.

ON THE EXISTENCE OF G-EQUIVARIANT MAPS

FRANCIELLE R. C. COELHO[1], DENISE DE MATTOS[2] AND EDIVALDO L. DOS SANTOS[3]

Abstract. Let G be a compact Lie group. Let X, Y be free G-spaces. In this paper, by using the numerical index $i(X; R)$, under cohomological conditions on the spaces X and Y, we consider the question of the existence of G-equivariant maps $f : X \to Y$.

Key words: Compact Lie group, Free G-actions, Index of G-spaces, Equivariant maps, Spectral Sequence.

1. Introduction

Results on the absence of G-maps often lead to important consequences. For example, the classical Borsuk-Ulam theorem, which states that each continuous map of an n-dimensional sphere to n-dimensional Euclidean space takes the same values at a pair of antipodal points, is equivalent to the assertion that there exists no $\mathbb{Z}_2$-map $S^n \to S^{n-1}$, where $\mathbb{Z}_2$ acts on spheres by antipodal involution. In [8], under certain cohomological conditions, this result was generalized for general topological spaces equipped with free involutions. On the other hand, results of this kind can be interesting in themselves. We can prove that one space cannot be embedded in another, it is sufficient to show that there exists no equivariant map between their deleted products considered with the natural action of a free involution; or one can consider, in a more general manner, other configuration spaces connected with the spaces in question and endowed with actions of suitable groups. One example of such a result is the well-known van Kampen-Flores theorem [10], which states that the k-dimensional skeleton of a $(2k + 2)$-dimensional simplex cannot be embedded in $\mathbb{R}^{2k}$.

2000 Mathematics Subject Classification. Primary 55M20; Secondary 55M35, 47H09.

[1]Universidade Federal de Uberlândia, Faculdade de Matemática, 38408-100, Uberlândia MG, Brazil, francielle@famat.ufu.br. The author was supported by CAPES.

[2]Universidade de São Paulo, Departamento de Matemática-ICMC, 13560-970, São Carlos SP, Brazil, deniseml@icmc.usp.br. The author was supported in part by CNPq of Brazil Grant number 308390/2008-3 and by FAPESP.

[3]Universidade Federal de São Carlos, Departamento de Matemática, 13565-905, São Carlos SP, Brazil, edivaldo@dm.ufscar.br. The author was supported in part by CNPq of Brazil Grant number 304480/2008-8 and by FAPESP.

Let R be a PID and G a compact Lie group. We denote by $\beta_i(X; R)$ the i-th Betti number of X.

In [1, Theorem 1.1], it was proved that if X, Y are free G-spaces, Hausdorff, pathwise connected and paracompact such that for some natural $m \geq 1$, $H^q(X; R) = 0$ for all q, $0 < q < m$, $H^{m+1}(Y/G; R) = 0$ and $\beta_m(X; R) < \beta_{m+1}(BG; R)$, then there is no G-equivariant map $f : X \to Y$, where Y/G is the orbit space of Y by G and BG denotes the classifying space of G.

In this paper, by using the numeral index $i(X; R)$ defined in [11], we prove the following theorem:

Theorem 1.1. *Let G be a compact Lie group and X, Y free G-spaces, Hausdorff, pathwise connected and paracompact. Suppose that for some natural $m \geq 1$, $i(X; R) \geq m + 1$ and $H^{k+1}(Y/G; R) = 0$ for some k, $1 \leq k \leq m$. Then*

(i) *if $k = m$ and $\beta_m(X; R) < \beta_{m+1}(BG; R)$, there is no G-equivariant map $f : X \to Y$;*

(ii) *if $1 \leq k < m$ and $0 < \beta_{k+1}(BG; R)$, there is no G-equivariant map $f : X \to Y$.*

Remark 1.2. Let us observe that if we consider the condition $H^{k+1}(Y/G; R) = 0$, for some k, $1 \leq k < m$ in [1, Theorem 1.1], since $H^q(X; R) = 0$, for $0 < q < m$, we have $\beta_k(X; R) = 0$, and the assumption $0 < \beta_{k+1}(BG; R)$ in Theorem 1.1(ii) is always valid.

Note that if Y is a topological manifold with a free action of a compact Lie group G, then $\dim(Y/G) = \dim(Y) - \dim(G)$, where *dim* denote the usual topological dimension. Thus, if $\dim(G) \geq 1$, one has that $\dim(Y/G) < \dim(Y)$. We have the following Corollary of Theorem 1.1.

