GEVREY VECTORS IN INVOLUTIVE TUBE STRUCTURES AND GEVREY REGULARITY FOR THE SOLUTIONS TO CERTAIN CLASSES OF SEMILINEAR SYSTEMS

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Abstract. In this work we introduce the notion of s-Gevrey vectors in locally integrable structures of tube type. Under the hypothesis of analytic hypoellipticity we study the Gevrey regularity of such vectors and also show how this notion can be applied to the study of the Gevrey regularity of solutions to certain classes of semilinear equations.

1. Introduction

Let $P = P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be an analytic linear partial differential operator, of order $m \geq 1$, defined in $\Omega \subset \mathbb{R}^N$ open. Given $s \geq 1$ we say that a distribution $u \in \mathcal{D}'(\Omega)$ is an *s-Gevrey vector for P* if $P^k u \in L^2_{\mathrm{loc}}(\Omega)$ for every $k = 0, 1, \dots$ and for every $K \subset \Omega$ compact there is $C = C(K) > 0$ such that

$$\|P^k u\|_{L^2(K)} \leq C^{k+1} k!^{ms}, \quad k = 0, 1, 2, \dots .$$

An 1-Gevrey vector for P is also referred to as an *analytic vector for P*.

We denote by $G^s(\Omega; P)$ the space of all s-Gevrey vectors for P in Ω. Since $\{u \in L^2_{\mathrm{loc}}(\Omega) : Pu \in G^s(\Omega)\} \subset G^s(\Omega; P)$ it is quite natural to investigate the regularity properties of the elements in $G^s(\Omega; P)$, as a generalization of the study of the regularity properties for the solutions to the equation $P(x, D)u = f$.

This problem has now a long history. If P is elliptic in Ω and if $s \geq 1$ then $G^s(\Omega; P) \subset G^s(\Omega)$ (cf. [KN, 1962], [LM, 1968]; for the microlocal version see [BCM, 1978]) A partial converse of this result holds: if there is $s > 1$ such that $G^s(\Omega; P) \subset G^s(\Omega)$ then P is elliptic [M, 1978].

A complete answer to this problem is also known in the case of principal type operators [BM, 1982]: if P is of principal type and hypoelliptic (and then subelliptic and analytic-hypoelliptic, according to [T, 1971]), if $U \subset\subset \Omega$ and if $s \geq 1$ then $G^s(U; P) \subset G^{s'}(U)$, where $s' = (sm - \delta)/(m - \delta)$. Here $0 \leq \delta < 1$ is the subellipticity index of P over U. Moreover this result is sharp.

At first glance one could expect that the inclusion $G^1(\Omega; P) \subset C^\omega(\Omega)$ holds when $P(x, D)$ is analytic hypoelliptic, but this is not true in general. An example is provided by the second order operator $M = \partial_t^2 + t^2\partial_x^2$ defined in $\mathbb{R}^2$. It is well know that M is analytic hypoelliptic in $\mathbb{R}^2$ [Mat, 1971] but we have $G^1(U; M) \not\subset G^\sigma(U)$ if $1 \leq \sigma < 2$ and U is an

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open neighborhood of the origin (cf. [G, 1969], where similar results concerning degenerate elliptic equations are presented). See also the survey [BCR, 1987] and references therein.

Similar results are known for certain classes of system of vector fields. Let $X_1, \dots, X_p$ be real-valued, analytic vector fields on $\Omega \subset \mathbb{R}^N$ open. Set $X^\alpha = X_{\alpha_1} \dots X_{\alpha_\ell}$ if $\alpha = (\alpha_1 \dots, \alpha_\ell)$, $\alpha_j \in \{1, \dots, p\}$, and also $|\alpha| = \ell$. If the system X_j is *elliptic*, which imposes necessarily that $p = N$, then it is a classical result [N, 1959] that any $u \in C^\infty(\Omega)$ which satisfies, on any compact subset K of Ω, estimates of the kind

$$\|X^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1} |\alpha|! \tag{$*$}$$

is a real-analytic function on Ω. The same conclusion can be derived if instead of $(*)$ we assume the validity of the following weaker property

$$\|X_\nu^j u\|_{L^2(K)} \leq C_K^{j+1} j!, \quad 1 \leq \nu \leq p, \quad j = 0, 1, 2, \dots \tag{$**$}$$

(cf. [Br, 1961], [Da, 1979]). These results can be extended in the following way: assume now that $X_1, \dots, X_p$ satisfy *Hörmander's condition*: there is $r < \infty$ such that their brackets of length $\leq r$ span $\mathrm{T}_x\Omega$, $\forall x \in \Omega$. Then, if $u \in C^\infty(\Omega)$ is such that estimates $(*)$ hold for every $K \subset\subset \Omega$, then we can also conclude that $u \in C^\omega(\Omega)$ [HM, 1980]. If, moreover, the vector fields $X_1, \dots, X_p$ generate a stratified nilpotent Lie algebra of rank 2 then, for $s \geq 1$, the inequalities

$$\|X^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1} \left(|\alpha|!\right)^s$$

imply that $u \in G^{1+(s-1)r}(\Omega)$ [DaH, 1980].

Motivated by these results we introduce, in the present work, the notion of s-Gevrey vectors for systems of complex vector fields associated to a real-analytic, locally integrable structure of tube type [T, 1992], [BCH, 2008] (cf. also Section 2 for the precise definition). We work under the hypothesis of analytic hypoellipticity, a property that for this particular class of structures has a simple characterization, thanks to a well know result due to Baouendi–Treves [BT, 1982]. We are able to prove that, under these conditions, every analytic vector (that is, 1-Gevrey vector) for such systems is a real-analytic function (Theorem 3.1 below).

On the other hand, the case $s > 1$ is more difficult. Although it is quite simple to extend the result of Metivier proved in [M, 1978] to our particular situation (Proposition 2.2), an example of Maire [Ma, 1980] treated in Theorem 3.2 shows that analytic hypoellipticity is not enough to guarantee some Gevrey regularity for the s-Gevrey vectors we study here. For this reason we assume, in the main result of this work (Theorem 4.1), that our locally integrable structure is analytic hypoelliptic and of corank one. Theorem 4.1 states that: (i) every such system is Gevrey hypoelliptic of order s, for every $s > 1$; (ii) every s-Gevrey vector for the system is a Gevrey function of order s', where $s' > s$ is given in terms of s and of an invariant attached to the structure. For the proof of (ii) we introduce a sequence of operators acting on the Fock spaces specially built for the structure, and construct for them left parametrices by integrating tensors over appropriate families of parametrized curves. We were very much inspired by the work [BM, 1982] in order to derive the correct estimates for such parametrices (cf. Proposition 8.1 below).

In Section 9 we prove the Gevrey regularity for solutions to certain semilinear systems built from the vector fields we work with. This is achieved by showing that such solutions are s-Gevrey vectors for our systems (cf. Corollary 9.1 below). Finally, in Section 10, we close the work with some remarks; of particular interest is the discussion of how our results connect with the very much recent work done in [JT, 2006], [D, 2006], [DH, 2008] on the subellipticity for such corank one systems.

2. Preliminaries - Gevrey vectors on tube structures

Let Θ be an open ball centered at the origin in $\mathbb{R}^n$. We shall assume given a real-analytic map defined in a neighborhood of the closure of Θ and valued in $\mathbb{R}^m$, $\Phi = (\phi_1, \dots, \phi_m)$, satisfying $\Phi(0) = 0$.

We introduce the complex vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(t) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, n. \tag{1}$$

Then $\mathfrak{L} \doteq \{\mathrm{L}_1, \dots, \mathrm{L}_n\}$ spans a complex vector subbundle of $\mathbb{C}\mathrm{T}(\mathbb{R}^m \times \Theta)$ of rank n. This vector subbundle defines an *involutive tube structure*[1] on $\mathbb{R}^m \times \Theta$; its orthogonal is the vector subbundle of $\mathbb{C}\mathrm{T}^*(\mathbb{R}^m \times \Theta)$ spanned by the differentials of the real-analytic functions

$$Z_k(x,t) = x_k + i\phi_k(t), \quad k = 1, \dots, m. \tag{2}$$

Notice that $[\mathrm{L}_j, \mathrm{L}_{j'}] = 0$ for all $j, j' = 1, \dots, n$; moreover

$$\mathbb{C}\mathrm{T}(\mathbb{R}^m \times \Theta) = \operatorname{span}\left\{\mathrm{L}_1, \dots, \mathrm{L}_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right\} \tag{3}$$

and

$$\left[\mathrm{L}_j, \frac{\partial}{\partial x_k}\right] = 0, \quad j = 1, \dots, n, \ k = 1, \dots, m.$$

We now introduce our main definition, in which we let $s \geq 1$ and write $\mathrm{L}^\alpha = \mathrm{L}_1^{\alpha_1} \dots \mathrm{L}_n^{\alpha_n}$ if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$.

Definition 2.1. *Let $\Omega = U \times V$ be an open subset of $\mathbb{R}^m \times \Theta$ and let $u \in \mathcal{D}'(\Omega)$. We say that u is an s-Gevrey vector for $\mathfrak{L}$ if $\mathrm{L}^\alpha u \in L^\infty_{\mathrm{loc}}(V, L^1_{\mathrm{loc}}(U))$ for all $\alpha \in \mathbb{Z}_+^n$ and, for every $K = K_1 \times K_2 \subset \Omega$ compact there is $C_K > 0$ such that*

$$\|\mathrm{L}^\alpha u\|_{L^\infty(K_2, L^1(K_1))} \leq C_K^{|\alpha|+1} \alpha!^s, \quad \alpha \in \mathbb{Z}_+^n. \tag{4}$$

[1] For the invariant characterization of an involutive tube structure see ([T, 1992], Theorem VI.9.3).

We shall denote by $G^s(\Omega;\mathfrak{L})$ the space of all s-Gevrey vectors for $\mathfrak{L}$ defined in Ω. An element in $G^1(\Omega;\mathfrak{L})$ is also called an *analytic vector for* $\mathfrak{L}$.

Proposition 2.1. *If $u \in L^\infty_{\mathrm{loc}}(V, L^1_{\mathrm{loc}}(U))$ is such that $\mathrm{L}_j u \in G^s(\Omega)$, $j = 1, \dots, n$, then $u \in G^s(\Omega;\mathfrak{L})$.*

Proof: Since the vector fields L_j are real-analytic, since (3) holds and since the vector fields in (3) are pairwise commuting we can characterize the Gevrey class of order $s \geq 1$ as the space of all smooth functions f in Ω such that given any compact set $K_1 \times K_2 \subset \Omega$ there is $A > 0$ such that

$$\|\mathrm{L}^\alpha \partial_x^\beta f\|_{L^\infty(K_1\times K_2)} \leq A^{|\alpha|+|\beta|+1}(\alpha!\beta!)^s, \quad \alpha \in \mathbb{Z}^n_+,\ \beta \in \mathbb{Z}^m_+.$$

In particular, for every $K_1 \times K_2 \subset \Omega$ compact there is $A_1 > 0$ such that

$$\|\mathrm{L}^\alpha \mathrm{L}_j u\|_{L^\infty(K_1\times K_2)} \leq A_1^{|\alpha|+1}\alpha!^s, \quad \alpha \in \mathbb{Z}^n_+,\ j = 1, \dots, n,$$

because $\mathrm{L}_j u \in G^s(\Omega)$ for all j. Since $\|\cdot\|_{L^\infty(K_2, L^1(K_1))} \leq |K_1|\,\|\cdot\|_{L^\infty(K_1\times K_2)}$ we obtain (4) when $|\alpha| > 0$ and $C_K = \max\{|K_1|, A_1\}$.

Finally we have $u|_{K_1\times K_2} \in L^\infty(K_2, L^1(K_1))$ and consequently (4) holds for every $\alpha \in \mathbb{Z}^n_+$ and for $C_K = \max\{\max\{|K_1|, A_1\}, \|u\|_{L^\infty(K_2, L^1(K_1))}\}$. ■

We denote the dual coordinates to $(x,t) \in \mathbb{R}^m \times \Theta$ as (ξ,τ), $\xi = (\xi_1, \dots, \xi_m)$, $\tau = (\tau_1, \dots, \tau_n)$. Under this notation we can write the principal symbol of L_j as

$$\sigma(\mathrm{L}_j)(x,t,\xi,\tau) = i\tau_j + \sum_{k=1}^m \frac{\partial \phi_k}{\partial t_j}(t)\,\xi_k\,.$$

If $u \in \mathcal{D}'(\Omega)$ is an s-Gevrey vector for $\mathfrak{L}$ then, "a fortiori", u is s-Gevrey vector for each L_j. It then follows from [BCM, 1978] that the G^s-wave-front of u is contained in the *characteristic set of $\mathfrak{L}$ over Ω*:

$$\begin{aligned} \mathcal{C}(\mathfrak{L})|_\Omega &= \bigcap_{j=1}^n \{(x,t,\xi,\tau) \in \Omega \times \left(\mathbb{R}^{m+n} \setminus \{0\}\right) :\ \sigma(\mathrm{L}_j)(x,t,\xi,\tau) = 0\} \\ &= \{(x,t,\xi,0) :\ x \in U,\, t \in V,\ \xi \neq 0,\ {}^tD\Phi(t)\cdot\xi = 0\}\,. \end{aligned}$$

Corollary 2.1. *If $\mathfrak{L}$ is elliptic and if $s \geq 1$ then $G^s(\Omega;\mathfrak{L}) = G^s(\Omega)$.*

Corollary 2.2. *Let u be defined in $\Omega = U \times V$ and belong to $G^s(\Omega;\mathfrak{L})$. Assume also that the following property holds, for some $s' \geq s$: given $x_0 \in U$ and $K_2 \subset V$ compact there are a sequence $\{\psi_N\} \subset C^\infty_c(U)$, each ψ_N equal to one in a neighborhood of x_0, and a constant $C > 0$ such that*

$$|\widehat{(\psi_N u)}(\xi,t)| \leq C^{N+1}N!^{s'}/|\xi|^N, \qquad t \in K_2,\ N \in \mathbb{Z}_+, \tag{5}$$

where the "hat" means partial Fourier transform with respect to x. Then $u \in G^{s'}(\Omega)$.

