STRONG UNIQUE CONTINUATION FOR SYSTEMS OF COMPLEX VECTOR FIELDS

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Abstract. In this work we study the property of strong unique continuation, at a given point, for Gevrey solutions to homogeneous systems of PDE defined by complex, real-analytic vector fields in involution. We show that when the system is minimal at the point then the strong unique continuation property holds for Gevrey solutions of order $\sigma \in [1, 2]$ and, furthermore, when the minimality property fails to hold then there are non trivial Gevrey flat solutions of any given order $\sigma > 1$. The case of Gevrey order $\sigma > 2$ is also studied for some particular classes of involutive systems.

Introduction

Let $\mathrm{L}_1, \dots, \mathrm{L}_n$ be real-analytic, complex vector fields defined on a real-analytic manifold Ω. Assume that $\mathrm{L}_1, \dots, \mathrm{L}_n$ are linearly independent at every point and that $[\mathrm{L}_j, \mathrm{L}_k] \in \mathrm{span}\{\mathrm{L}_1, \dots, \mathrm{L}_n\}$, $j, k = 1 \dots, n$. Given $p \in \Omega$ we say that the system $\mathbf{L} = \{\mathrm{L}_1, \dots, \mathrm{L}_n\}$ satisfies the *strong unique continuation property at* p if any smooth solution near p to the system

$$\mathrm{L}_j u = 0, \quad j = 1, \dots, n, \tag{1}$$

that vanishes to infinite order at p necessarily vanishes identically in a neighborhood of p.

For instance, if $\mathbf{L}$ is *analytic hypoelliptic at* p, that is, if every (weak) solution to (1) near p is automatically real-analytic in a neighborhood of p, then clearly $\mathbf{L}$ satisfies the strong unique continuation property at p. On the other hand, the single existence of a real-analytic solution W to (1) defined near p, not open at p and satisfying $\mathrm{d}W(p) \neq 0$ implies the existence of a non trivial smooth solution u to (1) which vanishes to infinite order at p [BT1, Theorem 2.6]. A closer look at the construction of such u gives a more precise information: for any $\sigma > 3$ we can find a non trivial solution to (1) which is Gevrey of order σ and vanishes to infinite order at p.

Such observation leads naturally to the question whether it is still possible to have the validity of the strong unique continuation property at p, for solutions in certain Gevrey classes, when this property fails to hold for general smooth solutions. In the present work we address to this question by characterizing the systems for which this is true. Indeed we prove that if the system is minimal at p (cf. Section 4) then the strong unique continuation property holds at p for Gevrey solutions of order $1 \leq \sigma \leq 2$ (Theorem 5.1) whereas, if the system is not minimal at p, then for

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Key words and phrases. Strong unique continuation property, Gevrey solutions, locally integrable structures.

2010 *Mathematics Subject Classification*: Primary: 35A02. Secondary: 35F05.

Partially supported by FAPESP. The second and third authors were also partially supported by CNPq.

every $\sigma > 1$ there is a non trivial Gevrey solution of order σ which vanishes to infinite order at p (Theorem 5.2).

These can be considered our two main results. Of course the case that is left out is what happens when $\sigma > 2$ and the system satisfies the minimality condition at p but is not analytic hypoelliptic at p. In fact it is plausible to conjecture that when analytic hypoellipticity fails to hold then the strong unique continuation property also fails to hold for Gevrey solutions of order $\sigma > 2$.

In this direction we have introduced in Section 5 a property called (†) and which imposes the existence of a solution W as alluded to above. Under the validity of such property we were able to prove that the strong unique continuation property at p fails to hold for Gevrey solutions of order $\sigma > 2$ (Theorem 5.3). When $\dim \Omega = n + 1$ or when the system is in the tube form (as described in Section 5), property (†) is equivalent to non analytic hypoellipticity, and this gives a positive answer to our conjecture in these cases.

Throughout the article we work in the set up of locally integrable structrures, as exposed in the books [BCH] and [T]. The main tools used here are a theorem due to Marson [Ma], which characterizes minimality in terms of wedge extendability, and the results on Gevrey asymptotic expansions described in [M], which inspired our work and led us to the statement of Theorem 5.1. Also, in order to prove Theorem 5.3 we derive, by functional analytic methods, the following necessary condition for the validity of the strong unique continuation property at p for Gevrey solutions of order $\sigma > 1$: the Borel property for Gevrey solutions of order σ at p must fail to hold (Theorem 2.1). The connection between strong unique continuation and the Borel property was first pointed out to us by B. Lamel.

1. The strong unique continuation property

Throughout this work Ω will stand for a real-analytic, paracompact manifold of dimension N. If $p \in \Omega$ we shall denote by C^k_p, $k = 0, 1, \ldots, \infty$, the space of germs of C^k functions at p, and for G^σ_p, $\sigma \geq 1$, the space of germs of Gevrey functions of order σ at p. Thus $G^1_p = C^\omega_p$ is the space of germs of real-analytic functions at p.

We shall assume that Ω is endowed with a real-analytic involutive structure $\mathcal{V}$ of rank $n \leq N$. Thus $\mathcal{V}$ is a real-analytic vector subbundle of $\mathbb{C}\mathrm{T}\Omega$ of rank n satisfying $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. Every such $\mathcal{V}$ is locally integrable, in the sense that its orthogonal bundle $\mathcal{V}^\perp \subset \mathbb{C}\mathrm{T}^*\Omega$ is locally spanned by the differential of $m \doteq N - n$ real-analytic functions.

A C^k solution for $\mathcal{V}$ in an open set $\Omega' \subset \Omega$ is a C^k function u defined in Ω' such that $\mathrm{L}u = 0$ for any local section L of $\mathcal{V}$ defined in an open subset of Ω'. This is equivalent to saying that $\mathrm{d}u$ is a continuous section of $\mathcal{V}^\perp|_{\Omega'}$. Denote by $\mathfrak{S}^k(\Omega')$ the space of C^k solutions for $\mathcal{V}$ on Ω'. The association $\Omega' \mapsto \mathfrak{S}^k(\Omega')$ is clearly a sheaf over Ω, whose stalk at a point $p \in \Omega$ will be denoted by $\mathfrak{S}^k_p(\mathcal{V})$. Any representative of an element of $\mathfrak{S}^k_p(\mathcal{V})$ will be referred to as a C^k solution for $\mathcal{V}$ near p.

