Submanifolds of products of space forms.

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Abstract

We give a complete classification of submanifolds with parallel second fundamental form of a product of two space forms. We also reduce the classification of umbilical submanifolds with dimension $m \geq 3$ of a product $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ of two space forms whose curvatures satisfy $k_1 + k_2 \neq 0$ to the classification of m-dimensional umbilical submanifolds of codimension two of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. The case of $\mathbb{S}^n \times \mathbb{R}$ was carried out in [13]. As a main tool we derive reduction of codimension theorems of independent interest for submanifolds of products of two space forms.

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1 Introduction

Let $f\colon M \to N$ be an isometric immersion between Riemannian manifolds and let $\alpha\colon TM \times TM \to N_f M$ be its second fundamental form with values in the normal bundle. Then f is said to have *parallel second fundamental form* if $\nabla\alpha = 0$, where $\nabla\alpha$ is the Van der Waerden-Bortolloti covariant derivative of α. One also says for short that *f is parallel*. Roughly speaking, this means that f has the same second fundamental form at any point of M, if tangent and normal spaces at any two distinct points are identified by means of parallel transport in the tangent and normal connections, respectively, along any curve joining them.

Parallel submanifolds of Euclidean space have been classified by Ferus [9]. He showed that all of them are products of an Euclidean factor and standard minimal embeddings into hyperspheres of symmetric R-spaces, which are orbits of special types of s-representations. The case of the sphere is an easy consequence of the Euclidean one, whereas the classification of parallel submanifolds of hyperbolic space was carried out independently by Backes–Reckziegel [1] and Takeuchi [17].

Apart from space forms, however, parallel submanifolds of a Riemannian manifold have been classified only in a few other cases, e.g. for simply connected rank one symmetric spaces (see, e.g., the discussion in Chapter 9 of [3]).

One of our main results is a complete classification of parallel submanifolds of a product of two space forms. We state separately the cases in which one of the factors is flat or not. First observe that, given $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 > 0$, the map

$$g\colon \mathbb{Q}_k^n \to \mathbb{Q}_{k_1}^n \times \mathbb{Q}_{k_2}^n, \quad g(x) = (ax, bx), \tag{1}$$

where $k = k_1 k_2/(k_1 + k_2)$, $a^2 = k_2/(k_1 + k_2)$ and $b^2 = k_1/(k_1 + k_2)$, is a totally geodesic isometric embedding (see Example 16 below).

We say that $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ is an isometric immersion into a slice of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ if there exist an isometric immersion $\tilde{f}\colon M^m \to \mathbb{Q}_{k_1}^{n_1}$ (resp., $\tilde{f}\colon M^m \to \mathbb{Q}_{k_2}^{n_2}$) and a totally geodesic inclusion $j\colon \mathbb{Q}_{k_1}^{n_1} \to \mathbb{Q}_{k_1}^{n_1} \times \{x_2\}$ (resp., $j\colon \mathbb{Q}_{k_2}^{n_2} \to \{x_1\} \times \mathbb{Q}_{k_2}^{n_2}$) such that $f = j \circ \tilde{f}$. All manifolds are assumed to be connected.

Theorem 1. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1 k_2 \neq 0$, be a parallel isometric immersion. Then one of the following possibilities holds:*

(i) f is a parallel isometric immersion into a slice of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$;

(ii) M^m is locally a Riemannian product $M^m = M_1^{m_1} \times M_2^{m_2}$ and $f = f_1 \times f_2$, where $f_i\colon M_i^{m_i} \to \mathbb{Q}_{k_i}^{n_i}$, $1 \leq i \leq 2$, is a parallel isometric immersion.

(iii) $k_1 k_2 > 0$ and there exists a parallel isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}_k^{m+\ell}$, with $k = k_1 k_2/(k_1 + k_2)$, such that $f = j \circ g \circ \bar{f}$, where $j\colon \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ is a totally geodesic inclusion and $g\colon \mathbb{Q}_k^{m+\ell} \to \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell}$ is as in (1).

Moreover, the second possibility holds globally if M^m is complete and simply connected.

Notice that the last possibility can occur only if both n_1 and n_2 are greater than or equal to m. Moreover, if this is the case and either n_1 or n_2 is equal to m then $\ell = 0$ and $\bar{f}$ is just a local isometry, in which case f is totally geodesic.