Corollary 1.3. *Let G be a compact Lie group of dimension p. Let X be a free G-space, Hausdorff, pathwise connected and paracompact such that for some $m \geq 1$, $i(X; R) \geq m+1$, and let Y be a $(k+p)$-dimensional topological manifold with a free action of G, for some k, $1 \leq k \leq m$. Then*

(i) *if $k = m$ and $\beta_m(X; R) < \beta_{m+1}(BG; R)$, there is no G-equivariant map $f : X \to Y$;*

(ii) *if $1 \leq k < m$ and $0 < \beta_{k+1}(BG; R)$, there is no G-equivariant map $f : X \to Y$.*

Proof of Corollary 1.3. Since Y is a $(k + p)$-dimensional manifold with a free action of G, for some k, $1 \leq k \leq m$, we have $\dim(Y/G) = k$ and therefore $H^{k+1}(Y/G; R) = 0$. It follows from Theorem 1.1 that there is no G-equivariant map $f : X \to Y$. $\square$

Remark 1.4. In [11], Volovikov obtained the following result directly from definition of this index:"*If* $H^q(X;R) = 0$ *for all* q, $0 < q < m$, *then* $i(X;R) \geq m+1$". This show that the condition $i(X;R) \geq m+1$ in Theorem 1.1 is weaker than cohomological condition in [1, Theorem 1.1]. However, to present this weakness in the context of Theorem 1.1 it is necessary to find examples which follow from Theorem 1.1 and not from the previous results. Next, we discuss the existence of such examples which involves the computation of the index $i(X;R)$ for a certain classes of spaces X.

Firstly, we consider the following specific example.

Example 1.5. Let $K = \Delta^{qs+q-2}_{s-1}$ be the $(s-1)$-dimensional skeleton of the $(qs+q-2)$-dimensional simplex Δ^{qs+q-2}, $q = p^n$, p prime and let $P^q_2(K) \subset K^q$ be the union of the subspaces of the form $\sigma_1 \times \sigma_2 \times \ldots \times \sigma_q$ in K^q, where $\sigma_1, \sigma_2, \ldots, \sigma_q$ are pairwise disjoint simplices of K. We have that $G = \mathbb{Z}^n_p = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ act freely on K and, as shown in [10] and [11], $i(P^q_2(K);\mathbb{Z}_p) \geq q(s-1)+1$ and $H^{s-1}(P^q_2(K);\mathbb{Z}_p) \neq 0$. In particular, if $G = \mathbb{Z}_p$ $(n=1)$, for the space $X = P^p_2(K)$ we have $i(X;\mathbb{Z}_p) \geq p(s-1)+1$ and it follows from Theorem 1.1 that there is no G-equivariant map $f : X \to S^k$, if $k < p(s-1)$. Let us observe that, if $s-1 < k < p(s-1)$, the previous example not follows from [1, Theorem 1.1], since the cohomological condition is not satisfied.

Now, let us consider $G = \mathbb{Z}_p$, p prime (resp. S^1, the circle group) act freely on a finitistic space X with mod$_p$ (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$; we abbreviate this as $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$) and recall that a paracompact Hausdorff space is finitistic if every open covering has a finite-dimensional refinement. Under these conditions, in [6] the authors proved a parametrized Borsuk-Ulam theorem for bundles whose fibre is a $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$) by using results in [5], where were determined the possible cohomology algebras of the orbit space X/G. The following result shows all possible values for the index $i(X;\mathbb{Z}_p)$ (resp. $i(X;\mathbb{Q})$) for this wide class of spaces.

Proposition 1.6. *Let* $G = \mathbb{Z}_p$, *p a prime* (*resp.* S^1), *act freely on a finitistic space* $X \sim_p S^m \times S^n$ (*resp.* $X \sim_{\mathbb{Q}} S^m \times S^n$), *where* $0 < m \leq n$, *and assume that* $H^*(X;\mathbb{Z})$ *is of finite type. Then* $i(X;\mathbb{Z}_p)$ (*resp.* $i(X;\mathbb{Q})$) *has one of the following values:*

(i) $m+1$, *where* m *must be odd, if* $G \neq \mathbb{Z}_2$;
(ii) $n+1$, *where* n *must be odd, if* $G \neq \mathbb{Z}_2$;
(iii) $m+n+1$, *where* m *must be even and* n *must be odd, if* $G \neq \mathbb{Z}_2$.

In the case that $G = \mathbb{Z}_2$ *there is no restriction on the parity of* m *and* n.

Remark 1.7. Let us observe that for $G = \mathbb{Z}_p$ (p a odd prime), $\mathbb{Z}_2$ and S^1 the items (i), (ii) and (iii) in Proposition 1.6 are related to possible cohomology algebras of the orbit space X/G given in the items (i), (iii) and (ii) of [5, Theorems 1, 2 and 3]. Moreover, in [5] are illustrated examples for possible cohomology algebras of the orbit space X/G.

