

# Primitivity of monodromy groups of branched coverings: a non-orientable case

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## Abstract

In the present work we characterize even-fold branched coverings over the projective plane with primitive monodromy groups. This answers the question on the decomposability of this class of maps, filling a gap for the non-orientable case. We also present similar results for the case where the degree is 9.

**Key words:** branched coverings, primitive groups, imprimitive groups, permutation groups, projective plane.

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## 1 Introduction

It is a natural and standard question to know whether a map in a given class of maps is a composition or not of maps in the same class. In 1957 Borsuk and Molski [9] asked about the existence of a continuous map of finite order<sup>1</sup>, which is not a composition of *simple maps* (maps of order  $\leq 2$ ). More recently, in 2002, Krzempek [16] constructed covering maps on locally arcwise connected continua that are not factorizable into covering maps of order  $\leq n - 1$ , for all  $n$ . Also in 2002, Bogataya, Bogatyř and Zieschang [5] gave an example of a 4-fold covering of a surface of genus 2 by a surface of genus 5 that cannot be represented as a composition of two non-trivial open maps.

The present work is a contribution to the study of this kind of problem in the class of branched coverings  $\phi : M \rightarrow N$  between connected closed surfaces. It consists in classifying which branch data (see definition below) can be realized

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<sup>1</sup>A continuous map  $\phi$  defined on a space  $X$  is said to be of *order*  $\leq k \in \mathbb{Z}^+$  if for any  $y \in \phi(X)$ ,  $\phi^{-1}(y)$  contains at most  $k$  points.

by an indecomposable branched covering and which ones can be realized by a decomposable branched covering. Recently in [4], this problem was translated in the construction of primitive and imprimitive permutation groups as monodromy groups of branched coverings, respectively. The problem was solved in the case where  $N$  is different from the sphere  $S^2$  and the projective plane  $\mathbb{R}P^2$ . The main result is:

**Theorem** ([4], Theorem 3.3). *Every non-trivial admissible data are realized on any  $N$  with  $\chi(N) \leq 0$ , by an indecomposable primitive branched covering.*

The purpose of this work is to study the same question for branched coverings over  $N = \mathbb{R}P^2$ .

A branched covering  $\phi : M \rightarrow N$  of degree  $d$  between closed connected surfaces determines a finite collection of partitions  $\mathcal{D}$  of  $d$ , the *branch datum*, in correspondence with the *branch point set*  $B_\phi \subset N$ . The *total defect* of  $\mathcal{D}$  is defined by  $\nu(\mathcal{D}) = \sum_{x \in B_\phi} d - \#\phi^{-1}(x)$ . Conversely, given a collection of partitions  $\mathcal{D}$  of  $d$  and  $N \neq S^2$ , the necessary conditions for  $\mathcal{D}$  to be a branch datum are sufficient to realize it. Let us point out that the realization problem for  $\mathbb{R}P^2$  has been solved in [11], in this case we will call  $\mathcal{D}$  *admissible datum*.

From now on let  $N = \mathbb{R}P^2$  and  $\phi : M \rightarrow \mathbb{R}P^2$  a primitive branched covering, i.e. the induced homomorphism  $\pi_1(M) \rightarrow \pi_1(\mathbb{R}P^2)$  is surjective. Since  $\phi$  is an orientation-true map then  $M$  is nonorientable, see [6]. If the homomorphism is not surjective the map admits an obvious decomposition.

The main result in the present work is:

**Theorem 3.7.** *Let  $d$  be even and  $\mathcal{D} = \{D_1, \dots, D_s\}$  an admissible datum. Then,  $\mathcal{D}$  is realizable by an indecomposable branched covering if, and only if, either:*

- (1)  $d = 2$ , or
- (2) There is  $i \in \{1, \dots, s\}$  such that  $D_i \neq [2, \dots, 2]$ , or
- (3)  $d > 4$  and  $s > 2$ .

In Proposition 2.6[4] the authors classify admissible data realizable by decomposable primitive branched coverings, for  $\chi(N) \leq 0$ . Notice that the same proof applies when  $\chi(N) = 1$ , i.e.  $N = \mathbb{R}P^2$ . Then, except for the case  $d = 2$ , where the branched covering clearly is never decomposable, we can assume in Theorem 3.7 that  $\mathcal{D}$  is realizable by a decomposable primitive branched covering. This is, if we join the main result in this paper with Proposition 2.6[4] we characterize admissible data realizable by both, decomposable and indecomposable primitive branched coverings.

The case where  $N = \mathbb{R}P^2$  and  $d$  odd looks more subtle and it is a work in progress.

The general problem of classifying the branched coverings from the viewpoint of decomposability is related with the Inverse Galois problem (see for example

the references [13] and [18]) and it is similar to the problem of primitive and imprimitive monodromy groups studied in [17]. Besides the facts mentioned above, the problem is interesting in its own right.