The structure $\mathcal{V}$ is said to satisfy the *strong unique continuation* property at $p \in \Omega$ if the following holds: if $u \in \mathfrak{S}^\infty_p(\mathcal{V})$ vanishes to infinite order at p then $u = 0$. As an example, if $\mathcal{V}$ is *analytic hypoelliptic* at p, that is, if $\mathfrak{S}^1_p(\mathcal{V}) \subset C^\omega_p$, then $\mathcal{V}$ satisfies the strong unique continuation property at p. In particular elliptic structures, that is, structures for which $\mathcal{V}^{\perp} \cap \mathrm{T}^*\Omega = 0$, satisfy the strong unique continuation property at every point of Ω.

In this work we shall be interested in studying the following more precise concept:

Definition 1.1. *The structure $\mathcal{V}$ is said to satisfy the strong unique continuation property at $p \in \Omega$ for Gevrey order $\sigma \geq 1$ if the only $u \in \mathfrak{S}_p^{\infty}(\mathcal{V}) \cap G_p^{\sigma}$ that vanishes to infinite order at p is $u = 0$.*

Local coordinates and generators. Following ([T, I.5] and [BCH, I.10]), each point of Ω is the center of a real-analytic coordinate system $(x_1, \dots, x_m, t_1, \dots, t_n)$, which can be assumed defined in a product $U = B \times \Theta$, where B (respectively Θ) is an open ball centered at the origin in $\mathbb{R}_x^m$ (respectively $\mathbb{R}_t^n$), over which there is defined a real-analytic, real vector-valued function $\Phi(x,t) = (\Phi_1(x,t), \dots, \Phi_m(x,t))$ satisfying $\Phi(0,0) = 0$, $\Phi_x(0,0) = 0$, such that the differential of the functions

$$Z_k(x,t) = x_k + i\Phi_k(x,t), \qquad k = 1, \dots, m,$$

span $\mathcal{V}^{\perp}$ over U. Moreover $\mathrm{d}Z_1, \dots, \mathrm{d}Z_m, \mathrm{d}t_1, \dots, \mathrm{d}t_n$ span $\mathbb{C}\mathrm{T}^*\Omega$ over U.

Over U we can define real-analytic vector fields

$$\mathrm{M}_k = \sum_{k'=1}^{m} \mu_{kk'}(x,t) \frac{\partial}{\partial x_{k'}}, \;\; k = 1, \dots, m$$

characterized by the rule

$$\mathrm{M}_k Z_{k'} = \delta_{k,k'}\,, \quad k, k' = 1, \dots, m.$$

It follows that the complex vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t)\mathrm{M}_k, \;\; j = 1, \dots, n,$$

span $\mathcal{V}|_U$. Moreover, $\mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m$ span $\mathbb{C}\mathrm{T}\Omega|_U$.

The following relations are easily checked, for every $j, j' = 1, \dots, n$, $k, k' = 1, \dots, m$:

$$\mathrm{d}Z_k(\mathrm{L}_j) = 0, \;\mathrm{d}Z_k(\mathrm{M}_{k'}) = \delta_{kk'}, \;\mathrm{d}t_j(\mathrm{L}_{j'}) = \delta_{jj'}, \;\mathrm{d}t_j(\mathrm{M}_k) = 0,$$

from which we conclude that $\mathrm{L}_1, \dots, \mathrm{L}_n, \mathrm{M}_1, \dots, \mathrm{M}_m$ are pairwise commuting.

Let $V \subset U$ be open and let $f \in \mathfrak{S}^{\infty}(V)$, that is, f is a smooth function on V which satisfies the system of homogeneous equations $\mathrm{L}_j f = 0$, $j = 1, \dots, n$, on V. Then f is Gevrey of order $\sigma \geq 1$ if for every $K \subset V$ compact there is $C = C(K) > 0$ such that

$$\sup_K |\mathrm{M}^{\alpha} f| \leq C^{|\alpha|+1} \alpha!^{\sigma}, \quad \alpha \in \mathbb{Z}_+^m\,.$$

Denote by $\mathbb{C}_{(\sigma)}[[Z_1, \dots, Z_m]]$ the space of all formal power series

$$\sum_{\alpha \in \mathbb{Z}_+^m} c_{\alpha} Z^{\alpha}/\alpha!$$

for which the following is true: there is a constant $A > 0$ such that $|c_{\alpha}| \leq A^{|\alpha|+1}\alpha!^{\sigma}$ for every $\alpha \in \mathbb{Z}_+^m$. We then obtain a ring homomorphism

$$\mathfrak{T}_{\sigma} : \mathfrak{S}_0^{\infty}(\mathcal{V}) \cap G_0^{\sigma} \longrightarrow \mathbb{C}_{(\sigma)}[[Z_1, \dots, Z_m]]$$

$$\mathfrak{T}_{\sigma}(f) = \sum_{\alpha \in \mathbb{Z}_+^m} \frac{(\mathrm{M}^{\alpha} f)(0,0)}{\alpha!} Z^{\alpha}.$$

It follows that $\mathcal{V}$ satisfies the strong unique continuation property at the origin for Gevrey order σ if and only if $\mathfrak{T}_\sigma$ is injective. As noticed before this is always the case when $\sigma = 1$ (indeed in this case $\mathfrak{T}_1$ is an isomorphism) or when $\sigma \geq 1$ is arbitrary and the structure is analytic hypoelliptic at the origin.

2. A necessary condition for strong unique continuation

In the next result, which gives a necessary condition for the validity of the strong uniqueness property at the origin for a fixed Gevrey order $\sigma > 1$, we consider Ω and $\mathcal{V}$ as in the preceding section.