Corollary 1.8. *Let $G = \mathbb{Z}_p$, p a prime (resp. S^1), act freely on a finitistic space $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$), where $0 < m \leq n$. Assume that $H^*(X;\mathbb{Z})$ is of finite type and that G act freely on a sphere S^k (where k must be odd, if $G \neq \mathbb{Z}_2$). Then*

(i) *if $i(X;\mathbb{Z}_p) = m+1$ (resp. $i(X;\mathbb{Q}) = m+1$) and $k < m$, then there is no G-equivariant map $f : X \to S^k$;*
(ii) *if $i(X;\mathbb{Z}_p) = n+1$ (resp. $i(X;\mathbb{Q}) = n+1$) and $k < n$ then there is no G-equivariant map $f : X \to S^k$;*
(iii) *if $i(X;\mathbb{Z}_p) = m+n+1$ (resp. $i(X;\mathbb{Q}) = m+n+1$) and $k < m+n$ then there is no G-equivariant map $f : X \to S^k$.*

Proof of Corollary 1.8. By Proposition 1.6, $i(X;\mathbb{Z}_p)$ (resp. $i(X;\mathbb{Q})$) assume one of the following values:

(i) $m+1$, where m must be odd, if $G \neq \mathbb{Z}_2$;
(ii) $n+1$, where n must be odd, if $G \neq \mathbb{Z}_2$;
(iii) $m+n+1$, where m must be even and n must be odd, if $G \neq \mathbb{Z}_2$.

We will show the case (iii) and the proof of the other cases follows analogously. For this case, we have $k < m+n$, $0 < \beta_{k+1}(B\mathbb{Z}_p;\mathbb{Z}_p) = 1$ (resp. $0 < \beta_{k+1}(BS^1;\mathbb{Q}) = 1$, since $k+1$ is even) and $H^{k+1}(S^k/\mathbb{Z}_p;\mathbb{Z}_p) = 0$ (resp. $H^{k+1}(S^k/S^1;\mathbb{Q}) = 0$). Then it follows from Theorem 1.1(ii) that there is no G-equivariant map $f : X \to S^k$. □

Remark 1.9. Since $H^m(X;\mathbb{Z}_p) \neq 0$ (resp. $H^m(X;\mathbb{Q}) \neq 0$), [1, Theorem 1.1] cannot be used to obtain the Corollary 1.8 (ii),(iii) and this class of examples also show that Theorem 1.1 generalizes [1, Theorem 1.1].

The paper is organized as follows. In Section 2, we recall definitions, fix notations and state results needed. In Section 3, we prove the main theorem and the Proposition 1.6. Finally, in Section 4, we show that Theorem 1.1 is related to Theorem 4.1 proved by Clapp and Puppe in [3] and we prove a similar result (Theorem 4.3) to Theorems 1.1 and 4.1.

2. Preliminaries

We start by introducing some basic notions and notations. We assume that all spaces under consideration are Hausdorff and paracompact spaces. Throughout this paper, H^* will always denote the cohomology groups. For a given space B, let $\mathcal{G}$ be a system of local coefficients for B. We will denote by $H^*(B;\mathcal{G})$ the cohomology groups of B with local coefficients in $\mathcal{G}$. The symbol $\cong$ will denote an appropriate isomorphism between algebraic objects.

Suppose that G is a compact Lie group which acts freely on a Hausdorff and paracompact space X, then $X \to X/G$ is a principal G-bundle [2, Theorem II.5.8] and one can take

$$h : X/G \to BG \tag{2.1}$$

a classifying map for the G-bundle $X \to X/G$.

Remark 2.1. Let us observe that if $\hat{h}$ is another classifying map for the principal G-bundle $X \to X/G$, then there is a homotopy between h and $\hat{h}$.

Given the G-space X, consider the product $EG \times X$ with the diagonal G-action given by $g(e,x) = (ge, gx)$ and let $EG \times_G X = (EG \times X)/G$ be its orbit space. The first projection $EG \times X \to EG$ induces a map

$$p_X : EG \times_G X \to (EG)/G = BG, \tag{2.2}$$

which is a fibration with fiber X and base space BG being the classifying space of G. This is called the *Borel construction*. It associates to each G-space X a space $EG \times_G X$, which will be denoted by X_G, over BG and to each G-map $X \to Y$ a fiber preserving map $EG \times_G X \to EG \times_G Y$ over BG.

Remark 2.2. If G acts freely on X, then the map

$$X_G \to X/G \tag{2.3}$$

induced by the second projection $EG \times X \to X$ is a fibration with a contractible fibre EG and therefore a homotopy equivalence (for details, see[4]).