Proof. According to our preceding discussion it suffices to show that given $(x_0,t_0) \in \Omega$ and $\xi_0 \neq 0$ then $(x_0,t_0,\xi_0,0) \notin WF_{s'}(u)$. Let then $\chi \in C^\infty_c(V)$ be identically one in a neighborhood of t_0, $0 \leq \chi \leq 1$, and select $\{\psi_N\}$ and $C > 0$ as in the statement, corresponding to $K_2 \doteq \operatorname{supp}\chi$. Then

$$|\mathcal{F}\left[(\psi_N \otimes \chi)u\right](\xi,\tau)| \le |K_2| \sup_{t\in K_2} |\widehat{(\psi_N u)}(\xi,t)| \le |K_2| C^{N+1} N!^{s'}/|\xi|^N .$$

Noticing finally that $(\xi_0, 0)$ belongs to the cone Γ defined by $|\tau| < |\xi|$ and noticing further that $|(\xi,\tau)| \le \sqrt{2}|\xi|$ when $(\xi,\tau)\in\Gamma$ we conclude that

$$|\mathcal{F}\left[(\psi_N \otimes \chi)u\right](\xi,\tau)| \le |K_2| 2^{N/2} C^{N+1} N!^{s'}/|(\xi,\tau)|^N, \quad (\xi,\tau)\in\Gamma,$$

which gives the conclusion (cf. [H, 1983], Definition 8.4.3). ■

When $s > 1$ there is a converse for the statement in Corollary 1, a result due to Metivier [M, 1978] in the case of scalar operators. More precisely:

Proposition 2.2. *Assume that $\mathfrak{L}$ is not elliptic. Then given s, s' satisfying*

$$1 < s \le s' < 2s - 1 \tag{6}$$

there is an s-Gevrey vector for $\mathfrak{L}$ which is not a Gevrey function of order s'.

Proof. By hypothesis there are $t_0 \in \Theta$ and $\xi_0 \in \mathbb{R}^m$, $|\xi_0| = 1$, such that ${}^t D\Phi(t_0)\cdot\xi_0 = 0$. Without lost of generality we can assume $t_0 = 0$. Hence $\mathrm{d}(\Phi\cdot\xi_0) = 0$ at $t = 0$ and consequently there is a constant $C > 0$ such that $|\Phi(t)\cdot\xi_0| \le C|t|^2$ when $t\in\Theta$.

Let $\alpha \in]0,1[$ be such that $s' < 1/\alpha < 2s-1$ and set $\sigma \doteq s - (1-\alpha)/(2\alpha)$. Then $\sigma > 1$. Select next $\zeta \in G^\sigma_c(\mathbb{R}^n)$ satisfying $\zeta(0) = 1$ and supported in the ball $|t| \le \rho$, with $\rho < \min\{r, 1/\sqrt{C}\}$, where r is the radius of Θ. We then set

$$u(x,t) = \int_1^\infty e^{i\lambda Z(x,t)\cdot\xi_0 - \lambda^\alpha}\zeta\left(\lambda^{(1-\alpha)/2}t\right)\mathrm{d}\lambda.$$

Notice that when $\lambda^{(1-\alpha)/2}t \in \operatorname{supp}\zeta$ then

$$\lambda|\phi(t)\cdot\xi_0| \le C\lambda\,|t|^2 \le C\rho^2\lambda^\alpha$$

and consequently u is well defined and smooth in $\mathbb{R}^m\times\Theta$.

We have

$$\begin{aligned}\left\{(\xi_0\cdot D_x)^k u\right\}(0,0) &= \int_1^\infty \lambda^k e^{-\lambda^\alpha}\,\mathrm{d}\lambda \\ &= \frac{1}{\alpha}\Gamma\left(\frac{k+1}{\alpha}\right) - \int_0^1 \lambda^k e^{-\lambda^\alpha}\,\mathrm{d}\lambda\,.\end{aligned}$$

Since the last term in the right is bounded in k it follows, from the asymptotic behaviour of the Gamma function, that u is not of Gevrey class τ near the origin for any $\tau < 1/\alpha$.

On the other hand, if $\gamma \in \mathbb{Z}^n_+$,

$$\mathrm{L}^\gamma u(x,t) = \int_1^\infty \lambda^{(1-\alpha)|\gamma|/2} e^{i\lambda Z(x,t)\cdot\xi_0 - \lambda^\alpha}\,\zeta^{(\gamma)}(\lambda^{(1-\alpha)/2}\,t)\,\mathrm{d}\lambda.$$

and hence

$$|\mathrm{L}^\gamma u(x,t)| \le C_1^{|\gamma|+1}\gamma!^\sigma \int_1^\infty \lambda^{(1-\alpha)|\gamma|/2} e^{-(1-C\rho^2)\lambda^\alpha}\,\mathrm{d}\lambda\ \le C_2^{|\gamma|+1}\gamma!^{\sigma+(1-\alpha)/(2\alpha)} = C_2^{|\gamma|+1}\gamma!^s$$

for suitable constants $C_1, C_2 > 0$.

Then $u \in G^s(\mathbb{R}^m \times \Theta; \mathfrak{L}) \setminus G^{s'}(\mathbb{R}^m \times \Theta)$ and the proof is complete. ■

3. Analytic vectors for $\mathfrak{L}$

We start by recalling a result due to M.S.Baouendi and F.Treves [BT, 1982]. If $\xi_0 \in \mathbb{R}^m \setminus 0$ then the origin is not a local minimum of the function $t \mapsto \Phi(t) \cdot \xi_0$ if and only if $(0, \xi_0) \notin WF_a(h(\cdot, 0))$ for every solution h to the system $\mathrm{L}_j h = 0$, $j = 1, \dots, n$, defined in a neighborhood of the origin. Motivated by this result we introduce the following property:

$$(\star) \qquad \forall \xi \in \mathbb{R}^m,\ \xi \neq 0, \text{ the map } \Phi \cdot \xi : \Theta \to \mathbb{R} \text{ is open.}$$

We now prove our first main result.

Theorem 3.1. *If $\mathfrak{L}$ satisfies property ($\star$) then every analytic vector for $\mathfrak{L}$ is a real-analytic function.*

Proof. Let Ω and u be as in Definition 1. If $y = (y_1, \dots, y_n)$ denotes a new variable in $\mathbb{R}^n$ we define

$$v(x, y, t) = \sum_{\alpha} \frac{(\mathrm{L}^\alpha u)(x, t)}{\alpha!} (-iy)^\alpha. \tag{7}$$

Let $U' \times V' \subset\subset U \times V$. By (4) there is $r > 0$ such that v defines a real-analytic function on $D = \{y \in \mathbb{R}^n : |y| < r\}$ valued in $L^\infty(V', L^1(U'))$ satisfying, for some constant $A > 0$,

$$\|\partial^\beta v(\cdot, y, \cdot)\|_{L^\infty(V', L^1(U'))} \leq A^{|\beta|+1} \beta!, \quad \beta \in \mathbb{Z}_+^n,\ y \in D. \tag{8}$$

Let $\mathcal{U} \doteq U' \times D \times V'$ and denote the dual coordinates to (x, y, t) by (ξ, η, τ). We now claim:

$$WF_a(v) \subset \mathcal{U} \times \{(\xi, 0, \tau) : (\xi, \tau) \neq 0\}. \tag{9}$$

Proof of (9). Fix $\chi \in C_c^\infty(U' \times V')$, $\chi = 1$ near (x_0, t_0) and let $\psi_N \in C_c^\infty(D)$, $N = 1, 2, \dots$, $\psi_N = 1$ near y_0, satisfy

$$|D^\alpha \psi_N| \leq (C_1 N)^{|\alpha|}, \quad |\alpha| \leq N.$$

Let $\beta \in \mathbb{Z}_+^n$ be such that $|\beta| = N$. Then

$$\eta^\beta \mathcal{F}(\chi(x,t)\psi_N(y)v)(\xi, \eta, \tau) = \int e^{-i(x\cdot\xi + y\cdot\eta + t\cdot\tau)} \chi(x,t) D_y^\beta(\psi_N(y) v(x, y, t))\, \mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$= \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int e^{-i(x\cdot\xi + y\cdot\eta + t\cdot\tau)} \chi(x,t) D_y^{\beta-\gamma}\psi_N(y) D_y^\gamma v(x, y, t)\, \mathrm{d}x\mathrm{d}y\mathrm{d}t,$$

which implies, for some constant $C_\bullet > 0$,

$$\begin{aligned}\left|\eta^\beta \mathcal{F}(\chi(x,t)\psi_N(y)v)(\xi,\eta,\tau)\right| &\leq C_\bullet \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} (C_1 N)^{|\beta-\gamma|} \sup_{y\in D} \|\partial^\gamma v(\cdot,y,\cdot)\|_{L^\infty(V',L^1(U'))} \\ &\leq C_\bullet \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} (C_1 N)^{|\beta-\gamma|} A^{|\gamma|+1}\gamma! \\ &\leq C_\bullet \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} (C_1 N)^{|\beta-\gamma|} A^{|\gamma|+1}|\gamma|^{|\gamma|}.\end{aligned}$$

Consequently, with a new constant $C_2 > 0$,

$$\left|\eta^\beta \mathcal{F}(\chi(x,t)\psi_N(y)v)(\xi,\eta,\tau)\right| \leq C_2^{N+1}N^N.$$

If for each $\eta \in \mathbb{R}^n \setminus 0$ we choose $\beta \in \mathbb{Z}_+^n$, $|\beta| = N$, such that $|\eta|^N \leq n^{N/2}|\eta^\beta|$ we conclude that

$$|\mathcal{F}(\chi(x,t)\psi_N(y)v)(\xi,\eta,\tau)| \leq (n^{1/2}C_2)^{N+1}N^N/|\eta|^N, \quad \eta \in \mathbb{R}^n,\ \eta \neq 0. \tag{10}$$

Let now $(\xi_0,\eta_0,\tau_0) \in \mathbb{R}^{m+2n}$ with $\eta_0 \neq 0$. There is $\rho > 0$ such that $(\xi_0,\eta_0,\tau_0) \in \Gamma_\rho$, where

$$\Gamma_\rho = \{(\xi,\eta,\tau) \in \mathbb{R}^{m+2n} : |\xi| + |\tau| < \rho|\eta|\}.$$

In Γ_ρ we have $|(\xi,\eta,\tau)| \leq \sigma|\eta|$, for some $\sigma > 0$, and thus (10) gives

$$|\mathcal{F}(\chi(x,t)\psi_N(y)v)(\xi,\eta,\tau)| \leq C_3^{N+1}N^N/|(\xi,\eta,\tau)|^N, \quad (\xi,\eta,\tau) \in \Gamma_\rho,$$

which shows that $(x_0,y_0,t_0;\xi_0,\eta_0,\tau_0) \notin WF_a(v)$ (cf. [H, 1983], Definition 8.4.3). This completes the proof of (9).

We can now complete the proof of Theorem 3.1. We consider a new tube structure $\tilde{\mathfrak{L}}$ on $\mathbb{R}^{m+n}_{(x,y)} \times \Theta$ defined by the $m+n$ first integrals $Z_1,\dots,Z_m$ and

$$W_\ell(x,y,t) = y_\ell + it_\ell, \quad \ell = 1,\dots,n.$$

According to our previous notation we now have $\tilde{\Phi} : \Theta \to \mathbb{R}^{m+n}$ given by

$$\tilde{\Phi}(t) = (\phi_1(t),\dots,\phi_m(t),t_1,\dots,t_n).$$

and the complex vector fields corresponding to this structure are [2]

$$\tilde{\mathrm{L}}_j = \mathrm{L}_j - i\frac{\partial}{\partial y_j}, \quad j = 1,\dots,n.$$

The reason for introducing this structure lies on the fact that

$$\tilde{\mathrm{L}}_j v = 0, \quad j = 1,\dots,n.$$

[2]It is easily seen that $\tilde{\mathfrak{L}}$ defines a CR structure. Its characteristic set, which is a vector subbundle of T^*U of rank m, is defined by the equations

$$\tau = 0, \quad \eta + {}^t\Phi'(t)\xi = 0.$$

Since $v(x,0,t) = u(x,t)$ the proof will be complete if we can show that v is real-analytic or, equivalently, thanks to the Cauchy-Kovalewsky and Hölmgren theorems, that the traces $v_t \doteq v(\cdot,\cdot,t)$ are real-analytic in $U' \times D$, for all $t \in V'$.