Theorem 2.1: *If the strong unique continuation property holds at the origin for Gevrey order $\sigma > 1$ then $\mathfrak{T}_\sigma$ is not surjective.*

Before the proof we recall some standard facts: a locally convex space E is called a (DFS) space if $E = \operatorname{ind}(E_k)$, for some injective sequence of Banach spaces $\iota_k : E_k \hookrightarrow E_{k+1}$, with each ι_k a compact inclusion. It is well known that E is Hausdorff and that it satisfies the following property: a set $B \subset E$ is bounded in E if and only if there is $k \in \mathbb{N}$ such that $B \subset E_k$ and B is bounded in E_k.

Consider now $F = \operatorname{ind}(F_k)$ another (DFS) space and denote by $\|\cdot\|_k$ (resp. $|\cdot|_k$) the norm in the Banach space E_k (resp. F_k). We state:

Lemma 2.1. *If $u \in L(E,F)$ is a bijection then for every $k \in \mathbb{N}$ there are $k' \in \mathbb{N}$ and $C_k > 0$ such that $u(x) \in F_k$ implies $x \in E_{k'}$ and*

$$\|x\|_{k'} \leq C_k |u(x)|_k, \quad u(x) \in F_k\,.$$

Proof. For each $k \in \mathbb{N}$ consider the linear map $\gamma_k \doteq u^{-1}|_{F_k} : F_k \to E$. It is easily seen that the graph of γ_k is sequentially closed: indeed, if $y_n \in F_k$, $y_n \to 0$ in F_k, and if $\gamma_k(y_n) \to x$ in E then $y_n = u(\gamma_k(y_n)) \to u(x)$ in F because u is continuous. Since also $y_n \to 0$ in F we obtain $u(x) = 0$ and hence $x = 0$. Since E is a webbed space [K, p.63 (8)], De Wilde's closed graph theorem [K, p.56 (1)] applies and we conclude that γ_k is continuous. Let now B_k denote the unit ball in F_k. Then $\gamma_k(B_k)$ is bounded in E and hence $\gamma_k(B_k) \subset E_{k'}$ for some $k' \in \mathbb{N}$. It follows that γ_k maps F_k into $E_{k'}$. Applying again the closed graph theorem, this time its version for Banach spaces, we conclude $\gamma_k : F_k \to E_{k'}$ is continuous and thus there is a constant $C_k > 0$ such that $y \in F_k$ implies $u^{-1}(y) \in E_{k'}$ and $\|u^{-1}(y)\|_{k'} \leq C_k |y|_k$. This concludes the proof of the lemma. ■

Proof of Theorem 2.1. Let $\{V_k\}_{k\in\mathbb{N}}$ be a fundamental system of neighborhoods of the origin in $\mathbb{R}^N$. We assume that each V_k is an open ball and that $V_{k+1} \Subset V_k$ for all $k \in \mathbb{N}$. For each k we consider the space E_k of all solutions u for $\mathcal{V}$ which belong to $C^\infty(\bar{V}_k)$ and satisfy

$$\|u\|_{(k)} \doteq \sup_\alpha \frac{\|\mathrm{M}^\alpha u\|_{L^\infty(\bar{V}_k)}}{k^{|\alpha|}\alpha!^\sigma} < \infty\,.$$

E_k is a Banach space with norm $\|\cdot\|_{(k)}$ and the linear maps induced by restriction $\lambda_k : E_k \to E_{k+1}$ are compact. Notice that each λ_k is injective, since we are assuming that the strong unique continuation property holds at the origin for Gevrey order σ. Hence we can view E_k as a subspace of E_{k+1} with compact inclusion, and it follows that $\mathfrak{S}_0^\infty(\mathcal{V}) \cap G_0^\sigma = \operatorname{ind}(E_k)$ is a (DFS) space.

Likewise, for each $k \in \mathbb{N}$ we let F_k be the space of all formal power series $\mathfrak{s} = \sum_\alpha a_\alpha Z^\alpha/\alpha!$ for which

$$|\mathfrak{s}|_{(k)} \doteq \sup_\alpha \frac{|a_\alpha|}{k^{|\alpha|}\alpha!^\sigma} < \infty.$$

Again, F_k is a Banach space with norm $|\cdot|_{(k)}$, the inclusions $\iota_k : F_k \hookrightarrow F_{k+1}$ are compact and $\mathbb{C}_{(\sigma)}[[Z_1, \dots, Z_m]]$ is the indutive limit ind (F_k), thus again a (DFS) space.

Now $\mathfrak{T}_\sigma : \mathfrak{S}_0^\infty(\mathcal{V}) \cap G_0^\sigma \to \mathbb{C}_{(\sigma)}[[Z_1, \dots, Z_m]]$ is continuous with respect to the inductive limit topologies just introduced. If $\mathfrak{T}_\sigma$ were bijective we could apply Lemma 2.1 and derive the following property: for every $k \in \mathbb{N}$ there are $k' \in \mathbb{N}$ and $C_k > 0$ such that if $\mathfrak{T}_\sigma(f) \in F_k$ then $f \in E_{k'}$ and

$$\sup_{V_{k'}} |f| \leq C_k |\mathfrak{T}_\sigma(f)|_{(k)}, \quad \mathfrak{T}_\sigma(f) \in F_k\,.$$

For each $\beta \in \mathbb{Z}_+^m$ we apply this property to $f(x,t) \doteq Z(x,t)^\beta/\beta!$. We have $\mathfrak{T}_\sigma(f) = Z^\beta/\beta!$ and $|\mathfrak{T}_\sigma(f)|_{(k)} = 1/(k^{|\beta|}\beta!^\sigma)$. Consequently, for every $(x,t) \in V_{k'}$,

$$|Z(x,t)^\beta| \leq \frac{C_k}{k^{|\beta|}\beta!^{\sigma-1}}, \quad \beta \in \mathbb{Z}_+^m\,.$$