The paper is divided into two sections apart from the introduction. In Section 2, we quote the main definitions and the results about realization of branched coverings over  $\mathbb{R}P^2$ . In Section 3, we characterize branch data realizable by indecomposable branched coverings with even degree and we analyze the case  $d = 9$ .

## 2 Preliminaries, terminology and notation

### 2.1 Permutation groups

We denote by  $\Sigma_d$  the symmetric group on a set  $\Omega$  with  $d$  elements and by  $1_d$  its identity element. If  $\alpha \in \Sigma_d$  and  $x \in \Omega$ ,  $x^\alpha$  is the image of  $x$  by  $\alpha$ . An explicit permutation  $\alpha$  will be written either as a product of disjoint cycles, i.e. its *cyclic decomposition*, or in the following way:

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & 2k+1 \\ 1^\alpha & 2^\alpha & \dots & (2k+1)^\alpha \end{pmatrix},$$

depending on what is more convenient. The set of lengths of the cycles in the cyclic decomposition of  $\alpha$ , including the trivial ones, defines a partition of  $d$ , say  $D_\alpha = [d_{\alpha_1}, \dots, d_{\alpha_t}]$ , called *the cyclic structure of  $\alpha$* . Define

$$\nu(\alpha) := \sum_{i=1}^t (d_{\alpha_i} - 1) = d - t, \quad (1)$$

then  $\alpha$  will be an *even permutation* if  $\nu(\alpha) \equiv 0 \pmod{2}$ . Given a partition  $D$  of  $d$ , we say  $\alpha \in D$  if the cyclic structure of  $\alpha$  is  $D$  and we put  $\nu(D) := \nu(\alpha)$ .

For  $1 < r \leq d$ , a permutation  $\alpha \in \Sigma_d$  is called a *r-cycle* if in its cyclic decomposition its unique non-trivial cycle has length  $r$ . Permutations  $\alpha, \beta \in \Sigma_d$  are *conjugate* if there is  $\lambda \in \Sigma_d$  such that  $\alpha^\lambda := \lambda\alpha\lambda^{-1} = \beta$ . It is a known fact that conjugate permutations have the same cyclic structure.

Given a permutation group  $G$  on  $\Omega$  and  $x \in \Omega$ , one defines the *isotropy subgroup of  $x$* ,  $G_x := \{g \in G : x^g = x\}$ , and the *orbit of  $x$  by  $G$* ,  $x^G := \{x^g : g \in G\}$ . For  $H \subset G$ , the subsets  $Supp(H) := \{x \in \Omega : x^h \neq x \text{ for some } h \in H\}$  and  $Fix(H) := \{x \in \Omega : x^h = x \text{ for all } h \in H\}$  are defined. For  $\Lambda \subset \Omega$  and  $g \in G$ ,  $\Lambda^g := \{y^g : y \in \Lambda\}$ .

The permutation group  $G$  is *transitive* if for all  $x, y \in \Omega$  there is  $g \in G$  such that  $x^g = y$ . A nonempty subset  $\Lambda \subset \Omega$  is a *block* of a transitive group  $G$  if for each  $g \in G$  either  $\Lambda^g = \Lambda$  or  $\Lambda^g \cap \Lambda = \emptyset$ . A block  $\Lambda$  is *trivial* if either  $\Lambda = \Omega$  or

$\Lambda = \{x\}$  for some  $x \in \Omega$ . Given a block  $\Lambda$  of  $G$ , the set  $\Gamma := \{\Lambda^\alpha : \alpha \in G\}$  defines a partition of  $\Omega$  in blocks. This set is called a *system of blocks containing  $\Lambda$*  and the cardinality of  $\Lambda$  divides the cardinality of  $\Omega$ .  $G$  acts naturally on  $\Gamma$ . A transitive permutation group is *primitive* if it admits only trivial blocks. Otherwise it is *imprimitive*.

**Example 1.** A transitive permutation group  $G < \Sigma_d$  containing a  $(d-1)$ -cycle is primitive. Without loss of generality let us suppose that  $g = (1 \dots d-1)(d) \in G$ . Then any proper subset  $\Lambda$  of  $\{1, \dots, d\}$  containing  $d$  and at least one more element satisfies  $\Lambda^g \neq \Lambda$  and  $\Lambda^g \cap \Lambda \neq \emptyset$ . Thus the blocks of  $G$  are trivial and  $G$  is primitive.