We fix $t \in V'$. According to ([H, 1983], Theorem 8.5.1), it follows from (9) that $(x,y,\xi,\eta) \notin WF_a(v_t)$ if $(x,y) \in U' \times D$ and $\eta \neq 0$. On the other hand we have $\tilde{\Phi}(t)\cdot(\xi,\eta) = \Phi(t)\cdot\xi + t\cdot\eta$ and consequently $\tilde{\Phi}(t)\cdot(\xi,0) = \Phi(t)\cdot\xi$. By the Baouendi-Treves result alluded to above in combination with property $(\star)$ it also follows that $(x,y,\xi,0) \notin WF_a(v_t)$ if $(x,y) \in U' \times D$ and $\xi \neq 0$. Thus $WF_a(v_t)$ is empty and the proof of Theorem 3.1 is complete. ■

When $s > 1$ the situation changes drastically. Even under condition $(\star)$ it can happen that s-Gevrey vectors for $\mathfrak{L}$ fail to be C^1. This phenomenum will be discussed in the next example.

Example. Assume that $m = n = 2$ and consider the tube structure (first introduced in [Ma, 1980]) defined by the first integrals

$$Z_1(x,t) = x_1 - 3it_1, \qquad Z_2(x,t) = x_2 + i(t_1t_2+1)t_1^3. \tag{11}$$

The corresponding vector fields in $\mathbb{R}^4$ are given by

$$\mathrm{L}_1 = \frac{\partial}{\partial t_1} + i\left\{3\frac{\partial}{\partial x_1} - (4t_1t_2+3)t_1^2\frac{\partial}{\partial x_2}\right\};$$

$$\mathrm{L}_2 = \frac{\partial}{\partial t_2} - it_1^4\frac{\partial}{\partial x_2}.$$

It is easily seen that property $(\star)$ holds in this case. Nevertheless we have

Theorem 3.2. *There is an open neighborhood of the origin $\Omega \subset \mathbb{R}^4$ satisfying the following property: given any open neighborhood of the origin $\Omega' \subset \Omega$ and any $s \geq 4$ there is $u \in C(\overline{\Omega})$ such that $\mathrm{L}_j u \in G^s(\overline{\Omega})$, $j = 1,2$, but $u|_{\Omega'} \notin C^1(\Omega')$.*

We shall closely follow the argument of the proof of Theorem 6.2 in [BT, 1981]. We begin by proving an auxiliary result.

Lemma 3.1. *Fix $s > 1$, let $\Omega' \subset \Omega \subset \mathbb{R}^m \times \Theta$ be open sets and assume that for every $v \in C(\overline{\Omega})$ satisfying $\mathrm{L}_j v \in G^s(\overline{\Omega})$, $j = 1,2$, its restriction to Ω' belongs to $C^1(\Omega')$. Then given $h > 0$ and given any compact set $K \subset \Omega'$ there is a constant $C > 0$ such that*

$$\sum_{|\alpha|=1} \sup_K |\partial_x^\alpha u| \leq C\left\{\sup_\Omega |u| + \sum_{k=1}^{2} \sup_{(\beta,\gamma)\in\mathbb{Z}_+^2\times\mathbb{Z}_+^2} \sup_\Omega \frac{|\mathrm{L}^\beta \partial_x^\gamma \mathrm{L}_k u|}{h^{|\beta|+|\gamma|}\beta!^s\,\gamma!^s}\right\} \tag{12}$$

is valid for every $u \in C^\infty(\overline{\Omega})$.

Proof. Denote by $G^{s,h}_\bullet(\overline{\Omega})$ the Banach space of all $u \in C^\infty(\overline{\Omega})$ for which

$$|||\,u\,|||_{s,h} \doteq \sup_{(\beta,\gamma)\in\mathbb{Z}_+^2\times\mathbb{Z}_+^2} \sup_{\overline{\Omega}} \frac{|\mathrm{L}^\beta \partial_x^\gamma u|}{h^{|\beta|+|\gamma|}\beta!^s\,\gamma!^s} < \infty.$$

We also consider the Banach space $E \doteq C(\overline{\Omega}) \times G^{s,h}_\bullet(\overline{\Omega}) \times G^{s,h}_\bullet(\overline{\Omega})$ and its subspace F formed by all $(u,f_1,f_2) \in E$ such that $\mathrm{L}_k u = f_k$ in Ω (in the distribution sense), $k = 1,2$. We claim that F is closed in E. Indeed, if $(u_j,f_{1j},f_{2j}) \in F$ and if $(u_j,f_{1j},f_{2j}) \to (u,f_1,f_2)$

in E then, in particular, $u_j \to u$ in $\mathcal{D}'(\Omega)$ which implies $\mathrm{L}_k u_j \to \mathrm{L}_k u$ in $\mathcal{D}'(\Omega)$, $k = 1, 2$. On other hand we also have $\mathrm{L}_k u_j = f_{kj} \to f_k$ in $\mathcal{D}'(\Omega)$ and hence $f_k = \mathrm{L}_k u$ in Ω, $k = 1, 2$. Thus $(u, f_1, f_2) \in F$ and F is closed in E. Hence F is a Banach space and our hypothesis gives a well defined linear map between Fréchet spaces

$$\lambda : F \to C^1(\Omega'), \quad \lambda(u, f_1, f_2) = u|_{\Omega'} .$$

Notice moreover that if $(u_j, f_{1j}, f_{2j}) \to 0$ in F and if $\lambda(u_j, f_{1j}, f_{2j}) \to w$ in $C^1(\Omega')$ then $w = 0$, which shows that λ has closed graph. Consequently λ is continuous, which gives (12). ■

Proof of Theorem 3.2. We will show the existence of an open neighborhood Ω of the origin in $\mathbb{R}^4$ such that given any $s \geq 4$ and any $\Omega' \subset \Omega$, open neighborhood of the origin, there is a compact set $K \subset \Omega'$ and a constant $h > 0$ such that (12) is not valid for any choice of $C > 0$.

We define

$$u(x,t) = g(t_1) e^{iZ(x,t)\cdot\xi}, \tag{13}$$

where $Z(x,t) = (Z_1(x,t), Z_2(x,t))$ is defined in (11), $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $g \in C_c^\infty(\mathbb{R})$ is a nonnegative cut-off function chosen below. We let $\xi_1 = c^2\rho$, $\xi_2 = \rho$, with $c > 0$ small $\rho > 0$ large. It follows from (11) that

$$|u(x,t)| = g(t_1) e^{-\phi(t)\cdot\xi} = g(t_1) e^{-\rho Q(t_1, t_2, c)}, \tag{14}$$

where $Q(\tau, a, c) = \tau^3 + a\tau^4 - 3c^2\tau$. We notice that

$$c^{-3} Q(c\tau, a, c) = \tau^3 + ac\tau^4 - 3\tau. \tag{15}$$

We shall then take $g(t_1) = g_0(t_1/c)$, with $g_0 \in G_c^\sigma(\mathbb{R})$ satisfying

$$\operatorname{supp} g_0 \subset \left]-\frac{3}{2}, 3\right[, \quad g_0 = 1 \text{ in } [-1, 2], \quad 0 \leq g_0 \leq 1. \tag{16}$$

The value of σ will be chosen below.

It follows from ([BT, 1981], formula (6.38)), the following property:

- There are $d_0 > 0$, $R_0 > 0$ such that

$$Q(c\tau, a, c) \geq d_0\, c^3, \quad \tau \in \operatorname{supp} g_0', \ |a| \leq R_0, \ c \text{ small}. \tag{17}$$

The value of R_0 depends only on property (16) and not on the particular choice of g_0.

We shall then take $\Omega = B_0 \times (]-R_0, R_0[\times]-R_0, R_0[)$, where B_0 is an open ball centered at the origin in $\mathbb{R}^2$, and $K = \{(0,0,t_1,0) : |t_1| \leq R_1\}$, where $R_1 > 0$ is choosen small enough in such a way that $K \subset \Omega'$. If we take $0 < c \leq R_1$ then $(0,0,c,0) \in K$; since, furthermore,

$$Q(t_1, 0, c) \geq Q(c, 0, c) = -2c^3, \ \ t_1 \in \operatorname{supp} g$$

we obtain

$$\sup_K \left| \frac{\partial u}{\partial x_2} \right| = \rho e^{2c^3\rho} . \tag{18}$$

On the other hand we also have (cf. [BT, 1981] formula (6.42))

$$\sup_{\Omega} |u| \leq e^{\rho\left(2c^3+\mathrm{O}(c^4)\right)} . \tag{19}$$

Notice also that we have $\mathrm{L}_1 u(x,t) = g'(t_1)\exp\{i\xi\cdot Z(x,t)\}$ and $\mathrm{L}_2 u = 0$.

Let us now fix $s \geq 4$. If (12) were true for some choice of $h > 0$ and the preceding choices of Ω and K, from (18) and (19) we could write, for some constant $d_1 > 0$,

$$\rho \leq C\left(e^{d_1 c^4 \rho} + e^{-2c^3\rho}\, |||\, \mathrm{L}_1 u\, |||_{s,h}\right) . \tag{20}$$

We must estimate the absolute value of

$$\left(\mathrm{L}^\beta \partial_x^\gamma \mathrm{L}_1 u\right)(x,t) = (i\xi)^\gamma c^{-\beta_1-1} g_0^{(\beta_1+1)}(t_1/c)\exp\{i\xi\cdot Z(x,t)\}$$

for $(x,t) \in A \doteq B \times (\operatorname{supp} g' \times [-R_0, R_0])$.

If $(x,t) \in A$ then

$$\Im Z(x,t)\cdot\xi = \rho Q(t_1,t_2,c) \geq \rho d_0 c^3 ,$$

thanks to (17), and thus, remembering that $g_0 \in G_c^\sigma(\mathbb{R})$ and that $\xi = \rho(c^2,1)$,

$$\left|\left(\mathrm{L}^\beta \partial_x^\gamma \mathrm{L}_1 u\right)(x,t)\right| \leq C^{\beta_1+1} c^{2\gamma_1-\beta_1-1}\rho^{|\gamma|}(\beta_1+1)!^\sigma e^{-\rho d_0 c^3} , \quad (x,t)\in\Omega.$$

Choosing $\rho = 1/c^4$ and writing $\lambda = 1/c$ give, for a new constant $C_1 > 0$,

$$\left|\left(\mathrm{L}^\beta \partial_x^\gamma \mathrm{L}_1 u\right)(x,t)\right| \leq C^{\beta_1+1}\beta!^\sigma \lambda^{4|\gamma|+|\beta|+1} e^{-d_0\lambda} \leq C_1^{|\beta|+|\gamma|+1}\beta!^{\sigma+1}\gamma!^4 , \quad (x,t)\in\Omega.$$

If we recall that $s \geq 4$ and if we choose $1 < \sigma \leq 3$ and $h \geq C_1$ we conclude that $|||\, \mathrm{L}_1 u\, |||_{s,h} \leq$ const. for any $\lambda > 0$ large. Hence (20) implies

$$\lambda^4 \leq C\left(e^{d_1} + \text{const.}\, e^{-2\lambda}\right) ,$$

which gives the sought contradiction after we let $\lambda \to \infty$. ∎

4. Tubular structures with corank one

A satisfying answer to this question can be given in the case when $m = 1$. Thus from now on we assume given a real-valued analytic function Φ defined in an open neighborhood of $\bar{\Theta}$ and satisfying $\Phi(0) = 0$. Our vector fields now read

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i\frac{\partial\Phi}{\partial t_j}(t)\,\frac{\partial}{\partial x}, \quad j = 1,\dots,n \tag{21}$$

The tube structure on $\mathbb{R}\times\Theta$ is given by the span of $\mathfrak{L} = \{\mathrm{L}_1,\dots,\mathrm{L}_n\}$. If we set

$$Z(x,t) = x + i\Phi(t)$$

then

$$\operatorname{span}\mathfrak{L} = \operatorname{span}\{\mathrm{d}Z\}^\perp .$$

Notice that property ($\star$), in this case, just means that $\Phi : \Theta \to \mathbb{R}$ is an open map.

