer, R.J. Fokkink, H. Vermeer, *Intersection properties for covering of G-spaces*, Topology Appl. 125 (2002) 249-261.

[3] S. Bogatyi, *Ljusternik-Schnirelman theorem and Bf* (in Russian), Fundam. Prikl. Mat. 4 (1998), no. 1, 11-38.

[4] Engelking, R. *Dimension Theory.* New York: North-Holland Pub. Co., 1978.

[5] D.L. Gonçalves, J. Jaworowski, P.L.Q. Pergher, A.Yu. Volovikov, *Coincidences for maps of spaces with finite group actions,* Topology Appl. 145 (2004), no. 1-3, 61-68.

[6] M. van Hartskamp, *Colorings of fixed-point free maps*, Thesis, Free University of Amsterdam, 1999.

[7] M.A. Krasnosel'skiĭ, *On special coverings of a finite-dimensional sphere*, Dokl. Akad. Nauk SSSR (N.S.) 103 (1955) 961-964 (in Russian).

[8] E.V. Shchepin, *On a problem of L. A. Tumarkin*, Dokl. Akad. Nauk SSSR 217 (1974), 42-43; English transl., Soviet Math. Dokl. 15 (1974), 1024-1026.

[9] A.S. Švarc, *Some estimates of the genus of a topological space in the sense of Krasnosel'skiĭ*. Uspekhi Mat. Nauk 12 : 4 (1957), 209-214. (Russian)

[10] H. Steinlein, *Some abstract generalizations of the Ljusternik-Schnirelman-Borsuk covering theorem.* Pacifc J. Math. (1979), 83, no. 1, 285-296.

[11] H. Steinlein, *On the theorems of Borsuk-Ulam and Ljusternik-Schnirelman-Borsuk.* Canad. Math. Bull. (1984) 27, 192-204.

[12] A.Yu. Volovikov, *Coincidence points of mappings of $\mathbb{Z}_p^n$-spaces*, Izv. Ross. Akad. Nauk Ser. Mat. 69 (2005), no. 5, 53-106, translation in Ivz. Math. 69 (2005), no. 5, 913-962.

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