**Proposition 2.1** ([10], Corollary 1.5A). Let  $G$  be a transitive permutation group on a set  $\Omega$  with at least two points. Then  $G$  is primitive if and only if each isotropy subgroup  $G_x$ , for  $x \in \Omega$ , is a maximal subgroup of  $G$ .  $\square$

## 2.2 Branched coverings on the projective plane

A surjective continuous open map  $\phi : M \rightarrow N$  between closed surfaces such that:

- for each  $x \in N$ ,  $\phi^{-1}(x)$  is a totally disconnected set, and
- there is a non-empty discrete set  $B_\phi \subset N$  such that the restriction  $\hat{\phi} := \phi|_{M-\phi^{-1}(B_\phi)}$  is an ordinary unbranched covering of degree  $d$ ,

is called a *branched covering of degree  $d$  over  $N$*  and it is denoted by  $(M, \phi, N, B_\phi, d)$ .  $N$  is the *base surface*,  $M$  is the *covering surface* and  $B_\phi$  is the *branch point set*. Its *associated unbranched covering* is denoted by  $(\widehat{M}, \hat{\phi}, \widehat{N}, d)$ , where  $\widehat{N} := N - B_\phi$  and  $\widehat{M} := M - \phi^{-1}(B_\phi)$ . It is known that  $\chi(\widehat{M}) = d\chi(\widehat{N})$ , equivalently

$$\chi(M) - \#\phi^{-1}(B_\phi) = d(\chi(N) - \#B_\phi). \quad (2)$$

The set  $B_\phi$  is just the image of the points in  $M$  in which  $\phi$  fails to be a local homeomorphism. Then each  $x \in B_\phi$  determines a non-trivial partition  $D_x$  of  $d$ , defined by the local degrees of  $\phi$  on each component in the preimage of a small disk  $U_x$  around  $x$ , with  $U_x \cap B_\phi = \{x\}$ . The collection  $\mathcal{D} := \{D_x\}_{x \in B_\phi}$  is called the *branch datum* and its *total defect* is the positive integer defined by  $\nu(\mathcal{D}) := \sum_{x \in B_\phi} \nu(D_x)$ . The total defect satisfies the *Riemann-Hurwitz formula*:

$$\nu(\mathcal{D}) = d\chi(N) - \chi(M). \quad (3)$$

Associated to  $(M, \phi, N, B_\phi, d)$  we have a permutation group, the *monodromy group of  $\phi$* , given by the image of the *Hurwitz's representation*

$$\rho_\phi : \pi_1(N - B_\phi, z) \rightarrow \Sigma_d, \quad (4)$$

which sends each class  $\alpha \in \pi_1(N - B_\phi, z)$  to a permutation of  $\phi^{-1}(z) = \{z_1, \dots, z_d\}$ , which indicates the terminal point of the lifting of a loop in  $\alpha$  after fixing the initial point. In particular, for  $x \in B_\phi$ , let  $c_x$  be a path from  $z$  to a small circle  $a_x$  about  $x$  and define the loop class  $\mathbf{u}_x := [c_x a_x c_x^{-1}]$ . Then the cyclic structure of the permutation  $\alpha_x := \rho_\phi(\mathbf{u}_x)$  is given by  $D_x$  and  $\nu(\prod_{x \in B_\phi} \alpha_x) \equiv \nu(\mathcal{D}) \pmod{2}$ . The problem of realization a branch datum is equivalent to an algebraic problem in terms of representation on the symmetric group. More precisely:

**Theorem 2.2** ([15], Theorem 3). *Let  $N$  be a surface,  $\mathcal{D}$  a finite collection of partitions of  $d$  and  $F \subset N$  a discrete subset such that there is a bijection  $b : F \rightarrow \mathcal{D}$ . If there is a representation  $\rho : \pi_1(N - F, z) \rightarrow \Sigma_d$  which for each  $x \in F$ ,  $\rho(\mathbf{u}_x)$  is a permutation with cyclic structure given by  $b(x)$ , then  $\mathcal{D}$  is realizable as branch datum of a branched covering on  $N$ . Moreover if the action is transitive, the covering surface will be connected.*  $\square$

**Remark.** *If  $N = \mathbb{R}P^2$  and  $\mathcal{D} = \{D_1, \dots, D_s\}$ , to define  $\rho_\phi$ , it is necessary and sufficient to have permutations  $\alpha_i \in D_i$ , for  $i = 1, \dots, s$ , such that  $\prod_{i=1}^s \alpha_i$  is a square and  $\langle \alpha_1, \dots, \alpha_s \rangle$  is transitive. This follows from the presentation  $\pi_1(\mathbb{R}P^2 - \{x_1, \dots, x_s\}) = \langle a, \mathbf{u}_1, \dots, \mathbf{u}_s \mid \prod_{i=1}^s \mathbf{u}_i = a^{-2} \rangle$ .*

**Example 2.** *If  $r > 0$  is an odd natural number then every  $r$ -cycle is the square of a permutation:*

$$\text{if } \alpha = (a_1 a_2 \dots a_r) \text{ then } \alpha = \beta^2 \text{ where } \beta = (a_1 a_{(\frac{r+1}{2})+1} a_2 a_{(\frac{r+1}{2})+2} \dots a_r a_{\frac{r+1}{2}}).$$