Next we recall the *Lojasiewicz gradient inequality*: after contracting the radius of Θ we can find constants $c \in [0,1[$, $C > 0$ such that $|\Phi(t)|^c \leq C|\vec{\nabla}\Phi(t)|$, $t \in \Theta$. The infimum of the values of c for which this property holds is called the *Lojasiewicz exponent* of Φ and will be denoted in the sequel by θ. We have $\theta = 0$ when $\mathrm{d}\Phi \neq 0$, which corresponds to the elliptic case. Otherwise we necessarily have $\theta \in [1/2, 1[$ and, furthermore, for a sufficiently contracted Θ around the origin, we have

$$|\Phi(t)|^{\theta} \leq C|\nabla\Phi(t)|, \quad t \in \Theta. \tag{22}$$

We are now in a position to state our second main result:

Theorem 4.1. *Assume that $m = 1$ and that $\mathfrak{L}$ satisfies property $(\star)$. Then*

1. *If $s \geq 1$ the system $\mathfrak{L}$ is G^s-hypoelliptic, that is, given $\Omega \subset \mathbb{R} \times \Theta$ open and $u \in \mathcal{D}'(\Omega)$ such that $\mathrm{L}_j u \in G^s(\Omega)$, $j = 1, \dots, n$, then $u \in G^s(\Omega)$.*
2. *If $s \geq 1$ and if $\Omega = J \times V$, where J (resp. V) is an open subset of $\mathbb{R}$ (resp. Θ) then $G^s(\Omega; \mathfrak{L}) \subset G^{s/(1-\theta)}(\Omega)$.*

Before we embark upon the proof we make some remarks. It follows from [BT, 1982] that $\mathfrak{L}$ is G^1-hypoelliptic when $(\star)$ is satisfied. On the other hand, it is a result in [Ma, 1980] that $\mathfrak{L}$ is C^∞-hypoelliptic again under condition $(\star)$. Hence, statement (1) fills in the gap when $1 < s < \infty$.

As far as statement (2) is concerned, we point out that our result is not sharp. Even when $s = 1$ we already know that every analytic vector for $\mathfrak{L}$ is indeed real-analytic. The case $s > 1$ is more involved. When $n = 1$ the best value of θ is $(k-1)/k$, where k is the order of the vanishing of the function Φ at the origin, and in [BM, 1982] it is proved that, when k is *odd*, every s-Gevrey vector for $\mathfrak{L}$ near the origin in $\mathbb{R}^2$ is of Gevrey class $s' = s + \theta(s-1)/(1-\theta) = s + (k-1)(s-1)$, this result being sharp.

5. Proof of Theorem 4.1 - Preliminaries

We begin by discussing some geometrical consequences of property $(\star)$. The results here described are due to Lojasiewicz [Lo, 1984] (see also [Ma, 1980]) and this argument is often referred to as the *Lojasiewicz argument*. Set

$$\Sigma = \{t \in \Theta : \ \nabla\Phi(t) = 0\}$$

and for $t \in \Theta \setminus \Sigma$ we solve the Cauchy problem

$$\frac{\mathrm{d}\alpha_t}{\mathrm{d}\tau}(\tau) = -\frac{\nabla\Phi}{|\nabla\Phi|}(\alpha_t(\tau)), \quad \alpha_t(0) = t.$$

The function α_t is defined for $\tau \in [0, \delta(t)[$ and satisfies, for some constant $c > 0$ and $\sigma = (1-\theta)^{-1}$,

$$\Phi(\alpha_t(\tau_1)) - \Phi(\alpha_t(\tau_2)) \geq c(\tau_2 - \tau_1)^{\sigma}, \quad \tau_1, \tau_2 \in [0, \delta(t)[, \ \tau_2 > \tau_1. \tag{23}$$

In particular it follows that $\sup_{t\in\Theta} \delta(t) < \infty$.

It is easily seen that, for any $t \in \Theta$, there exists the limit $\ell(t) = \lim_{\tau\to\delta(t)} \alpha_t(\tau)$.

We shall now introduce the curves γ_t that will appear in the construction of our parametrices.

If $\ell(t) \in \partial\Theta$ we set $\gamma_t \doteq \alpha_t$. If otherwise $\ell(t) \in \Sigma$ we notice then that $\Phi(\ell(t)) = 0$ and $\Phi(t) > 0$. We can, thanks to condition $(\star)$, choose $t_0 \in \Theta$ arbitrarily close to $\ell(t)$ such that $\Phi(t_0) < 0$ and $\Phi < \Phi(t)$ on the segment $[\ell(t), t_0]$ (Φ has no local minimum at $\ell(t)$); a fortiori we must then have $\ell(t_0) \in \partial\Theta$. In this case we shall set $\gamma_t \doteq \alpha_t \bullet [\ell(t), t_0] \bullet \alpha_{t_0}$.

We emphasize the following key properties:

$$\Phi(t) - \Phi(s) > 0, \quad s \in \gamma_t,\ s \neq t; \tag{24}$$

$$\text{if } C \doteq \max\{\mathrm{diam}(\Theta), 1\} \text{ then } |\gamma_t'(s)| \leq C \text{ for all } t \in \Theta \setminus \Sigma,\ s \in \gamma_t. \tag{25}$$

Moreover, if we denote by $t_\sharp \in \partial\Theta$ the end-point of γ_t and if take $\Theta' \subset\subset \Theta$ an open ball also centered at 0 there is $d > 0$ such that

$$\Phi(t) - \Phi(t_\sharp) \geq d, \quad t \in \Theta'. \tag{26}$$

6. Proof of Theorem 4.1 - beginning

We now prove Theorem 4.1(1). Assume $s > 1$ and let $u \in \mathcal{D}'(\Omega)$ be such that $\mathrm{L}_j u \in G^s(\Omega)$, $j = 1, \dots, n$. By the result of Maire mentioned before we can assume $u \in C^\infty(\Omega)$ and our goal is to show that u is Gevrey of class s in a neighborhood of an arbitrary point $(x_0, t_0) \in \Omega$. We assume, without lost of generality, that (x_0, t_0) is the origin and assume that $u \in C^\infty(\mathbb{R} \times \bar{\Theta})$ is such that $\mathrm{L}_j u \in G^s(J \times \Theta)$, $j = 1, \dots, n$, where J is an open interval centered at $0 \in \mathbb{R}$ and Θ has been conveniently shrunk around $0 \in \mathbb{R}^n$.

Our goal is then to show that u is Gevrey of class s in some neighborhood of the origin.

Let $\psi \in G^s_c(J)$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on an open interval $J_1 \subset J$ also centered at the origin and also $\chi \in G^s(J_1)$, with χ identically equal to one in some neighborhood of the origin. With the notation $\mathbb{L}v = \sum_{j=1}^n (\mathrm{L}_j v)\mathrm{d}t_j$, taking Fourier transform with respect to x gives

$$e^{\xi\Phi(t)}\widehat{\mathbb{L}(\psi u)}(\xi, t) = \mathrm{d}_t \left\{ e^{\xi\Phi}\widehat{(\psi u)} \right\}(\xi, t)$$

and thus, for $t \in \Theta \setminus \Sigma$,

$$\widehat{(\psi u)}(\xi, t) = \widehat{(\psi u)}(\xi, t_\sharp)e^{-\xi(\Phi(t) - \Phi(t_\sharp))} - \int_{\gamma_t} e^{-\xi(\Phi(t)-\Phi(s))}\widehat{\mathbb{L}(\psi u)}(\xi, s)\,\mathrm{d}s.$$

If H denotes the Heaviside function we can write, where the convolution is in the ξ variable,

$$\hat{\chi}(\xi) \star \left\{ H(\xi)\widehat{(\psi u)}(\xi, t) \right\} =$$

$$\hat{\chi}(\xi) \star \left\{ H(\xi)\widehat{(\psi u)}(\xi, t_\sharp)e^{-\xi(\Phi(t)-\Phi(t_\sharp))} \right\} - \hat{\chi}(\xi) \star \left\{ H(\xi) \int_{\gamma_t} e^{-\xi(\Phi(t)-\Phi(s))}\widehat{\mathbb{L}(\psi u)}(\xi, s)\,\mathrm{d}s \right\}.$$

If we note that, since $\chi\psi = \chi$,

$$\hat{\chi}(\xi) \star \left\{ H(\xi)\widehat{(\psi u)}(\xi, t) \right\} = \widehat{(\chi u)}(\xi, t) + \hat{\chi}(\xi) \star \left\{ (H(\xi) - 1)\widehat{(\psi u)}(\xi, t) \right\}$$

we can write, for $t \in \Theta \setminus \Sigma$,

$$\widehat{(\chi u)}(\xi, t) = \hat{\chi}(\xi) \star \left\{ H(\xi)\widehat{(\psi u)}(\xi, t_\sharp)e^{-\xi(\Phi(t)-\Phi(t_\sharp))} \right\} + \tag{27}$$

$$\hat{\chi}(\xi) \star \left\{ (1 - H(\xi))\widehat{(\psi u)}(\xi, t) \right\} - \hat{\chi}(\xi) \star \left\{ H(\xi) \int_{\gamma_t} e^{-\xi(\Phi(t)-\Phi(s))}\widehat{(\psi \mathbb{L} u)}(\xi, s)\,\mathrm{d}s \right\}$$

$$-\hat{\chi}(\xi)\star\left\{H(\xi)\int_{\gamma_t} e^{-\xi(\Phi(t)-\Phi(s))}\widehat{(u\mathbb{L}\psi)}(\xi,s)\,\mathrm{d}s\right\}.$$

Our goal is to estimate the right end side of (27) for $t\in\Theta'\backslash\Sigma$ and $\xi>0$, where $\Theta'\subset\subset\Theta$ is an open ball also centered at the origin of $\mathbb{R}^n$. We write

$$\widehat{(\chi u)}(\xi,t)\doteq F_1(\xi,t)+F_2(\xi,t)+F_3(\xi,t)+F_4(\xi,t)$$

and estimate each term separately.

Firstly we note that

$$|\widehat{(\psi u)}(\xi,t)|\leq\|u(\cdot,t)\|_{L^1(J)}$$

and hence (26) gives

$$|F_1(\xi,t)|\leq\left(\int|\hat{\chi}(\xi-\eta)|H(\eta)e^{-d\eta}\mathrm{d}\eta\right)\|u\|_{L^\infty(\Theta,L^1(J))}\,.$$

Now write

$$\xi^N\int|\hat{\chi}(\xi-\eta)|H(\eta)e^{-d\eta}\mathrm{d}\eta=\int_0^{\xi/2}\xi^N|\hat{\chi}(\xi-\eta)|e^{-d\eta}\mathrm{d}\eta+\int_{\xi/2}^{\infty}\xi^N|\hat{\chi}(\xi-\eta)|e^{-d\eta}\mathrm{d}\eta.$$

In the first integral we use that $\xi\leq 2(\xi-\eta)$ if $0\leq\eta\leq\xi/2$. In the second we estimate $\xi\leq 2\eta$. Thus

$$\xi^N|F_1(\xi,t)|\leq$$
$$\left\{\int_0^\infty(2|\xi-\eta|)^N|\hat{\chi}(\xi-\eta)|e^{-d\eta}\mathrm{d}\eta+\int_0^\infty|\hat{\chi}(\xi-\eta)|(2\eta)^Ne^{-d\eta}\mathrm{d}\eta\right\}\|u\|_{L^\infty(\Theta,L^1(J))}\,.$$

Now there is a constant $A>0$ such that $|\xi^N\hat{\chi}(\xi)|\leq A^{N+1}N!^s$. Since we also have $(2\eta)^N\exp\{-d\eta\}\leq(2N/ed)^N$ we conclude that there exists $A_1>0$ such that

$$\xi^N|F_1(\xi,t)|\leq A_1^{N+1}N!^s\|u\|_{L^\infty(\Theta,L^1(J))}. \tag{28}$$

Likewise we select $A_\flat>0$ such that $(1+|\xi|)^{N+2}|\hat{\chi}(\xi)|\leq A_\flat^{N+1}N!^s$. Since

$$|F_2(\xi,t)|\leq\left\{\int_{-\infty}^0|\hat{\chi}(\xi-\eta)|\mathrm{d}\eta\right\}\|u\|_{L^\infty(\Theta,L^1(J))}$$

we obtain

$$\begin{aligned}\xi^N|F_2(\xi,t)|&\leq\left\{\int_{-\infty}^0(\xi-\eta)^N|\hat{\chi}(\xi-\eta)|\mathrm{d}\eta\right\}\|u\|_{L^\infty(\Theta,L^1(J))}\\&\leq A_\flat^{N+1}N!^s\left\{\int_{-\infty}^\infty\frac{1}{(1+|\xi-\eta|)^2}\mathrm{d}\eta\right\}\|u\|_{L^\infty(\Theta,L^1(J))}\,.\end{aligned}$$

Hence there is $A_2>0$ such that

$$\xi^N|F_2(\xi,t)|\leq A_2^{N+1}N!^s\|u\|_{L^\infty(\Theta,L^1(J))}. \tag{29}$$

Now we estimate $F_3(\xi,t)$. Firstly we observe that

$$|F_3(\xi,t)|\leq\int_0^\infty|\hat{\chi}(\xi-\eta)|\,|R(\eta,t)|\,\mathrm{d}\eta,$$

where
$$R(\xi,t)=\int_{\gamma_t} e^{-\xi(\Phi(t)-\Phi(s))}\widehat{(\psi\mathbb{L}u)}(\xi,s)\,\mathrm{d}s\,.$$