We point out that, after a preliminary version of this article was completed, we learned that the case $k_1 = k_2 \neq 0$ of Theorem 1 was independently obtained with a different approach by Jentsch [11] as a consequence of his classification of parallel submanifolds of the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes of $\mathbb{R}^{n+2}$ and of its noncompact dual.

In case one of the factors is flat, the classification of parallel submanifolds of a product of space forms reads as follows. We recall that a parallel unit speed curve $\gamma\colon \mathbb{R} \to M$ on a Riemannian manifold is also called an *extrinsic circle*. Thus, γ is an extrinsic circle if its curvature vector $\nabla_{\gamma'}\gamma'$ is parallel with respect to its normal connection. By an extrinsic circle $\gamma\colon \mathbb{R} \to \mathbb{Q}_{k_1}^2 \times \mathbb{R}$ being *full* we mean that $\gamma(\mathbb{R})$ does not lie in a totally geodesic surface of $\mathbb{Q}_{k_1}^2 \times \mathbb{R}$.

Theorem 2. *Let $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$, $k_1 \neq 0$, be a parallel isometric immersion. Then one of the following possibilities holds:*

(*i*) *f is a parallel isometric immersion into a slice of $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$;*

(*ii*) *M^m is locally a Riemannian product $M^m = M_1^{m_1} \times M_2^{m_2}$ and $f = f_1 \times f_2$, where $f_1\colon M_1^{m_1} \to \mathbb{Q}^{n_1}_{k_1}$ and $f_2\colon M_1^{m_2} \to \mathbb{R}^{n_2}$ are parallel isometric immersions.*

(*iii*) *$k_1 > 0$ (resp., $k_1 < 0$) and $f \circ \tilde{\Pi} = j \circ \Pi \circ \tilde{f}$ (resp., $f = j \circ \Pi \circ \tilde{f}$), where $\tilde{\Pi}\colon \tilde{M}^m \to M^m$ is the universal covering of M^m, $\tilde{f}\colon \tilde{M}^m \to \mathbb{R}^{n_2+1}$ (resp., $\tilde{f}\colon M^m \to \mathbb{R}^{n_2+1}$) is a parallel isometric immersion, $j\colon \mathbb{Q}^1_{k_1} \times \mathbb{R}^{n_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$ is totally geodesic and $\Pi\colon \mathbb{R}^{n_2+1} \to \mathbb{Q}^1_{k_1} \times \mathbb{R}^{n_2}$ is a locally isometric covering map (resp., isometry).*

(*iv*) *M^m is locally a Riemannian product $M^m = \mathbb{R} \times N^{m-1}$ and $f = j \circ (\gamma \times \tilde{f})$, where $j\colon \mathbb{Q}^2_{k_1} \times \mathbb{R}^{n_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$ is a totally geodesic inclusion, $\gamma\colon \mathbb{R} \to \mathbb{Q}^2_{k_1} \times \mathbb{R}$ is a full extrinsic circle and $\tilde{f}\colon N^{m-1} \to \mathbb{R}^{n_2-1}$ is a parallel isometric immersion.*

Moreover, the second and fourth possibilities hold globally if M^m is complete and simply connected.

Notice that case (*iv*) (respectively, (*iii*)) can occur only if $n_2 \geq m$ (respectively, $n_2 \geq m-1$), and f must be totally geodesic if equality holds. Therefore, a nontotally geodesic parallel isometric immersion $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$, $k_1 \neq 0$, $n_2 \leq m-1$, must be either as in (*ii*) or a parallel isometric immersion into a slice $\mathbb{Q}^{n_1}_{k_1} \times \{x_2\} \subset \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$. Moreover, in the last case one must have $n_1 \geq m+1$. This extends the results in [4] and [18] for the case of hypersurfaces of $\mathbb{Q}^n_k \times \mathbb{R}$.

As a consequence of Theorems 1 and 2, we obtain the classification of totally geodesic submanifolds of $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$.