The branch data which can be realized by branched coverings over  $\mathbb{R}P^2$  are given by:

**Theorem 2.3** ([11], Theorem 5.1). *Let  $\mathcal{D}$  be a collection of partitions of  $d$ . Then there is a branched covering  $\phi : M \rightarrow \mathbb{R}P^2$  of degree  $d$ , with  $M$  connected and with branch datum  $\mathcal{D}$  if and only if*

$$d - 1 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}. \quad (5)$$

*Moreover,  $M$  can be chosen to be nonorientable.*  $\square$

**Remark.** *The realization result above does not tell which branch data can be realized by an orientable covering. In fact it is not hard to show that there is a bijection between the set of branched coverings over  $\mathbb{R}P^2$  where the covering surface is orientable and the set of branched coverings over the sphere  $S^2$  which have an even number of branched points. It is certainly an interesting problem to classify such realizable brached data over the sphere  $S^2$  from the viewpoint of decomposibility.*

**Definition 2.4.** *A collection of partitions  $\mathcal{D}$  of  $d$  satisfying (5) will be called an admissible datum.*

### 3 Decomposability

Given a covering, it is *decomposable* if it can be written as a composition of two non-trivial coverings (i.e., both with degree bigger than 1), otherwise it is called *indecomposable*. In a decomposition of a branched covering at least one of its components is a branched covering having proper branching. Moreover, since the degree of a decomposable covering is the product of the degrees of its components (see [5], Theorem 2.3), we are interested in branched coverings with non-prime degree.

From now on we will consider *primitive maps* (i.e the induced homomorphisms are surjections) because if not they have an obvious decomposition.

**Proposition 3.1** ([4], Proposition 1.8). *A primitive branched covering is decomposable if and only if its monodromy group is imprimitive.*  $\square$

**Remark.** *Notice that a primitive branched covering does not have, necessarily, a primitive monodromy group.*

In [4] Proposition 2.6, the authors classify admissible data realizable by decomposable primitive branched coverings, for  $\chi(N) \leq 0$ . Moreover in [4] Theorem 3.3, they showed that any such branch datum is also realized by an indecomposable primitive branched covering. This means that decomposable and indecomposable realizations may coexist when  $\chi(N) \leq 0$ .

For the case  $\chi(N) = 1$ , i.e.  $N = \mathbb{R}P^2$ , Proposition 2.6 [4] remains true (with almost the same proof) but Theorem 3.3 [4] is not true, as shows the following example.

**Example 3.** *If  $d$  is non-prime, the branched covering  $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$  is decomposable. In fact, since there is only one branch point, the branch datum has only one partition  $D$  of  $d$ . Thus by (3)  $\nu(D) = d - 1$ , by (5)  $d$  is odd and by (1)  $D = [d]$ . Then the representation*

$$\begin{aligned} \rho : \pi_1(\mathbb{R}P^2 - \{x\}) = \langle a, \mathbf{u}_x \mid a^2 u_x = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha, \\ \mathbf{u}_x &\longmapsto \gamma, \end{aligned}$$

*where  $\alpha^2 = \gamma^{-1}$  is a  $d$ -cycle, implies that  $\alpha$  is a  $d$ -cycle. Therefore every isotropy subgroup of the monodromy group  $G := \text{Im} \rho = \langle \gamma, \alpha \rangle = \langle \alpha \rangle$  is trivial and, since  $d$  is non-prime, it is contained in a proper subgroup of  $G$ . Then by Proposition 2.1,  $G$  is imprimitive.*

The problem of realizing a collection of partitions as branch datum of a branched covering over the projective plane was solved in [11]. Using Lemma 4.5 [11], arguments in the proof of Theorem 5.1 therein, and translating them to our framework, yields the following propositions.

**Remark.** Given  $(M, \phi, \mathbb{R}P^2, B_\phi, d)$  primitive with branch datum  $\mathcal{D} = \{D_1, \dots, D_s\}$ , notice that if  $d$  is even, by (3) and (5),  $\chi(M)$  is even and since  $M$  is non-orientable  $\chi(M) \leq 0$ . Then by (2),  $s > 1$ .

**Proposition 3.2.** Let  $d > 2$  be an even number and  $\mathcal{D} = \{D_1, \dots, D_s\}$  an admissible datum. If  $\mathcal{D}$  contains a partition different from  $[2, \dots, 2]$ , then there is an indecomposable branched covering  $(M, \phi, \mathbb{R}P^2, B_\phi, d)$  realizing  $\mathcal{D}$ .