Thanks to our hypothesis and properties (24) and (25) we have, for some constant $A_3>0$,
$$|R(\xi,t)|\leq A_3^{N+1}N!^s/\xi^N,\quad N\in\mathbb{Z}_+,\ t\in\Theta'\setminus\Sigma,$$
and thus
$$\begin{aligned}\xi^N|F_3(\xi,t)| &\leq 2^N\int_0^{\xi/2}|\xi-\eta|^N|\hat\chi(\xi-\eta)||R(\eta,t)|\mathrm{d}\eta+2^N\int_{\xi/2}^\infty|\hat\chi(\xi-\eta)|\eta^N|R(\eta,t)|\mathrm{d}\eta\\ &\leq 2^NA_3\int_{-\infty}^\infty|\eta|^N|\hat\chi(\eta)|\mathrm{d}\eta+2^NA_3^{N+1}N!^s\int_{-\infty}^\infty|\hat\chi(\eta)|\mathrm{d}\eta.\end{aligned}$$

In conclusion, we derive the existence of a constant $A_4>0$ such that
$$\xi^N|F_3(\xi,t)|\leq A_4^{N+1}N!^s,\quad t\in\Theta'\setminus\Sigma,\ \xi>0. \tag{30}$$

It remains to estimate $F_4(\xi,t)$. We can write
$$F_4(\xi,t)=i\lim_{\epsilon\to0^+}\int_0^\infty\int_{\gamma_t}\int_J\hat\chi(\xi-\eta)e^{-\eta(\Phi(t)-\Phi(s))-i\eta y-\epsilon\eta^2}\psi'(y)u(y,s)\Phi_t(s)\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\eta.$$

Thanks to the properties of ψ there is $\delta>0$ such that $|y|\geq\delta$ in the y-integral. We then deform the η integral to the complex integral over the chain
$$\eta\mapsto\left(1-\frac{i}{2}\frac{y}{|y|}\right)\eta,\quad\eta>0.$$

Making use of (24) we see that the real part of the exponent will the bounded by $-\delta\eta/2$ uniformly in y,t,s and ϵ. Hence we have, for some constant $A_5>0$,
$$\xi^N|F_4(\xi,t)|\leq A_5\left\{\xi^N\max_{\omega=\pm\frac12}\int_0^\infty|\hat\chi(\xi-\eta+i\omega\eta)|e^{-\delta\eta/2}\,\mathrm{d}\eta\right\}\|u\|_{L^\infty(\Theta,L^1(J))}\,.$$

Now if we write $J_1=]-a,a[$ then $a<\delta$. Again as before we estimate ξ^N times the integral as being
$$\leq 2^N\int_0^\infty|(\xi-\eta)^N\hat\chi(\xi-\eta+i\omega\eta)|e^{-\delta\eta/2}\,\mathrm{d}\eta+2^N\int_0^\infty|\hat\chi(\xi-\eta+i\omega\eta)|\eta^Ne^{-d\eta/2}\mathrm{d}\eta.$$

At this point we recall the Paley-Wiener-Schwartz theorem for the Gevrey class: there are constants $C>0$, $h>0$ such that
$$|\hat\chi(\xi+i\eta)|\leq Ce^{-h|\zeta|^{1/s}+a|\eta|/2},\quad\zeta=\xi+i\eta\in\mathbb{C}.$$

Then, since $a<\delta$ we have
$$\int_0^\infty|(\xi-\eta)^N\hat\chi(\xi-\eta+i\omega\eta)|e^{-\delta\eta/2}\,\mathrm{d}\eta\leq\text{const.}\left\{\sup_{t>0}\left(t^Ne^{-ht^{1/s}}\right)\right\}=\text{const.}\left(\frac{sN}{eh}\right)^{sN}.$$

For the last integral we estimate
$$\int_0^\infty|\hat\chi(\xi-\eta+i\omega\eta)|\eta^Ne^{-d\eta/2}\mathrm{d}\eta\leq C\left\{\sup_{t>0}\left(t^Ne^{-(\delta-a)t}\right)\right\}\int_0^\infty e^{-h|\xi-\eta|^{1/s}}\mathrm{d}\eta=$$

$$C\left(\frac{N}{e(\delta-a)}\right)^N\int_0^\infty e^{-h|\xi-\eta|^{1/s}}\mathrm{d}\eta\,.$$

Summing up we conclude the existence of a constant $A_6>0$ such that

$$\xi^N|F_4(\xi,t)|\le A_6^{N+1}N!^s\|u\|_{L^\infty(\Theta,L^1(J))}. \tag{31}$$

In view of (27), (28), (29), (30) and (31) and making a similar analysis for $\xi<0$ allows us to conclude that, for some constant $A_7>0$

$$|\widehat{(\chi u)}(\xi,t)|\le A_7^{N+1}N!^s/|\xi|^N,\quad t\in\Theta',\ N\in\mathbb{Z}_+\,. \tag{32}$$

By Corollary 2.2 it then follows that $u\in G^s(J\times\Theta')$, and the proof of Theorem 4.1(1) is complete. ■

Remark . The argument in the preceding proof is not perhaps the most direct one. We have chosen to follow it as a preparatory step for the much more involved proof of Theorem 4.1(2). For the latter we shall need to study parametrices for the iterates of the operator $\mathbb{L}$, and to them we turn our attention in the next section.

7. The operators $\mathbb{L}^N$ and their parametrices

Let Ω be an open subset of $\Theta\times\mathbb{R}$ and let $N\in\mathbb{Z}_+$. We shall denote by $C^\infty(\Omega)^{(N)}$ the space of all smooth tensor fields f of type $(0,N)$ on Ω of the form

$$f=\sum_{i_1,\dots,i_N=1}^n f_{i_1,\dots,i_N}(x,t)\,\mathrm{d}t_{i_1}\otimes\cdots\otimes\mathrm{d}t_{i_N}\,, \tag{33}$$

where $f_{i_1,\dots,i_N}\in C^\infty(\Omega)$. Likewise, if $E(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$, we denote by $E(\Omega)^{(N)}$ the space of all tensor fields of the form (33), where $f_{i_1,\dots,i_N}\in E(\Omega)$. One particular case that will be of most importance for us are the Banach spaces $L^\infty(\Theta,L^1(J))^{(N)}$, $J\subset\mathbb{R}$ an open interval, with norm

$$\|f\|_{L^\infty(\Theta,L^1(J))}=\sum_{i_1,\dots,i_N=1}^n\|f_{i_1,\dots,i_N}\|_{L^\infty(\Theta,L^1(J))}.$$

We shall make use of the linear operators $D_t:C^\infty(\Omega)^{(N)}\to C^\infty(\Omega)^{(N+1)}$, $\mathbb{L}:C^\infty(\Omega)^{(N)}\to C^\infty(\Omega)^{(N+1)}$ defined by

$$D_tf=\sum_{j=1}^n\frac{\partial f}{\partial t_j}\otimes\mathrm{d}t_j=\sum_{i_1,\dots,i_N=1}^n\sum_{j=1}^n\frac{\partial f_{i_1,\dots,i_N}}{\partial t_j}(x,t)\,\mathrm{d}t_{i_1}\otimes\cdots\otimes\mathrm{d}t_{i_N}\otimes\mathrm{d}t_j\,, \tag{34}$$

$$\mathbb{L}f=\sum_{j=1}^n\mathrm{L}_jf\otimes\mathrm{d}t_j=\sum_{i_1,\dots,i_N=1}^n\sum_{j=1}^n(\mathrm{L}_jf_{i_1,\dots,i_N})(x,t)\,\mathrm{d}t_{i_1}\otimes\cdots\otimes\mathrm{d}t_{i_N}\otimes\mathrm{d}t_j\,. \tag{35}$$

We can then write

$$\mathbb{L}f=D_tf-i\frac{\partial f}{\partial x}\otimes D_t\Phi.$$

Notice that if $N \in \mathbb{N}$ and if $u \in \mathcal{D}'(J \times \Theta)$ is such that $\mathbb{L}^N u \in L^\infty(\Theta, L^1(J))^{(N)}$ then

$$\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))} = \sum_{|\alpha|=N} \frac{N!}{\alpha_1! \cdots \alpha_n!} \|\mathrm{L}^\alpha u\|_{L^\infty(\Theta,L^1(J))} \leq n^N \sum_{|\alpha|=N} \|\mathrm{L}^\alpha u\|_{L^\infty(\Theta,L^1(J))}.$$

Now assume that $f \in C^\infty(\theta, C_c^\infty(\mathbb{R}))^{(N)}$. Then, after taking partial Fourier transform with respect to x,

$$e^{\xi\Phi(t)}\widehat{\mathbb{L}f}(\xi,t) = D_t \left\{ e^{\xi\Phi}\hat{f} \right\}(\xi,t).$$

Let now γ_t, $t \in \Theta \setminus \Sigma$, be the family of curves considered previously. If $s \in \gamma_t$ we denote by $\gamma_{t,s}$ the piece of γ_t joining s to $t_\sharp \in \partial\Theta$, under the same parametrization as γ_t.

If $u \in C^\infty(\overline{\theta}, \mathcal{E}'(J))$ and if $t \in \Theta \setminus \Sigma$ we can write [3]

$$e^{\xi\Phi(t_\sharp)}\hat{u}(\xi,t_\sharp) - e^{\xi\Phi(t)}\hat{u}(\xi,t) = \int_{s_1\in\gamma_t} e^{\xi\Phi(s_1)}\widehat{(\mathbb{L}u)}(\xi,s_1);$$

and hence

$$\hat{u}(\xi,t) = -\int_{s_1\in\gamma_t} e^{-\xi(\Phi(t)-\Phi(s_1))}\widehat{(\mathbb{L}u)}(\xi,s_1) + e^{-\xi(\Phi(t)-\Phi(t_\sharp))}\hat{u}(\xi,t_\sharp)\,. \tag{36}$$

Analogously

$$\widehat{(\mathbb{L}u)}(\xi,s_1) = -\int_{s_2\in\gamma_{t,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\widehat{(\mathbb{L}^2u)}(\xi,s_2) + e^{-\xi(\Phi(s_1))-\Phi(t_\sharp)}\widehat{(\mathbb{L}u)}(\xi,t_\sharp)\,.$$

Hence

$$\hat{u}(\xi,t) = \int_{s_1\in\gamma_t} e^{-\xi(\Phi(t)-\Phi(s_1))}\int_{s_2\in\gamma_{t,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\widehat{(\mathbb{L}^2u)}(\xi,s_2) -$$
$$-e^{-\xi(\Phi(t)-\Phi(t_\sharp))}\int_{s_1\in\gamma_t}\widehat{(\mathbb{L}u)}(\xi,t_\sharp) + e^{-\xi(\Phi(t)-\Phi(t_\sharp))}\hat{u}(\xi,t_\sharp)$$

---

[3]A tensor of type $(0,N)$ on Θ

$$F(t) = \sum_{i_1,\dots,i_N=1}^{n} f_{i_1,\dots,i_N}(t)\,\mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_N}$$

can be identified to a one-form whose coefficients are tensors of type $(0,N-1)$ in the following way:

$$F(t) = \sum_{j=1}^{n} F_{\{j\}}\,\mathrm{d}t_j\,,$$

where

$$F_{\{j\}} = \sum_{i_1,\dots,i_{N-1}=1}^{n} f_{i_1,\dots,i_{N-1},j}(t)\,\mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_{N-1}}\,.$$

Hence, for a parametrized curve $\gamma : [a,b] \to \Theta$ we can define $\int_\gamma F$ by

$$\int_\gamma F \doteq \sum_{i_1,\dots,i_{N-1}=1}^{n} \left\{ \int_a^b f_{i_1,\dots,i_{N-1},j}(\gamma(s))\gamma_j'(s)\mathrm{d}s \right\} \mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_{N-1}}\,.$$

In particular, if G is a tensor of type $(0,N-1)$ on Θ, then $\int_\gamma D_tG = G(b) - G(a)$.

and by iteration we obtain, for every N,

$$\hat{u}(\xi,t) = \mathrm{K}_N(\mathbb{L}^N u) + \tag{37}$$

$$+e^{-\xi(\Phi(t)-\Phi(t_\sharp))}\left\{\hat{u}(\xi,t_\sharp) + \sum_{k=1}^{N-1}(-1)^k \int_{s_1\in\gamma_t}\cdots\int_{s_k\in\gamma_{t,s_{k-1}}} \widehat{(\mathbb{L}^k u)}(\xi,t_\sharp)\right\},$$

where

$$\mathrm{K}_N(g) =$$

$$(-1)^N \int_{s_1\in\gamma_t} e^{-\xi(\Phi(t)-\Phi(s_1))}\int_{s_2\in\gamma_{t,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\cdots\int_{s_N\in\gamma_{t,s_{N-1}}} e^{-\xi(\Phi(s_{N-1})-\Phi(s_N))}\widehat{g}(\xi,s_N).$$

8. Proof of Theorem 4.1(2)

In the argument we assume $\theta > 0$, since when $\theta = 0$ (the elliptic case) the result has already been established.

Let J be an open interval centered at the origin in $\mathbb{R}$ and let $u \in \mathcal{D}'(\Theta\times J)$ be such that $\mathrm{L}^\alpha u \in L^\infty(\Theta, L^1(J))$ for all α. In particular $u \in C^\infty(\bar{\Theta}, \mathcal{D}'(J))$

Fix $N \in \mathbb{N}$ and $\tau > 1$. First we select a cut-off function $\chi \in G^\tau_c(J)$, satisfying $\chi = 1$ in some neighborhood of the origin. We then take another cut-off function $\psi \in G^\tau_c(J)$, $0 \le \psi \le 1$, with $\psi = 1$ on a neighborhood of the support of χ.