Corollary 3. *Let $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$, $(k_1, k_2) \neq (0,0)$, be a totally geodesic isometric immersion. Then one of the following possibilities holds:*

(*i*) *f is a totally geodesic isometric immersion into a slice of $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$;*

(*ii*) *There exist a local isometry $\phi\colon M^m \to \mathbb{Q}^{m_1}_{k_1} \times \mathbb{Q}^{m_2}_{k_2}$ and a totally geodesic inclusion $j = j_1 \times j_2\colon \mathbb{Q}^{m_1}_{k_1} \times \mathbb{Q}^{m_2}_{k_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ such that $f = j \circ \phi$.*

(*iii*) *$k_1 k_2 > 0$ and there exist a local isometry $\phi\colon M^m \to \mathbb{Q}^m_k$, $k = k_1 k_2/(k_1 + k_2)$, and a totally geodesic inclusion $j = j_1 \times j_2\colon \mathbb{Q}^m_{k_1} \times \mathbb{Q}^m_{k_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ such that $f = j \circ g \circ \phi$, where $g\colon \mathbb{Q}^m_k \to \mathbb{Q}^m_{k_1} \times \mathbb{Q}^m_{k_2}$ is as in (1).*

(*iv*) *$k_1 k_2 = 0$, say, $k_2 = 0$, and there exist a local isometry $\phi\colon M^m \to \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$, a unit-speed geodesic $\gamma\colon \mathbb{R} \to \mathbb{Q}^1_{k_1} \times \mathbb{R}$ and a totally geodesic inclusion $j\colon \mathbb{Q}^1_{k_1} \times \mathbb{R}^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$ such that $f = j \circ (\gamma \times id) \circ \phi$, where $id\colon \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ is the identity.*

Throughout the paper we use the framework introduced in [12] for studying submanifolds of products of space forms. As a main tool for the proofs of Theorems 1 and 2 we derive reduction of codimension theorems of independent interest for arbitrary submanifolds of products of space forms, some of which extend well-known results for submanifolds of space forms.

In the last part of the paper we apply them to study umbilical submanifolds of a product of two space forms. Recall that an isometric immersion $f\colon M \to N$ between Riemannian manifolds is *umbilical* if there exists a normal vector field ζ along f such that its second fundamental form satisfies $\alpha(X,Y) = \langle X,Y\rangle\zeta$ for all $X, Y \in TM$. One main motivation for this study is a theorem by Nikolayevsky (see Theorem 1 of [15]), which states that any umbilical submanifold of a symmetric space N is an umbilical submanifold of a product of space forms totally geodesically embedded in N.

We prove the following result, which reduces the classification of m-dimensional umbilical submanifolds of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $m \geq 3$ and $k_1 + k_2 \neq 0$, to the cases in which either $n_1 \in \{m, m+1\}$ and $n_2 = 1$ or $n_2 \in \{m, m+1\}$ and $n_1 = 1$, or equivalently, by passing to the universal coverings, to the classification of m-dimensional umbilical submanifolds of codimension two of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. Here $\mathbb{S}^n$ and $\mathbb{H}^n$ stand for the sphere and hyperbolic space, respectively. The case of $\mathbb{S}^n \times \mathbb{R}$ was carried out in [13], extending previous results in [16] and [18] for hypersurfaces.

Theorem 4. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, with $m \geq 3$ and $k_1 + k_2 \neq 0$, be an umbilical nontotally geodesic isometric immersion. Then one of the following possibilities holds:*

- *(i) f is an umbilical isometric immersion into a slice of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$;*
- *(ii) there exist umbilical isometric immersions $f_i\colon M^m \to \mathbb{Q}_{\tilde{k}_i}^{n_i}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1\cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$ for some $\theta \in (0, \pi/2)$, such that $f = (\cos\theta f_1, \sin\theta f_2)$.*
- *(iii) after possibly reordering the factors, we have $k_1 > 0$ (resp., $k_1 \leq 0$) and $f \circ \tilde{\Pi} = j \circ \Pi \circ \tilde{f}$ (resp., $f = j \circ \Pi \circ \tilde{f}$), where $\tilde{\Pi}\colon \tilde{M}^m \to M^m$ is the universal covering of M^m, $\tilde{f}\colon \tilde{M}^m \to \mathbb{R} \times \mathbb{Q}_{k_2}^{m+\delta}$ (resp., $\tilde{f}\colon M^m \to \mathbb{R} \times \mathbb{Q}_{k_2}^{m+\delta}$) is an umbilical isometric immersion with $\delta \in \{0,1\}$, $j\colon \mathbb{Q}_{k_1}^1 \times \mathbb{Q}_{k_2}^{m+\delta} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ is totally geodesic and $\Pi\colon \mathbb{R} \times \mathbb{Q}_{k_2}^{m+\delta} \to \mathbb{Q}_{k_1}^1 \times \mathbb{Q}_{k_2}^{m+\delta}$ is a locally isometric covering map (resp., isometry).*

In particular, it follows from Theorem 4 that there does not exist an umbilical nontotally geodesic m-dimensional submanifold of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1 + k_2 \neq 0$, if m is greater than both n_1 and n_2.