Fixing k and taking $\beta = (\beta_1, 0, \dots, 0)$ we obtain

$$(k|Z_1(x,t)|)^{\beta_1}(\beta_1)!^{\sigma-1} \leq C_k, \quad \beta_1 \geq 0.$$

Taking $(x,t) \in V_{k'}$ with $x_1 \neq 0$ implies that the left-hand side of this last inequality goes to infinity when $\beta_1 \to \infty$. This contradiction completes the proof of the theorem. ■

As a consequence we study the case $m = 1$. Write a corresponding first integral as $Z(x,t) = x + i\Phi(x,t)$, $(x,t) \in U = J \times \Theta \subset \mathbb{R} \times \mathbb{R}^n$, where J (resp. Θ) is an open interval (resp. ball) centered at the origin of $\mathbb{R}$ (resp. $\mathbb{R}^n$), and $\Phi(x,t)$ is real valued, real-analytic and satisfies $\Phi(0,0) = \Phi_x(0,0) = 0$. According to [BT1], $\mathcal{V}$ is analytic hypoelliptic at the origin if and only if Z is open at the origin.

Theorem 2.2. *If $m = 1$ and $\mathcal{V}$ is not analytic hypoelliptic at the origin then the strong unique continuation property does not hold at the origin for any Gevrey order $\sigma > 2$.*

Proof. By hypothesis Z is not open at the origin and thus we can assume that

$$\Phi(0,t) \geq 0, \quad t \in \Theta.$$

Since $\Phi_x(0,0) = 0$ given $\lambda > 0$ there is an open neighborhood V_λ of the origin such that

$$|\Phi(x,t) - \Phi(x',t)| \leq \lambda|x - x'|/2 \quad (x,t),\, (x',t) \in V_\lambda\,.$$

Hence

$$\Phi(x,t) \geq \Phi(x,t) - \Phi(0,t) \geq -\lambda|x|/2 > -\lambda|x|, \quad (x,t) \in V_\lambda,\ x \neq 0.$$

In particular it follows that Z maps the set $V_\lambda \setminus Z^{-1}\{0\}$ into a truncated sector

$$S_\lambda = \{w \in \mathbb{C} : \mathsf{Im}\, w + \lambda|\mathsf{Re}\, w| > 0,\ |w| < \rho\}.$$

Notice that S_λ has opening equal to $\pi + 2\arctan\lambda$.

Let $\sigma > 2$. Taking $\lambda > 0$ such that $\pi + 2\arctan\lambda < (\sigma - 1)\pi$ it follows that there is a truncated sector $S^\bullet$ with opening equal to $(\sigma - 1)\pi$ so that $S_\lambda \subset\subset S^\bullet$, and consequently [M, Théorème 2.1.3.1] given $\mathfrak{s} = \sum_{k=0}^\infty a_k Z^k/k! \in \mathbb{C}_{(\sigma)}[[Z]]$ there is $H \in \mathcal{O}(S^\bullet)$ satisfying:

- For some constant $C > 0$ the following estimates hold:

$$|H^{(k)}(w)| \leq C^{k+1}k!^\sigma,\ z \in S_\lambda,\ k \in \mathbb{N};$$

- $\lim_{w \to 0, w \in S_\lambda} H^{(k)}(w) = a_k,\ k \in \mathbb{N}$.

Define $u = H \circ Z$ on $V_\lambda \setminus Z^{-1}\{0\}$. It is clear that

$$|(\mathrm{M}^k u)(x,t)| \leq C^{k+1} k!^\sigma,\ (x,t) \in V_\lambda \setminus Z^{-1}\{0\},\ k \in \mathbb{N}.$$

Furthermore

$$\lim_{(x,t)\to Z^{-1}\{0\}} (\mathrm{M}^k u)(x,t) = a_k,$$

and hence each of the functions $\mathrm{M}^k u$ on $V_\lambda \setminus Z^{-1}\{0\}$ has a continuous extension to V_λ, simply by setting it equal to a_k on $Z^{-1}\{0\}$. We denote by $v_k \in C(V_\lambda)$ each of such extensions. Since $Z^{-1}\{0\} \subset \{(x,t) \in V_\lambda : x = 0\}$ it follows that, in the distribution sense, $\mathrm{L}_j v_k = 0$, $j = 1, \dots, n$, $\mathrm{M} v_k = v_{k+1}$. Consequently u is a smooth solution for $\mathcal{V}$ defined in V_λ (cf. [H, Theorem 3.1.7]) and it satisfies $(\mathrm{M}^k u)|_{Z^{-1}\{0\}} = a_k$ for every $k \in \mathbb{Z}_+$. Since it is also clear from the argument that u is Gevrey of order σ, and since $\mathfrak{T}_\sigma(u) = \mathfrak{s}$, we have shown that $\mathfrak{T}_\sigma$ is surjective and hence the result is a consequence of Theorem 2.1. ■

3. The key lemma

For $r > 0$ we shall let

$$D = \{w \in \mathbb{C} : |w| < r,\ \mathrm{Re}\, w > 0\}, \quad \gamma = \{w \in \mathbb{C} : |\mathrm{Im}\, w| < r,\ \mathrm{Re}\, w = 0\}.$$

Lemma 3.1. ([M],[Hr]) *Let $h \in \mathcal{O}(D)$ be continuous up to γ and assume that $h|_\gamma$ is of Gevrey order 2. If $h|_\gamma$ vanishes to infinite order at the origin then h vanishes identically.*

Proof: Since h is harmonic in D, continuous up to γ and its trace on γ is G^2, it follows from the regularity results for boundary value problems for elliptic operators that h is of Gevrey order 2 up to γ (cf. [LM, Ch. 8, Théorème 1.3]). Decreasing $r > 0$ if necessary it follows that

$$|h^{(n)}(w)| \leq C^{n+1} n!^2,\ n \geq 0,\ w \in D,$$

for some constant $C > 0$.