*Proof.* We know that  $s \geq 2$ . In the case  $s = 2$  the result follows from Lemma 4.5 [11] together with the above Example 1. If  $s > 2$  from the proof of the Theorem 5.1 [11] we associate a new branch datum with  $s = 2$  which satisfies the hypothesis of the proposition. Furthermore in the proof of Theorem 5.1 [11] a realization of the given branch datum is constructed such that its monodromy contains the monodromy of the realization of the new branch datum with  $s = 2$ . Hence we have an indecomposable realization and this completes the proof.  $\square$

**Proposition 3.3.** Let  $d > 4$  be even and  $\mathcal{D} = \{D_1, \dots, D_s\}$  an admissible datum such that  $D_i = [2, \dots, 2]$  for  $i = 1, \dots, s$ . If  $s > 2$ , there is an indecomposable branched covering  $(M, \phi, \mathbb{R}P^2, B_\phi, d)$  realizing  $\mathcal{D}$ .

*Proof.* Since  $\nu(D_1) + \nu(D_2) = d$ , by Lemma 4.5 [11] there are permutations  $\gamma_1 \in D_1$ ,  $\gamma_2 \in D_2$  such that  $\gamma_1\gamma_2 \in D_{12} := [d/2, d/2]$ . The new collection of partitions  $\{D_{12}, D_3, \dots, D_s\}$ , satisfies the hypothesis of the previous proposition. Let  $H = \langle \gamma, \gamma_3, \dots, \gamma_s \rangle$  the monodromy of branched covering given by proposition above, with  $\gamma \in [d/2, d/2]$  and  $\gamma_i \in D_i$ , for  $i = 3, \dots, s$ . Notice that there is  $\lambda \in \text{Sym}_d$  such that  $\gamma = \lambda\gamma_1\lambda^{-1}$ , then  $G = \langle \lambda\gamma_1\lambda^{-1}, \lambda\gamma_2\lambda^{-1}, \gamma_3, \dots, \gamma_s \rangle$  is a monodromy for a indecomposable realization of the given datum.  $\square$

The remaining cases, for the even degree case, are covered by Proposition 3.5 and Proposition 3.6. Before state them, let us consider the following lemma.

**Lemma 3.4.** Let  $d \neq 2$  be even and  $\alpha, \beta \in [2, \dots, 2]$ , such that  $G := \langle \alpha, \beta \rangle$  is transitive. Then  $G$  is imprimitive and unique up to conjugation.

*Proof.* Let us suppose  $d = 2k$ , with  $1 < k \in \mathbb{N}$ . To obtain  $\langle \alpha, \beta \rangle$  transitive, it is necessary that transpositions in  $\beta$  “link”  $k$  transpositions in  $\alpha$ . For that, we need  $k - 1$  transpositions and thus  $\beta$  is automatically defined. Then, up to conjugation, we can consider  $\alpha = (1\ 2)(3\ 4) \dots (d-1\ d)$  and  $\beta = (2\ 3)(4\ 5) \dots (d-2\ d-1)(d\ 1)$ . Thus  $\alpha\beta = (1\ 3 \dots d-1)(2\ 4 \dots d)$  and the set  $B = \{1, 3, \dots, d-1\}$  will be a nontrivial block of  $G = \langle \alpha, \beta \rangle$ , which makes it imprimitive.  $\square$

**Proposition 3.5.** A primitive branched covering realizing  $\{[2, \dots, 2], [2, \dots, 2]\}$  is decomposable.

*Proof.* Let  $(M, \phi, \mathbb{R}P^2, \{x, y\}, d)$  be a primitive branched covering with branch datum  $\{[2, \dots, 2], [2, \dots, 2]\}$ . If

$$\begin{aligned} \rho : \langle a, \mathbf{u}_1, \mathbf{u}_2 | a^2 \mathbf{u}_1 \mathbf{u}_2 = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha, \\ \mathbf{u}_1 &\longmapsto \gamma_1, \\ \mathbf{u}_2 &\longmapsto \gamma_2, \end{aligned}$$

is its Hurwitz's representation,  $G := \text{Im} \rho = \langle \alpha, \gamma_1, \gamma_2 | \gamma_1^2 = \gamma_2^2 = 1, \alpha^2 \gamma_1 \gamma_2 = 1 \rangle$  is its monodromy group with  $\gamma_1, \gamma_2 \in [2, \dots, 2]$ .

If  $\langle \gamma_1, \gamma_2 \rangle$  is transitive,  $G$  is imprimitive by Lemma 3.4. Thus, by Proposition 3.1 the branched covering is decomposable.