We point out that umbilical nontotally geodesic submanifolds of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ have been alternatively described by Nikolayevsky as intersections of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ with its osculating spaces at generic points in the flat underlying ambient space (see Theorem 2 of [15]).

2 Preliminaries

Let $\pi_i\colon \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2} \to \mathbb{Q}_{k_i}^{n_i}$ denote the canonical projection, $1 \leq i \leq 2$. By abuse of notation, we denote by the same letter its derivative, which we regard as a section of either $T(\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2})^* \otimes T\mathbb{Q}_{k_i}^{n_i}$ or $T(\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2})^* \otimes T(\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2})$. Then, the curvature tensor $\bar{\mathcal{R}}$ of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ can be written as

$$\bar{\mathcal{R}}(X, Y) = k_1(X \wedge Y - X \wedge \pi_2 Y - \pi_2 X \wedge Y) + (k_1 + k_2)\pi_2 X \wedge \pi_2 Y,$$

where $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$.

Let $f\colon M \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion of a Riemannian manifold. Denote by $\mathcal{R}$ and $\mathcal{R}^\perp$ the curvature tensors of the tangent and normal bundles TM and $N_f M$, respectively, by $\alpha = \alpha_f \in \Gamma(T^*M \otimes T^*M \otimes N_f M)$ the second fundamental form of f and by $A_\eta = A_\eta^f$ its shape operator in the normal direction η, given by

$$\langle A_\eta X, Y\rangle = \langle \alpha(X, Y), \eta\rangle$$

for all $X, Y \in TM$. Set

$$L = L^f := \pi_2 \circ f_* \in \Gamma(T^*M \otimes T\mathbb{Q}_{k_2}^{n_2}) \;\text{ and }\; K = K^f := \pi_2|_{N_f M} \in \Gamma((N_f M)^* \otimes T\mathbb{Q}_{k_2}^{n_2}).$$

We can write

$$L = f_* R + S \quad\text{and}\quad K = f_* S^t + T, \tag{2}$$

where

$$R = R^f := L^t L, \quad S = S^f := K^t L \quad\text{and}\quad T = T^f := K^t K.$$

The tensors R, S and T were introduced in [12]. Note that R and T are symmetric. Using (2), one can check by applying $\pi_2^2 = \pi_2$ to tangent and normal vectors, and then taking tangent and normal components, that they satisfy the algebraic relations

$$S^t S = R(I - R), \quad TS = S(I - R) \quad\text{and}\quad SS^t = T(I - T). \tag{3}$$

In particular, from the first and third equations, respectively, it follows that R and T are in fact nonnegative operators whose eigenvalues lie in $[0, 1]$. On the other hand, taking tangent and normal components in $\nabla \pi_2 = 0$ and using the Gauss and Weingarten formulae yields the differential equations

$$(\nabla_X R)Y = A_{SY} X + S^t \alpha(X, Y), \tag{4}$$

$$(\nabla_X S)Y = T\alpha(X, Y) - \alpha(X, RY) \tag{5}$$

and

$$(\nabla_X T)\eta = -S A_\eta X - \alpha(X, S^t \eta). \tag{6}$$

The Gauss, Codazzi and Ricci equations of f are, respectively,

$$\begin{aligned}\mathcal{R}(X,Y)Z = (k_1(X\wedge Y - X\wedge RY - RX\wedge Y) + (k_1+k_2)RX\wedge RY)Z \\ +A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y,\end{aligned} \tag{7}$$

$$(\nabla^{\perp}_X\alpha)(Y,Z) - (\nabla^{\perp}_Y\alpha)(X,Z) = \langle \Phi X, Z\rangle SY - \langle \Phi Y, Z\rangle SX \tag{8}$$

and

$$\mathcal{R}^{\perp}(X,Y)\eta = \alpha(X, A_\eta Y) - \alpha(A_\eta X, Y) + (k_1+k_2)(SX\wedge SY)\eta. \tag{9}$$

where $\Phi = k_1 I - (k_1+k_2)R$. The Codazzi equation (8) can also be written as

$$(\nabla_Y A)(X,\xi) - (\nabla_X A)(Y,\xi) = \langle SX,\xi\rangle \Phi Y - \langle SY,\xi\rangle \Phi X. \tag{10}$$