If $t \in \Theta\setminus\Sigma$ we obtain

$$\widehat{(\psi u)}(\xi,t) = \mathrm{K}_N(\mathbb{L}^N(\psi u)) + \tag{38}$$

$$+e^{-\xi(\Phi(t)-\Phi(t_\sharp))}\left\{\widehat{(\psi u)}(\xi,t_\sharp) + \sum_{k=1}^{N-1}(-1)^k \int_{s_1\in\gamma_t}\cdots\int_{s_k\in\gamma_{t,s_{k-1}}} \widehat{\mathbb{L}^k(\psi u)}(\xi,t_\sharp)\right\}.$$

If H denotes the Heaviside function we can write, where the convolution is in the ξ variable,

$$\widehat{\chi}(\xi)\star\left\{H(\xi)\widehat{(\psi u)}(\xi,t)\right\} = \widehat{\chi}(\xi)\star\{H(\xi)\mathrm{K}_N(\mathbb{L}(\psi u))(\xi,t)\} +$$

$$\widehat{\chi}(\xi)\star\left\{e^{-\xi(\Phi(t)-\Phi(t_\sharp))}H(\xi)\left[\widehat{(\psi u)}(\xi,t_\sharp) + \sum_{k=1}^{N-1}(-1)^k\int_{s_1\in\gamma_t}\cdots\int_{s_k\in\gamma_{t,s_{k-1}}}\widehat{\mathbb{L}^k(\psi u)}(\xi,t_\sharp)\right]\right\}.$$

If we note that, since $\chi\psi = \chi$,

$$\widehat{\chi}(\xi)\star\left\{H(\xi)\widehat{(\psi u)}(\xi,t)\right\} = \widehat{(\chi u)}(\xi,t) + \hat{\chi}(\xi)\star\left\{(H(\xi)-1)\widehat{(\psi u)}(\xi,t)\right\}$$

we can write, for $t \in \Theta\setminus\Sigma$,

$$\widehat{(\chi u)}(\xi,t) = \widehat{\chi}(\xi)\star\left\{(1-H(\xi))\widehat{(\psi u)}(\xi,t)\right\} + \tag{39}$$

$$\widehat{\chi}(\xi)\star\left\{e^{-\xi(\Phi(t)-\Phi(t_\sharp))}H(\xi)\left[\widehat{(\psi u)}(\xi,t_\sharp) + \sum_{k=1}^{N-1}(-1)^k\int_{s_1\in\gamma_t}\cdots\int_{s_k\in\gamma_{t,s_{k-1}}}\widehat{\mathbb{L}^k(\psi u)}(\xi,t_\sharp)\right]\right\} +$$

$$\widehat{\chi}(\xi)\star\left\{H(\xi)\mathrm{K}_N(\psi\mathbb{L}^N u)(\xi,t)\right\} + \widehat{\chi}(\xi)\star\{H(\xi)\mathrm{K}_N(u_N)(\xi,t)\},$$

where we have written $\mathbb{L}^N(\psi u) = \psi \mathbb{L}^N u + u_N$.

Our goal is to estimate the right end side of (39) for $t \in \Theta' \setminus \Sigma$ and $\xi > 0$. We write

$$\widehat{(\chi u)}(\xi,t) \doteq F_1(\xi,t) + F_{2N}(\xi,t) + F_{3N}(\xi,t) + F_{4N}(\xi,t)$$

and estimate each term separately.

The term $F_1(\xi,t)$ was already estimated in Section 6. We have the following property: there is $A_1 > 0$ such that

$$\xi^N |F_1(\xi,t)| \leq A_1^{N+1} N!^\tau \|u\|_{L^\infty(\Theta, L^1(J))}, \qquad \xi > 0. \tag{40}$$

Next we note that

$$|\widehat{\mathbb{L}^k(\psi u)}(\xi,t)| \leq \|\mathbb{L}^k(\psi u)(\cdot,t)\|_{L^1(J)}$$

and hence (26) together with Lemma 8.1 below give

$$|F_{2N}(\xi,t)| \leq \left(\int |\widehat{\chi}(\xi-\eta)| H(\eta) e^{-d\eta} \mathrm{d}\eta \right) \sum_{j=0}^{N-1} \frac{C^j}{j!} \|\mathbb{L}^j(\psi u)\|_{L^\infty(\Theta, L^1(J))} ,$$

where $C > 0$ is a constant (here we have made use of the fact that $t \mapsto \delta(t)$ is bounded in Θ). Working as in Section 6 we obtain the existence of a constant $A_2 > 0$ such that

$$\begin{aligned} \xi^N |F_{2N}(\xi,t)| &\leq A_2^{N+1} N!^\tau \sum_{|\alpha| \leq N-1} \frac{1}{|\alpha|!} \|\mathrm{L}^\alpha(\psi u)\|_{L^\infty(\Theta, L^1(J))} \\ &\leq A_2^{N+1} N!^\tau \sum_{|\alpha| \leq N-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\beta!\gamma!} \|\mathrm{L}^\beta(\psi)\mathrm{L}^\gamma(u)\|_{L^\infty(\Theta, L^1(J))} , \end{aligned}$$

where we have also made use of the Leibniz rule. Since ψ is Gevrey of order τ we obtain, after redefining A_2,

$$\xi^N |F_{2N}(\xi,t)| \leq A_2^{N+1} N!^{2\tau-1} \sum_{|\gamma| \leq N-1} \frac{1}{\gamma!} \|\mathrm{L}^\gamma u\|_{L^\infty(\Theta, L^1(J))}, \;\; \xi > 0, \;\; N \in \mathbb{N}. \tag{41}$$

A similar property holds for the term $F_{4N}(\xi,t)$. Indeed we can write

$$i(-1)^{N+1} F_{4N}(\xi,t) =$$

$$\lim_{\epsilon \to 0} \int_0^\infty \int_{s_1 \in \Gamma_t} \cdots \int_{s_N \in \gamma_{t,s_{N-1}}} \int_J \widehat{\chi}(\xi-\eta) e^{-\eta(\Phi(t)-\Phi(s_N)) - i\eta y - \epsilon\eta^2} u_N(y,s_N) \mathrm{d}y \mathrm{d}y_N \ldots \mathrm{d}y_1 \mathrm{d}\eta \,.$$

If we proceed as in the study of the term $F_4(\xi,t)$ in Section 6 we obtain the existence of a constant $D > 0$ such that

$$\xi^N |F_{4N}(\xi,t)| \leq D^{N+1} N!^{\tau-1} \|u_N\|_{L^\infty(\Theta, L^1(J))}$$

(cf. Lemma 8.1 below). Now we have, again by the Leibniz rule,

$$\begin{aligned}
\|u_N\|_{L^\infty(\Theta,L^1(J))} &= \sum_{i_1,\cdots,i_N=1}^{N} \|\mathrm{L}_{i_1}\cdots\mathrm{L}_{i_N}(\psi u)-\psi\mathrm{L}_{i_1}\cdots\mathrm{L}_{i_N}u\|_{L^\infty(\Theta,L^1(J))}\\
&\leq n^N\sum_{|\alpha|=N}\|\mathrm{L}^\alpha(\psi u)-\psi\mathrm{L}^\alpha u\|_{L^\infty(\Theta,L^1(J))}\\
&\leq n^N\sum_{|\alpha|=N}\sum_{\beta+\gamma=\alpha,\,\gamma<\alpha}\frac{\alpha!}{\beta!\gamma!}\|\mathrm{L}^\beta(\psi)\mathrm{L}^\gamma(u)\|_{L^\infty(\Theta,L^1(J))}\\
&\leq D_1^{N+1}N!^\tau\sum_{|\alpha|=N}\sum_{\beta+\gamma=\alpha,\,\gamma<\alpha}\frac{1}{\gamma!}\|\mathrm{L}^\gamma(u)\|_{L^\infty(\Theta,L^1(J))}\,,
\end{aligned}$$

for a convenient constant $D_1>0$. Hence for some constant $A_4>0$ we obtain

$$\xi^N|F_{4N}(\xi,t)|\leq A_4^{N+1}N!^{2\tau-1}\sum_{|\gamma|\leq N-1}\frac{1}{\gamma!}\|\mathrm{L}^\gamma u\|_{L^\infty(\Theta,L^1(J))},\quad \xi>0,\ N\in\mathbb{N}. \tag{42}$$

The term that remains to be estimated is the crucial one: $F_{3N}(\xi,t)$.

For this firstly we observe that

$$|F_{3N}(\xi,t)|\leq\int_0^\infty|\widehat{\chi}(\xi-\eta)|\,|\mathrm{K}_N(\psi\mathbb{L}^N u)(\eta,t)|\,\mathrm{d}\eta\,.$$

We first assume that $\gamma_t=\alpha_t$. Then (23) gives

$$|\mathrm{K}_N(\psi\mathbb{L}^N u)(\eta,t)|\leq$$

$$\underbrace{\left\{\int_0^{\delta(t)}e^{-c\eta\tau_1^\sigma}\int_{\tau_1}^{\delta(t)}e^{-c\eta(\tau_2-\tau_1)^\sigma}\cdots\int_{\tau_{N-1}}^{\delta(t)}e^{-c\eta(\tau_N-\tau_{N-1})^\sigma}\mathrm{d}\tau_N\cdots\mathrm{d}\tau_1\right\}}_{\doteq R_N(\eta,t)}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))}\,.$$

We now prove an auxiliary result:

Lemma 8.1. *There is a constant $C_1>0$, which does not depend on N, such that*

$$R_N(\eta,t)\leq\frac{C_1^N}{N!},\quad \eta>0,\ t\in\Theta. \tag{43}$$

$$\eta^{N/\sigma}R_N(\eta,t)\leq C_1^N,\quad \eta>0,\ t\in\Theta. \tag{44}$$

Proof. (43) follows immediately from the identity

$$\int_0^{\delta(t)}\int_{\tau_1}^{\delta(t)}\cdots\int_{\tau_{N-1}}^{\delta(t)}\mathrm{d}\tau_N=\frac{\delta(t)^N}{N!}$$

and from the fact that $\sup_{t\in\Theta}\delta(t)<\infty$.

Next we see that the change of variables $(c\eta)^{1/\sigma}(\tau_j-\tau_{j-1})=s_j$ gives

$$(c\eta)^{N/\sigma}R_N(\eta,t)\leq\left(\int_0^\infty e^{-s^\sigma}\,\mathrm{d}s\right)^N$$

which proves (44). ■

Consequently,

$$\xi^{N/\sigma}|F_{3N}(\xi,t)| \le \left\{\xi^{N/\sigma}\int_0^\infty |\widehat{\chi}(\xi-\eta)|R_N(\eta,t)\mathrm{d}\eta\right\}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))} \le$$

$$2^{N/\sigma}\left\{\int_0^{\xi/2}(\xi-\eta)^{N/\sigma}|\widehat{\chi}(\xi-\eta)|R_N(\eta,t)\mathrm{d}\eta + \right.$$

$$\left. + \int_{\xi/2}^\infty |\widehat{\chi}(\xi-\eta)|\eta^{N/\sigma}R_N(\eta,t)\mathrm{d}\eta\right\}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))} \le$$

$$2^{N/\sigma}\left\{\frac{\delta(t)^N}{N!}\int_{-\infty}^\infty |\xi|^{N/\sigma}|\widehat{\chi}(\xi)|\mathrm{d}\xi + \int_{\xi/2}^\infty |\widehat{\chi}(\xi-\eta)|\eta^{N/\sigma}R_N(\eta,t)\mathrm{d}\eta\right\}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))}.$$

Now, by the Paley-Wiener-Schwartz Theorem we have, for some $c>0$,

$$\int_{-\infty}^\infty |\xi|^{N/\sigma}|\widehat{\chi}(\xi)|\mathrm{d}\xi \le \sup_{w>0}\left\{w^{N/\sigma}e^{-cw^{1/\tau}}\right\}\int_{-\infty}^\infty e^{-c|\xi|^{1/\tau}}\,\mathrm{d}\xi \le \left(\frac{\tau N}{c\sigma}\right)^{\tau N/\sigma}\int_{-\infty}^\infty e^{-c|\xi|^{1/\tau}}\,\mathrm{d}\xi\,.$$

If τ is chosen such that $1<\tau\le\sigma=1/(1-\theta)$ we conclude that there is a constant $A_3>0$ such that

$$\xi^{N(1-\theta)}|F_{3N}(\xi,t)| \le A_3^{N+1}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))}, \quad t\in\Theta',\ \xi>0. \tag{45}$$

We now assume that $\gamma_t = \alpha_t \bullet [\ell(t),t_0]\bullet\alpha_{t_0}$. According to our definitions we have

$$\gamma_{t,s} = \begin{cases} \alpha_{t,s}\bullet[\ell(t),t_0]\bullet\alpha_{t_0} & \text{if } s\in\alpha_t;\\ [s,t_0]\bullet\alpha_{t_0} & \text{if } s\in[\ell(t),t_0];\\ \alpha_{t_0,s} & \text{if } s\in\alpha_{t_0}.\end{cases} \tag{46}$$