If  $\langle \gamma_1, \gamma_2 \rangle$  is not transitive, using relations in  $G$ , is easy to see that  $[\text{Fix}(\gamma_1 \gamma_2)]^{\gamma_i} \subset \text{Fix}(\gamma_1 \gamma_2)$ , for  $i = 1, 2$ , and  $[\text{Fix}(\gamma_1 \gamma_2)]^\alpha \subset \text{Fix}(\gamma_1 \gamma_2)$ . Then

$$\text{Fix}(\gamma_1 \gamma_2) = [\text{Fix}(\gamma_1 \gamma_2)]^{\gamma_i} = [\text{Fix}(\gamma_1 \gamma_2)]^\alpha,$$

because  $\gamma_i$ , for  $i = 1, 2$ , and  $\alpha$  are bijections. Then for all  $g \in G$  we have  $[\text{Fix}(\gamma_1 \gamma_2)]^g = \text{Fix}(\gamma_1 \gamma_2)$ . If  $\text{Fix}(\gamma_1 \gamma_2) \neq \emptyset$ , since  $G$  is transitive, then  $\gamma_1 \gamma_2 = 1$ . Up to conjugation,  $\gamma_1 = \gamma_2 = (1\ 2)(3\ 4) \dots (d-3\ d-2)(d-1\ d)$ . By the relation, the options for  $\alpha$  are either  $\alpha := (2\ 3)(4\ 5) \dots (d-2\ d-1)(d\ 1)$  or  $\alpha := (1)(2\ 3)(4\ 5) \dots (d-2\ d-1)(d)$ . The first option implies  $G$  equal to the group in Lemma 3.4, therefore it is imprimitive. The second one produce the block  $\{1, d\}$ , and  $G$  will be also imprimitive. If  $\text{Fix}(\gamma_1 \gamma_2) = \emptyset$  then every cycle of  $\alpha$  has length  $\geq 3$  and  $\gamma_1, \gamma_2$  have not common cycles. Let  $O_1, \dots, O_k$  be the orbits of the action of  $\langle \gamma_1, \gamma_2 \rangle$  on  $\{1, \dots, d\}$ , with  $k > 1$ . Notice that, for  $i = 1, \dots, k$ ,  $\#O_i \geq 4$  is even, because each transposition of  $\gamma_2$  links two transpositions of  $\gamma_1$ , then it connects an even number of elements. On the other hand, if  $\langle \gamma_1, \gamma_2 \rangle_i$  denotes the restriction of the action of  $\langle \gamma_1, \gamma_2 \rangle$  on  $O_i$ , we are in the situation of Lemma 3.4, therefore  $\langle \gamma_1, \gamma_2 \rangle_i$  is imprimitive. If  $\#O_i = 2n$  for  $n \in \mathbb{Z}^+$ , then its elements appear in  $\gamma_1 \gamma_2$  in the form  $(a_{i_1} \dots a_{i_n})(a_{i_{n+1}} \dots a_{i_{2n}})$ . Considering the relation  $\gamma_1 \gamma_2 = \alpha^{-2}$ , we conclude that  $\alpha$  will connect two orbits,  $O_i$  and  $O_j$ , only if  $\#O_i = \#O_j$ . Moreover  $\alpha$  makes the group  $G$  transitive, implying that all the orbits have cardinality  $2n$ . For example, if  $i \neq j$  and the elements of  $O_j$  are in  $\gamma_1 \gamma_2$  in the form  $(b_{j_1} \dots b_{j_n})(b_{j_{n+1}} \dots b_{j_{2n}})$ , whitout loss of generality  $(a_{i_1} b_{j_1} a_{i_2} b_{j_2} \dots a_{i_n} b_{j_n})$  is a cycle of  $\alpha$  and thus, the blocks of  $\langle \gamma_1, \gamma_2 \rangle_i$ , for  $i = 1, \dots, k$ , become blocks for  $G$ . Then  $G$  is imprimitive and the branched covering is decomposable.  $\square$

**Proposition 3.6.** *A primitive branched covering of degree 4 realizing the finite collection  $\mathcal{D} = \{[2, 2], \dots, [2, 2]\}$  is decomposable.*



*Proof.* Let  $(M, \phi, \mathbb{R}P^2, B_\phi, 4)$  be a primitive branched covering with branch datum  $\mathcal{D}$ . Suppose  $\nu(\mathcal{D}) = 2s$ ,  $2 \leq s \in \mathbb{Z}^+$ . Let

$$\begin{aligned} \rho : \langle a, \mathbf{u}_1, \dots, \mathbf{u}_t | a^2 \prod_{i=1}^s \mathbf{u}_i = 1 \rangle &\longrightarrow \Sigma_4 \\ a &\longmapsto \alpha \\ \mathbf{u}_i &\longmapsto \gamma_i. \end{aligned}$$

