R TORSION AND ANALYTIC TORSION OF A CONICAL FRUSTUM

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1. Introduction

Recently important advances have been made in the description of the analytic torsion of compact Riemannian manifolds with boundary [18] [2] [3]. In particular, in the last two works a formula for the analytic torsion of a compact oriented Riemannian manifold with boundary and absolute or relative boundary conditions was given. On the other side, in a series of works [14] [11] [10] [12], we presented explicit calculations of the analytic torsion of some class of manifolds and pseudomanifolds. In particular formulas for the torsion of a cone over a compact manifold where given. When working with cones, a natural question arises: if we truncate the cone we end up with a manifold. Does the analytic torsion of the cone coincide with some limit of the analytic torsion of the truncated cone? This question was suggested to us by W. Müller. The answer is given in the Section 5 below, for an odd dimensional section, and is positive after a suitable regularization. As an important related problem, we tackle and answer the same question for Ray and Singer metrics [1]. In the last section we describe in details a particular case, namely the frustum over a circle. It is clear that the technique used for the circle admits a straightforward generalization to the case of any section. In the appendix, we present explicit calculation of the intersection torsion of a cone, that was proved to coincide with its analytic torsion [13].

2. Geometric setting

Let W be a compact connected oriented m dimensional Riemannian manifold with metric g. The conical frustum (or truncated cone) over W is the product manifold $\mathsf{F}W = [l_1, l_2] \times W$, where $0 < l_1 < l_2$, with metric (in the local coordinates (x, y), where y is a local system on W)

$$g_{\mathsf{F}} = dx \otimes dx + x^2 g.$$

The boundary of $\mathsf{F}W$ is the disjoint union $\partial \mathsf{F}W = W_1 \sqcup W_2$ of two copies of W with metric $l_j^2 g$, that we denote by $W_j = (W, l_j^2 g)$.

3. Reidemeister torsion and Reidemeister metric on the frustum

In this section we prove relations between the R torsion and the R metric of the frustum $\mathsf{F}W$ and of its section W. For we first review some necessary notation.

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3.1. **R torsion of a chain complex.** Let V be a real vector space of dimension n. Denote the one dimensional vector space $\Lambda^n V$ by $\det V$. For $V = \{0\}$, set $\det\{0\} = \mathbb{R}$. For a finite dimensional real graded vector space $V_\bullet = \bigoplus_{q\in\mathbb{Z}} V_q$, set

$$\det V_\bullet = \bigotimes_{q\in\mathbb{Z}} (\det V_q)^{(-1)^q},$$

where L^{-1} denotes the dual line of a one dimensional vector space L. A set of $\dim V_q$ linear independent vectors $\mathsf{v}_q = \{v_{q,1}, \dots, v_{q,\dim V_q}\}$ of V_q induces a non zero vector $\det \mathsf{v}_q = \Lambda_{j=1}^{\dim V_q} v_{q,j}$ in $\det V_q$, and a non zero vector $\det \mathsf{v} = \bigotimes_{q=0}^{\dim V_q} (\det \mathsf{v}_q)^{(-1)^q}$ in $\det V_\bullet$, where for a non zero vector l in a one dimensional vector space L, $l^{-1} \in L^{-1}$ is defined by $l^{-1}(l) = 1$.

Let C be a finite dimensional complex of finite dimensional vector spaces,

$$\mathsf{C}: \qquad \mathsf{C}_m \xrightarrow{\partial_m} \mathsf{C}_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} \mathsf{C}_1 \xrightarrow{\partial_1} \mathsf{C}_0.$$

There is a natural isomorphism called Euler isomorphism (see for example [9] pg. 481)

$$\mathcal{E} : \det \mathsf{C} \to \det H(\mathsf{C}), \tag{1}$$

where we denote by $H(\mathsf{C})$ the homology of the complex C. Let $\mathsf{c}_q = \{c_{q,j}\}$ and $\mathsf{h}_q = \{h_{q,j}\}$ be given preferred bases for C_q and $H_q(\mathsf{C})$, respectively, for each q. These bases induce bases $\det \mathsf{c} = \bigotimes_{q=0}^m (\det \mathsf{c}_q)^{(-1)^q}$ of $\det \mathsf{C}$, and $\det \mathsf{h} = \bigotimes_{q=0}^m (\det \mathsf{h}_q)^{(-1)^q}$ of $\det H(\mathsf{C})$. Then:

$$\mathcal{E}(\det \mathsf{c}) = \mathcal{E}^{(\det \mathsf{h}/\det \mathsf{c})} \det \mathsf{h},$$

where $(\mathcal{E}^{(\det \mathsf{h}/\det \mathsf{c})})^{-1} \in \mathbb{R}^+$ is the torsion of C in the basis c and h, and is defined as follows. Let $Z_q = \ker \partial_q$, $B_q = \mathrm{Im}\partial_{q+1}$, and $H_q(\mathsf{C}) = Z_q/B_q$. Let $\mathsf{b}_q = \{b_{q,j}\}$ be a set of independent vectors in C_q with $\partial_q(\mathsf{b}_q) \neq 0$, and let $\mathsf{z}_q = \{z_{q,j}\}$ be a set of independent vectors in Z_q with $p(z_{q,j}) = h_{q,j}$. Then, considering the sequence

$$0 \longrightarrow B_q \longrightarrow Z_q \xrightarrow[p]{} H_q(\mathsf{C}) \longrightarrow 0,$$

a basis for Z_q is given by the basis $\partial_{q+1}(\mathsf{b}_{q+1})$ of B_q and the set z_q. We denote this basis by $\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q$ (see [15] for details). By the same argument, the sequence

$$0 \longrightarrow Z_q \longrightarrow \mathsf{C}_q \xrightarrow[\partial_q]{} B_{q-1} \longrightarrow 0,$$

determines the basis $\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q$ of C_q. Let $(\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)$ denote the matrix of the change of basis. Then, the torsion of C is the class

$$\tau(\mathsf{C}; \mathsf{c}, \mathsf{h}) = \prod_{q=0}^{n} |\det(\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)|^{(-1)^q},$$

in $\mathbb{R}^+$. It is easy to see that the torsion is independent of the graded bases $\mathsf{b} = \{\mathsf{b}_q\}$ and on the lifts $\mathsf{z} = \{\mathsf{z}_q\}$, but depends on the graded basis $\mathsf{c} = \{\mathsf{c}_q\}$ and $\mathsf{h} = \{\mathsf{h}_q\}$. More precisely, $\tau(\mathsf{C}; \mathsf{c}, \mathsf{h})$ depends on the volume elements $\det \mathsf{c} = \otimes_{q=0}^m (\Lambda_j c_{q,j})^{(-1)^q}$ in $\otimes_{q=0}^m \Lambda^{\dim \mathsf{C}_q} \mathsf{C}_q$, and $\det \mathsf{h} = \otimes_{q=0}^m (\Lambda_j h_{q,j})^{(-1)^q}$ in $\otimes_{q=0}^m \Lambda^{r_q} H_q(\mathsf{C})$, where $r_q = \mathrm{rk} H_q(\mathsf{C})$ (see for example [17]).

We remark that $\tau(\mathsf{C}; \mathsf{c}, \mathsf{h})$ is invariant under 'algebraic' subdivision of the complex C (see [15] Section 5 for definition and details). In particular, this means that if the complex C is associated to some simplicial or cellular complex, and the basis c

is the basis of the simplices (cells), then $\mathcal{E}^{(\mathsf{h}/\mathsf{c})}$ is invariant if we change the basis c by the simplicial (cellular) basis of some subdivision of the simplicial (cellular) complex.

3.2. **R torsion of the cylinder of a chain complex.** Recall that the cylinder of the complex C is the mapping cylinder of the identity $id : \mathsf{C} \to \mathsf{C}$, i.e. the complex $\mathsf{C}_q(Cyl(\mathsf{C})) = \mathsf{C}_q \oplus \mathsf{C}_{q-1} \oplus \mathsf{C}_q$ with boundary

$$\ddot{\partial} = \begin{pmatrix} \partial & 1 & 0 \\ 0 & -\partial & 0 \\ 0 & -1 & \partial \end{pmatrix}.$$

A preferred basis for $\mathsf{C}_q(Cyl(\mathsf{C}))$ is $\ddot{\mathsf{c}}_q = \{c_{q,j} \oplus 0 \oplus 0, 0 \oplus c_{q-1,k} \oplus 0, 0 \oplus 0 \oplus c_{q,l}\}$. By construction, $Cyl(\mathsf{C})$ has an homology graded preferred basis, and therefore its Whitehead torsion is well defined. We denote the preferred basis of $H_q(Cyl(\mathsf{C}))$ by $\ddot{\mathsf{h}}_q$, and we let $\ddot{\mathsf{z}}_q$ denotes a lift of $\ddot{\mathsf{h}}_q$. Now we have the decomposition $B_q(Cyl(\mathsf{C})) = \mathrm{Im}\ddot{\partial}_{q+1} = (\mathrm{Im}\partial_{q+1} + \ker\partial_q) \oplus \mathrm{Im}\partial_q \oplus (\mathrm{Im}\partial_{q+1} + \ker\partial_q)$, and hence a set of independent elements in $\mathsf{C}_q(Cyl(\mathsf{C}))$ with non trivial image is $\ddot{\mathsf{b}}_q = \{b_{q,j} \oplus 0 \oplus 0, 0 \oplus 0 \oplus b_{q,k}, 0 \oplus b_{q-1,l} \oplus 0, 0 \oplus z_{q-1,i} \oplus 0, 0 \oplus \partial_q(b_{q,h}) \oplus 0\}$, that we denote by: $\mathsf{b}_q \oplus 0 \oplus 0, 0 \oplus 0 \oplus \mathsf{b}_q, 0 \oplus \mathsf{b}_{q-1} \oplus 0, 0 \oplus \mathsf{z}_{q-1} \oplus 0, 0 \oplus \partial_q(\mathsf{b}_q) \oplus 0$. A basis for $Z_q(Cyl(\mathsf{C}))$ is then $\ddot{\mathsf{b}}_q, \ddot{\mathsf{z}}_q$, and a basis for $\mathsf{C}_q(Cyl(\mathsf{C}))$ is $\ddot{\partial}_{q+1}(\ddot{\mathsf{b}}_{q+1}), \ddot{\mathsf{z}}_q, \ddot{\mathsf{b}}_q$. Reordering and simplifying, we obtain the new basis

$$\begin{aligned} \ddot{\partial}_{q+1}(\ddot{\mathsf{b}}_{q+1}), \ddot{\mathsf{z}}_q, \ddot{\mathsf{b}}_q =& \partial_{q+1}(\mathsf{b}_{q+1}) \oplus 0 \oplus 0, \ddot{\mathsf{z}}_q, \mathsf{b}_q \oplus 0 \oplus 0, \\ & 0 \oplus \partial_q(\mathsf{b}_q) \oplus 0, 0 \oplus \mathsf{z}_{q-1} \oplus 0, 0 \oplus \mathsf{b}_{q-1} \oplus 0, \\ & 0 \oplus 0 \oplus, \partial_{q+1}(\mathsf{b}_{q+1}), 0 \oplus 0 \oplus \mathsf{z}_q, 0 \oplus 0 \oplus \mathsf{b}_q. \end{aligned}$$

This gives (with some care at the high dimensions)

$$\begin{aligned} &|\det(\ddot{\partial}_{q+1}(\ddot{\mathsf{b}}_{q+1}), \ddot{\mathsf{z}}_q, \ddot{\mathsf{b}}_q/\ddot{\mathsf{c}}_q)| \\ &\quad = |(i^{-1}_{*,q}(\ddot{\mathsf{h}}_q)/\mathsf{h}_q)||\det(\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)|^2 |\det(\partial_q(\mathsf{b}_q), \mathsf{z}_{q-1}, \mathsf{b}_{q-1}/\mathsf{c}_{q-1})|, \end{aligned}$$

where $i : \mathsf{C} \to Cyl(\mathsf{C})$ denotes the inclusion, and for the torsion

$$\tau(Cyl(\mathsf{C}); \ddot{\mathsf{c}}, \ddot{\mathsf{h}}) = \frac{\tau(\mathsf{C}; \mathsf{c}, \mathsf{h})}{|\det(i_*)|}.$$

3.3. **R torsion of a manifold.** Let (K, L) be a pair of connected finite cell complexes of dimension m, and $(\tilde{K}, \tilde{L})$ its universal covering complex pair, and identify the fundamental group $\pi = \pi_1(K)$ with the group of the covering transformations of $\tilde{K}$. Note that covering transformations are cellular. Let $\mathsf{C}((\tilde{K}, \tilde{L}); \mathbb{Z})$ be the chain complex of $(\tilde{K}, \tilde{L})$ with integer coefficients. The action of the group of covering transformations makes each chain group $\mathsf{C}_q((\tilde{K}, \tilde{L}); \mathbb{Z})$ into a module over the group ring $\mathbb{Z}\pi$, and each of these modules is $\mathbb{Z}\pi$-free and finitely generated with preferred basis given by the natural choice of the q-cells of $K - L$. Since K is finite it follows that $\mathsf{C}((\tilde{K}, \tilde{L}); \mathbb{Z})$ is free and finitely generated over $\mathbb{Z}\pi$. We obtain a complex of free finitely generated modules over $\mathbb{Z}\pi$ that we denote by $\mathsf{C}((K, L); \mathbb{Z}\pi)$. Let $\rho : \pi \to O(\mathbb{R}, k)$ be an orthogonal representation of the fundamental group on a real vector space V of dimension k, and consider the twisted complex $\mathsf{C}((K, L); V_\rho) = V \otimes_{\mathbb{Z}\pi} \mathsf{C}((K, L); \mathbb{Z}\pi)$. Bases and volume elements are given in the obvious way by taking tensor product of a fixed basis of V and basis of C and $H(\mathsf{C})$. In particular, we fix the chain basis to be the preferred basis of the cells, and therefore we omit c from the notation. Then, the torsion of (K, L) with respect to the representation ρ is the class

$$\tau((K, L); \rho, \mathsf{h}) = \tau(\mathsf{C}((K, L); V_\rho); \mathsf{h}),$$

of $\mathbb{R}^+$. Next, let M be an n dimensional orientable compact connected Riemannian manifold with metric g and possible boundary ∂M. The torsion of M can be defined taking any smooth triangulation or cellular decomposition of M. Moreover, the volume element $\det \mathsf{h}$ can also be fixed by using the metric structure. More precisely, given a graded orthonormal basis $\mathsf{a} = \{\mathsf{a}_q\}$ for the space of harmonic forms with twisted coefficients in V_ρ, $\mathcal{H}(M, V_\rho) = \bigoplus \mathcal{H}_q(M, V_\rho)$, either with absolute or relative BC, and applying the Hodge-de Rham map

$$\mathcal{A}_{\text{abs},q} = (-1)^q \mathcal{P}^{-1} \mathcal{R}_{\text{rel}}^{n+1-q} \star : \mathcal{H}^q_{\text{abs}}(M, V_\rho) \to H_q(M, V_\rho), \tag{2}$$

where $\mathcal{P}$ is the Poincarè duality isomorphism, $\star$ is the Hodge star, and $\mathcal{R}^q_{\text{rel}}$ is the de Rham map (see for example [21], pg. 164, for details), we obtain a preferred homology graded basis $\mathcal{A}_{\text{abs}}(\mathsf{a})$, that fix the volume element $\det \mathcal{A}_{\text{abs}}(\mathsf{a})$. With this notation, the R torsion of (M, g), and the relative R torsion of $(M, \partial M, g)$ are:

$$\tau_{\rm R}((M, g); \rho) = \tau(\mathsf{C}(M; V_\rho); \mathcal{A}_{\text{abs}}(\mathsf{a})),$$
$$\tau_{\rm R}((M, \partial M, g); \rho) = \tau(\mathsf{C}((M, \partial M); V_\rho); \mathcal{A}_{\text{abs}}(\mathsf{a})).$$