Since $h|_\gamma$ is flat at the origin, by Taylor's formula we obtain a constant $B > 0$ such that

$$|h(w)| \leq B^{n+1} n! |w|^n, \quad n \geq 0,\ w \in D,$$

and hence

$$|h(w)| \leq B \min_{n\geq 0} \{B^n n^n |w|^n\}, \quad w \in D.$$

Now for $w \neq 0$ the minimum of the function $\lambda(\tau) = (B|w|\tau)^\tau$, $\tau > 0$, is attained at the point $\tau_0 = (eB|w|)^{-1}$. Consequently

$$\min_{n\geq 0} \{B^n n^n |w|^n\} = \lambda(\tau_0^*) = e^{\tau_0^* \log\{B|w|\tau_0^*\}},$$

where either $\tau_0^* = [\tau_0]$ or $\tau_0^* = [\tau_0] + 1$. Since $\tau_0^* \leq \tau_0 + 1$ we have, in the region

$$D' = \{w \in D : |w| \leq 1/(eB)\},$$

the estimate $\tau_0^* \leq 2(eB|w|)^{-1}$. Hence, when $w \in D'$,

$$\log\{B|w|(\tau_0^*)\} \leq \log\{2/e\} \doteq -a < 0.$$

Thus

$$|h(w)| \leq B e^{-a\tau_0^*}, \quad w \in D',$$

Since $\tau_0^* \geq [\tau_0] \geq \tau_0 - 1$ we obtain

$$|h(w)| \leq B e^a e^{-a\tau_0} = B' e^{-c|w|^{-1}}, \quad w \in D'.$$

Using now the fact that $e^{c|w|^{-1}}h(w)$ is bounded in $D \setminus D'$ we finally conclude that, for a new constant $C > 0$,

$$|h(w)| \le Ce^{-c|w|^{-1}}, \quad w \in D.$$

Let now $\mathcal{W}$ be the image of D under the inversion $w \mapsto 1/w$. Then $g(\zeta) = h(1/\zeta)$ is holomorphic in $\mathcal{W}$ and satisfies

$$|g(\zeta)| \le C\exp\{-c|\zeta|\}, \quad \zeta \in \mathcal{W}. \tag{2}$$

Since $\mathcal{W}$ contains a semi-space (2) implies that g, and hence h, vanishes identically, according to Watson's lemma [M, Lemma 1.2.3.3]. This completes the argument. ■

4. Minimal structures and Marson's theorem

We continue to work on a real analytic manifold Ω of dimension N over which it is defined a real analytic involutive structure $\mathcal{V}$ of rank $n \le N$. Denote by $\mathfrak{g}(\mathcal{V})$ the (real) Lie algebra spanned by all vectors fields $\mathsf{Re}\,\mathrm{L}$, where L is a section of $\mathcal{V}$. By Nagano's theorem (see e.g. [BER], [MP]) given $p \in \Omega$ there is a real-analytic submanifold $\mathcal{N}$ in Ω, with $p \in \mathcal{N}$, such that $\mathrm{T}_q\mathcal{N} = \mathfrak{g}(\mathcal{V})_q$, for all $q \in \mathcal{N}$. The germ of such $\mathcal{N}$ at p is uniquely determined and it is referred to as the *local Nagano leaf of* $\mathcal{V}$ *at* p. Finally we shall say that $\mathcal{V}$ is *minimal at* p if the local Nagano leaf $\mathcal{N}$ of $\mathcal{V}$ at p is an open neighborhood of p (that is, $\operatorname{codim}\mathcal{N} = 0$). This property is equivalent to $\mathcal{V}$ being of *finite type at* p, that is, to the existence of an open neighborhood $\omega \subset \Omega$ of p such that $\mathfrak{g}(\mathcal{V})|_\omega = \mathrm{T}\Omega|_\omega$.

In order to state Marson's theorem we shall introduce, near a fixed point $p \in \Omega$, more suitable coordinates and generators which bring to the picture the characteristic set of $\mathcal{V}$ at p:

$$\mathrm{T}_p^\circ \doteq \mathcal{V}_p^\perp \cap \mathrm{T}_p^*\Omega.$$

We apply [BCH, Theorem I.10.1]: there is a coordinate system for Ω centered at p, written in the form $(x_1, \dots, x_\alpha, y_1, \dots, y_\alpha, s_1, \dots, s_d, t_1, \dots, t_\beta)$ and assumed defined in an open neighborhood U of the origin in $\mathbb{R}^N$, where $2\alpha + \beta + d = N$, such that $\mathcal{V}^\perp$ is spanned over U by the differential of the real-analytic functions

$$\begin{cases} Z_j = z_j = x_j + iy_j, \quad j = 1, \dots, \alpha; \\ \\ W_k(x,y,s,t) = s_k + i\Psi_k(x,y,s,t), \quad k = 1, \dots, d. \end{cases}$$

Here Ψ_k are real-valued and satisfy $\Psi_k(0,0,0,0) = 0$, $\mathrm{d}\Psi_k(0,0,0,0) = 0$ for all $k = 1, \dots, d$. As usual we shall write $Z = (Z_1, \dots, Z_\alpha)$, $\Psi = (\Psi_1, \dots, \Psi_d)$ and $W = (W_1, \dots, W_d)$. In our previous notation we have

$$m = \alpha + d, \quad n = \alpha + \beta\,.$$

Moreover the characteristic set of $\mathcal{V}$ at the origin is $\mathrm{T}_0^\circ = \mathrm{span}\,\{\mathrm{d}s_1|_0, \dots, \mathrm{d}s_d|_0\}$.