We shall also adopt the following notation: we shall write $A\sim B$ if $A-B=\mathrm{O}(|t-t_0|)$. Thanks to the definition of K_N we have

$$\mathrm{K}_N(\psi\mathbb{L}u)\sim$$

$$(-1)^N\int_{s_1\in\alpha_t} e^{-\xi(\Phi(t)-\Phi(s_1))}\int_{s_2\in\gamma_{t,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\cdots\int_{s_N\in\gamma_{t,s_{N-1}}} e^{-\xi(\Phi(s_{N-1})-\Phi(s_N))}(\widehat{\psi\mathbb{L}^N u})(\xi,s_N)+$$

$$(-1)^N\int_{s_1\in\alpha_{t_0}} e^{-\xi(\Phi(t)-\Phi(s_1))}\int_{s_2\in\alpha_{t_0,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\cdots\int_{s_N\in\alpha_{t_0,s_{N-1}}} e^{-\xi(\Phi(s_{N-1})-\Phi(s_N))}(\widehat{\psi\mathbb{L}^N u})(\xi,s_N).$$

If we observe that, in the last integral in the right, the bound

$$e^{-\xi(\Phi(t)-\Phi(s_1))} \le e^{-\xi(\Phi(t_0)-\Phi(s_1))}$$

and if we denote this last term by $Q_1(\xi,t)$, our preceding argument gives a bound

$$\xi^{N(1-\theta)}|\left[\hat\chi\star(H(\cdot)Q_1(\cdot,t))\right](\xi,t)| \le A_3^{N+1}\|\mathbb{L}^N u\|_{L^\infty(\Theta,L^1(J))},\ t\in\Theta',\ \xi>0\,, \tag{47}$$

for a conveniently enlarged A_3 which, we emphasize, does not depend also on the choice of t_0. Next, after replacing, in the first integral in the right, γ_{t,s_1} by the decomposition (46), we obtain

$$\mathrm{K}_N(g)\sim Q_1(t,\xi)+$$

$$(-1)^N\int_{s_1\in\alpha_t} e^{-\xi(\Phi(t)-\Phi(s_1))}\int_{s_2\in\alpha_{t,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))}\int_{s_3\in\gamma_{t,s_2}}\cdots\int_{s_N\in\gamma_{t,s_{N-1}}} e^{-\xi(\Phi(s_{N-1})-\Phi(s_N))}\widehat{g}(\xi,s_N)+$$

$$(-1)^N \int_{s_1\in\alpha_t} e^{-\xi(\Phi(t)-\Phi(s_1))} \int_{s_2\in\alpha_{t_0,s_1}} e^{-\xi(\Phi(s_1)-\Phi(s_2))} \cdots \int_{s_N\in\alpha_{t_0,s_{N-1}}} e^{-\xi(\Phi(s_{N-1})-\Phi(s_N))}\widehat{g}(\xi,s_N)\,.$$

Now, in the second integral in the right we have $\Phi(s_1) > 0$; since $\Phi(t_0) < 0$ we must have

$$e^{-\xi(\Phi(s_1)-\Phi(s_2))} \le e^{-\xi(\Phi(t_0)-\Phi(s_2))},$$

and then, if we denote this last term by $Q_2(\xi,t)$, we derive the validity of (47) with Q_1 replaced by Q_2. Repeating the argument $N-2$ more times we will end up with a sum

$$\mathrm{K}_N(g) \sim Q_1(t,\xi) + Q_2(\xi,t) + \ldots + Q_N(\xi,t),$$

where, for each j, property (47) holds with Q_1 replaced by Q_j. After we let $t \to t_0$ we then conclude that (45) holds in general.

Summing up, taking into account (40), (41), (42) and (45), we have proved:

Proposition 8.1. *Assume condition ($\star$) and let $\Theta' \subset\subset \Theta$ (resp. J) be an open ball (resp. interval) centered at the origin of $\mathbb{R}^n$ (resp. $\mathbb{R}$). Let τ satisfy $1 < \tau \le 1/(1-\theta)$. Then given $\chi \in G^\tau_c(J)$ identically one in some neighborhood of the origin there is a constant $A > 0$ such that, for all $\xi \neq 0$ and all $N \in \mathbb{N}$ the estimate*

$$\left|\widehat{(\chi u)}(\xi,t)\right| \le \tag{48}$$

$$A^{N+1}\left\{\frac{1}{|\xi|^{N(1-\theta)}}\sum_{|\alpha|=N}\|\mathrm{L}^\alpha u\|_{L^\infty(\Theta,L^1(J))} + \frac{N!^{2\tau-1}}{|\xi|^N}\sum_{|\gamma|\le N-1}\frac{1}{\gamma!}\|\mathrm{L}^\gamma u\|_{L^\infty(\Theta,L^1(J))}\right\}$$

holds for every $t \in \Theta'$ and every $u \in \mathcal{D}'(J\times\Theta)$ such that $\mathrm{L}^\alpha u \in L^\infty(\Theta, L^1(J))$ for all α.

We can now complete the proof of Theorem 4.1(2). Assuming that u satisfies, for some constant $C > 0$,

$$\|\mathrm{L}^\alpha u\|_{L^\infty(\Theta,L^1(J))} \le C^{|\alpha|+1}\alpha!^s, \qquad \alpha \in \mathbb{Z}^n_+\,,$$

we must show that u is Gevrey of order $s' = s/(1-\theta)$ near the origin. By (48) we obtain, with a new constant $A_1 \ge 1$,

$$\left|\widehat{(\chi u)}(\xi,t)\right| \le A_1^{N+1}\left\{\frac{N!^s}{|\xi|^{N(1-\theta)}} + \frac{N!^{s+2\tau-2}}{|\xi|^N}\right\}.$$

But

$$s + 2\tau - 2 \le s'$$

if we further require that

$$\tau - 1 \le \frac{s'-s}{2} = \frac{s\theta}{2(1-\theta)}.$$

Hence

$$\left|\widehat{(\chi u)}(\xi,t)\right| \le A_1^{N+1}\left\{\left(\frac{N^{s'}}{|\xi|}\right)^{N(1-\theta)} + \left(\frac{N^{s'}}{|\xi|}\right)^N\right\}, \quad N \in \mathbb{N},\ t \in \Theta'. \tag{49}$$

We claim that (49) implies

$$\left|\widehat{(\chi u)}(\xi,t)\right| \le B^{N+1}\left(\frac{N^{s'}}{|\xi|}\right)^N, \quad N \in \mathbb{N},\ t \in \Theta' \tag{50}$$

for some constant $B > 0$, and then Corollary 2.2 will imply that u is $G^{s'}$ in a neighborhood of the origin.

We prove (50). Since $|\widehat{(\chi u)}(\xi,t)| \leq B_1$ if $t \in \Theta'$ and $\xi \neq 0$, (50) is certainly true if $N^{s'} > |\xi|$. We then assume $|\xi| \geq N^{s'}$. From (49) we derive, for every $N, p \in \mathbb{N}$,

$$\begin{aligned}\left|\widehat{(\chi u)}(\xi,t)\right| &\leq 2A_1^{N+p+1}\left\{\frac{(N+p)^{s'}}{|\xi|}\right\}^{(N+p)(1-\theta)} \\ &= 2A_1^{N+p+1}\left(\frac{N+p}{N}\right)^{s'(N+p)(1-\theta)}\left(\frac{N^{s'}}{|\xi|}\right)^{(N+p)(1-\theta)}.\end{aligned}$$

If we select p such that

$$\frac{\theta}{1-\theta}N \leq p < \frac{\theta}{1-\theta}N + 1$$

we get $(N+p)(1-\theta) \geq N$ and thus

$$\left|\widehat{(\chi u)}(\xi,t)\right| \leq 2A_1^{N+\theta N/(1-\theta)+2}\left(2+\frac{\theta}{1-\theta}\right)^{s'(N+1-\theta)}\left(\frac{N^{s'}}{|\xi|}\right)^{N},$$

which completes the proof of (50). ■

9. Application to semilinear systems

We shall now study the regularity of the solutions to semilinear systems associated to vector fields $\mathfrak{L}$ given by (1). Our first result deals with analytic regularity.

Theorem 9.1. *Assume that $\mathfrak{L}$ satisfies property $(\star)$ and let u be a C^1 function defined in an open neighborhood of the origin in $\mathbb{R}^m \times \Theta$ and satisfying the system*

$$(\mathrm{L}_j u)(x,t) = g_j(x,t,u(x,t)), \quad j = 1,\dots,n, \tag{51}$$

where the functions $g_j(x,t,\zeta)$ are real-analytic for (x,t) near the origin in $\mathbb{R}^m \times \Theta$ and entire holomorphic with respect to $\zeta \in \mathbb{C}$. We also assume that the vector fields

$$\mathcal{L}_j \doteq L_j + g_j(x,t,\zeta)\frac{\partial}{\partial\zeta}, \quad j = 1,\dots,n,$$

are pairwise commuting, that is,

$$\mathcal{L}_k g_j = \mathcal{L}_j g_k, \quad j,k = 1,\dots,n.$$

Then u is real-analytic near the origin.

Proof. Consider the solution $H(x,t,\zeta)$ of the Cauchy problem

$$\mathcal{L}_j H = 0, \;\; j = 1,\dots,n, \qquad H(x,0,\zeta) = \zeta.$$

The solution H, whose existence is guaranteed thanks to our involution hypothesis, is a holomorphic function defined in an open neighborhood of the origin in $\mathbb{C}^{m+n+1}$. Notice that by the implicit function theorem we can solve the equation $w = H(x,t,\zeta)$ in the form $\zeta = G(x,t,w)$, where G is also holomorphic in an open neighborhood of the origin in $\mathbb{C}^{m+n+1}$.

Now, the chain rule and (51) give

$$\mathrm{L}_j\{H(x,t,u(x,t))\} = (\mathcal{L}_j H)(x,t,u(x,t)) = 0, \quad j = 1,\dots,n.$$

By property $(\star)$ we conclude that $v(x,t) \doteq H(x,t,u(x,t))$ is real-analytic. Since $u(x,t) = G(x,t,v(x,t))$ it follows that u is real-analytic near the origin. ■

In order to study the Gevrey regularity for this kind of systems we pause to prove a result for a slightly more general class of systems of vector fields. We let Ω be an open subset of $\mathbb{R}^N$, where we assume defined real-analytic, pairwise commuting complex vector fields $\mathrm{Z}_1, \dots, \mathrm{Z}_n$.

Proposition 9.1. *Let $u \in C^1(\Omega)$ satisfy the semilinear system*

$$\left(\mathrm{Z}_j u\right)(x) = f_j(x, u(x)), \quad x \in \Omega,\ j = 1, \dots, n, \tag{52}$$

where $f_j(x,\zeta)$, $j = 1, \dots, n$, are Gevrey functions of order $s \geq 1$ in $x \in \Omega$, valued in the space of entire functions of $\zeta \in \mathbb{C}$. Then $\mathrm{Z}^\alpha u \in C^1(\Omega)$ for every α and for every compact $K \subset \Omega$ there is a constant $C = C(K) > 0$ such that

$$\|\mathrm{Z}^\alpha u\|_{L^\infty(K)} \leq C^{|\alpha|+1} |\alpha|!^s, \quad \alpha \in \mathbb{Z}_+^n. \tag{53}$$

Proof. First we select, for each compact set $K \subset \Omega$, a constant $A(K) \geq 1$ such that

$$\| \left(\mathrm{Z}^\alpha \partial_\zeta^p f_j\right)(x,\zeta) ||_{L^\infty(K \times u(K))} \leq A(K)^{|\alpha|+p+1} |\alpha|!^s p!,\ \alpha \in \mathbb{Z}_+^n,\ p \in \mathbb{Z}_+,\ j = 1, \dots, n. \tag{54}$$

We now recall some useful tools which are important when dealing with the particular kind of computation that follows (cf., e.g., [AM, 1984]). For $s \geq 1$ and $c > 0$ we consider the sequence

$$m_p = \frac{c p!^s}{(p+1)^{n+1}}.$$

Also, for $\epsilon > 0$ and $p \geq 0$, we set

$$M_p = \frac{m_p}{\epsilon^p}.$$

For a convenient choice of $c > 0$ one obtains the validity of the following inequality:

$$\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} m_{|\beta|}\, m_{|\gamma|} \leq m_{|\alpha|}\,, \tag{55}$$

from which it also follows

$$\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} M_{|\beta|}\, M_{|\gamma|} \leq M_{|\alpha|}\,, \ \ \epsilon > 0. \tag{56}$$

Furthermore we also have

$$M_p \leq 2^{n+2} \epsilon M_{p+1}, \quad p \geq 1, \tag{57}$$

and, for some $0 < \epsilon_0 \leq 1$,

$$A(K)^{p+1} p!^s \leq M_p/2, \quad p \geq 1,\ 0 < \epsilon \leq \epsilon_0, \tag{58}$$

where $A(K)$ comes from (54).