3.4. **Reidemeister metric.** An inner product on a finite real vector space V induces a norm $\|\ \ \|_{\det V}$ on $\det V$, and a dual norm $\|\ \ \|_{(\det V)^{-1}}$ on $(\det V)^{-1}$. For a finite complex C of finite dimensional real vector spaces with inner product, as in Section 3.1, we denote by $\|\ \ \|_{\det \mathsf{C}}$ the norm on $\det \mathsf{C}$ defined by (see [1] (1.5), this is the metric defined by requiring that the cells have unitary norm)

$$\|\ \ \|_{\det \mathsf{C}} = \bigotimes_{q \in \mathbb{Z}} \|\ \ \|_{(\det \mathsf{C}_q)^{(-1)^q}}.$$

We denote by $\|\ \ \|_{\det H(\mathsf{C})}$ the norm on $\det H(\mathsf{C})$ induced by $\|\ \ \|_{\det \mathsf{C}}$ via the Euler isomorphism, equation (1), namely

$$\|\ \ \|_{\det H(\mathsf{C})} = \|\mathcal{E}^{-1}(\ \ )\|_{\det \mathsf{C}}. \tag{3}$$

Equivalent definition is by requiring either that the image $\mathcal{E}(\det \mathsf{c})$ of a a unitary element $\det \mathsf{c} \in \det \mathsf{C}$ is unitary, or that $\mathcal{E}$ is an isometry [8].

If $(M, \partial M)$ is a compact connected oriented manifold with possible boundary, then there is a class of distinguished CW decompositions (given by the smooth triangulations) of $(M, \partial M)$. Let (K, L) one of these CW decompositions, and let c the preferred graded basis of the chain complex $\mathsf{C}((K, L); V_\rho)$ given by the cells of (K, L), as described in Section 3.3. If an inner product is defined on V, denoting the induced norm on $\det V$ by $\|\ \ \|_{\det V}$, a norm is defined on $\det \mathsf{C}((K, L); V_\rho)$ by setting

$$\|\mathsf{c} \otimes v\|_{\det \mathsf{C}((K,L);V_\rho)} = \|v\|_{\det V},$$

in other words by setting that the cells have unitary norm. Then, the Euler isomorphism, applied either on the absolute or on the relative complex, determines either a norm on $\det H(M; V_\rho) = \det H(\mathsf{C}(K; V_\rho))$ or a norm on $\det H((M, \partial M); V_\rho) = \det H(\mathsf{C}((K, L); V_\rho))$ by (3). Following standard notation, we denote these norms by

$$\|\ \ \|^R_{\det H(M;V_\rho)}, \qquad \|\ \ \|^R_{\det H((M,\partial M);V_\rho)},$$

and we call them Reidemeister metric on $\det H(M;V_\rho)$ and on $\det H((M,\partial M);V_\rho)$, respectively [1], Definition 1.4. It is clear that the preferred element $\mathcal{E}(\det \mathsf{c})$ of $\det H(M;V_\rho)$ ($\det H((M,\partial M);V_\rho)$) determined by the Euler isomorphism is unitary in this norm. It is also clear that these norms are independent on the triangulation.

3.5. **R torsion of the frustum.** From now on, we assume that suitable compatible cell decompositions has been fixed for W and $\mathsf{F}W$, and we use the shorter notation $\mathsf{C}(W)$ for the associated chain complexes, $H(W)$ for the homology and $\mathcal{H}(W)$ for the harmonic forms, and similarly for the frustum. It is clear that $\mathsf{C}(\mathsf{F}W) = \mathsf{C}(W\times I) = Cyl(\mathsf{C}(W))$, however in order to compute the R torsion we need to give to each complex the graded homology basis induced by the geometry. For we have at least two approaches: first apply the definition, and second use the exact sequence of the pair $(\mathsf{F}W, W_2)$. We start with the first approach, and we will sketch the second one at the end of the section.

In order to deal with the R torsion of $\mathsf{F}W$, we need some results on harmonic forms. By Hodge theory, it is clear that the space of harmonic q forms on the frustum $\mathcal{H}^q(\mathsf{F}W)$ with absolute BC is isomorphic to the space of harmonic q forms on W, $\mathcal{H}^q(W)$ (see Lemma 2). We need an explicit map. For we study the harmonic forms using the approach of [4] (see also [19]). It is clear that the formal solutions of the eigenvalues equation on $\mathsf{F}W$ and on the cone over W are the same, hence the next lemma follows (see [12] Sections 3.3 and 8.1 for more details and for the notation).

Lemma 1. *Let $\{\varphi^{(q)}_{\rm har},\varphi^{(q)}_{{\rm cl},n},\varphi^{(q)}_{{\rm ccl},n}\}$ be an orthonormal base of $\Gamma(W,\Lambda^{(q)}T^*W)$ consisting of harmonic, closed and coclosed eigenforms of $\tilde\Delta^{(q)}$ on W. Let $\lambda_{q,n}$ denotes the eigenvalue of $\varphi^{(q)}_{{\rm ccl},n}$ and $m_{{\rm ccl},q,n}$ its multiplicity. Define $\alpha_q=\frac{1}{2}(1+2q-m)$, $\mu_{q,n}=\sqrt{\lambda_{q,n}+\alpha_q^2}$, and $a_{\pm,q,n}=\alpha_q\pm\mu_{q,n}$. Then, all the solutions of the harmonic equation $\Delta u=0$ on $\mathsf{F}W$ are convergent sums of forms of the following four types (here a tilda denotes intrinsic operations on the section W as in [6], beginning of Section 3):*

$$
\begin{aligned}
\psi^{(q)}_{\pm,1,n} =& x^{a_{\pm,q,n}}\varphi^{(q)}_{{\rm ccl},n},\\
\psi^{(q)}_{\pm,2,n} =& x^{a_{\pm,q-1,n}}\tilde d\varphi^{(q-1)}_{{\rm ccl},n}+a_{\pm,q-1,n}x^{a_{\pm,q-1,n}-1}dx\wedge\varphi^{(q-1)}_{{\rm ccl},n},\\
\psi^{(q)}_{\pm,3,n} =& x^{a_{\pm,q-1,n}+2}\tilde d\varphi^{(q-1)}_{{\rm ccl},n}+a_{\mp,q-1,n}x^{a_{\pm,q-1,n}+1}dx\wedge\varphi^{(q-1)}_{{\rm ccl},n},\\
\psi^{(q)}_{\pm,4,n} =& x^{a_{\pm,q-2,n}+1}dx\wedge\tilde d\varphi^{(q-2)}_{{\rm ccl},n}.
\end{aligned}
$$

Next, introducing absolute BC (as defined in [21, Section 4] , or see [11, Section 2]): $\mathcal{B}_{\rm abs}(\omega)=0$ if and only if $\omega_{norm}|_{\partial W}=0$ and $(d\omega)_{norm}|_{\partial W}=0$, we have the following result, whose proof is by direct verification: namely take the four types of forms as given in Lemma 1 and apply absolute BC to each. The unique forms that satisfy the absolute BC are the $\psi^{(q)}_{-,1,0}$, where $\lambda_{q,0}=0$ by definition. Sufficiency is easily verified: for $(\psi^{(q)}_{-,1,0})_{norm}=0$, and $(d\psi^{(q)}_{-,1,0})_{norm}=a_{-,q,n}x^{a_{-,q,n}-1}\psi^{(q)}_{-,1,0}$, and this vanishes at $x=l_1$ and $x=l_2$ if and only if $a_{-,q,n}=q-\sqrt{\lambda_{q,n}+q^2}=0$. The result for relative BC is similar.

Lemma 2. *The space of harmonic forms with absolute BC, $\mathcal{H}^q_{\rm abs}(\mathsf{F}W)$, coincides with the constant normal extension of the forms in $\mathcal{H}^q(W)$. The map $\omega\mapsto$*

$(-1)^{m-q}x^{2q-m}dx\wedge\omega$ *defines an isomorphism of* $\mathcal{H}^q(W)$ *onto* $\mathcal{H}^{q+1}_{\rm rel}(\mathsf{F}W)$*, the space of harmonic forms with relative BC.*

Next, let denote by $\star_g$ the Hodge operator in the metric g. Let $g_2 = l_2^2 g$. It is clear that $\star_{g_2} = l_2^{m-2q}\star_g$. A q-form ω on $\mathsf{F}W$ decomposes as $\omega = \omega_1 + dx\wedge\omega_2$. Writing $\omega(x,y) = f_1(x)\omega_1(y) + f_2(x)dx\wedge\omega_2(y)$, a simple calculation gives

$$\star_{g_F}\omega = x^{m-2q+2}f_2(x)\star_g\omega_2(y) + (-1)^q x^{m-2q}f_1(x)dx\wedge\star_g\omega_1(y).$$

As vector spaces $\mathcal{H}^q(W) = \mathcal{H}^q(W_2)$. If ω is q form in $\mathcal{H}^q(W)$, denote the the constant extension of ω in $\mathcal{H}^q_{\rm abs}(\mathsf{F}W)$ by $\ddot\omega$. Then:

$$||\omega||^2_{l_2^2 g} = \int_W \omega\wedge\star_{l_2^2 g}\omega = l_2^{m-2q}\int_W\omega\wedge\star_g\omega = l_2^{m-2q}||\omega||^2_g,$$

and

$$||\ddot\omega||^2_{g_{\mathsf{F}}} = \int_{\mathsf{F}W}\ddot\omega\wedge\star_{g_{\mathsf{F}}}\ddot\omega = \int_{l_1}^{l_2}x^{m-2q}dx\int_W\omega\wedge\star_g\omega = \Gamma_q||\omega||^2_g,$$

where

$$\Gamma_q = \begin{cases} \frac{1}{m+1-2q}\left(l_2^{m+1-2q} - l_1^{m+1-2q}\right), & \text{if} \quad |m+1-2q|\neq 0,\\ \ln\frac{l_2}{l_1}, & \text{if} \quad m+1-2q = 0.\end{cases}$$

By the definition of the de Rham map $\mathcal{A}^{\rm abs}_{g_{\mathsf{F}},q} = (-1)^q\mathcal{P}^{-1}\mathcal{A}^{m+1-q}_{g_{\mathsf{F}},{\rm rel}}\star_{g_{\mathsf{F}}}$ on $\mathsf{F}W$ (see [21] Section 3), we have the following commutative diagram of isometries of vectors spaces, where the $\hat{}$ denotes the dual block complex (we use here only the first line of squares of the diagram, the second line will be used at the end of this section, in the second proof)

$$\begin{array}{ccccccc}
\mathcal{H}^q_{\rm abs}(\mathsf{F}W) & \xrightarrow{\star_{g_{\mathsf{F}}}} & \mathcal{H}^{m-q+1}_{\rm rel}(\mathsf{F}W) & \xrightarrow{\mathcal{A}^{m-q+1}_{g_{\mathsf{F}},{\rm rel}}} & H^{m-q+1}(\widehat{\mathsf{F}W},\widehat{\partial\mathsf{F}W}) & \xleftarrow{\mathcal{P}} & H_q(\mathsf{F}W)\\
\uparrow{\scriptstyle\frac{(\ddot{_})}{\sqrt{\Gamma_q}}} & & \uparrow{\scriptstyle\frac{(-1)^q x^{m-2q}dx\wedge_}{\sqrt{\Gamma_q}}} & & \uparrow & & \uparrow{\scriptstyle(-1)^q\sqrt{\Gamma_q}}\\
\mathcal{H}^q(W) & \xrightarrow{\star_g} & \mathcal{H}^{m-q}(W) & \xrightarrow{\mathcal{A}^{m-q}_g} & H^{m-q}(\widehat{W}) & \xleftarrow{\mathcal{P}} & H_q(W)\\
\downarrow{\scriptstyle\frac{1}{\sqrt{l_2^{m-2q}}}} & & \downarrow{\scriptstyle\sqrt{l_2^{m-2q}}} & & \downarrow & & \downarrow{\scriptstyle\sqrt{l_2^{m-2q}}}\\
\mathcal{H}^q(W_2) & \xrightarrow{\star_{g_2}} & \mathcal{H}^{m-q}(W_2) & \xrightarrow{\mathcal{A}^{m-q}_{g_2}} & H^{m-q}(\widehat{W}_2) & \xleftarrow{\mathcal{P}} & H_q(W_2)
\end{array}$$

Commutativity of the first square follows by the given formula for the Hodge operator and Lemma 2. Commutativity of the other squares follows by construction. For suppose a cell decomposition of W is fixed. Then, a cell decomposition of $\mathsf{F}W$ is determined with q-cells either the q-cells of W or the product $I\times c$, where c is a $(q-1)$-cell of W. Namely, we are using the direct sum decomposition of the cellular chain complex $\mathsf{C}_q(\mathsf{F}W) = \mathsf{C}_q(W)\oplus\mathsf{C}_{q-1}(W)\oplus\mathsf{C}_q(W)$. This fix a preferred basis for the chain vector spaces. It is clear that the dual block in $\mathsf{F}W$ of $\{0\}\times c$ is $[0,1/2]\times\hat c$, where $\hat c$ is the dual block of c in W. Next, recall the de Rham maps $\mathcal{A}^q_g$ and $\mathcal{A}^q_{g_{\mathsf{F}},{\rm rel}}$ on W and on $\mathsf{F}W$ are defined respectively by

$$\mathcal{A}^q_g(\omega)(\hat c) = \int_{\hat c}\omega, \qquad \mathcal{A}^q_{g_{\mathsf{F}},{\rm rel}}(\omega)(\hat d) = \int_{\hat d}\omega,$$

where $\hat c$ denotes the dual block in W of a cell c of W, and $\hat d$ the dual block in $\mathsf{F}W$ of a cell d of $\mathsf{F}W - \partial\mathsf{F}W$ It follows that $\mathcal{A}^{m-q+1}_{g_{\mathsf{F}},{\rm rel}}(\omega)(\hat d)$ is non vanishing only on the dual block $\hat d$ of $(q+1)$-cells d of type $\{0\}\times c$. Hence, ω must have a non trivial normal component, i.e. $\omega = f_2(x)dx\wedge\omega_2(y)$, and the unique contributions are

$$\mathcal{A}^q_{g_{\mathsf F},\mathrm{rel}}(\omega)(\hat d)=\int_{\widehat{(c,0)}} f_2(x)dx\wedge\omega_2(y)=\int_{l_1}^{l_2} f_2(x)dx\int_{\hat c}\omega_2(y).$$

This gives the isomorphisms in the vertical lines of the last square, and their coefficients.