be its Hurwitz's representation. Note that the possible images for  $\prod_{i=1}^s \mathbf{u}_i$  are, without loss of generality, either  $(1)(2)(3)(4)$  or  $(12)(34)$ . Define  $U := \langle \gamma_1, \dots, \gamma_s \rangle$ . If  $U$  is transitive, then  $U \cong \langle (12)(34), (13)(24) \rangle$  is imprimitive, because each pair of elements is a block. Thus, if  $\rho(\prod_{i=1}^s \mathbf{u}_i) = 1$ , the group  $G := \text{Im} \rho = \langle U, \alpha | \alpha^2 \prod_{i=1}^s \gamma_i = 1 \rangle$  is imprimitive, for all  $\alpha$ . On the other hand, if  $\rho(\prod_{i=1}^t \mathbf{u}_i) = (12)(34)$  then, either  $\alpha = (1324)$  or  $\alpha = (1423)$ . Whichever the case,  $\{1, 2\}$  is a block. If  $U$  is not transitive, then  $U \cong \langle (12)(34) \rangle$  and, for guarantee the transitivity of  $G$ , we have  $\alpha = (1324)$ . Thus  $\{1, 2\}$  is a block and  $G$  is imprimitive.  $\square$

We summarize the even degree case in the following theorem:

**Theorem 3.7.** *Let  $d$  be even and  $\mathcal{D} = \{D_1, \dots, D_s\}$  an admissible datum. Then,  $\mathcal{D}$  is realizable by an indecomposable branched covering if, and only if, either:*

- (1)  $d = 2$ , or
- (2) There is  $i \in \{1, \dots, s\}$  such that  $D_i \neq [2, \dots, 2]$ , or
- (3)  $d > 4$  and  $s > 2$ .  $\square$

The odd degree case looks more subtle and it is a work in progress. By brute force we solved the first non-trivial case  $d = 9$ . The table below list all possible branch data which can be realized by decomposable. Among them, the cases in 8, 19, 20 and 26 can be realized only by decomposable branched coverings, and they all have defect equal to  $d - 1 = 8$ . For the remaining cases we provided an indecomposable realization. This follow from the fact that for each one of these cases, there is  $\gamma$  such that  $\alpha\beta = \gamma^2$  (by Example 2). Moreover notice that the possible non-trivial blocks are given by the orbits of  $(\alpha\beta)^3$ , but they are not invariant either by  $\alpha$  or  $\beta$ . Then  $G = \langle \alpha, \beta, \gamma | \alpha\beta = \gamma^2 \rangle$  is primitive.

Finally we observe that if an admissible branch datum is not part of the table, that means that it can be realized only by indecomposable branched coverings.