We use the fact that $\mathbb{Q}_k^N$, $k\neq 0$, admits a canonical isometric embedding in $\mathbb{R}^{N+1}_{\sigma(k)}$ as (a connected component of, if $k<0$)

$$\mathbb{Q}_k^N = \{X \in \mathbb{R}^{N+1}_{\sigma(k)} \,:\, \langle X, X\rangle = 1/k\}.$$

Here, for $k\in\mathbb{R}$ we set $\sigma(k)=1$ if $k<0$ and $\sigma(k)=0$ otherwise, and as a subscript of an Euclidean space it means the index of the corresponding flat metric. Thus, $\mathbb{Q}_{k_1}^{n_1}\times\mathbb{Q}_{k_2}^{n_2}$ admits a canonical isometric embedding

$$h\colon \mathbb{Q}_{k_1}^{n_1}\times\mathbb{Q}_{k_2}^{n_2} \to \mathbb{R}^{N_1}_{\sigma(k_1)}\times\mathbb{R}^{N_2}_{\sigma(k_2)} = \mathbb{R}^{N_1+N_2}_{\mu}, \tag{11}$$

with $\mu = \sigma(k_1)+\sigma(k_2)$, $N_i = n_i+1$ if $k_i\neq 0$ and $N_i = n_i$ otherwise, in which case $\mathbb{Q}_{k_i}^{n_i}$ stands for $\mathbb{R}^{n_i}$.

Denote by $\tilde{\pi}_i\colon \mathbb{R}^{N_1+N_2}_{\mu}\to\mathbb{R}^{N_i}_{\sigma(k_i)}$ the canonical projection, $1\leq i\leq 2$. Then, the normal space of h at each point $z\in\mathbb{Q}_{k_1}^{n_1}\times\mathbb{Q}_{k_2}^{n_2}$ is spanned by $k_1\tilde{\pi}_1(h(z))$ and $k_2\tilde{\pi}_2(h(z))$, and the second fundamental form of h is given by

$$\alpha_h(X,Y) = -k_1\langle \pi_1 X, Y\rangle \tilde{\pi}_1\circ h - k_2\langle \pi_2 X, Y\rangle\tilde{\pi}_2\circ h. \tag{12}$$

Given an isometric immersion $f\colon M\to\mathbb{Q}_{k_1}^{n_1}\times\mathbb{Q}_{k_2}^{n_2}$, set $F = h\circ f$. If $k_i\neq 0$, write $k_i = \epsilon_i/r_i^2$, where ϵ_i is either 1 or -1, according as $k_i>0$ or $k_i<0$, respectively. If $k_1\neq 0$, then the unit vector field $\nu_1 = \nu_1^F = \frac{1}{r_1}\tilde{\pi}_1\circ F$ is normal to F, and

$$\tilde{\nabla}_X\nu_1 = \frac{1}{r_1}\tilde{\pi}_1 F_*X = \frac{1}{r_1}(F_*X - h_*LX),$$

where $\tilde{\nabla}$ stands for the derivative in $\mathbb{R}^{N_1+N_2}_{\mu}$. Hence

$${}^F\nabla^{\perp}_X\nu_1 = -\frac{1}{r_1}h_*SX \quad\text{and}\quad A^F_{\nu_1} = -\frac{1}{r_1}(I-R). \tag{13}$$

If $k_2 \neq 0$, then $\nu_2 = \nu_2^{\tilde{f}} = \frac{1}{r_2}\tilde{\pi}_2 \circ F$ is also a unit normal vector field to F such that

$$\tilde{\nabla}_X \nu_2 = \frac{1}{r_2}\tilde{\pi}_2 F_* X = \frac{1}{r_2} h_* LX.$$

Thus

$$ {}^F\nabla^{\perp}_X \nu_2 = \frac{1}{r_2} h_* SX \quad \text{and} \quad A^F_{\nu_2} = -\frac{1}{r_2} R. \tag{14}$$

Set

$$\vartheta = -\frac{\epsilon_1}{r_1}\nu_1 + \frac{\epsilon_2}{r_2}\nu_2. \tag{15}$$

Then $\langle \vartheta, \vartheta \rangle = \frac{\epsilon_1}{r_1^2} + \frac{\epsilon_2}{r_2^2} = k_1 + k_2$ and $\langle \vartheta, F \rangle = 0$. Moreover,

$$A^F_\vartheta = \Phi \quad \text{and} \quad {}^F\nabla^{\perp}_X \vartheta = (k_1 + k_2) h_* SX. \tag{16}$$

For later use we prove the following fact.