Now realize the imbedding of W in $\mathsf F W$ as W_2. Let α_q be an orthonormal basis for $\mathcal H^q(W)$. Then, an orthonormal base for $\mathcal H^q(W_2)$ is $l_2^{-\frac{m-2q}{2}}\alpha_q$, and applying the de Rham maps we obtain

$$\begin{aligned}\mathcal A_{g_2,q}(l_2^{-\frac{m-2q}{2}}\alpha_{q,j})&=l_2^{-\frac{m-2q}{2}}\mathcal A_{g_2,q}(\alpha_{q,j})=l_2^{-\frac{m-2q}{2}}\mathcal P_q^{-1}\mathcal A^{m-q}_{g_2}\star_{g_2}(\alpha_{q,j})\\&=l_2^{-\frac{m-2q}{2}}l_2^{m-2q}\mathcal P_q^{-1}\mathcal A^{m-q}_{g}\star_{g}(\alpha_{q,j})=l_2^{\frac{m-2q}{2}}\mathcal A_{g,q}(\alpha_{q,j}),\end{aligned}$$

and hence a basis h_q for $H_q(W_2)$ is $l_2^{\frac{m-2q}{2}}\mathcal A_{g,q}(\alpha_q)$, and $z_q=l_2^{\frac{m-2q}{2}}\mathcal A_{g,q}(\alpha_q)$. Next, consider the conical frustum $(\mathsf F W, g_{\mathsf F})$. An orthonormal basis for $\mathcal H^q_{\mathrm{abs}}(\mathsf F W)$ is $\Gamma_q^{-\frac12}\ddot\alpha_q$, and

$$\begin{aligned}\mathcal A_{g_{\mathsf F},\mathrm{abs},q}\left(\Gamma_q^{-\frac12}\ddot\alpha_{q,j}\right)&=\Gamma_q^{-\frac12}\mathcal A_{g_{\mathsf F},\mathrm{abs},q}(\ddot\alpha_{q,j})=\Gamma_q^{-\frac12}\mathcal P_q^{-1}\mathcal A^{m+1-q}_{g_{\mathsf F},\mathrm{rel}}\star_{g_{\mathsf F}}(\ddot\alpha_{q,j})\\&=\Gamma_q^{-\frac12}\Gamma_q\mathcal P_q^{-1}\mathcal A^{m-q}_{g}\star_g(\alpha_{q,j})=\Gamma_q^{\frac12}\mathcal A_{g,q}(\alpha_{q,j}).\end{aligned}$$

Then the basis $\ddot h_q$ is $\Gamma_q^{\frac12}(h_q\oplus 0\oplus 0)$, and hence $\ddot z_q=\Gamma_q^{\frac12}(z_q\oplus 0\oplus 0)$. Thus

$$\begin{aligned}|\det(\ddot\partial_{q+1}(\ddot b_{q+1}),\ddot z_q,\ddot b_q/\ddot c_q)|=&\Gamma_q^{\frac{r_q}{2}\mathrm{rk}(\rho)}|\det\rho(\partial_{q+1}(b_{q+1}),\mathcal A_g(\alpha_q),b_q/c_q)|^2\\&\times|\det\rho(\partial_q(b_q),\mathcal A_g(\alpha_{q-1}),b_{q-1}/c_{q-1})|,\end{aligned}$$

and

$$\begin{aligned}\tau_{\mathrm R}((\mathsf F W,g_{\mathsf F});\rho)&=\prod_{q=0}^m\Gamma_q^{(-1)^q\frac{r_q}{2}\mathrm{rk}(\rho)}|\det\rho(\partial_{q+1}(b_{q+1}),\mathcal A_g(\alpha_q),b_q/c_q)|^{(-1)^q}\\&=\prod_{q=0}^m\Gamma_q^{(-1)^q\frac{r_q}{2}\mathrm{rk}(\rho)}\tau_{\mathrm R}((W,g);\rho).\end{aligned}$$

We have proved the following proposition.

Proposition 1. *The R torsion of the conical frustum with respect to the homology basis induced by the orthonormal harmonic basis is:*

$$\log\tau_{\mathrm R}((\mathsf F W,g_{\mathsf F});\rho)=\log\tau_{\mathrm R}((W,g);\rho)+\mathrm{rk}(\rho)\log\tau(\mathcal T),$$

where

$$\begin{aligned}\log\tau(\mathcal T)=&\frac12\sum_{q=0}^{2p}(-1)^q r_q\log\frac{l_2^{2p+1-2q}-l_1^{2p+1-2q}}{(2p+1-2q)}, && m=2p,p\geq 0,\\ \log\tau(\mathcal T)=&\frac12\sum_{q=0,q\neq p}^{2p-1}(-1)^q r_q\log\frac{l_2^{2p-2q}-l_1^{2p-2q}}{(2p-2q)} && m=2p-1,p\geq 1,\\&+\frac{(-1)^p}{2}r_p\log\log\frac{l_2}{l_1},\end{aligned}$$

where $r_q=\mathrm{rk}H_q(W)$.

We conclude this section with an other proof of Proposition 1. Consider the short exact sequence of chain complexes associated to the pair $(\mathsf{F}W, W_2)$,

$$0 \to \mathsf{C}(W_2) \to \mathsf{C}(\mathsf{F}W) \to \mathsf{C}(\mathsf{F}W, W_2) \to 0,$$

by Milnor [15, Section 3], we have

$$\log \tau_{\rm R}((\mathsf{F}W, g_{\mathsf{F}}); \rho) = \log \tau_{\rm R}((W, l_2^2 g); \rho) + \log \tau_{\rm R}((\mathsf{F}W, W_2), g_{\mathsf{F}}); \rho) + {\rm rk}(\rho) \log \tau(\mathcal{T}_2),$$

where the complex $\mathcal{T}_2$ is defined by the long exact homology sequence of the pair, namely

$$\mathcal{T}_2 : \qquad \ldots \longrightarrow H_q(W_2) \longrightarrow H_q(\mathsf{F}W) \longrightarrow H_q(\mathsf{F}W, W_2) \longrightarrow \ldots, \tag{4}$$

with $\mathcal{T}_{2,3q+2} = H_q(W_2)$, $\mathcal{T}_{2,3q+1} = H_q(\mathsf{F}W)$ and $\mathcal{T}_{2,3q} = H_q(\mathsf{F}W, W_2)$. It is clear that both the relative homology and the relative torsion are trivial (for the second is the torsion of a simple homotopy trivial pair). Therefore the torsion of $\mathcal{T}_2$ is given by the graded product of the torsions of the isomorphisms: $i_{2*,q} : H_q(W) \to H_q(\mathsf{F}W)$. Using the graded homology basis given above, we can now compute the determinants of the change of basis in the vector spaces of the sequence in equation (4). At $H_q(W, W_2)$ the determinant is 1, at $H_q(W_2)$ is 1 and at $H_q(\mathsf{F}W)$ is $\left(\frac{l_2^{m-2q}}{\Gamma_q}\right)^{\frac{r_q}{2}}$, where $r_q = {\rm rk} H_q(W)$. Applying the definition of Reidemeister torsion to the complex $\mathcal{T}_2$, we obtain (where D denotes the class of the determinant of the matrix of the change of basis)

$$\begin{aligned}
\log \tau(\mathcal{T}_2) &= \sum_{q=0}^{3m} (-1)^q \log D(\mathcal{T}_{2,q}) \\
&= \sum_{q=0}^{m} (-1)^{3q} \log D(H_q(\mathsf{F}W, W_2)) + \sum_{q=0}^{m} (-1)^{3q+1} \log D(H_q(\mathsf{F}W)) \\
&\quad + \sum_{q=0}^{m} (-1)^{3q+2} \log D(H_q(W_2)) \\
&= \sum_{q=0}^{m} (-1)^{3q+1} \log D(H_q(\mathsf{F}W)) \\
&= \sum_{q=0}^{m} (-1)^{q+1} \frac{r_q}{2} \log \frac{l_2^{m-2q}}{\Gamma_q}.
\end{aligned}$$

This agrees with Proposition 1 since a simple calculation (using for example the variational formula for the torsion) shows that

$$\log \tau_{\rm R}((W, l^2 g); \rho) = \log \tau_{\rm R}((W, g); \rho) + \frac{1}{2} {\rm rk}(\rho) \sum_{q=0}^{m} (-1)^q r_q (m - 2q) \log l. \tag{5}$$

3.6. **R metric on the frustum.** We now read the result of Proposition 1 in term of Reidemeister metrics. The direct sum decomposition of the chain complex of the frustum determines an homomorphism

$$i_{\#} : \mathsf{C}(W) \to \mathsf{C}(\mathsf{F}W),$$

that induces an isomorphism on homology: $i_* : H(W) \to H(\mathsf{F}W)$. In particular, explicit calculation shows that given cycles representing the homology classes of

$H(W)$, we can chose the same cycles to represent the homology classes of $H(\mathsf{F}W)$. In other words the map i_* is an identification. Then, we have that the following square commutes if φ is the map that sends the preferred cell basis $\det \mathsf{c}$ of $\det \mathsf{C}(W)$ onto the preferred cell basis $\det \check{\mathsf{c}}$ of $\det \mathsf{C}(\mathsf{F}W)$, (note that φ is not induced by i)

$$\begin{array}{ccc} \det \mathsf{C}(W) & \xrightarrow{\varphi} & \det \mathsf{C}(\mathsf{F}W) \\ \downarrow{\scriptstyle \mathcal{E}_W} & & \downarrow{\scriptstyle \mathcal{E}_{\mathsf{F}W}} \\ \det H(W) & \xrightarrow[\det(i_*)]{} & \det H(\mathsf{F}W) \end{array}$$

This gives the following corollary of Proposition 1.

Corollary 1. *The isomorphism* $\det(i_*): \det H(W) \to \det H(\mathsf{F}W)$ *is an isometry with respect to the R metrics. In other words, identifying* $\det H(W)$ *and* $\det H(\mathsf{F}W)$ *by* $\det(i_*)$*, we have*

$$\|\det(i_*)(\quad)\|^{\rm R}_{\det H(\mathsf{F}W)} = \|\quad\|^{\rm R}_{\det H(W)}.$$

4. Analytic torsion and Ray and Singer metric

In this section we express the analytic torsion and the Ray and Singer metric of the frustum $\mathsf{F}W$ in term of those of the section W.

4.1. Analytic torsion.

The analytic torsion is defined starting with the de Rham complex associated to the compact connected oriented Riemannian n-manifold (M, g) with twisted coefficients in V_ρ [21]. The zeta function of Laplace operator $\Delta^{(q)}$ on q forms in $\Omega^q(M, V_\rho)$ is defined by the meromorphic extension (analytic at $s = 0$) of the series

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in {\rm Sp}_+ \Delta^{(q)}} \lambda^{-s},$$

convergent for ${\rm Re}(s) > \frac{n}{2}$, and where ${\rm Sp}_+$ denotes the positive part of the spectrum. If M has no boundary, the analytic torsion of (M, g) is

$$\log T((M,g);\rho) = \frac{1}{2} \sum_{q=1}^{n} (-1)^q q \zeta'(0, \Delta^{(q)}). \tag{6}$$

If M has a boundary, we denote by $T_{\rm abs}((M,g);\rho)$ the number defined by equation (6) with Δ satisfying absolute BC, and by $T_{\rm rel}((M,g);\rho)$ the number defined by the same equation with Δ satisfying relative BC.

The analytic torsion can be computed either by the definition or by using the suitable extension of the Cheeger Müller theorem, that gives the analytic torsion in terms of the R torsion computed in the basis induced by the orthonormal basis of harmonic forms. In this section we will follow the second approach, while in the last section we will show how to compute the torsion applying the definition in a particular case.