We recall that for $G = \mathbb{Z}_p$,

$$H^*(B\mathbb{Z}_p;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p[t], & \deg t = 1, p = 2, \\ \Lambda(s) \otimes \mathbb{Z}_p[t], & \deg s = 1, t = \beta(s), p > 2. \end{cases}$$

where β is the mod p Bockstein cohomology operation associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ and for $G = S^1$,

$$H^*(BS^1;\mathbb{Q}) \cong \mathbb{Q}[t], \quad \deg t = 2.$$

Now, let us recall the following theorem of Leray-Serre for fibrations, as given in [7, Theorem 5.2].

Theorem 2.3. (The cohomology Leray-Serre Spectral Sequence) *Let R be a commutative ring with unit. Given a fibration $F \hookrightarrow E \xrightarrow{p} B$ where B is pathwise connected, there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$, with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R)), \tag{2.4}$$

the cohomology of B with local coefficients in the cohomology of F, the fibre of p, and coverging to $H^*(E; R)$ as an algebra. Furthermore, this spectral sequence is natural with respect to fibre-preserving maps of fibrations.

Let us now recall one of the numerical indices defined by Volovikov in [11]. This is a function on G-spaces whose value is either a positive integer or ∞. The definition of $i(X; R)$ uses the spectral sequence of the bundle $p_X : X_G \to BG$ with fibre X (the Borel construction). The spectral sequence converges to the equivariant cohomology $H^*(X_G; R)$. Let Λ^* be the equivariant cohomology algebra $H^*(pt_G; R) = H^*(BG; R)$ of a point. Suppose that X is path connected. Then $E_2^{*,0} = \Lambda^*$. Assume that $E_2^{*,0} = \cdots = E_s^{*,0} \neq E_{s+1}^{*,0}$. Then, by definition, $i(X; R) = s$. If $E_2^{*,0} = \cdots = E_\infty^{*,0}$ then, by definition, $i(X; R) = \infty$.

We state some properties of the index (see [11]).

Proposition 2.4. *Let X, Y and Z be G-spaces.*

(i) *If there is a G-equivariant map from X to Y, then $i(X; R) \leq i(Y; R)$.*
(ii) *If $\tilde{H}^j(X; R) = 0$ for all $i \leq n$, then $i(X; R) \geq n + 1$.*
(iii) *If $H^j(Z; R) = 0$ for all $j \geq n-1$ and $i(Z; R) < \infty$, then $i(Z; R) \leq n$.*

3. Proof of Theorem 1.1 and Proposition 1.6

The proof of Theorem 1.1 will follow from the following lemmas

Lemma 3.1. *Let R be a PID, G a compact Lie group and X a free G-space Hausdorff, pathwise connected and paracompact. Suppose that for some natural $m \geq 1$, $i(X; R) \geq m + 1$. Then, for each k, $1 \leq k \leq m$, there exists the following exact sequence with coefficients in R,*

$$E_{k+1}^{0,k} \to H^{k+1}(BG; R) \stackrel{p_X^*}{\to} H^{k+1}(X_G; R). \tag{3.1}$$

Proof. Given the fibration $p_X : X_G = EG \times_G X \to BG$, it follows from Theorem 2.3 that there exists a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$, with

$$E_2^{p,q} \cong H^p(BG; \mathcal{H}^q(F)), \tag{3.2}$$

the cohomology of BG with local coefficients in the cohomology of X, the fibre of p_X, and coverging to $H^*(X_G; R)$.

Since X is pathwise connected the local coefficients system $\mathcal{H}^0(X)$ over BG is trivial and follows from [7, Proposition 5.18] that for all $p \geq 0$,

$$E_2^{p,0} \cong H^p(BG; \mathcal{H}^0(X)) = H^p(BG; H^0(X)) = H^p(BG; R). \tag{3.3}$$

On the other hand, since $i(X;R) \geq m+1$ follows that

$$E_2^{p,0} \cong E_3^{p,0} \cong \cdots \cong E_{m+1}^{p,0}, \text{ for all } p \geq 0.$$

Thus, in particular, for each k, $1 \leq k \leq m$,

$$H^{k+1}(BG; R) = E_2^{k+1,0} = E_3^{k+1,0} = \cdots = E_{k+1}^{k+1,0}. \tag{3.4}$$

Now, we consider the exact sequence

$$0 \to Ker\, d_r \to E_r^{0,r-1} \xrightarrow{d_r} E_r^{r,0} \to \frac{E_r^{r,0}}{Im\, d_r} \to 0,$$

where $Im(d_r : E_r^{0,r-1} \to E_r^{r,0})$. We have that,

$$Ker\, d_r = E_\infty^{0,r-1} \quad and \quad \frac{E_r^{r,0}}{Im\, d_r} = E_\infty^{r,0}.$$

Therefore, we obtain the sequence,

$$0 \to E_\infty^{0,r-1} \to E_r^{0,r-1} \xrightarrow{d_r} E_r^{r,0} \to E_\infty^{r,0} \to 0 \tag{3.5}$$