	$\mathcal{D}$	$\nu(\mathcal{D})$	$\alpha$	$\beta$	$\alpha\beta$
1	{[6, 3], [6, 3]}	14	(1 2 3 4 5 6)(7 8 9)	(1 7 3)(2 4 5 6 8 9)	(1 4 6 7 9 3 5 8 2)
2	{[6, 3], [6, 1, 1, 1]}	12	(1 2 3 4 5 6)(7)(8)(9)	(1 7 8)(2 3 4 5 6 9)	(1 3 5 9 2 4 6 7 8)
3	{[6, 3], [4, 2, 2, 1]}	12	(1 2 3 4)(5 6)(7 8)(9)	(1 5 7)(2 9 3 4 6 8)	(1 9 3 6 7 2 4 5 8)
4	{[6, 3], [3, 3, 2, 1]}	12	(1 2 3)(4 5 6)(7 8)(9)	(1 4 7)(2 9 3 5 6 8)	(1 9 3 4 6 7 2 5 8)
5	{[6, 3], [3, 2, 2, 2]}	12	(1 2 3)(4 5)(6 7)(8 9)	(1 4 6)(2 8 3 5 7 9)	(1 8 2 5 6 9 3 4 7)
6	{[6, 3], [3, 2, 1, 1, 1, 1]}	10	(1 2 3)(4 5)(6)(7)(8)(9)	(1 4 6)(2 7 8 9 3 5)	(1 7 8 9 3 4 2 5 6)
7	{[6, 3], [2, 2, 2, 1, 1, 1]}	10	(1 2)(3)(4 5)(6)(7 8)(9)	(1 3 4)(2 6 7 9 5 8)	(1 6 7 2 3 4 8 9 5)
8	{[6, 3], [2, 1, 1, 1, 1, 1, 1]}	8	X	X	X
9	{[1, 2, 6], [1, 2, 6]}	12	(1 2 3 4 5 6)(7 8)(9)	(1)(2 7)(3 9 4 5 6 8)	(1 7 3 5 8 2 9 4 6)
10	{[1, 2, 6], [2, 3, 4]}	12	(1 2 3 4 5 6)(7 8)(9)	(1 7)(2 9 4)(3 5 6 8)	(1 9 4 6 7 3 2 5 8)
11	{[2, 3, 4], [2, 3, 4]}	12	(1 2 3 4)(5 6 7)(8 9)	(1 5)(2 4 8)(3 6 7 9)	(1 8 3 2 6 9 4 5 7)
12	{[3 3 3], [3, 3, 3]}	12	(1 2 3)(4 5 6)(7 8 9)	(1 4 7)(2 3 5)(6 8 9)	(1 3 4 2 5 8 6 7 9)
13	{[1, 2, 6], [4, 2, 1, 1, 1]}	10	(1 2 3 4 5 6)(7 8)(9)	(1)(2)(3)(4 7)(5 9 6 8)	(1 2 3 7 5 8 4 9 6)
14	{[1, 2, 6], [2, 2, 2, 2, 1]}	10	(1 2 3 4 5 6)(7 8)(9)	(1)(2 7)(3 9)(4 6)(5 8)	(1 7 5 4 8 2 9 3 6)
15	{[2, 3, 4], [4, 2, 1, 1, 1]}	10	(1 2 3 4)(5 6 7)(8 9)	(1)(2)(3)(5 4)(6 8 7 9)	(1 2 3 5 8 6 9 7 4)
16	{[2, 3, 4], [2, 2, 2, 2, 1]}	10	(1 2)(3 4)(5 6)(7 8)(9)	(1 3)(2 5 7)(4 9 6 8)	(1 5 8 2 3 9 6 7 4)
17	{[3, 3, 3], [3, 2, 2, 1, 1]}	10	(1 2 3)(4 5 6)(7 8 9)	(1)(2)(4 7)(5 3)(6 8 9)	(1 2 5 8 6 7 9 4 3)
18	{[3, 3, 3], [3, 3, 1, 1, 1]}	10	(1 2 3)(4 5 6)(7 8 9)	(1)(2)(4)(5 7 3)(6 8 9)	(1 2 5 8 6 4 7 9 3)
19	{[3, 3, 3], [3, 1, 1, 1, 1, 1, 1]}	8	X	X	X
20	{[3, 3, 3], [2, 2, 1, 1, 1, 1, 1]}	8	X	X	X
21	{[6, 1, 1, 1], [2, 2, 2, 1, 1, 1]}	8	(1 2 3 4 5 6)(7)(8)(9)	(1)(2)(3)(4 7)(5 8)(6 9)	(1 2 3 7 4 8 5 9 6)
22	{[4, 2, 2, 1], [2, 2, 2, 1, 1, 1]}	8	(1 2)(3 4)(5 6)(7)(8)(9)	(1)(3 2)(5 4)(6 7 8 9)	(1 3 5 7 8 9 6 4 2)
23	{[3, 2, 2, 2], [2, 2, 2, 1, 1, 1]}	8	(1 2)(3 4)(5 6)(7)(8)(9)	(1 3)(2 5)(4 7)(6 8 9)	(1 5 8 9 6 2 3 7 4)
24	{[4, 2, 1, 1, 1], [4, 2, 1, 1, 1]}	8	(1 2 3 4)(5 6)(7)(8)(9)	(1)(2)(3)(5 7)(6 8 9 4)	(1 2 3 6 7 5 8 9 4)
25	{[4, 2, 1, 1, 1], [2, 2, 2, 2, 1]}	8	(1 2 3 4)(5 6)(7)(8)(9)	(1)(2 5)(3 7)(4 8)(6 9)	(1 5 9 6 2 7 3 8 4)
26	{[2, 2, 2, 2, 1], [2, 2, 2, 2, 1]}	8	X	X	X
27	{[6, 1, 1, 1], [6, 1, 1, 1]}	10	(1 2 3 4 5 6)(7)(8)(9)	(1)(2)(3)(4 7 8 9 5 6)	(1 2 3 7 8 9 5 4 6)
28	{[6, 1, 1, 1], [4, 2, 2, 1]}	10	(1 2 3 4)(5 6)(7 8)(9)	(1)(2)(3)(4 5 7 9 6 8)	(1 2 3 5 8 9 6 7 4)
29	{[6, 1, 1, 1], [3, 2, 2, 2]}	10	(1 2 3 4 5 6)(7)(8)(9)	(1 7)(2 8)(3 9)(4 5 6)	(1 8 2 9 3 5 4 6 7)
30	{[4, 2, 2, 1], [4, 2, 2, 1]}	10	(1 2 3 4)(5 6)(7 8)(9)	(1)(2 5)(3 7)(4 9 6 8)	(1 5 8 3 9 6 2 7 4)
31	{[4, 2, 2, 1], [3, 2, 2, 2]}	10	(1 2 3 4)(5 6)(7 8)(9)	(1 5)(2 7)(3 9)(4 6 8)	(1 7 4 5 8 2 9 3 6)
32	{[3, 2, 2, 2], [3, 2, 2, 2]}	10	(1 2)(3 4)(5 6)(7 8 9)	(1 3)(2 5)(4 7)(6 8 9)	(1 5 8 6 2 3 7 9 4)

For  $d$  odd, in general, some exceptional cases can be analyzed. For example when the admissible branch datum contains one partition given by  $[d]$ .

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