4.2. Ray and Singer metric.

Analytic torsion permits to define a new metric on the space $\det H(M)$. Let M be an n dimensional orientable compact connected Riemannian manifold with metric g and possible boundary ∂M. Let V_ρ a real flat vector bundle, and consider twisted coefficients as in Section 3.3. Then, the L^2 norm on the space of harmonic forms $\mathcal{H}_{\rm abs}(M)$ induces a norm on the line $\det\mathcal{H}_{\rm abs}(M)$, and we define the Ray and Singer metric on $\det H(M)$ by [1, (2.7)]

$$\|\det\mathcal{A}_{\rm abs}(\quad)\|^{RS}_{\det H(M)} = T_{\rm abs}((M,g);\rho)\|\quad\|_{\det\mathcal{H}_{\rm abs}(M)},$$

where $\det\mathcal{A}_{\rm abs}$ is induced by the de Rham map (see equation (2) in Section 3.3)

$$\det\mathcal{A}_{\rm abs} = \bigotimes_{q=0}^{m}(\det\mathcal{A}_{\rm abs,q})^{(-1)^q},$$

and $(\det\mathcal{A}_{\rm abs,q})^{-1}$ denotes the inverse of the dual of $\mathcal{A}_{\rm abs,q}$.

4.3. **The analytic torsion of the frustum.** Using recent works of Brüning and Ma [2] [3], and classic work of Lück [18], the Cheeger Müller theorem for an oriented compact connected Riemannian n-manifold (M,g) with boundary reads [3] Theorem 3.4 (see [11, Section 6] or [12, Section 2.3] for details on our notation)

$$\log T_{\rm abs}((M,g);\rho) = \log\tau_{\rm R}((M,g);\rho) + \frac{{\rm rk}(\rho)}{4}\chi(\partial M)\log 2 + {\rm rk}(\rho)A_{\rm BM,abs}(\partial M),$$

$$\log T_{\rm rel}((M,g);\rho) = \log\tau_{\rm R}((M,\partial M,g);\rho) + \frac{{\rm rk}(\rho)}{4}\chi(\partial M)\log 2 + {\rm rk}(\rho)A_{\rm BM,rel}(\partial M),$$

where ρ is an orthogonal representation of the fundamental group, and where the boundary anomaly term of Brüning and Ma is defined as follows. Using the notation of [2] (see [12, Section 2.2] for more details) for $\mathbb{Z}/2$ graded algebras, we identify an antisymmetric endomorphism ϕ of a finite dimensional vector space V (over a field of characteristic zero) with the element $\hat\phi = \frac{1}{2}\sum_{j,k=1}^n\langle\phi(v_j),v_k\rangle\hat v_j\wedge\hat v_k$, of $\widehat{\Lambda^2 V}$. For the elements $\langle\phi(v_j),v_k\rangle$ are the entries of the tensor representing ϕ in the base $\{v_k\}$, and this is an antisymmetric matrix. Now assume that r is an antisymmetric endomorphism of V with values in $\Lambda^2 V$. Then, $(R_{jk} = \langle r(v_j),v_k\rangle)$ is a tensor of two forms in $\Lambda^2 V$. We extend the above construction identifying R with the element

$$\hat R = \frac{1}{2}\sum_{j,k=1}^n\langle r(v_j),v_k\rangle\wedge\hat v_j\wedge\hat v_k,$$

of $\Lambda^2 V\wedge\widehat{\Lambda^2 V}$. This can be generalized to higher dimensions. In particular, all the construction can be done taking the dual V^* instead of V. Accordingly to [2], we define the following forms (where $i:\partial M\to M$ denotes the inclusion)

$$\mathcal{S} = \frac{1}{2}\sum_{k=1}^{n-1}(i^*\omega - i^*\omega_0)_{0k}\wedge\hat e_k^*$$

$$\widehat{i^*\Omega} = \frac{1}{2}\sum_{k,h=1}^{n-1}i^*\Omega_{k,h}\wedge\hat e_k^*\wedge\hat e_h^*, \qquad \hat\Theta = \frac{1}{2}\sum_{k,h=1}^{n-1}\Theta_{k,h}\wedge\hat e_k^*\wedge\hat e_h^*.$$

Here, ω and ω_0 are the connection one forms associated to the metrics g_0 and $g_1 = g$, respectively, where g_0 is a suitable deformation of g that is a product near the boundary. Ω is the curvature two form of g, Θ is the curvature two form of the boundary (with the metric induced by the inclusion), and $\{e_k\}_{k=0}^{n-1}$ is an orthonormal base of TM (with respect to the metric g). Then, setting

$$B = \frac{1}{2}\int_0^1\int^B {\rm e}^{-\frac{1}{2}\hat\Theta - u^2\mathcal{S}^2}\sum_{k=1}^\infty\frac{1}{\Gamma\left(\frac{k}{2}+1\right)}u^{k-1}\mathcal{S}^k du,$$

the anomaly boundary term is

$$A_{\rm BM,abs}(\partial M) = (-1)^{n+1} A_{\rm BM,rel}(\partial M) = \frac{1}{2}\int_{\partial M} B.$$

It is not too difficult to see that in the case of the frustum $\mathsf{F}W$, the boundary term $A_{\rm BM}(W_j)$ is independent on l_j, either with absolute or relative BC. For let $\{b_k\}_{k=1}^m$ and $\{e_k\}_{k=0}^m$ denote local orthonormal bases of TW and $T\mathsf{F}W$ respectively. Then, direct calculations (see [12, Section 3.2], see also Section 6 for an example) give $\Theta_{j,k} = \tilde\Omega_{j,k}$ (recall that a tilda denotes intrinsic operators on the section W, see Lemma 2), and

$$\mathcal{S}|_{W_j} = \frac{(-1)^{j+1}}{2l_j}\sum_{k=1}^m e_k^*\wedge \hat e_k^* = \frac{(-1)^{j+1}l_j}{2}\sum_{k=1}^m b_k^*\wedge \hat b_k^* = \frac{(-1)^{j+1}}{2}\sum_{k=1}^m b_k^*\wedge \hat e_k^*.$$

We have the following result.

Lemma 3. *The anomaly boundary term on the frustum is: if m is odd*

$$A_{\rm BM,abs}(\partial\mathsf{F}W) = -A_{\rm BM,rel}(\partial\mathsf{F}W) = 0, \qquad A_{{\rm BM,abs}W_1,{\rm rel}W_2}(\partial\mathsf{F}W) = \int_W B;$$

if m is even

$$A_{\rm BM,abs}(\partial\mathsf{F}W) = A_{\rm BM,rel}(\partial\mathsf{F}W) = A_{{\rm BM,abs}W_1,{\rm rel}W_2}(\partial\mathsf{F}W) = \int_W B.$$

Proposition 2. *The analytic torsion of the conical frustum is*

$$\begin{aligned}\log T_{\rm abs}((\mathsf{F}W,g_\mathsf{F});\rho) =&\log\tau_{\rm R}((\mathsf{F}W,g_\mathsf{F});\rho)+\frac{1}{2}{\rm rk}(\rho)\chi(W)\log 2\\ &+\frac{1-(-1)^{m+1}}{2}{\rm rk}(\rho)\int_W B,\\ \log T_{{\rm abs}W_1,{\rm rel}W_2}((\mathsf{F}W,g_\mathsf{F});\rho) =&\frac{1}{2}{\rm rk}(\rho)\chi(W)\log 2+{\rm rk}(\rho)\int_W B.\end{aligned}$$

4.4. **The Ray and Singer metric on the frustum.** In terms of metrics, Theorem 3.4 of [3] combined with Corollary 5.1 of [18] reads

$$\|\det\mathcal{A}_{\rm abs}(\quad)\|^{RS}_{\det H(M)} = 2^{\frac{1}{4}{\rm rk}(\rho)\chi(\partial M)}{\rm e}^{{\rm rk}(\rho)A_{\rm BM,abs}(M)}\|\quad\|^{R}_{\det H(M)}.$$

Applying this equation to the torsion of the frustum, we have the following result, that is a second corollary of Proposition 1.

Corollary 2. *The Ray Singer metrics on the frustum and on its section are related by the following equation:*

$$\|\det(i_*)(\quad)\|^{\rm RS}_{\det H(\mathsf{F}W)} = 2^{\frac{1}{2}{\rm rk}(\rho)\chi(W)}{\rm e}^{{\rm rk}(\rho)A_{\rm BM,abs}(\partial\mathsf{F}W)}\|\quad\|^{\rm RS}_{\det H(W)}.$$

This result can also be obtained applying the definition. For

$$\|\det(\mathcal{A}_{g_\mathsf{F},{\rm abs}})(\quad)\|^{\rm RS}_{\det H(\mathsf{F}W)} = T_{\rm abs}(\mathsf{F}W,g_\mathsf{F})\|\quad\|_{\det\mathcal{H}_{\rm abs}(\mathsf{F}W)},$$

and this gives the formula in the corollary, since from one side:

$$\begin{aligned}T_{\rm abs}(\mathsf{F}W,g_\mathsf{F}) &= \tau_R(\mathsf{F}W,g_\mathsf{F})2^{\frac{1}{2}{\rm rk}(\rho)\chi(W)}{\rm e}^{{\rm rk}(\rho)A_{\rm BM,abs}(\partial\mathsf{F}W)}\\ &=\tau(\mathcal{T})^{{\rm rk}(\rho)}\tau_R(W,g)2^{\frac{1}{2}{\rm rk}(\rho)\chi(W)}{\rm e}^{{\rm rk}(\rho)A_{\rm BM,abs}(\partial\mathsf{F}W)},\end{aligned}$$

and from the other, by simple calculation like the ones in Section 3.5, we have that

$$\|\det(i^*)^{-1}(\quad)\|_{\det\mathcal{H}_{\rm abs}(\mathsf{F}W)} = \tau(\mathcal{T})^{{\rm rk}(\rho)}\|\quad\|_{\det\mathcal{H}(W)}.$$

5. Limit case

In this section we study the limit case $l_1 \to 0^+$, and the relation with the torsion of the cone $C_{l_2}W = \mathsf{F}W/W_1$. For we first recall the formula for the analytic torsion of the cone (formulas for relative BC follow by duality, as proved in Theorems 1.1 and 1.2 of [12]). In this section we analyze the case of odd dimensional section, so we assume $m = 2p-1$, $p \geq 1$; we also assume that ρ is the trivial representation of rank one (ρ_0), and hence we omit it from the notation.

Theorem 1 ([12])**.** *The analytic torsion of the cone C_lW on an orientable compact connected Riemannian manifold (W,g) of odd dimension $2p-1$ is*

$$\begin{aligned}\log T_{\rm abs}(C_lW, g_C) =& \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}{\rm rk}H_q(W;\mathbb{Q})\log\frac{2(p-q)}{l}\\ &+\frac{1}{2}\log T((W,l^2g);\rho_0) + A_{\rm BM,abs}(\partial C_lW).\end{aligned}$$

Using equation (5) and duality

$$\sum_{q=p}^{2p-1}(-1)^q r_q(2p-1-2q) = \sum_{q=0}^{p-1}(-1)^q r_q(2p-1-2q) = \frac{1}{2}\sum_{q=0}^{m}(-1)^q r_q(m-2q),$$

the formula for the analytic torsion of the cone $C_{l_2}W$ reads:

$$\begin{aligned}\log T_{\rm abs}(C_{l_2}W, g_C) =& \frac{1}{2}\log \tau_{\rm R}((W,g);\rho_0) + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q \log\frac{l_2^{2(p-q)}}{2(p-q)}\\ &+ A_{\rm BM,abs}(\partial C_{l_2}W).\end{aligned}$$

Consider the formula for the R torsion of the frustum given in Proposition 1. It is clear that in the limit $l_1 \to 0^+$ the last p terms diverge. This suggests the following approach.

Let (W,g) be an oriented compact connected Riemannian manifold of dimension m. Such a space has a class of distinguished CW decompositions (given by the smooth triangulations). Let K one of these CW decompositions, and let c the preferred graded basis of the chain complex $\mathsf{C}(W;V_\rho)$ given by the cells, as described in Section 3.3. Fix the sets b and z as in Section 3.1, and let $\mathcal{A}(\alpha)$ denote the graded basis for homology induced by the metric structure as in Section 3.3, and use the notation

$$D_q = |\det(\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)| \in \mathbb{R}^+.$$

Let $\hat{K}$ denote the dual block complex, and $K_{(q)}$ the q-skeleton. It is clear that the $p-1$ homology of $K_{(p-1)}$ coincides with the cycles of K, and the bijection is cellular. Consider the torsion

$$\tau(K_{(p-1)};\mathsf{x}) = \tau(\mathsf{C}(K_{(p-1)};V_\rho),\mathsf{x}) = \prod_{q=0}^{p-1}|\det(\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)| \in \mathbb{R}^+,$$

where the z_q are cycles projecting onto some fixed basis x_q for $H_q(K_{(p-1)})$, as in Section 3.3. We can fix x using the geometry. Since $\hat{K}$ is another decomposition of W, there is a common subdivision T of K and $\hat{K}$. The identity maps $id : W = |K| \to W = |T|$ and $id : W = |\hat{K}| \to W = |T|$ are cellular, and hence restrict to maps $id_{(q)} : K_{(q)} \to T_{(q)}$ and $id_{(q)} : (\hat{K})_{(q)} \to T_{(q)}$, i.e. $T_{(q)}$ is a common subdivision of $K_{(q)}$ and $(\hat{K})_{(q)}$. It follows that $K_{(q)}$ and $(\hat{K})_{(q)}$ have the same torsion up to the choice of the homology volume elements, by [15, 7.1]. Consider the chain complex associated to $(\hat{K})_{(p-1)}$. Let

$$\hat{D}_q = |\det \rho(\hat{\partial}_{q+1}(\hat{\mathsf{b}}_{q+1}), \hat{\mathsf{z}}_q, \hat{\mathsf{b}}_q/\hat{\mathsf{c}}_q)| \in \mathbb{R}^+.$$

By duality $\hat{\mathsf{h}}_q = \mathsf{h}^\dagger_{2p-1-q}$, $\hat{D}_q = D^{-1}_{2p-1-q}$, and hence

$$\prod_{q=0}^{p-1} \hat{D}_q^{(-1)^q} = \prod_{q=p}^{2p-1} D_q^{(-1)^q},$$

and

$$\prod_{q=0}^{2p-1} D_q^{(-1)^q} = \prod_{q=0}^{p-1} D_q^{(-1)^q} \prod_{q=0}^{p-1} \hat{D}_q^{(-1)^q}.$$