Also, we prove that there exists a natural injection,

$$0 \to E_\infty^{k+1,0} \hookrightarrow H^{k+1}(X_G), \text{ for all } k,\ 1 \leq k \leq m. \tag{3.6}$$

For this we consider the following decreasing filtration of $H^r(X_G, R)$,

$$0 = F^{r+1}(H^r(X_G)) \subset F^r(H^r(X_G)) \subset \ldots \subset F^1(H^r(X_G)) \subset F^0(H^r(X_G)) = H^r(X_G).$$

Since the spectral sequence $\{E_r^{*,*}; d_r\}$ converges to $H^*(X_G; R)$ as an algebra, we have that

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*(X_G)) = F^p(H^{p+q}(X_G))/F^{p+1}(H^{p+q}(X_G)),$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence and $E_0^{*,*}(H^r(X_G))$ is the module associated bigraduade.
Since $F^{r+1}(H^r(X_G)) = \{0\}$, it follows that

$$E_\infty^{r,0} \cong F^r(H^r(X_G))/F^{r+1}(H^r(X_G)) \cong F^r(H^r(X_G)) \subset H^r(X_G; R),$$

for any r, and this completes the existence of sequence in (3.6).
Taking $r = k+1$, for $1 \leq k \leq m$, and putting together (3.5) and (3.6), one obtains the exact sequence

$$E_{k+1}^{0,k} \xrightarrow{d_{k+1}} E_{k+1}^{k+1,0} \to H^{k+1}(X_G; R), \tag{3.7}$$

where d_{k+1} is the differential.
Since $H^{k+1}(BG; R) = E_{k+1}^{k+1,0}$ (eq.(3.4)), of the sequence (3.7) we obtain the desired sequence

$$E_{k+1}^{0,k} \xrightarrow{d_{k+1}} H^{k+1}(BG; R) \xrightarrow{p_X^*} H^{k+1}(X_G; R).$$

□

Lemma 3.2. *Let X be a free G-space, Hausdorff, pathwise connected and paracompact. Suppose that for some natural $m \geq 1$, $i(X; R) \geq m+1$ and let k be such that $1 \leq k \leq m$. Then*

(i) *If $k = m$ and if $\beta_m(X; R) < \beta_{m+1}(BG; R)$ then the homomorphism $h^* : H^{m+1}(BG; R) \to H^{m+1}(X/G; R)$ is nontrivial;*
(ii) *If $1 \leq k < m$ and if $0 < \beta_{k+1}(BG; R)$ then the homomorphism $h^* : H^{k+1}(BG; R) \to H^{k+1}(X/G; R)$ is nontrivial,*

where $h : X/G \to BG$ is a classifying map for the principal G-bundle $X \to X/G$.

Proof. Let $EG \to BG$ be the universal G-bundle and $h : X/G \to BG$ a classifying map for the principal G-bundle $X \to X/G$. Let $p_X : X_G \to BG$ the Borel-fibration associated to the G-space X, where X_G is the Borel space, as in (2.2). It follows from Remark (2.2) that the map $X_G \to X/G$ is a homotopy equivalence. Let $r : X/G \to X_G$ be its inverse homotopic. Then $p_X \circ r : X/G \to BG$ also classifies the principal G-bundle $X \to X/G$, and follows from Remark (2.1) that the map $(p_X \circ r)$ is homotopic to h. Since $r^* : H^{k+1}(X_G; R) \to H^{k+1}(X/G; R)$ is an isomorphism, it suffices to prove that $p_X^* : H^{k+1}(BG; R) \to H^{k+1}(X_G; R)$ is nontrivial for $1 \leq k \leq m$. In fact, since $i(X; R) \geq m+1$, it follows from Lemma 3.1 that for $1 \leq k \leq m$ there exists an exact sequence with coefficients in R,

$$E_{k+1}^{0,k} \xrightarrow{d_{k+1}} H^{k+1}(BG; R) \xrightarrow{p_X^*} H^{k+1}(X_G; R). \tag{3.8}$$

Suppose that $p_X^* : H^{k+1}(BG; R) \to H^{k+1}(X_G; R)$ is the zero homomorphism for $1 \leq k \leq m$. From (3.8), we have that $d_{k+1} : E_{k+1}^{0,k} \longrightarrow H^{k+1}(BG)$ is a surjective homomorphism for $1 \leq k \leq m$.