In a contracted U we can introduce the real-analytic vector fields

$$\mathrm{N}_k = \sum_{k'=1}^{d} \mu_{kk'}(x,y,s,t)\frac{\partial}{\partial s_{k'}}, \;\; k = 1, \dots, d$$

characterized by the rule

$$\mathrm{N}_k W_{k'} = \delta_{k,k'}\,, \quad k, k' = 1, \dots, d.$$

It follows that the complex vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial \bar{z}_j} - i\sum_{k=1}^{d} \frac{\partial \Psi_k}{\partial \bar{z}_j}(x,y,s,t)\mathrm{N}_k, \;\; j = 1, \dots, \alpha,$$

$$\mathrm{L}_\ell^\bullet = \frac{\partial}{\partial t_\ell} - i \sum_{k=1}^{d} \frac{\partial \Psi_k}{\partial t_\ell}(x, y, s, t)\mathrm{N}_k, \;\; \ell = 1, \dots, \beta,$$

span $\mathcal{V}|_U$. If we further introduce the vector fields

$$\tilde{\mathrm{L}}_j = \frac{\partial}{\partial z_j} - i \sum_{k=1}^{d} \frac{\partial \Psi_k}{\partial z_j}(x, y, s, t)\mathrm{N}_k, \;\; j = 1, \dots, \alpha,$$

then the following holds:

- $\mathrm{L}_1, \dots, \mathrm{L}_\alpha, \mathrm{L}_1^\bullet, \dots, \mathrm{L}_\beta^\bullet, \tilde{\mathrm{L}}_1, \dots, \tilde{\mathrm{L}}_\alpha, \mathrm{N}_1, \dots \mathrm{N}_d$ are pairwise commuting and span $\mathbb{C}T\mathbb{R}^N$ over U.

Remark 4.1. There is a germ of biholomorphism G at the origin in $\mathbb{C}^d$ such that $G(0) = 0$, $G'(0)$ is the identity and $G(w + i\Psi(0, 0, w, 0)) = w$ (here of course the map $s \mapsto \Psi(0, 0, s, 0)$ has been extended holomorphically to an open neighborhood of the origin in $\mathbb{C}^d$). Writing, for $k = 1, \dots, d$,

$$\tilde{W}_k(x, y, s, t) = G_k(W(x, y, s, t)) = A_k(x, y, s, t) + iB_k(x, y, s, t),$$

then $A(0, 0, s, 0)$ is the identity map, and $B(0, 0, s, 0) = 0$. Taking $s' = A(x, y, s, t)$ as a new variable s allows us to assume, after a suitable contraction of U around the origin, that

$$\Psi_k(0, 0, s, 0) = 0, \quad k = 1 \dots, d. \qquad \blacksquare \tag{3}$$

We now introduce some additional notation. We denote by $\lambda : U \to \mathbb{C}^m$ the map

$$\lambda(x, y, s, t) = (Z(x, y, s, t), W(x, y, s, t)) = (z_1, \dots, z_\alpha, s_1 + i\Psi_1(x, y, s, t), \dots, s_d + i\Psi_d(x, y, s, t))$$

and, for a strictly convex open cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R}^\beta$ near the origin, we set

$$\mathcal{W}_a(\Gamma; t) \doteq \Big\{(z, w) \in \mathbb{C}^{\alpha+d} : |z| < a, |\mathsf{Re}\, w| < a, \mathsf{Im}\, w = \Psi(\mathsf{Re}\, z, \mathsf{Im}\, z, \mathsf{Re}\, w, t) + v, v \in \Gamma, |v| < a\Big\}.$$

Finally, if $a > 0$ is small we also set

$$\mathcal{W}_a(\Gamma) = \bigcup_{|t|<a} \mathcal{W}_a(\Gamma; t).$$

Notice that each $\mathcal{W}_a(\Gamma; t)$ is an open subset of $\mathbb{C}^m$ and so the same is true for $\mathcal{W}_a(\Gamma)$, which is called *a wedge with edge*

$$\mathcal{X}_a \doteq \bigcup_{|t|<a} \Big\{(z, w) \in \mathbb{C}^{\alpha+d} : |z| < a, |\mathsf{Re}\, w| < a, \mathsf{Im}\, w = \Psi(\mathsf{Re}\, z, \mathsf{Im}\, z, \mathsf{Re}\, w, t)\Big\} \subset \lambda(U)$$

Notice also that (3) implies that

$$\mathcal{W}_{a,0}(\Gamma) \doteq \mathcal{W}_a(\Gamma; 0) \cap \{z = 0\} = \{(0, w) : \; |\mathsf{Re}\, w| < a, |\mathsf{Im}\, w| < a, \; \mathsf{Im}\, w \in \Gamma\}, \tag{4}$$

a property that will be important in what follows.

According to a theorem of Marson [Ma, Theorem 1], *$\mathcal{V}$ is minimal at the origin if and only if the following holds: there is a strictly convex open cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ such that if u is a smooth solution for $\mathcal{V}$ near the origin then there is $a > 0$ and a (unique) holomorphic function $H(z, w) \in \mathcal{O}(\mathcal{W}_a(\Gamma))$, continuous up to $\mathcal{X}_a$, such that $u = H \circ \lambda$ on $\lambda^{-1}(\mathcal{X}_a)$.*

Remark 4.2. In the statement of Marson's theorem, taking into account that u is smooth, we can assert that all derivatives of H are indeed continuous up to $\mathcal{X}_a$ (see e.g. the argument in [BER, pp. 202-203]). In particular, we obtain the relations

$$\tilde{\mathrm{L}}_j u = (\partial H/\partial z_j) \circ \lambda, \quad \mathrm{N}_k u = (\partial H/\partial w_k) \circ \lambda,$$

which hold for $j = 1 \dots, \alpha$, $k = 1, \dots, d$. ■

5. Strong unique continuation versus minimality

We keep the notation established in the previous section and start by proving the following result:

Theorem 5.1. *If $\mathcal{V}$ is minimal at the origin then the strong unique continuation property holds at the origin for any Gevrey order $\sigma \in [1, 2]$.*

Proof. Let u be a G^2 solution for $\mathcal{V}$ near the origin. We assume that u vanishes to infinite order at the origin and will show that u vanishes identically in some neighborhood of the origin. Take Γ and $a > 0$ according to Marson's theorem and let $H(z, w) \in \mathcal{O}(\mathcal{W}_a(\Gamma))$, continuous up to $\mathcal{X}_a$, be such that $u = H \circ \lambda$ on $\lambda^{-1}(\mathcal{X}_a)$. Fix $v \in \Gamma$ with $|v| < a$ and set $P_0 = (0, iv) \in \mathcal{W}_{a,0}(\Gamma)$. We claim that it suffices to show that $H(P_0) = 0$.