It is clear that the basis $\mathcal{A}(\alpha_q)$ gives an homology basis for $H_q(K_{(p-1)})$ for all $q < p-1$. Moreover, $\mathsf{z}_{p-1} = \partial_p(\mathsf{b}_p), \mathcal{A}(\alpha_{p-1})$ gives a basis for $H_{p-1}(K_{(p-1)})$. This basis depends on the b_p, however, if we change the set b_p by b'_p, we have

$$D'_{p-1} = kD_{p-1},$$

for some real constant $k \neq 0$. Also, the dual basis change gives the change

$$\hat{D}'_{p-1} = k^{-1}\hat{D}_{p-1}.$$

It follows that there exists a family $\mathcal{B}$ of homology basis w_{p-1} of $H_{p-1}(K_{(p-1)})$, but a unique volume element $\det \mathsf{w}_{p-1}$, such that

$$1 = \frac{\tau(K_{(p-1)}, \mathsf{w})}{\tau((K^\dagger)_{(p-1)}; \mathsf{w}^\dagger)}.$$

This fix the basis in the class $\mathcal{B}$, i.e $\mathsf{x} = \mathsf{w}$, and with this choice

$$\tau_{\rm R}(W, g) = (\tau(K_{(p-1)}; \mathsf{w}))^2.$$

Back to the frustum, by Proposition 1, we have that

$$\begin{aligned}\tau_{\rm R}(\mathsf{F}W, g_{\mathsf{F}}) &= \prod_{q=0}^{2p-1} \Gamma_q^{(-1)^q \frac{r_q}{2}} \tau_{\rm R}(W, g)\\ &= \prod_{q=0}^{2p-1} \Gamma_q^{(-1)^q \frac{r_q}{2}} (\tau(K_{(p-1)}; \mathsf{w}))^2.\end{aligned}$$

This suggests to consider the factor

$$\begin{aligned}\log \frac{\tau_{\rm R}(\mathsf{F}W, g_{\mathsf{F}})}{\tau(K_{(p-1)}; \mathsf{w}) \prod_{q=p}^{2p-1} \Gamma_q^{(-1)^q \frac{r_q}{2}}} &= \log \frac{\tau_{\rm R}(W, g)}{\tau(K_{(p-1)}; \mathsf{w})}\\ &\quad + \frac{1}{2}\sum_{q=0}^{p-1} (-1)^q r_q \log \frac{l_2^{2p-2q} - l_1^{2p-2q}}{(2p-2q)}.\end{aligned}$$

For simplicity, we call the above fraction the *geometrically regularized* R *torsion* of $\mathsf{F}W$, and we use the notation

$$\Upsilon_{\rm R}(\mathsf{F}W, g_{\mathsf{F}}) = \frac{\tau_{\rm R}(\mathsf{F}W, g_{\mathsf{F}})}{\sqrt{\tau_{\rm R}(W, g)} \prod_{q=p}^{2p-1} \Gamma_q^{(-1)^q \frac{r_q}{2}}}.$$

It is easy to see that the limit is

$$\lim_{l_1\to 0^+} \log \Upsilon_{\rm R}(\mathsf{F}W, g_{\mathsf{F}}) = \frac{1}{2}\log \tau_{\rm R}(W,g) + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q \log \frac{l_2^{2(p-q)}}{2(p-q)},$$

and this coincides precisely with the analytic torsion of the cone up to the boundary term. Comparing with Proposition 4.1 of [12], we see that

$$\lim_{l_1\to 0^+} \frac{\tau_{\rm R}(\mathsf{F}W, g_{\mathsf{F}})}{\sqrt{\tau_{\rm R}(W,g)}\prod_{q=p}^{2p-1}\Gamma_q^{(-1)^q\frac{r_q}{2}}} = I\tau_{\rm R}(C_{l_2}W),$$

where the right end side is the intersection R torsion of the cone [7] [13] (see also the appendix). We have proved the following result:

Theorem 2. *In the limit $\mathsf{F}W \to CW$, the geometrically regularized R torsion of the conical frustum $\mathsf{F}W$ over an oriented compact connected odd dimensional manifold W gives the intersection torsion of the cone CW over W.*

Remark 1. *We resist the temptation of writing a formal result for the Reidemeister metrics, since it would require a rigorous description of the intersection Reidemeister metric on the cone. However, formally the result would have the following form*

$$\frac{\|\det(i_*)(\quad)\|^{\rm R}_{\det H(\mathsf{F}W)}}{\sqrt{\|\quad\|^{\rm R}_{\det H(W)}}} = \|\det(j_*)(\quad)\|^{\rm R}_{\det I^{\mathfrak{m}}H(CW)}.$$

By analogy, we define the *geometrically regularized analytic torsion* of $\mathsf{F}W$ by

$$Y_{\rm abs}(\mathsf{F}W, g_{\mathsf{F}}) = \frac{T_{\rm abs}(\mathsf{F}W, g_{\mathsf{F}})}{\sqrt{T(W,g)}\prod_{q=p}^{2p-1}\Gamma_q^{(-1)^q\frac{r_q}{2}}}.$$

Then,

$$Y_{\rm abs}(\mathsf{F}W, g_{\mathsf{F}}) = \frac{\tau_{\rm R}(\mathsf{F}W, g_{\mathsf{F}})}{\sqrt{\tau_{\rm R}(W,g)}\prod_{q=p}^{2p-1}\Gamma_q^{(-1)^q\frac{r_q}{2}}} {\rm e}^{A_{\rm BM,abs}(\partial \mathsf{F}W)}.$$

In order to complete the analysis of the limit case, we need to consider the boundary term. From Section 4 the boundary term of the frustum is

$$A_{\rm BM,abs}(\partial\mathsf{F}W) = \frac{1}{2}\int_{W_2} B + \frac{1}{2}\int_{W_1} B,$$

and hence there is a jump discontinuity and

$$A_{\rm BM,abs}(\partial\mathsf{F}W)|_{l_1=0} = \frac{1}{2}\int_{W_2} B = A_{\rm BM,abs}(\partial C_{l_2}W),$$

this completes the proof of the following result.

Theorem 3. *In the "limit case" $FW \to CW$, i.e. $l_1 = 0$, the geometrically regularized analytic torsion of the conical frustum FW over an oriented compact connected odd dimensional manifold W gives the analytic torsion of the cone CW over W.*

In terms of Ray and Singer metrics, this gives the following corollary.

Corollary 3. *Let $i : W \to \mathsf{F}W$ and $j : W \to CW$ denote the inclusions, then*

$$\frac{\|\det(i_*)(\quad)\|^{\rm RS}_{\det H(\mathsf{F}W)}}{\sqrt{\|\quad\|^{\rm RS}_{\det H(W)}}} = \|\det(\bar{j}_*)(\quad)\|^{\rm RS}_{\det I^{\mathfrak{m}}H(C_{l_2}W)}.$$

Proof. By Corollary 2

$$\frac{\|\det(i_*\mathcal{A}_g)(\quad)\|^{\rm RS}_{\det H(\mathsf{F}W)}}{\sqrt{\|\det(\mathcal{A}_g)(\quad)\|^{\rm RS}_{\det H(W)}}} = {\rm e}^{A_{\rm BM,abs}(\partial \mathsf{F}W)}\sqrt{\|\det(\mathcal{A}_g)(\quad)\|^{\rm RS}_{\det H(W)}}.$$

Hodge duality implies that the orthonormal graded basis $\mathsf{a} = \{\mathsf{a}_q = a_{q,1} \wedge \cdots \wedge a_{q,\dim \mathcal{H}^q(W)}\}$ of $\det \mathcal{H}(W)$ satisfies the relation

$$a_{q,j} = \star a_{2p-1-q,j},$$

and hence

$$\det \mathsf{a} = \bigotimes_{q=0}^{p-1} (\det \mathsf{a}_q)^{(-1)^q} \bigotimes_{q=0}^{p-1} (\det \star \mathsf{a}_q)^{(-1)^{q+1}}.$$

Identifying $\star\mathcal{H}^{2p-1-q}$ with $(\mathcal{H}^q)^\dagger$, we write

$$\det \mathsf{a} = \left(\bigotimes_{q=0}^{p-1} (\det \mathsf{a}_q)^{(-1)^q} \right)^2.$$

If $\bar{\mathcal{H}}(W)$ denotes the truncated complex $\bar{\mathcal{H}}(W) = \bigoplus_{q=0}^{p-1} \mathcal{H}^q(W)$. Then, $\bar{\mathsf{a}} = \bigoplus_{q=0}^{p-1} \mathsf{a}_q$ is a graded basis for $\bar{\mathcal{H}}(W)$, $\det \bar{\mathsf{a}}$ is a basis for $\det \bar{\mathcal{H}}(W)$, and we have

$$\det \mathsf{a} = (\det \bar{\mathsf{a}})^2.$$

Moreover, the map of complex

$$\bar{j}^* := j^* : \mathcal{H}(CW) \to \bar{\mathcal{H}}(W),$$

is an isomorphism. Whence the map $\det(\bar{j}^*)$ is well defined and simple calculation shows that

$$\|\det(\bar{j}^*)^{-1}(\quad)\|_{\det \mathcal{H}(C_{l_2}W)} = \chi \|\quad\|_{\det \bar{\mathcal{H}}(W)} = \chi \sqrt{\|\quad\|_{\det \mathcal{H}(W)}},$$

where

$$\chi = \prod_{q=0}^{p-1} \Gamma_q^{(-1)^q \frac{r_q}{2}} \Big|_{l_1 = 0}.$$

Thus, by definition and Theorem 1,

$$\begin{aligned}
\|\det(\bar{j}_*\mathcal{A}_g)(\quad)\|^{\rm RS}_{\det I^{\mathfrak{m}}H(C_{l_2}W)} &= \chi^{-2} \|\det(\mathcal{A}_{g_C,{\rm abs}}(\bar{j}^*)^{-1})(\quad)\|^{\rm RS}_{\det I^{\mathfrak{m}}H(C_{l_2}W)} \\
&= \chi^{-2} T(C_l W, g_C) \|\det(\bar{j}^*)^{-1}(\quad)\|_{\det \mathcal{H}(C_{l_2}W)} \\
&= \sqrt{T(W,g)} {\rm e}^{A_{\rm BM,abs}(\partial C_l W)} \sqrt{\|\quad\|_{\det \mathcal{H}(W)}} \\
&= {\rm e}^{A_{\rm BM,abs}(\partial C_l W)} \sqrt{\|\det(\mathcal{A}_g)(\quad)\|^{\rm RS}_{\det H(W)}}.
\end{aligned}$$

□

6. The case of a circle

In this section we develop explicit calculations on a simple example, namely the case $W = S^1_{\sin\alpha}$, the circle of radius $\sin\alpha$. Set $\mathsf{F} = \mathsf{F}S^1_{\sin\alpha}$, then F is the finite surface in $\mathbb{R}^3$ parametrized by

$$\mathsf{F} = \begin{cases} x_1 = & r\sin\alpha\cos\theta \\ x_2 = & r\sin\alpha\sin\theta \\ x_3 = & r\cos\alpha \end{cases}$$

with $(r,\theta) \in [l_1, l_2] \times [0, 2\pi]$, and the metric is the metric induced by the immersion $g_{\mathsf{F}} = dr \otimes dr + (\sin^2\alpha) r^2 d\theta \otimes d\theta$.

6.1. **R torsion.** Let K denotes the cellular decomposition of F described in Figure 1, with subcomplexes L_j that are cellular decompositions of W_j. Let $\tilde{K}$ be the universal covering complex (that is a cellular decomposition of the space $\tilde{\mathsf{F}} = [l_1, l_2] \times \mathbb{R}$). It is easy to see that the integral cellular chain complex of $\tilde{K}$ with the fundamental group acting by covering transformations gives the following chain complex of $\mathbb{Z}\pi$-modules, where $\pi = \pi_1(\mathsf{F}) = \mathbb{Z}$,

$$\mathsf{C}(\mathsf{F};\mathbb{Z}\pi): \qquad 0 \longrightarrow \mathbb{Z}\pi[c_2] \longrightarrow \mathbb{Z}\pi[c_{1,0}, c_{1,1}, c_{1,2}] \longrightarrow \mathbb{Z}\pi[c_{0,0}, c_{0,1}] \longrightarrow 0,$$

with boundaries (where we are representing $\mathbb{Z}\pi$ by $\mathbb{Z}[t]$)

$$\partial_2(c_2) = c_{1,1} + c_{1,0} - tc_{1,1} - c_{1,2},$$
$$\partial_1(c_{1,0}) = tc_{0,1} - c_{0,1}, \qquad \partial_1(c_{1,1}) = c_{0,1} - c_{0,0}, \qquad \partial_1(c_{1,2}) = tc_{0,0} - c_{0,0}.$$

Taking the trivial representation $\rho_0 : \pi \to O(1, \mathbb{R})$, and considering the complex of vector spaces $\mathsf{C}(\mathsf{F};\mathbb{R}_\rho) = \mathbb{R} \times_\rho \mathsf{C}(\mathsf{F};\mathbb{Z}\pi)$, we have $H_0(\mathsf{F};\mathbb{R}_\rho) = H_1(\mathsf{F};\mathbb{R}_\rho) = \mathbb{R}$, $H_2(\mathsf{F};\mathbb{R}_\rho) = 0$. In order to compute the R torsion, we fix bases for homology. In dimension zero, take $1 \in \mathcal{H}^0(F)$. Then, $\|1\| = \sqrt{\mathrm{Vol}(\mathsf{F})} = \sqrt{(l_2^2 - l_1^2)\pi\sin\alpha}$, and using the de Rham map $\mathcal{A}_0^{\mathrm{abs}} = \mathcal{P}_0^{-1}\mathcal{A}^2_{\mathrm{rel}}\star : \mathcal{H}^0(\mathsf{F}) \to C_0(K, E_{\rho_0})$ we get

$$\begin{aligned} A_0^{\mathrm{abs}}\left(\frac{1}{\|1\|}\right) &= \mathcal{P}_0^{-1}\mathcal{A}^2_{\mathrm{rel}}\left(\frac{r\sin\alpha dr\wedge d\theta}{\|1\|}\right) = \mathcal{P}_0^{-1}(\sqrt{\mathrm{Vol}(\mathsf{F})}(\hat{c}_{0,1} + \hat{c}_{0,0})^*) \\ &= \sqrt{\mathrm{Vol}(\mathsf{F})}(c_{0,1} + c_{0,0}). \end{aligned}$$