If $k = m$, since $d_{m+1} : E_{m+1}^{0,m} \longrightarrow H^{m+1}(BG)$ is a surjective homomorphism we have that

$$\text{rank } (E_{m+1}^{0,m}) \geq \text{rank } (H^{m+1}(BG; R)) = \beta_{m+1}(BG; R). \tag{3.9}$$

On the other hand, since $E_{m+1}^{0,m}$ is isomorphic to a submodule of $H^0(BG;\mathcal{H}^m(X))$ and $H^0(BG;\mathcal{H}^m(X))$ is isomorphic to a submodule of $H^m(X;R)$ [12, Theorem 3.2] then $E_{m+1}^{0,m}$ is isomorphic to a submodule of $H^m(X;R)$. Therefore,

$$\beta_m(X;R) = \text{rank } (H^m(X;R)) \geq \text{rank } (E_{m+1}^{0,m}) \overset{(3.9)}{\geq} \beta_{m+1}(BG;R),$$

which contradicts the hypothesis $\beta_m(X;R) < \beta_{m+1}(BG;R)$.

If $1 \leq k < m$ and $0 < \beta_{k+1}(BG;R)$, since

$$d_{k+1} : E_{k+1}^{0,k} \to E_{k+1}^{k+1,0} = H^{k+1}(BG;R)$$

is surjective, it follows that $d_{k+1} : E_{k+1}^{0,k} \to E_{k+1}^{k+1,0}$ is nontrivial, which contradicts the hypothesis $i(X;R) \geq m+1$.

Thus, the homomorphism $h^* : H^{k+1}(BG;R) \to H^{k+1}(X/G;R)$ is nontrivial for $1 \leq k \leq m$. □

Proof of Theorem 1.1. Suppose that $f : X \to Y$ is a G-equivariant map. Since Y is a Hausdorff paracompact space, one can take a classifying map $g : Y/G \to BG$ for the principal G-bundle $Y \to Y/G$. Then the map $h = g \circ \bar{f} : X/G \to BG$ can be taken as a classifying map for the principal G-bundle $X \to X/G$, where $\bar{f} : X/G \to Y/G$ is the map induced by f between the orbit spaces. Since by hypothesis $H^{k+1}(Y/G;R) = 0$ for some k, $1 \leq k \leq m$, one has that $g^* : H^{k+1}(BG;R) \to H^{k+1}(Y/G;R)$ is trivial and consequently $h^* : H^{k+1}(BG;R) \to H^{k+1}(X/G;R)$ is the zero homomorphism for some k, $1 \leq k \leq m$, which contradicts Lemma 3.2. □

Next, we present the computation of the index $i(X;\mathbb{Z}_p)$ (resp. $i(X;\mathbb{Q})$) for the class of spaces $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$) by using the same arguments to that in the proof of [5, Theorems 1, 2, and 3].

Proof of Proposition 1.6. Consider $G = \mathbb{Z}_p$, p odd prime. For $G = \mathbb{Z}_2$ or S^1 the proof is analogous by using similar arguments to that of [5, Theorems 2 and 3] . Since there are no fixed points, it follows from assumptions that the Leray-Serre spectral sequence of the map $p_X : X_G \to BG$ does not collapse at the E_2-term and $E_2^{k,l} = H^k(BG) \otimes H^l(X)$. Let $r \geq 2$ be the smallest integer such that $d_r \neq 0$ and, in particular, $E_2^{*,0} = \cdots = E_r^{*,0}$. By the multiplicative properties of the spectral sequence, we have $d_r(1 \otimes v_1) \neq 0$ or $d_r(1 \otimes v_2) \neq 0$ and the only possibilities for values of r are $r = m+1$, $r = n+1$ or $r = n-m+1$. Suppose, first, that $d_r(1 \otimes v_1) \neq 0$. Then $r = m+1$ and m must be odd. So we can write $d_{m+1}(1 \otimes v_1) = t^{(m+1)/2} \otimes 1$. Now, we either have $d_{m+1}(1 \otimes v_2) = 0$ or $n = m$ and $d_{m+1}(1 \otimes v_2) = at^{m+1} \otimes 1$, $0 \neq a \in \mathbb{Z}_p$.

For $n \neq m$, obviously, $d_{m+1}(1 \otimes v_2) = 0$ and $d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes v_2$. Thus the differentials

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}, \text{ and}$$
$$d_{m+1} : E_{m+1}^{k,m+n} \to E_{m+1}^{k+m+1,n}$$

are isomorphisms.

If $n = m$ and $d_{m+1}(1 \otimes v_2) = at^{(m+1)/2} \otimes 1$, $a \in \mathbb{Z}_p$, then $d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes (v_2 - av_1)$. So the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is surjective. In both cases, for all $k \geq 0$

$$E_{m+2}^{k+m+1,0} = \frac{\ker d_{m+1}}{Im\, d_{m+1}} = \frac{E_{m+1}^{k+m+1,0}}{E_{m+1}^{k+m+1,0}} = 0 \neq E_{m+1}^{k+m+1,0}.$$

Thus,

$$E_2^{*,0} = \cdots = E_{m+1}^{*,0} \neq E_{m+2}^{*,0},$$

which implies $i(X;\mathbb{Z}_p) = m+1$ and we are in case (i).