Indeed, since $\tilde{\mathrm{L}}^\gamma \mathrm{N}^\delta u$, $\gamma \in \mathbb{Z}^\alpha_+$, $\delta \in \mathbb{Z}^d_+$, are also G^2 solutions for $\mathcal{V}$ vanishing to infinite order at the origin (we are writing $\tilde{\mathrm{L}}^\gamma = \tilde{\mathbf{L}}_1^{\gamma_1} \cdots \tilde{\mathrm{L}}_\alpha^{\gamma_\alpha}$, $\mathrm{N}^\delta = \mathrm{N}_1^{\delta_1} \cdots \mathrm{N}_d^{\delta_d}$), Remark 4.2 and the uniqueness property pointed out in Marson's theorem implies that all derivatives of H also vanish at P_0. Hence H vanishes identically on $\mathcal{W}_a(\Gamma; 0)$, since the latter is connected. We then conclude that u vanishes on the submanifold $t = 0$, which is strongly noncharacteristic with respect to $\mathcal{V}$, and then that u vanishes identically in a neighborhood of the origin [T, Corollary II.3.7].

In order to show that $H(P_0) = 0$ we first observe that the open set of the complex plane $A(P_0) = \{\zeta \in \mathbb{C} : \zeta P_0 \in \mathcal{W}_{a,0}(\Gamma)\}$ contains the set $D = \{\zeta : |\zeta| \le 1, \mathsf{Re}\,\zeta > 0\}$. Indeed if $\zeta = \rho e^{i\theta}$, with $0 < \rho \le 1$ and $-\pi/2 < \theta < \pi/2$ then $\zeta P_0 = (0, i\rho e^{i\theta} v) = (0, -\rho(\sin\theta)v) + i(0, \rho(\cos\theta)v)$, and hence $|\mathsf{Re}\,(\zeta P_0)| < a$, $|\mathsf{Im}\,(\zeta P_0)| < a$ and the imaginary part of the w-component of ζP_0 belongs to Γ, for $\rho\cos\theta > 0$ and $v \in \Gamma$.

Let $h(\zeta) = H(\zeta P_0)$, $\zeta \in D$. Then h is holomorphic in the interior of D and continuous on its closure. We can say more: the restriction of h to $\gamma = \{i\tau : \tau \in \mathbb{R}, |\tau| < 1\} \subset \partial D$ is of Gevrey order 2. Indeed, this follows from the fact that $u(0, 0, -\tau v, 0) = H(\lambda(0, 0, -\tau v, 0)) = H(0, -\tau v + i\Phi(0, 0, -\tau v, 0)) = H(0, -\tau v) = h(i\tau)$. Since, furthermore, $h|_\gamma$ is flat at the origin, Lemma 3.1 implies that h vanishes identically. In particular we obtain $H(P_0) = h(1) = 0$, which concludes the proof. ■

Corollary 5.1. *Assume $m = 1$. If $\mathcal{V}$ is minimal but not analytic hypoelliptic at the origin then the strong unique continuation property holds at the origin for Gevrey order σ if and only if $\sigma \in [1, 2]$.*

Proof. Combine theorems 5.1 and 2.2. ■

Next we prove:

Theorem 5.2. *If $\mathcal{V}$ is not minimal at the origin then the strong unique continuation property does not hold at the origin for any Gevrey order $\sigma > 1$.*

Proof. We shall make use of the local coordinates $(x_1, \dots, x_m, t_1, \dots, t_n)$ and the generators L_j and Z_k for $\mathcal{V}$ as described in Section 1. By hypothesis the local Nagano leaf $\mathcal{N}$ of $\mathcal{V}$ through the origin has codimension $M \ge 1$. Since $\mathcal{V}|_\mathcal{N} \subset \mathbb{C}\mathrm{T}\mathcal{N}$ we must have $M \le m$. Select real analytic functions $\rho_1, \dots, \rho_N$ defined near the origin such that $\mathcal{N}$ is defined by the equations $\rho_1 = \ldots = \rho_M = 0$, with $\mathrm{d}\rho_1 \wedge \ldots \wedge \mathrm{d}\rho_M \neq 0$ near the origin. Since $\mathrm{L}_j \rho_\ell = 0$ it follows in particular that the rank of the matrix $\{\mathrm{M}_k \rho_{k'}(0, 0)\}$ is equal to M. For each $k = 1, \dots, M$ we solve the Cauchy problem

$$\mathrm{L}_j u_k = 0, \quad j = 1, \dots, n, \;\; u_k|_{\{t=0\}} = \rho_k. \tag{5}$$

Since $\mathcal{N} \cap \{t = 0\}$ is also maximally real for the structure $\mathcal{V}|_{\mathcal{N}}$ on $\mathcal{N}$, and since each $u_k|_{\mathcal{N}}$ is a solution for $\mathcal{V}|_{\mathcal{N}}$, it follows from the uniqueness in the Cauchy problem that $u_k|_{\mathcal{N}} = 0$. Hence we can write

$$u_k(x,t) = \sum_{r=1}^{M} v_{k,r}(x,t)\, \rho_r(x,t),$$

where $v_{k,\ell}$ are real-analytic functions. Moreover (5) gives

$$\mathrm{M}_\ell u_k(0,0) = \mathrm{M}_\ell \rho_k(0,0)$$

and consequently

$$\mathrm{M}_\ell \rho_k(0,0) = \sum_{r=1}^{M} v_{k,r}(0,0)\, \mathrm{M}_\ell \rho_r(0,0), \quad \ell = 1, \dots, m, \;\; k = 1 \dots, M.$$