In dimension one, consider $d\theta \in \mathcal{H}^1(\mathsf{F})$, satisfying absolute BC. We have $\|d\theta\|^2 = \frac{2\pi}{\sin\alpha}\log\frac{l_2}{l_1}$, and we want to apply the de Rham map $\mathcal{A}_1^{\mathrm{abs}} = -\mathcal{P}_1^{-1}\mathcal{A}^1_{\mathrm{rel}}\star$. Now, $\mathcal{A}^1_{\mathrm{rel}} : \mathcal{H}^1(\mathsf{F}) \to C^1(\hat{K}, E_\rho)$ is defined by $\mathcal{A}^1_{\mathrm{rel}}(h)(\hat{c}) = \int_{\hat{c}} h$. Using the basis of $C^1(\hat{K}, E_\rho)$, we have $\mathcal{A}^1_{\mathrm{rel}}(dr)(\hat{c}_{1,1}) = 0$, since $(\hat{c})^1_1$ is a circle with constant r, and

$$\mathcal{A}^1_{\mathrm{rel}}\left(-\frac{1}{r\sin\alpha\|d\theta\|}dr\right)(\hat{c}_{1,0} - \hat{c}_{1,2}) = \int_{l_1}^{l_2} -\frac{1}{r\sin\alpha\|d\theta\|}dr = -\frac{(\log\frac{l_2}{l_1})^{\frac{1}{2}}}{(2\pi\sin\alpha)^{\frac{1}{2}}}.$$

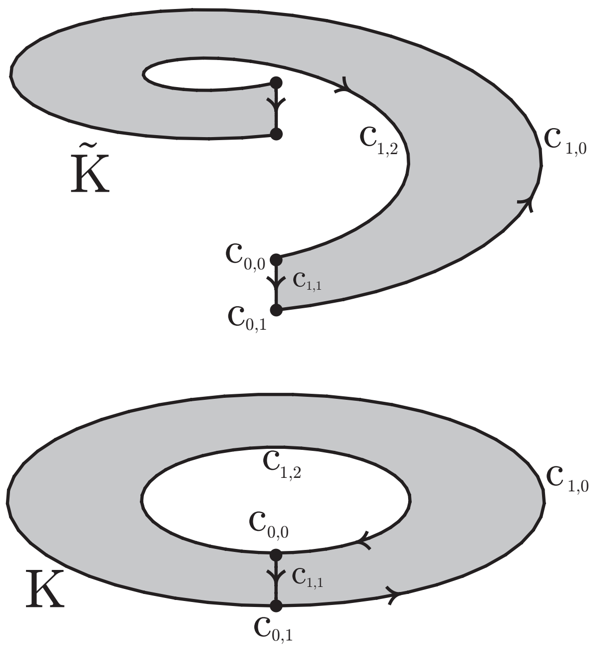

FIGURE 1. Cell decomposition of F.

This gives $\mathcal{A}_1^{\mathrm{abs}}\left(\frac{d\theta}{\|d\theta\|}\right) = \frac{(\log \frac{l_2}{l_1})^{\frac{1}{2}}}{(2\pi\sin\alpha)^{\frac{1}{2}}}(c_{1,0}-c_{1,2})$. Therefore: taking the following bases b_q for boundary and h_q for homology

$$\begin{aligned}
\{\partial_1(b_1), z_0, b_0\} &= \{c_{0,1}-c_{0,0}, \sqrt{\mathrm{Vol}(\mathsf{F})}(c_{0,0}+c_{0,1}), \emptyset\},\\
\{\partial_2(b_2), z_1, b_1,\} &= \{c_{1,0}+c_{1,2}, \frac{(\log \frac{l_2}{l_1})^{\frac{1}{2}}}{(2\pi\sin\alpha)^{\frac{1}{2}}}(c_{1,0}-c_{1,2}), c_{1,1}\},\\
\{\partial_3(b_3), z_2, b_2\} &= \{\emptyset, \emptyset, c_2\},
\end{aligned}$$

the torsion is

$$\tau_{\mathrm{R}}(\mathsf{F}, g_{\mathsf{F}}) = \prod_{q=1}^{2} |\det(\partial_{q+1}(b_{q+1}), z_q, b_q/c_q)| = \frac{\sqrt{\mathrm{Vol}(\mathsf{F})}}{\frac{(\log \frac{l_2}{l_1})^{\frac{1}{2}}}{(2\pi\sin\alpha)^{\frac{1}{2}}}} = \frac{\pi\sin\alpha(2(l_2^2-l_1^2))^{\frac{1}{2}}}{(\log\frac{l_2}{l_1})^{\frac{1}{2}}}.$$

Similar calculations for the pairs $(\mathsf{F}, \partial\mathsf{F})$ and (F, W_2) give, respectively,

$$\begin{aligned}
\tau_{\mathrm{R}}(\mathsf{F}, \partial\mathsf{F}, g_{\mathsf{F}}) &= \frac{\frac{1}{\sqrt{\mathrm{Vol}(\mathsf{F})}}}{\frac{(2\pi\sin\alpha)^{\frac{1}{2}}}{(\log\frac{l_2}{l_1})^{\frac{1}{2}}}} = \frac{(\log\frac{l_2}{l_1})^{\frac{1}{2}}}{\pi\sin\alpha(2(l_2^2-l_1^2))^{\frac{1}{2}}};\\
\tau_{\mathrm{R}}(\mathsf{F}, W_2, g_{\mathsf{F}}) &= 1.
\end{aligned}$$

Triviality of the last is expected since this corresponds to the cone relative to a point, up to simple homotopy type.

6.2. **Anomaly boundary term.** We determine the forms $\mathcal{S}$ and B appearing in the definition of $A_{\rm BM,abs}(\partial \mathsf{F})$. Since the last is local, and the boundary $\partial\mathsf{F} = W_1 \sqcup W_2$ is non connected, we consider the three metrics: g_{F}, and $g_j = dr\otimes dr + \sin^2\alpha l_j^2 d\theta\otimes d\theta$. In the first metric, an orthonormal basis is $e_r = \frac{\partial}{\partial r}$, $e^r = dr$, $e_{\theta_1} = \frac{1}{r\sin\alpha}\frac{\partial}{\partial\theta}$, and $e^\theta = r\sin\alpha d\theta$. The non trivial Christoffel symbols are: ${\Gamma_\theta}^r{}_\theta = -\frac{1}{r}$ and ${\Gamma_\theta}^\theta{}_r = \frac{1}{r}$, and the connection one form is

$$(\omega)^r{}_\theta = -\frac{1}{r}e^\theta = -\sin\alpha d\theta,$$

implying the vanishing of the curvature. In the metric g_j it is easy to see that the connection one form vanishes identically. Applying the definition

$$-\mathcal{S}_{W_1} = \mathcal{S}_{W_2} = \frac{1}{2}(i_j^*\omega_1 - i_j^*\omega_{0,l_1})^r{}_\theta\hat{e}^\theta = -\frac{1}{2}\sin\alpha d\theta\hat{e}^\theta,$$

(where i_j^* is the inclusion of the boundary W_j), giving

$$B_{W_j}(\nabla^{T\mathsf{F}}) = \frac{1}{2}\int_0^1\int^B\frac{1}{\Gamma(1+\frac{1}{2})}\mathcal{S}_{W_j}du = \frac{(-1)^{j+1}}{2\sqrt{\pi}}\int^B \sin\alpha d\theta\hat{e}^\theta = (-1)^j\frac{\sin\alpha}{2\pi}d\theta,$$

and hence

$$\begin{aligned}
A_{\rm BM,abs}(W_j) &= \frac{1}{2}\int_{S^1_{l_j\sin\alpha}} B_{W_j}(\nabla_1^{T\mathsf{F}}) = \frac{(-1)^j}{2}\sin\alpha,\\
A_{\rm BM,rel}(W_j) &= \frac{(-1)^{j+1}}{2}\sin\alpha.
\end{aligned}$$

6.3. **Analytic torsion.** In this section we compute the analytic torsion of F in the trivial representation. The technique is the one used in [11], and we refer to that work for details. We write $\frac{1}{\nu} = \sin\alpha$, for convenience. So the Hodge operator is $\star 1 = \frac{r}{\nu}dr\wedge d\theta$, $\star dr = \frac{r}{\nu}d\theta$, $\star d\theta = -\frac{\nu}{r}dr$, $\star dr\wedge d\theta = \frac{\nu}{r}$, and the Laplace operator reads

$$\begin{aligned}
\Delta^{(0)}(f) =& -\partial_r^2 f - \frac{1}{r}\partial_r f - \frac{\nu^2}{r^2}\partial_\theta^2 f,\\
\Delta^{(1)}(f_r dr + f_\theta d_\theta) =& \left(-\partial_r^2 f_r - \frac{\nu^2}{r^2}\partial_\theta^2 f_r + \frac{1}{r^2}f_r - \frac{1}{r}\partial_r f_r + \frac{2\nu^2}{r^3}\partial_\theta f_\theta\right)dr\\
&+\left(-\partial_r^2 f_\theta - \frac{\nu^2}{r^2}\partial_\theta^2 f_\theta + \frac{1}{r}\partial_r f_\theta - \frac{2}{r}\partial_\theta f_r\right)d\theta,\\
\Delta^{(2)}(f dr\wedge d\theta) =& -\partial_r^2 f + \frac{1}{r}\partial_r f - \frac{2\nu^2}{r^2}f - \frac{1}{r^2}\partial_\theta^2 f.
\end{aligned}$$

Proceeding as in [11, Section 3] or [10, Lemma 3], we find a complete system of eigenforms for Δ, and imposing absolute and relative BC respectively, we obtain the spectrum

$$
\begin{aligned}
\mathrm{Sp}\Delta^{(0)}_{\mathrm{abs}} &= \left\{2 : \tilde{a}^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{\tilde{a}^2_{0,k}\right\}_{k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(1)}_{\mathrm{abs}} &= \left\{2 : \tilde{a}^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{2 : a^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{\tilde{a}^2_{0,k}\right\}_{k=1}^{\infty} \cup \left\{a^2_{0,k}\right\}_{k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(2)}_{\mathrm{abs}} &= \left\{2 : a^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{a^2_{0,k}\right\}_{k=1}^{\infty};\\
\mathrm{Sp}\Delta^{(0)}_{\mathrm{rel}} &= \left\{2 : a^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{a^2_{0,k}\right\}_{k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(1)}_{\mathrm{rel}} &= \left\{2 : a^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{2 : \tilde{a}^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{\tilde{a}^2_{0,k}\right\}_{k=1}^{\infty} \cup \left\{a^2_{0,k}\right\}_{k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(2)}_{\mathrm{rel}} &= \left\{2 : \tilde{a}^2_{\nu n,k}\right\}_{n,k=1}^{\infty} \cup \left\{\tilde{a}^2_{0,k}\right\}_{k=1}^{\infty},
\end{aligned}
$$

where the $a_{\nu n,k}, \tilde{a}_{\nu n,k}$ are the zeros of the function $F_{\nu n}(z), \tilde{F}_{\nu n,k}$, respectively (here J_{-0} is replaced by Y_0):

$$
\begin{aligned}
F_{\nu n}(z) &= J_{\nu n}(l_2 z) J_{-\nu n}(l_1 z) - J_{\nu n}(l_1 z) J_{-\nu n}(l_2 z),\\
\tilde{F}_{\nu n}(z) &= J'_{\nu n}(l_1 z) J'_{-\nu n}(l_2 z) - J'_{\nu n}(l_2 z) J'_{-\nu n}(l_1 z).
\end{aligned}
$$

The torsion zeta function is

$$
t_{\mathrm{abs/rel}}(s) = \frac{1}{2}\sum_{q=1}^{2}(-1)^q \zeta(s, \Delta^{(q)}_{\mathrm{abs/rel}}),
$$

and using the above description of the spectra, after some simplification, we obtain

$$
t_{\mathrm{abs}}(s) = -t_{\mathrm{rel}}(s) = \sum_{n,k=1}^{\infty} a^{-2s}_{\nu n,k} - \sum_{n,k=1}^{\infty} \tilde{a}^{-2s}_{\nu n,k} + \frac{1}{2}\sum_{k=1}^{\infty} a^{-2s}_{0,k} - \frac{1}{2}\sum_{k=1}^{\infty} \tilde{a}^{-2s}_{0,k},. \tag{7}
$$

To compute the derivative at zero of the last two functions, $Z(s, S_0) = \sum_{k=1}^{\infty} a^{-2s}_{0,k}$, $Z(s, \tilde{S}_0) = \sum_{k=1}^{\infty} \tilde{a}^{-2s}_{0,k}$, we use Proposition 2.4 of [22]. For we need the asymptotic expansion for large λ of the Gamma functions associated to the sequences $S_0 = \{a_{0,k}\}_{k=1}^{\infty}$ and $\tilde{S}_0 = \{\tilde{a}_{0,k}\}_{k=1}^{\infty}$ (see [22, Sec. 2.1]). Proceeding as in [11, Section 5.2], we have the product representations (for $-\pi < \arg(z) < \frac{\pi}{2}$)

$$
\begin{aligned}
G_0(z) = F_0(iz) &= \frac{2}{\pi}\log\frac{l_2}{l_1}\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{a^2_{0,k}}\right),\\
\tilde{G}_0(z) = \tilde{F}_0(iz) &= \frac{1}{\pi}\left(\frac{l_2^2 - l_1^2}{l_1 l_2}\right)\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{(\tilde{a}_{0,k})^2}\right),
\end{aligned}
$$

that give

$$
\begin{aligned}
\log\Gamma(-\lambda, S_0) &= -\log\prod_{k=1}^{\infty}\left(1 + \frac{(-\lambda)}{a^2_{0,k}}\right) = -\log G_0(\sqrt{-\lambda}) + \log\frac{2}{\pi} + \log\log\frac{l_2}{l_1},\\
\log\Gamma(-\lambda, \tilde{S}_0) &= -\log\prod_{k=1}^{\infty}\left(1 + \frac{(-\lambda)}{(\tilde{a}_{0,k})^2}\right) = -\log\tilde{G}_0(\sqrt{-\lambda}) - \log\pi + \log\frac{l_2^2 - l_1^2}{l_1 l_2}.
\end{aligned}
$$