Suppose, now, that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$. We then have either $r = n-m+1$ and $d_r(1 \otimes v_2) = A \otimes v_1$ or $r = n+1$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(BG)$. In the former case, we must have m even and n odd (for details, see [5, proof of Theorem 1, p. 924] so that we can write $d_{n-m+1}(1 \otimes v_2) = t^{(n-m+1)/2} \otimes v_1$. It follows that the differential

$$d_{n-m+1} : E_{n-m+1}^{*,n} \to E_{n-m+1}^{*,m}$$

is an isomorphism and $d_{n-m+1}(E_{n-m+1}^{*,m}) = 0 = d_{n-m+1}(E_{n-m+1}^{*,m+n})$. So we have $E_r^{k,n} = 0 = E_r^{k+n-m+1,m}$, $E_r^{k,m+n} = E_2^{k,m+n}$ and $E_r^{k,0} = E_2^{k,0}$ for all $k \geq 0$ and $r = n-m+2$. It is easily seen that the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is trivial for $0 \leq k \leq n-m$, since $E_{m+1}^{k,m} = E_2^{k,m}$. Because there are no fixed points, the differential $d_{n+m+1} : E_{n+m+1}^{0,m+n} \to E_{n+m+1}^{n+m+1,0}$ must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_3) = t^{(n+m+1)/2} \otimes 1$. Then, the differential

$$d_{n+m+1} : E_{n+m+1}^{k,m+n} \to E_{n+m+1}^{k+m+n+1,0}$$

is an isomorphism and for all $k \geq 0$

$$E_{m+n+2}^{k+m+n+1,0} = \frac{\ker d_{m+n+1}}{Im\, d_{m+n+1}} = \frac{E_{m+n+1}^{k+m+n+1,0}}{E_{m+n+1}^{k+m+n+1,0}} = 0 \neq E_{m+n+1}^{k+m+n+1,0}.$$

Thus,

$$E_2^{*,0} = \cdots = E_{m+n+1}^{*,0} \neq E_{m+n+2}^{*,0},$$

which implies $i(X;\mathbb{Z}_p) = m+n+1$ and we are in case (iii).

Finally, consider the possibility $r = n+1$, and in this case $d_r(1\otimes v_1) = 0$ and $d_r(1\otimes v_2) = A\otimes 1$, $0 \neq A \in H^*(BG)$. Then n must be odd and we can set $d_{n+1}(1\otimes v_2) = t^{(n+1)/2}\otimes 1$. So $d_{n+1}(1\otimes v_3) = \pm t^{(n+1)/2}\otimes v_1$; consequently the differentials

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0};$$
$$d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k+n+1,m}$$

are isomorphisms and for all $k \geq 0$

$$E_{n+2}^{k+n+1,0} = \frac{\ker d_{n+1}}{Im\, d_{n+1}} = \frac{E_{n+1}^{k+n+1,0}}{E_{n+1}^{k+n+1,0}} = 0 \neq E_{n+1}^{k+n+1,0}.$$

Thus,

$$E_2^{*,0} = \cdots = E_{n+1}^{*,0} \neq E_{n+2}^{*,0},$$

which implies $i(X;\mathbb{Z}_p) = n+1$ and we are in case (ii).

□

4. Some considerations on Theorem 1.1

Clapp and Puppe proved in [3] the following theorem:

Theorem 4.1. *Let G be a p-torus or a torus (that is, $G = \mathbb{Z}_p^n$ or $G = S^1 \times \ldots \times S^1$). Let X and Y be G-spaces with fixed-points-free actions; moreover, in the case of a torus action assume additionally that Y has finitely many orbit types. Suppose that $\tilde{H}^i(X) = 0$ for $i < N$, Y is compact or paracompact and finite-dimensional, and $H^i(Y) = 0$ for $i \geq N$; here cohomology is considered with coefficients in $\mathbb{Z}_p$ in the case of a p-torus action and in $\mathbb{Q}$ in the case of a torus action. Then there exists no G-equivariant map of X into Y.*

Remark 4.2. Let $G = S^1 \times S^1$, $X = S^5 \times S^5$ and $Y = S^3 \times S^3$, which admit free action of G. One has that $H^q(X;\mathbb{Z}) = 0$, for $0 < q < m = 5$ and thus $i(X) \geq 5+1$. Moreover, $H^6(Y/G;\mathbb{Z}) = 0$, since dim(Y/G) = 4, and $B(S^1\times S^1) = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, which implies $\beta_5(X;\mathbb{Z}) = 2 < \beta_6(BG;\mathbb{Z}) = 4$. It follows from Corollary 1.3 that there is no G-equivariant map $f : X \to Y$. One has that $\tilde{H}^i(X;\mathbb{Q}) = 0$, for $i < N = 5$ and $H^6(Y;\mathbb{Q}) \neq 0$. Then, in this example, it does not valid $H^i(Y;\mathbb{Q}) = 0$ for $i \geq 5$. Therefore, we can not say that there is no G-equivariant map from X to Y, which does not happen if we use the Corollary 1.3. This shows that, in some situations, the Theorem 1.1 is more efficient than the Theorem 4.1, although the results are not comparable due to their hypothesis.