Since $\{\mathrm{M}_k \rho_{k'}(0,0)\}$ has rank M we conclude that $\Lambda \doteq \{v_{k,r}(0)\}$ is invertible. Set $\mathbf{u} = (u_1, \dots, u_M)$, $\mathbf{v} = \Lambda^{-1}\mathbf{u}$ and $\vec{\rho} = (\rho_1, \dots, \rho_M)$. Then

$$\mathbf{v}(x,t) = \vec{\rho} + \Theta(x,t)\vec{\rho},$$

where $\Theta(x,t)$ is a real-analytic $M \times M$ matrix, $\Theta(0,0) = 0$. If we set [1]

$$W(x,t) \doteq \mathbf{v}(x,t) \cdot \mathbf{v}(x,t)$$

then W is a real-analytic solution for $\mathcal{V}$ near the origin. Moreover

$$W(x,t) = |\vec{\rho}|^2 + 2(\Theta(x,t)\vec{\rho}) \cdot \vec{\rho} + (\Theta(x,t)\vec{\rho} \cdot (\Theta(x,t)\vec{\rho})$$

and consequently given $\varepsilon > 0$ there is an open neighborhood U_ε of the origin in $\mathbb{R}^N$ such that $(x,t) \in U_\varepsilon$ implies

$$\mathsf{Re}\, W(x,t) \geq |\vec{\rho}|^2/2, \quad |\mathsf{Im}\, W(x,t)| \leq \varepsilon |\vec{\rho}|^2/4.$$

Then $W(\{(x,t) \in U_\varepsilon : \vec{\rho}(x,t) \neq 0\}) \subset S_\varepsilon \doteq \{w \in \mathbb{C} : |\mathsf{Im}\, w| < \varepsilon \mathsf{Re}\, w\}$. Let $\sigma > 1$. Taking $\varepsilon > 0$ appropriately small we can ensure that $H(w) = e^{-|w|^{-1/(\sigma-1)}}$ is holomorphic in the sector S_ε. If we define $v : U_\varepsilon \to \mathbb{C}$ by the rule $v(x,t) = H(W(x,t))$ when $\vec{\rho}(x,t) \neq 0$, $v(x,t) = 0$ when $\vec{\rho}(x,t) = 0$ we obtain a non trivial solution for $\mathcal{V}$ which is Gevrey of order σ and vanishes to infinite order at the origin. ■

Corollary 5.2. *The real-analytic, involutive structure $\mathcal{V}$ is minimal at the origin if and only if there is $\sigma > 1$ such that the strong unique continuation property holds at the origin for Gevrey order σ .*

In our next result we shall adopt the following definition: we say that $\mathcal{V}$ *satisfies property (†) at the origin if there is $W \in \mathfrak{S}_0^\infty(\mathcal{V}) \cap C_0^\omega$ not open at 0 and satisfying $\mathrm{d}W|_0 \neq 0$.*

Theorem 5.3. *If $\mathcal{V}$ satisfies property (†) at the origin then the strong unique continuation property does not hold at the origin for any Gevrey order $\sigma > 2$.*

Proof. Consider the real-analytic locally integrable structure $\mathcal{V}_\bullet$ whose orthogonal bundle is spanned by $\mathrm{d}W$. Since every solution for $\mathcal{V}_\bullet$ is *a fortiori* a solution for $\mathcal{V}$ and since (†) implies

[1]Here we are considering, in $\mathbb{C}^M$, the bilinear form

$$(z,w) \mapsto z \cdot w = z_1 w_1 + \ldots + z_M w_M.$$

that $\mathcal{V}_\bullet$ is not analytic hypoelliptic at the origin [BT1] it follows from Theorem 2.2 that for every $\sigma > 2$ there is $u \in \mathfrak{S}_0^\infty(\mathcal{V})$ of Gevrey order σ and not identically zero which vanishes to infinite order at the origin. ■

We have a result similar than that stated in Corollary 5.1 for real-analytic tubular structures, that is, involutive structures $\mathcal{V}$ on $\mathbb{R}^N$ whose orthogonal is spanned by the differentials of the first integrals $Z_j(x,t) = x_j + i\Phi_j(t)$, $j = 1, \dots, m$. Here $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$ and the real-valued, real-analytic functions Φ_j are defined in an open neighborhood of the origin in $\mathbb{R}^n$ and satisfy $\Phi_j(0) = 0$.

Corollary 5.3. *If $\mathcal{V}$ is a tubular real-analytic structure which is minimal but not analytic hypoelliptic at the origin then the strong unique continuation property holds at the origin for Gevrey order σ if and only if $\sigma \in [1,2]$.*

Proof. By theorems 5.1 and 5.3 it suffices to show that $\mathcal{V}$ satisfies (†) at the origin. According to [BT2, Theorem 2.1], $\mathcal{V}$ is analytic hypoelliptic at the origin if and only if for every $\xi \in \mathbb{R}^m \setminus \{0\}$ the origin is not a local extremum of the function $t \mapsto \Phi(t) \cdot \xi$. Hence, under the hypothesis of non analytic hypoellipticity at the origin we conclude the existence of $\xi_0 \in \mathbb{R}^m \setminus \{0\}$ such that $\Phi(t) \cdot \xi_0 \geq 0$ for t near the origin. Define $W(x,t) = x \cdot \xi_0 + i\Phi(t) \cdot \xi_0$. It follows that W is a real-analytic solution for $\mathcal{V}$ satisfying $\mathrm{d}W(0,0) \neq 0$ and $\mathsf{Im}\, W \geq 0$. ■

Finally we turn our attention to nondegenerate structures.

Theorem 5.4. *Assume that $\mathcal{V}$ is minimal at the origin and that for some $\theta \in \mathrm{T}_0^\circ$ the Levi form of $\mathcal{V}$ at θ is definite positive. Then the strong unique continuation property holds at the origin for Gevrey order σ if and only if $\sigma \in [1,2]$.*

Proof. As proved in [BCP, Proposition 4], the existence of such θ implies the validity of property (†). Hence the result is a consequence of theorems 5.1 and 5.3. ■

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