Using classical expansions for the Bessel functions, we have the expansions for large λ

$$\log\Gamma(-\lambda,S_0)\sim\log\sqrt{-\lambda}+\frac{1}{2}\log l_1l_2+\log\log\frac{l_2}{l_1}+\log 2-(l_2-l_1)\sqrt{-\lambda},$$
$$\log\Gamma(-\lambda,\tilde S_0)\sim\log\sqrt{-\lambda}-\frac{1}{2}\log l_1l_2+\log(l_2^2-l_1^2)-(l_2-l_1)\sqrt{-\lambda},$$

that give

$$Z'(0,S_0)=-\frac{1}{2}\log l_1l_2-\log\log\frac{l_2}{l_1}-\log 2,$$
$$Z_0'(0,\tilde S_0)=\frac{1}{2}\log l_1l_2-\log(l_2^2-l_1^2).$$

For the double series we use Theorem 3 of [11] and its corollary. For we consider the zeta functions: $Z(s,S)=\sum_{n,k=1}^\infty a_{\nu n,k}^{-2s}$, and $Z(s,\tilde S)=\sum_{n,k=1}^\infty \tilde a_{\nu n,k}^{-2s}$, associated to the double series $S=\{a_{\nu n,k}^{-2s}\}_{n,k=1}^\infty$, $\tilde S=\{\tilde a_{\nu n,k}^{-2s}\}_{n,k=1}^\infty$, respectively. We first prove that the two double sequences are spectrally decomposable over the sequence $U=\{(\nu n)^2\}_{n=1}^\infty$, with power 2 and length 3 according to Definition 1 of [11] (for the proof see [10, Section 5.5]). Next, we need uniform asymptotic expansion of the associated Gamma functions $\Gamma(\lambda,S_n/(\nu n)^2)$, $\Gamma(\lambda,\tilde S_n/(\nu n)^2)$, for large n, and expansions for large λ. We have the product representations ($-\pi<\arg(z)<\frac{\pi}{2}$) [11, Section 5.2]

$$G_\nu(z)=F_\nu(iz)=\frac{\sin\pi\nu}{\nu\pi}\left(\frac{l_2^\nu}{l_1^\nu}-\frac{l_1^\nu}{l_2^\nu}\right)\prod_{k=1}^{+\infty}\left(1+\frac{z^2}{a_{\nu,k}^2}\right),$$
$$\tilde G_\nu(z)=i^2\tilde F_\nu(iz)=\frac{\nu\sin\pi\nu}{\pi}\left(\frac{l_2^\nu}{l_1^\nu}-\frac{l_1^\nu}{l_2^\nu}\right)\frac{1}{l_1l_2z^2}\prod_{k=1}^{+\infty}\left(1+\frac{z^2}{\tilde a_{\nu,k}^2}\right).$$

This gives

$$\log\Gamma(-\lambda,S_n/(\nu n)^2)=-\log G_{\nu n}(\nu n\sqrt{-\lambda})+\log\frac{\sin(\nu n\pi)}{\nu n\pi}+\log\left(\frac{l_2^{\nu n}}{l_1^{\nu n}}-\frac{l_1^{\nu n}}{l_2^{\nu n}}\right),$$
$$\log\Gamma(-\lambda,\tilde S_n/(\nu n)^2)=-\log\tilde G_{\nu n}(\nu n\sqrt{-\lambda})+\log\frac{\sin(\nu n\pi)}{\nu n\pi}+\log\left(\frac{l_2^{\nu n}}{l_1^{\nu n}}-\frac{l_1^{\nu n}}{l_2^{\nu n}}\right)$$
$$-\log(-\lambda l_1l_2),$$

and using the asymptotic expansions of the Bessel functions and of their derivatives for large index [20, (7.18), Ex. 7.2] we have the desired expansions. According to equation (7), we just need to work with the difference $Z(s,S)-Z(s,\tilde S)$. After some computations, we obtain the asymptotic expansion for large n (uniformly in λ)

$$\log\Gamma(-\lambda,\tilde S_n/(\nu n)^2)-\log\Gamma(-\lambda,S_n/(\nu n)^2)$$
$$=-\frac{1}{2}\log(1-\lambda l_1^2)(1-\lambda l_2^2)+\phi_1(\lambda)\frac{1}{\nu n}+O\left(\frac{1}{(\nu n)^2}\right),$$

where $\phi_1(\lambda)=V_1(l_1\sqrt{-\lambda})-U_1(l_1\sqrt{-\lambda})-V_1(l_2\sqrt{-\lambda})+U_1(l_2\sqrt{-\lambda})$, and using the coefficients in the expansions of the Bessel functions given in [20, (7.18), Ex. 7.2]

$$\phi_1(\lambda)=\frac{1}{2}\left(\frac{1}{(1-l_2^2\lambda)^{\frac{1}{2}}}-\frac{1}{(1-l_1^2\lambda)^{\frac{1}{2}}}\right)-\frac{1}{2}\left(\frac{1}{(1-l_2^2\lambda)^{\frac{3}{2}}}-\frac{1}{(1-l_1^2\lambda)^{\frac{3}{2}}}\right).$$

Applying the definition [11, (11)]

$$\begin{aligned}\Phi_1(s) &= \int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\phi_1(\lambda)d\lambda dt\\ &= (l_2^{2s}-l_1^{2s})\left(\frac{1}{2}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})s}-\frac{1}{2}\frac{\Gamma(s+\frac{3}{2})}{\Gamma(\frac{3}{2})s}\right),\end{aligned}$$

and $\mathrm{Res}_{0\,s=0}\,\Phi_1(s)=\mathrm{Res}_{1\,s=0}\,\Phi_1(s)=0$. On the other side, the expansion for large λ is

$$\log\Gamma(-\lambda,\tilde{S}_n/(\nu n)^2)-\log\Gamma(-\lambda,S_n/(\nu n)^2)=-\log(-\lambda)-\log l_1l_2+O\left(\frac{1}{\sqrt{-\lambda}}\right).$$

Recalling the definition in [11, (13)], we have $a_{0,0,n}=-\log l_1l_2$, $a_{0,1,n}=-1$, $b_{1,0,0}=b_{1,0,1}=0$, that gives

$$A_{0,0}(s)=-\sum_{n=1}^\infty\frac{\log l_1l_2}{(\nu n)^{2s}},\qquad A_{0,1}(s)=-\sum_{n=1}^\infty\frac{1}{(\nu n)^{2s}},$$

and

$$A_{0,0}(0)=\frac{1}{2}\log l_1l_2,\qquad A_{0,1}(0)=\frac{1}{2},\qquad A_{0,1}'(0)=-\log\nu+\log 2\pi.$$

Collecting all these results and applying Theorem 3 of [11] we obtain

$$Z'(0,\tilde{S})-Z'(0,S)=-\frac{1}{2}\log l_1l_2+\log\nu-\log 2\pi,$$

that by equation (7) gives

$$\log T_{\mathrm{abs}}(\mathsf{F},g_{\mathsf{F}})=-\log T_{\mathrm{rel}}(\mathsf{F},g_{\mathsf{F}})=t_{\mathrm{abs}}'(0)=\log\frac{\pi(2(l_2^2-l_1^2))^{\frac{1}{2}}}{(\log\frac{l_2}{l_1})^{\frac{1}{2}}\nu}.$$

6.4. **Some limits.** The geometry of F has at least two natural limit cases: the cone over W_2, reached for $l_1\to 0^+$, and the cylinder over W_2, reached for $l_1=l_2$. We investigate in this section the value of the torsion of F in these two limit cases. The first case is an instance of the general case discussed in Section 5. It is easy to realize that in the limit for $l_1\to 0^+$, the torsion of the conical frustum diverges. So consider the geometric regularized R torsion (the extension to analytic torsion is straightforward). Then, $W=S^1_{l_2\sin\alpha}$, the circle of radius $l_2\sin\alpha$, and $W_{(0)}$ is its preferred 0 cell. Since $m=1=p$, the volume element $\det\mathsf{w}$ is fixed by the Ray and Singer basis for the homology of S^1. Thus,

$$\tau_{\mathrm{R}}((S^1_{l_2\sin\alpha})_{(0)};\mathsf{w})=|\det(h_0/c_0)|.$$

Looking at the calculation of the torsion of a sphere in [14], the harmonic basis in dimension 0 is $1/\sqrt{\mathrm{Vol}(S^1_{l_2\sin\alpha})}$, and applying the de Rham map $h_0=\left\{\sqrt{\mathrm{Vol}(S^1_{l_2\sin\alpha})}c_{0,0}\right\}$. Since $\rho=\rho_0$, then $\tau_{\mathrm{R}}((S^1_{l_2\sin\alpha})_{(0)};\rho,\mathrm{x})=\sqrt{\mathrm{Vol}(S^1_{l_2\sin\alpha})}=\sqrt{2\pi l_2\sin\alpha}$. Therefore,

$$\Upsilon_{\mathrm{R}}(\mathsf{F},g_F)=\frac{\tau_{\mathrm{R}}(\mathsf{F},g_F)}{\tau_{\mathrm{R}}((S^1_{l_2\sin\alpha})_{(0)};\mathrm{x})\left(\frac{\Gamma_1}{l_2^{-1}}\right)^{-\frac{r_1}{2}}}=\frac{\tau_{\mathrm{R}}(\mathsf{F},g_F)}{\sqrt{\frac{\pi(l_2^2-l_1^2)\sin\alpha}{l_2\log\frac{l_2}{l_1}}}}.$$

This gives

$$\lim_{l_1\to 0^+}\Upsilon_{\rm R}(\mathsf{F},g_F)=\sqrt{{\rm Vol}(CS^1_{l_2\sin\alpha})}=\sqrt{\pi l_2^2\sin\alpha}=\tau_R(CS^1_{l_2\sin\alpha},g_C).$$

Next, consider the cylinder. This is reached by fixing $b_1=l_1\sin\alpha$ and $h=(l_2-l_1)\cos\alpha$, and taking the limit for $\alpha\to 0$. The result is

$$\lim_{b_1,h\ \text{fixed},\alpha\to 0^+}T_{\rm abs}(\mathsf{F},g_{\mathsf{F}})=2\pi b_1={\rm Vol}(S^1_{b_1})=\tau_{\rm R}(S^1_{b_1},g_{S^1_{b_1}}),$$

where the $g_{S^1_{b_1}}$ is the standard metric (see [14]), consistently with the fact that a cylinder has the same simple homotopy type of a circle.

7. Appendix: the intersection torsion of a cone

In this appendix, based on some work in progress by the same authors, we sketch the calculation of the intersection torsion of the cone CW. For we review some classical fact about intersection torsion. Let W be a compact connected manifold without boundary of dimension m. Let K a regular cellular decomposition of W (CW or simplicial). Let $CK=(K\times I)/(K\times\{0\})$ be the cone on K. Then, CK is a cellular decomposition of CW. CW is an $m+1$ dimensional pseudomanifold with boundary, either in the CW or simplicial category. A perversity $\mathsf{p}=\{\mathsf{p}_j\}_{j=2}^n$ is a sequence of integers such that $\mathsf{p}_2=0$ and $\mathsf{p}_{j+1}=\mathsf{p}_j$ or p_j+1. The null perversity is $0_j=0$, and the top perversity is $\mathsf{t}_j=j-2$. Given a perversity p, the complementary perversity p^c is $\mathsf{p}^c_j=\mathsf{t}_j-\mathsf{p}_j=j-\mathsf{p}_j-2$. The perversity: $\mathsf{m}=\{\mathsf{m}_j=[j/2]-1\}$ is called lower middle perversity. Let N be a finite m dimensional regular cellular complex. If q is an integer and p a perversity, a cell e of N_q is said (p,q)-allowable if $\dim(\bar e\cap\Sigma)\leq q-n+\mathsf{p}_n$. The intersection cellular family of perversity p associated to N is the subfamily of the (p,q)-allowable cells of N, namely $\{I^{\mathsf{p}}N_{(q)}\}_{q=0}^m$, where $I^{\mathsf{p}}N_{(q)}$ is the subcomplex

$$I^{\mathsf{p}}N_{(q)}=\{e\in N_{(q)}\mid e\text{ is }(\mathsf{p},q)\text{−allowable}\}.$$

It is easy to see that

$$I^{\mathsf{p}}(CK)_{(q)}=\begin{cases}K_{(q)}, & q<n-\mathsf{p}_n,\\ (CK)_{(q)}, & q\geq n-\mathsf{p}_n.\end{cases}$$

The q intersection cellular chain module of N with perversity p is the module

$$I^{\mathsf{p}}\mathsf{C}_q(N)=H_q(I^{\mathsf{p}}N_{(q)},I^{\mathsf{p}}N_{(q-1)}).$$

The intersection cellular chain complex of perversity p of N is the complex $I^{\mathsf{p}}\mathsf{C}(N)$ of modules $I^{\mathsf{p}}\mathsf{C}_q(N)$, with the boundary operator naturally induced by the restriction of the boundary operator of the relative cellular chain complex. The intersection homology of N is the homology of the intersection chain complex $I^{\mathsf{p}}\mathsf{C}(N)$. It is not difficult to see that

$$I^{\mathsf{p}}\mathsf{C}_q(CK)=\begin{cases}\mathsf{C}_q(K) & q<a,\\ H_a(CK_{(a)},K_{(a-1)}), & q=a,\\ \mathsf{C}_q(CN) & q>a.\end{cases}$$

where $a=m+1-\mathsf{p}_{m+1}$. A direct calculation gives the intersection homology of CK, for example in the lower middle perversity:

$$I^{\mathsf{m}}H_q(CK)=I^{\mathsf{m}^c}H_q(CK)=\begin{cases}H_q(K), & q<a-1=p,\\ 0, & q\geq a-1=p,\end{cases}$$

Next, recalling that the cone of a chain complex C is the algebraic mapping cone of the chain identity of the augmentation $\tilde{\mathsf{C}}$, i.e. the chain complex $\dot{\mathsf{C}} = C\mathsf{C}$ of length $m+1$ with

$$\dot{\mathsf{C}}_q = \begin{cases} \mathsf{C}_q \oplus \mathsf{C}_{q-1}, & 0 < q \leq m+1, \\ \mathsf{C}_0 \oplus \mathbb{Z}, & q = 0, \end{cases}$$

and boundary operator

$$\dot{\partial}_q = \begin{cases} \begin{pmatrix} \partial_q & 1 \\ 0 & -\tilde{\partial}_{q-1} \end{pmatrix}, & q > 0, \\ \quad 0, & q = 0, \end{cases}$$

and using the notation $\dot{\mathsf{C}} = C\mathsf{C}(K)$, $\mathsf{C} = \mathsf{C}(K) = \mathsf{C}(K; \mathbb{Z}\pi_1(K))$ we obtain that

$I^{\mathsf{p}}\mathsf{C}(CK)$:

$$\dot{\mathsf{C}}_{m+1} \xrightarrow{\dot{\partial}_{m+1}} \dots \xrightarrow{\dot{\partial}_{a+2}} \dot{\mathsf{C}}_{a+1} \xrightarrow{\dot{\partial}_{a+1}} \mathsf{C}_a \oplus Z_{a-1} \xrightarrow{\dot{\partial}_a} \mathsf{C}_{a-1} \xrightarrow{\partial_{a-1}} \dots \xrightarrow{\partial_1} \mathsf{C}_0.$$

It is clear that the cell bases c of $\mathsf{C}(K)$ and $\dot{\mathsf{c}}$ of $\mathsf{C}(CK)$, determine chain bases for the complex $I^{\mathsf{p}}\mathsf{C}(CK)$ in all dimension $q \neq a$. Moreover, a calculation similar to the one of Section 3.2 shows that the matrix of the change of basis for the complex $\dot{\mathsf{C}}$ are

$$(\dot{\partial}_{q+1}(\dot{\mathsf{b}}_{q+1}), \dot{\mathsf{b}}_q/\dot{\mathsf{c}}_q) = (\partial_{q+1}(\mathsf{b}_{q+1}), \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)(\partial_q(\mathsf{b}_q), \mathsf{z}_{q-1}, \mathsf{b}_{q-1}/\mathsf{c}_{q-1}),$$

where the z_q are cycles projecting on some fixed homology basis h_q of $H_q(K)$.