More generally, in the special case where $G = \mathbb{Z}_p^n$, X and Y are free G-spaces, the Theorem 1.1 extends Theorem 4.1 since the condition $\tilde{H}^i(X;\mathbb{Z}_p) = 0$ for $i < N$ implies $i(X;\mathbb{Z}_p) \geq N+1 > N$ and $\beta_{N-1}(X;\mathbb{Z}_p) = 0 < \beta_N(BG;\mathbb{Z}_p)$, the condition Y compact or paracompact and finite-dimensional and $H^i(Y;\mathbb{Z}_p) = 0$ for $i \geq N$ implies $H^N(Y/G;\mathbb{Z}_p) = 0$ ([9, Proposition A.11]). Thus, taking $m = N-1$ in the Theorem 1.1, we conclude that there is no G-equivariant map of X into Y.

A similar result to Theorems 1.1 and 4.1 is the following theorem:

Theorem 4.3. *Let R be a PID, G a compact Lie group, X a free G-space Hausdorff paracompact and Y a G-space Hausdorff compact or paracompact finite-dimensional which admits a free G-action. Suppose that $i(X;R) \geq m+1$, $i(Y;R) < \infty$ and $H^i(Y;R) = 0$, for $i \geq m$. Then, there is no G-equivariant map $f : X \to Y$.*

Proof. Suppose that there exists G-equivariant map $f : X \to Y$. Then, by Proposition 2.4(i), $i(X;R) \leq i(Y;R)$. Since $i(Y;R) < \infty$ and $H^i(Y;R) = 0$ for $i \geq m$, it follows from Proposition 2.4(iii) that $i(Y;R) \leq m$. Therefore, $i(X;R) \leq i(Y;R) \leq m$, which is a contradiction. □

Acknowledgements. We are grateful to the referee for his careful reading and helpful comments and suggestions concerning the presentation of this article, which led to this present version. The authors express their thanks to Professor Carlos Biasi of the University of São Paulo and to Professor Pedro L. Q. Pergher of the Federal University of São Carlos for their helpful comments.

References

[1] C. Biasi, D. de Mattos, *A Borsuk-Ulam theorem for compact Lie group actions*, Bull Braz Math Soc, New Series 37(1) (2006) 127-137.

[2] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, INC., New York and London (1972).

[3] M. Clapp and D. Puppe, *Critical point theory with symmetries*, J. Reine Angew. Math. 418 (1991), 1-29.

[4] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin-New York (1987).

[5] Dotzel, R.M., Singh, T.B. and Tripathi S.P., *The cohomology Rings of the orbit spaces of free transformation groups of the Product of Two Spheres*, Proc. of the Amer. Math. Soc., **129** (2000) 921-930.

[6] De Mattos, D. and Dos Santos, E.L. *A parametrized Borsuk-Ulam theorem for a product of spheres with free $\mathbb{Z}_p$-action and free S^1-action.* Algebraic and Geometric Topology **7** 1791-1804, 2007.

[7] J. McCleary, *User's Guide to Spectral Sequences*, Mathematics Lectures Series, Publish or Perish, Inc., Wilmington, Delaware (U.S.A.) (1985).

[8] Pergher, P.L.Q., De Mattos, D. and Dos Santos, E.L. *The Borsuk-Ulam Theorem for General Spaces.* Arch. Math. **81** n.1 (2003) 96-102.

[9] D. Quillen, *The spectrum of an equivariant cohomology ring: I*, Ann. of Math., 94 n.3 549-572, (1971).

[10] A. Yu. Volovikov, *On the van Kampen-Flores theorem*, Mat. Zametki 59 (1996), 663-670; English transl., Math. Notes 59 (1996), 477-481.

[11] A. Yu. Volovikov, *On the index of G-spaces*, Sb. Math. 191 (9-10) (2000) 1259-1277.

[12] G.W. Whitehead, *Elements of Homotopy Theory*, Springer Verlag, New York, Heidelberg, Berlin (1978).

E-mail address: `francielle@famat.ufu.br`

E-mail address: `deniseml@icmc.usp.br`

E-mail address: `edivaldo@dm.ufscar.br`