Next we pause for a small digression. Given two formal power series $\mathbf{f}, \mathbf{g} \in \mathbb{R}[[X_1, \dots, X_n]]$ we shall write $\mathbf{f} \ll \mathbf{g}$ if $\mathbf{f}^{(\alpha)}(0) \leq \mathbf{g}^{(\alpha)}(0)$ for all $\alpha \in \mathbb{Z}_+^n$. We shall consider the formal power series $\mathfrak{T} \in \mathbb{R}[[X_1, \dots, X_n]]$ defined by

$$\mathfrak{T}(X) = \sum_{|\alpha|>0} \frac{M_{|\alpha|}}{\alpha!} X^\alpha, \quad X \in \mathbb{R}^n \,.$$

Inequality (56) implies that

$$\mathfrak{T}(X)^2 \ll \epsilon \mathfrak{T}(X) \tag{59}$$

and hence an elementar induction argument gives

$$\mathfrak{T}(X)^p \ll \epsilon^{p-1} \mathfrak{T}(X), \; p \geq 1. \tag{60}$$

Let $\rho > 0$. If $\epsilon \rho A(K) < 1$ and if we consider

$$\mathfrak{t}_\rho(X) = \sum_{p>0} \rho^p A(K)^p \mathfrak{T}(X)^p \in \mathbb{R}[[X_1, \dots, X_n]]$$

then

$$\mathfrak{t}_\rho\,(X) \ll \frac{A(K)\rho}{1 - \epsilon A(K)\rho}\, \mathfrak{T}(X). \tag{61}$$

Proof of (61): We have, from (60),

$$(\partial^\alpha \mathfrak{t}_\rho)(0) =$$

$$\sum_{p>0} \rho^p A(K)^p \left\{ \partial^\alpha \left(\mathfrak{T}(X)^p \right) \right\}(0) \leq \sum_{p>0} \rho^p \epsilon^{p-1} A(K)^p \left(\partial^\alpha \mathfrak{T} \right)(0) = \frac{A(K)\rho}{1 - \epsilon A(K)\rho} (\partial^\alpha \mathfrak{T})(0). \; \blacksquare$$

We now return to the proof of Proposition 9.1. We set

$$M \doteq \max\{1, \frac{1}{c} \sup_K |u|, \frac{2^{n+1}}{c} \sup_K |\mathrm{Z}_j u|, \; j = 1, \dots, n\}$$

(notice that $\mathrm{Z}_j u \in C^1(\Omega)$, for all $j = 1, \dots, n$, as a consequence of (52)). The result will be proved if we can prove that $\mathrm{Z}^\alpha u \in C^1(\Omega)$ for every α and that, for some $0 < \epsilon \leq 1$, the inequalities

$$\sup_K |\mathrm{Z}^\alpha u| \leq M M_{|\alpha|}, \quad \alpha \in \mathbb{Z}^n_+, \; |\alpha| \geq 0, \tag{62}$$

hold. Thanks to our choice of M we have, for every $0 < \epsilon \leq 1$,

$$\sup_K |\mathrm{Z}^\alpha u| \leq M\, M_{|\alpha|}, \quad |\alpha| \leq 1.$$

We shall prove both claims by induction on the length of α. We fix $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \geq 2$, and assume that both properties have been proved for all multi-indices with length $< |\alpha|$. There is no lost of generality in assuming that $\alpha = \beta + (0, \dots, 0, 1)$, with $|\beta| = |\alpha| - 1$. Then $\mathrm{Z}^\alpha u = \mathrm{Z}^\beta \mathrm{Z}_n u = Z^\beta f_n(x, u)$ and hence

$$Z^\alpha u(x) = (\mathrm{Z}^\beta f_n)(x, u) + H(x),$$

where, thanks to (54) and (58),

$$\sup_K |(\mathrm{Z}^\beta f_n)(\cdot, u)| \leq A(K)^{|\alpha|+1} |\alpha|!^s \leq \frac{1}{2}\, M_{|\alpha|} \leq \frac{1}{2} M\, M_{|\alpha|}\,, \tag{63}$$

and [4]

$$H(x)=\sum_{0\neq\tilde\beta\leq\beta}\binom{\beta}{\tilde\beta}\sum_{\substack{\gamma_1+\ldots+\gamma_q=\tilde\beta\\|\gamma_j|>0}}C^{\tilde\beta}_{q,\gamma}\left(\partial^q_\zeta \mathrm{Z}^{\beta-\tilde\beta}f_n\right)(x,u(x))\mathrm{Z}^{\gamma_1}u(x)\cdots\mathrm{Z}^{\gamma_q}u(x)\,.$$

The induction hypothesis shows that $\mathrm{Z}^\alpha u\in C^1(\Omega)$. Moreover, (54) and (61) imply

$$\begin{aligned}\sup_{x\in K}|H(x)| &\leq \sum_{0\neq\tilde\beta\leq\beta}\binom{\beta}{\tilde\beta}A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s\sum_{\substack{\gamma_1+\ldots+\gamma_q=\tilde\beta\\|\gamma_j|>0}}C^{\tilde\beta}_{q,\gamma}A(K)^q q!M^q M_{|\gamma_1|}\cdots M_{|\gamma_q|}\\
&= \sum_{0\neq\tilde\beta\leq\beta}\binom{\beta}{\tilde\beta}A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s\sum_{\substack{\gamma_1+\ldots+\gamma_q=\tilde\beta\\|\gamma_j|>0}}C^{\tilde\beta}_{q,\gamma}A(K)^q q!M^q(\partial^{\gamma_1}\mathfrak{T})(0)\cdots(\partial^{\gamma_q}\mathfrak{T})(0)\\
&= \sum_{0\neq\tilde\beta\leq\beta}\binom{\beta}{\tilde\beta}A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s(\partial^{\tilde\beta}\mathfrak{t}_M)(0)\\
&\leq \frac{A(K)M}{1-\epsilon A(K)M}\sum_{0\neq\tilde\beta\leq\beta}\binom{\beta}{\tilde\beta}A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s M_{|\tilde\beta|}\\
&= \frac{A(K)M}{1-\epsilon A(K)M}\left\{A(K)M_{|\beta|}+\sum_{0\neq\tilde\beta\leq\beta,\,\tilde\beta\neq\beta}\binom{\beta}{\tilde\beta}A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s M_{|\tilde\beta|}\right\}.\end{aligned}$$

Now (58) gives

$$A(K)^{|\beta-\tilde\beta|+1}|\beta-\tilde\beta|!^s\leq\frac{1}{2}\,M_{|\beta-\tilde\beta|},\quad |\beta|>|\tilde\beta|, \tag{64}$$

and hence, (56) implies

$$\begin{aligned}\sup_{x\in K}|H(x)| &\leq \frac{A(K)M}{(1-\epsilon A(K)M)}\left\{A(K)M_{|\beta|}+\frac{1}{2}\sum_{0\neq\tilde\beta\leq\beta,\,\tilde\beta\neq\beta}\binom{\beta}{\tilde\beta}M_{|\beta-\tilde\beta|}\,M_{|\tilde\beta|}\right\}\\
&\leq \frac{A(K)M}{(1-\epsilon A(K)M)}\left\{A(K)M_{|\beta|}+\frac{1}{2}M_{|\beta|}\right\}.\end{aligned}$$

Finally, noticing that (57) implies $M_{|\beta|}\leq 2^{n+2}\epsilon M_{|\alpha|}$ (recall that $|\beta|\geq 1$) we conclude that

$$\sup_{x\in K}|H(x)|\leq\frac{1}{2}M\,M_\alpha$$

[4]Here we recall the Faà di Bruno formula

$$\partial^\eta_x\phi(w(x))=\sum C^\eta_{q,\gamma}\phi^{(q)}(w)\partial^{\gamma_1}_x w(x)\ldots\partial^{\gamma_q}_x w(x),$$

where the sum is performed over all multi-indices $\gamma_1,\ldots,\gamma_q$ with $|\gamma_j|>0$ and $\gamma_1+\ldots+\gamma_q=\eta$ and the constants $C^\eta_{q,\gamma}$ are universal.

if $\epsilon > 0$ is chosen small (depending only on K). This, together with (63), concludes the proof of the induction step for verifying (62) and hence the proof of Proposition 9.1. ■

If we return to the set up and notation employed in Theorems 3.1 and 4.1 we can state:

Corollary 9.1. *Assume that $\mathfrak{L}$ satisfies property $(\star)$ and let u be a C^1 function defined in an open neighborhood of the origin in $\mathbb{R}^m \times \Theta$ and satisfying the system*

$$(\mathrm{L}_j u)(x,t) = g_j(x,t,u(x,t)), \quad j = 1, \dots, n, \tag{65}$$

where the functions $g_j(x,t,\zeta)$ are Gevrey functions of order $s \geq 1$ for (x,t) near the origin in $\mathbb{R}^m \times \Theta$ and valued in the space of entire holomorphic with respect to $\zeta \in \mathbb{C}$.

1. *If $s = 1$ then u is real-analytic near the origin.*
2. *If $s > 1$ and if $m = 1$ then u is Gevrey of order $s/(1-\theta)$ near the origin.*

10. Final remarks

We close this work with some important remarks.

A. Assume $m = 1$. Although under condition $(\star)$ the system $\mathcal{L}$ is C^∞-hypoelliptic [Ma, 1980] $\mathcal{L}$ is not, in general, subelliptic. The most general regularity result known for this class of systems is proved in [JT, 2006]: given $s \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ then $\mathrm{L}_j u \in H^s$ near the origin, $j = 1, \dots, n$, implies $u \in H^{s-n/2}$ near the origin. Moreover, given $\rho > -(n-1)/4$, in the same article the authors provided examples of systems $\mathcal{L}$ satisfying $(\star)$ and for which there are $s \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ such that $\mathrm{L}_j u \in H^s$ near the origin, $j = 1, \dots, n$, but $u \notin H^{s+\rho}$ in any neighborhood of the origin. This fact suggests that it is likely that a similar result to that obtained in [BM, 1982] for principal type scalar operators does not hold for the systems studied in this work.

B. Still in the case $m = 1$ M. Derridj [D, 2006] has established a criterium which implies the subellipticity for the system $\mathcal{L}$ with loss of $\delta \geq 0$ derivatives. It is based on a geometric escape condition for the function ϕ and it is fair to conjecture that, under such condition, every s-Gevrey vector for $\mathcal{L}$ would belong to $G^{s'}$, where $s' = (s-\delta)/(1-\delta)$. In particular we would recover, when $n = 1$, the result in [BM, 1982], and we would also obtain a sharper regularity result for the classes of systems $\mathcal{L}$ discussed in [DH, 2008].

C. It was due to the lack of subellipticity that our definition of s-Gevrey vectors (cf. Definition 1.1) is given in terms of the L^∞-norm in the t-variables. It is not clear if it can be replaced by a weaker norm in t-space, although it is possible to replace the L^1-norm in the x-space by any negative Sobolev norm (cf. [BM, 1982], Remark 5.1]).

D. Again assume $m = 1$. If $|\Phi(t)| = \mathrm{O}(|t|^k)$, $k \geq 2$, arguing as in the proof of Proposition 2.2 shows that for any $1 < s \leq s' < ks - (k-1)$ there is a smooth function which is an s-Gevrey vector for $\mathfrak{L}$ but is not Gevrey of order s' in any neighborhood of the origin. Let then

$$d \doteq \min\{k : \text{ the } k\text{-jet of } \Phi \text{ at } 0 \text{ is not } 0\}.$$

If $\mathfrak{L}$ is not elliptic then $d \geq 2$, from what we obtain $|\Phi(t)| = \mathrm{O}(|t|^d)$ and $\theta_\phi \geq (d-1)/d$ (this last statement follows from a direct argument). Notice that, in this situation, $s' = ds$.

E. We emphasize that the result stated in Corollary 9.1(1) is stronger than the one proved in Theorem 9.1. Regarding the latter, it is possible, for some particular cases, to apply the same technique in order to obtain precise Gevrey regularity results. For instance, assume $m = 1$, the validity of condition $(\star)$, the commutation relations $[\mathcal{L}_j, \mathcal{L}_k] = 0$ and that the functions g_j are independent of x, Gevrey of order $s \geq 1$ in t and entire holomorphic with respect to $\zeta \in \mathbb{C}$. Under such hypotheses it is possible to prove that the Cauchy problem

$$\mathcal{L}_j H = 0, \ \ j = 1, \dots, n, \qquad H(0, \zeta) = \zeta,$$

has a solution $H(t, \zeta)$ near the origin in $\mathbb{R}^n \times \mathbb{C}$, which is Gevrey of order s in t and holomorphic in ζ. By the same argument used in the proof of Theorem 9.1, in conjunction with Theorem 4.1(1), it follows that any C^1 solution to the system $(\mathrm{L}_j u)(x,t) = g_j(t, u(x,t))$, $j = 1, \dots, n$, is Gevrey of order s near the origin.

F. The Gevrey regularity of the solution u, as stated in Corollary 9.1(2), can be improved when we also assume $n = 1$ (cf. [BM, 1982]).

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