Denote by $I^{\mathsf{p}}\mathsf{z}_a$ a fixed basis for Z_{a-1}. This together with c and $\dot{\mathsf{c}}$ completes a chain basis for $I^{\mathsf{p}}\mathsf{C}(CK)$, that we denote by $I^{\mathsf{p}}\mathsf{c}$. A new graded basis for $I^{\mathsf{p}}\mathsf{C}(CK)$ is given in all dimensions up to a and $a-1$, by those of $\mathsf{C}(CK)$ and of $\mathsf{C}(K)$, respectively. It remains to deal with the following part of $I^{\mathsf{p}}\mathsf{C}(CK)$:

$$\dots \xrightarrow{I^{\mathsf{p}}\partial_{a+1} = \dot{\partial}_{a+1}} I^{\mathsf{p}}\mathsf{C}(CK)_a = \mathsf{C}_a \oplus Z_{a-1} \xrightarrow{I^{\mathsf{p}}\partial_a = \dot{\partial}_a} I^{\mathsf{p}}(CK)_{a-1} = \mathsf{C}_{a-1} \xrightarrow{I^{\mathsf{p}}\partial_{a-1} = \partial_{a-1}} \dots .$$

We denote by $I^{\mathsf{p}}\mathsf{b}$ and $I^{\mathsf{p}}\mathsf{z}$ the basis for the boundary and for the cycles of the complex $I^{\mathsf{p}}\mathsf{C}(CK)$. Since for each $x \oplus z \in \mathsf{C}_a \oplus Z_{a-1}$, $\dot{\partial}_a(x \oplus z) = \partial_a(x) + z \oplus 0$, it follows that $\mathrm{Im} I^{\mathsf{p}}\partial_a = Z_{a-1} \oplus 0$, and we can choose as set $I^{\mathsf{p}}\mathsf{b}_a = 0 \oplus \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\}$. The image will be $I^{\mathsf{p}}\partial_a(I^{\mathsf{p}}\mathsf{b}_a) = \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\} \oplus 0$.

At $q = a$, it is easy to see that the homology $I^{\mathsf{p}}H_a$ is trivial. Therefore, $I^{\mathsf{p}}\mathsf{z}_a = \emptyset$. Also, the basis for the boundary is given as in the cone, i.e.

$$I^{\mathsf{p}}\partial_{a+1}(I^{\mathsf{p}}\mathsf{b}_{a+1}) = \dot{\partial}_{a+1}(\dot{\mathsf{b}}_{a+1}) = \{\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a\} \oplus 0, \mathsf{b}_a \oplus \partial_a(\mathsf{b}_a).$$

Hence the new basis at $q = a$ is:

$$I^{\mathsf{p}}\partial_{a+1}(I^{\mathsf{p}}\mathsf{b}_{a+1}), I^{\mathsf{p}}\mathsf{b}_a = \{\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a, \mathsf{b}_a\} \oplus 0, \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\} \oplus 0.$$

The change of basis is

$$\begin{aligned} &(I^{\mathsf{p}}\partial_{a+1}(I^{\mathsf{p}}\mathsf{b}_{a+1}), I^{\mathsf{p}}\mathsf{b}_a/I^{\mathsf{p}}\mathsf{c}_a) \\ &\qquad = (\{\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a, \mathsf{b}_a\} \oplus 0, \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\} \oplus 0/\mathsf{c}_a \oplus I^{\mathsf{p}}\mathsf{z}_{a-1}) \\ &\qquad = (\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a, \mathsf{b}_a/\mathsf{c}_a)(\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}). \end{aligned}$$

Observing that

$$\begin{aligned} (\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}) &= (\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}, \mathsf{b}_{a-1}) \\ &= (\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/\mathsf{c}_{a-1})(\mathsf{c}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}, \mathsf{b}_{a-1}), \end{aligned}$$

the change of basis is

$$(I^{\mathsf{p}}\partial_{a+1}(I^{\mathsf{p}}\mathsf{b}_{a+1}), I^{\mathsf{p}}\mathsf{b}_a/I^{\mathsf{p}}\mathsf{c}_a) = (\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a, \mathsf{b}_a/\mathsf{c}_a)(\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/\mathsf{c}_{a-1})(\mathsf{c}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}, \mathsf{b}_{a-1}).$$

In dimension $q = a-1$, we have seen that $I^{\mathsf{p}}\partial_a(I^{\mathsf{p}}\mathsf{b}_a) = \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\} \oplus 0$, and that $I^{\mathsf{p}}\mathsf{b}_{a-1} = \mathsf{b}_{a-1}$. Now, it is easy to see that $I^{\mathsf{p}}H_{a-1} = 0$. Hence, $I^{\mathsf{p}}\mathsf{z}_{a-1} = \emptyset$, and the new basis at $q = a-1$ is:

$$I^{\mathsf{p}}\partial_a(I^{\mathsf{p}}\mathsf{b}_a), I^{\mathsf{p}}\mathsf{b}_{a-1} = \{\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}\} \oplus 0, \mathsf{b}_{a-1} \oplus 0,$$

and the change of basis

$$(I^{\mathsf{p}}\partial_a(I^{\mathsf{p}}\mathsf{b}_a), I^{\mathsf{p}}\mathsf{b}_{a-1}/I^{\mathsf{p}}\mathsf{c}_{a-1}) = (\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/\mathsf{c}_{a-1}).$$

We define the intersection torsion of CK with respect to the perversity p and the intersection chain and homology bases $I^{\mathsf{p}}\mathsf{c}$, $I^{\mathsf{p}}\mathsf{h}$, to be the following class in the Whitehead group of $\mathbb{Z}\pi_1(K)$

$$\tau(I^{\mathsf{p}}\mathsf{C}(CK); I^{\mathsf{p}}\mathsf{h}, I^{\mathsf{p}}\mathsf{c}) = \prod_{q=0}^{m+1}[(I^{\mathsf{p}}\partial_{q+1}(I^{\mathsf{p}}\mathsf{b}_{q+1}), I^{\mathsf{p}}\mathsf{h}_q, I^{\mathsf{p}}\mathsf{b}_q/I^{\mathsf{p}}\mathsf{c}_q)]^{(-1)^q}.$$

Using the previous calculations we find that

$$\begin{aligned}
&\tau(I^{\mathsf{p}}\mathsf{C}(CK); I^{\mathsf{p}}\mathsf{h}, I^{\mathsf{p}}\mathsf{c})\\
&= \prod_{q=0}^{a-2}[(I^{\mathsf{p}}\mathsf{h}_q/j_*(\mathsf{h}_q))]^{(-1)^q}[(\partial_{q+1}(\mathsf{b}_{q+1}, \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)]^{(-1)^q}[(\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/\mathsf{c}_{a-1})]^{(-1)^{a-1}}\\
&\quad [(\partial_{a+1}(\mathsf{b}_{a+1}), \mathsf{z}_a, \mathsf{b}_a/\mathsf{c}_a)]^{(-1)^a}[(\partial_a(\mathsf{b}_a), \mathsf{z}_{a-1}, \mathsf{b}_{a-1}/\mathsf{c}_{a-1})]^{(-1)^a}\\
&\quad [(\mathsf{c}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}, \mathsf{b}_{a-1})]^{(-1)^a}\prod_{q=a+1}^{m+1}[(\dot\partial_{q+1}(\dot{\mathsf{b}}_{q+1}, \dot{\mathsf{b}}_q/\dot{\mathsf{c}}_q)]^{(-1)^q}\\
&= \prod_{q=0}^{a-2}[(I^{\mathsf{p}}\mathsf{h}_q/j_*(\mathsf{h}_q))]^{(-1)^q}[(\partial_{q+1}(\mathsf{b}_{q+1}, \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)]^{(-1)^q}[(\mathsf{c}_{a-1}/I^{\mathsf{p}}\mathsf{z}_{a-1}, \mathsf{b}_{a-1})]^{(-1)^a}.
\end{aligned}$$

In order to proceed we fix $m = 2p-1$, odd, and $\mathsf{p} = \mathsf{m}$, the lower middle perversity. Then, $\mathsf{m} = \mathsf{m}^c$, and $a = p+1$.

Then

$$\tau(I^{\mathsf{m}}\mathsf{C}(CK); I^{\mathsf{m}}\mathsf{h}, I^{\mathsf{m}}\mathsf{c}) = \prod_{q=0}^{p-1}[(I^{\mathsf{m}}\mathsf{h}_q/j_*(\mathsf{h}_q))]^{(-1)^q}[(\partial_{q+1}(\mathsf{b}_{q+1}, \mathsf{z}_q, \mathsf{b}_q/\mathsf{c}_q)]^{(-1)^q}[(I^{\mathsf{m}}\mathsf{z}_p, \mathsf{b}_p/\mathsf{c}_p)]^{(-1)^p}.$$

This class depends on the basis $I^{\mathsf{m}}\mathsf{z}_p$, however there is a natural way to fix this basis. First, we twist the complexes by the representation ρ, as above, and we consider torsion of the complex of vector spaces $I^{\mathsf{m}}\mathsf{C}(CK; V_\rho)$, i.e. the positive real number $\tau(I^{\mathsf{m}}\mathsf{C}(CK; V_\rho); I^{\mathsf{m}}\mathsf{h}, I^{\mathsf{m}}\mathsf{c})$.

Next, by the same argument of Section 5, i.e. by duality, we fix the basis b_p. Whence this basis is fixed, we fix the basis $I^{\mathsf{m}}\mathsf{z}_p$ by requiring

$$[(I^{\mathsf{m}}\mathsf{z}_p, \mathsf{b}_p/\mathsf{c}_p)] = 1.$$

With this choices, $\tau(I^{\mathsf{m}}\mathsf{C}(CK); I^{\mathsf{m}}\mathsf{h}, I^{\mathsf{m}}\mathsf{c})$ only depends on the basis $I^{\mathsf{m}}\mathsf{h}$, and moreover satisfies the equation

$$\begin{aligned}\tau(I^{\mathsf{m}}\mathsf{C}(CK); I^{\mathsf{m}}\mathsf{h}, I^{\mathsf{m}}\mathsf{c}) &= \prod_{q=0}^{p-1}[(I^{\mathsf{m}}\mathsf{h}_q/j_*(\mathsf{h}_q))]^{(-1)^q}\sqrt{\tau(\mathsf{C}(K);\mathsf{h},\mathsf{c})}\\ &= \frac{\sqrt{\tau(\mathsf{C}(K);\mathsf{h},\mathsf{c})}}{[j_*]}.\end{aligned}$$

Here the notation $\bar{j}_*$ is for the restriction of the map $j*$ onto the truncated graded complex $\bigoplus_{q=0}^{p-1} H_q(W)$, while the square brackets denote the Whitehead class of the matrix of the map. As in the proof of Corollary 3, $\bar{j}_*$ is an isomorphism, and hence the Whitehead class as well as the determinant below are well defined.

Assuming a Riemannian metric on W and the usual metric on the (non singular part of) the cone [12, Section 1] and fixing the the homology bases by the de Rham map as in Section 3.3 (this is possible since by means of the Riemannian structure defined on the non singular part of CW, L^2 forms can be used to extend the construction of Ray and Singer and to define suitable de Rham maps from L^2 harmonic forms to intersection homology [6]), we obtain what we call the intersection R torsion of the cone CW:

$$I^{\mathsf{m}}\tau_R((CW, g_C);\rho) = \tau(I^{\mathsf{m}}\mathsf{C}(CK;V_\rho); I^{\mathsf{m}}\mathcal{A}(\mathsf{a}), I^{\mathsf{m}}\mathsf{c}).$$

By the previous result, we have that

$$I^{\mathsf{m}}\tau_R((C_lW, g_C);\rho) = \frac{\sqrt{\tau_R((W,g);\rho,\mathsf{h})}}{|\det(j_*)|}.$$

It remains to compute $|\det(j_*)|$. This clearly follows by the choice of the homology basis on W and CW, and can be obtained by a calculation similar to the one used for the cylinder in Section 3.5. The result is

$$|\det(j_*)| = \prod_{q=0}^{p-1}\left(\frac{2(p-q)}{l^{2(p-q)}}\right)^{\frac{(-1)^q}{2}r_q}.$$

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