Proposition 5. *Let $f\colon M \to N$ and $g\colon N \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be isometric immersions. Then the tensors R^g, S^g and T^g of g and R^F, S^F and T^F of $F = g \circ f$ are related by*

$$\langle R^F X, Y \rangle = \langle R^g f_* X, f_* Y \rangle, \quad \langle S^F X, g_* \xi \rangle = \langle R^g f_* X, \xi \rangle, \quad \langle S^F X, \zeta \rangle = \langle S^g f_* X, \zeta \rangle, \tag{17}$$

$$\langle T^F g_* \eta, g_* \xi \rangle = \langle R^g \eta, \xi \rangle, \quad \langle T^F g_* \eta, \zeta \rangle = \langle S^g \eta, \zeta \rangle, \quad \langle T^F \zeta, \beta \rangle = \langle T^g \zeta, \beta \rangle \tag{18}$$

for all $X, Y \in TM$, $\xi, \eta \in N_f M$ and $\zeta, \beta \in N_g N$.

Proof: Notice that $N_F M = g_* N_f M \oplus N_g N$. Given $X \in TM$ we have

$$g_* f_* R^F X + S^F X = F_* R^F X + S^F X = \pi_2 F_* X = \pi_2 g_* f_* X = g_* R^g f_* X + S^g f_* X.$$

Taking the inner product of both sides of the preceding equation with $F_* Y = g_* f_* Y$, $g_* \xi$ and ζ, respectively, gives the three equations in (17). For $\eta \in N_f M$ we have

$$g_* R^g \eta + S^g \eta = \pi_2 g_* \eta = F_* (S^F)^t g_* \eta + T^F g_* \eta = g_* f_* (S^F)^t g_* \eta + T^F g_* \eta.$$

Taking the inner product of the above equation with $F_* Y = g_* f_* Y$ gives again the second equation in (17). Taking the inner product with $g_* \xi$ and ζ gives the first two relations in (18). To prove the last equation in (18), for any $\zeta \in N_g N$ write

$$g_* (S^g)^t \zeta + T^g \zeta = \pi_2 \zeta = F_* (S^F)^t \zeta + T^F \zeta = g_* f_* (S^F)^t \zeta + T^F \zeta$$

and take the inner product of both sides with β. Notice that taking the inner product with $F_* Y$ and $g_* \eta$ gives again the third equation in (17) and the second one in (18). ∎

We also state as a lemma the following observation.

Lemma 6. *Let $f\colon M \to \mathbb{Q}_{k_1}^{m_1} \times \mathbb{Q}_{k_2}^{m_2}$ be an isometric immersion and let $j\colon \mathbb{Q}_{k_1}^{m_1} \times \mathbb{Q}_{k_2}^{m_2} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be a totally geodesic inclusion. Set $F = j \circ f$. Then the tensors R^f, S^f and T^f of f and R^F, S^F and T^F of F are related by*

$$R^F = R^f, \quad S^F = j_* S^f \ \text{ and } \ T^F j_* = j_* T^f \tag{19}$$

We conclude this section by listing well known formulae for the second fundamental form and normal connection of a composition of isometric immersions. We omit the proofs, which follow by a straightforward application of the Gauss and Weingarten formulae.

Proposition 7. *Let $f\colon M \to N$ and $g\colon N \to P$ be isometric immersions. Set $F = g \circ f$. Then $N_F M = g_* N_f M \oplus N_g N$ and the second fundamental forms and normal connections of f, g and F are related by*

$$\alpha_F(X,Y) = g_* \alpha_f(X,Y) + \alpha_g(f_*X, f_*Y), \tag{20}$$

$${}^F\nabla_X^\perp g_*\xi = g_* {}^f\nabla_X^\perp \xi + \alpha_g(f_*X, \xi), \quad ({}^F\nabla_X^\perp \zeta)_{N_gN} = {}^g\nabla_{f_*X}^\perp \zeta \tag{21}$$

and

$$\langle {}^F\nabla_X^\perp \zeta, g_*\xi\rangle = -\langle A_\zeta^g f_*X, \xi\rangle \tag{22}$$

for all $X \in TM, \xi \in N_f M$ and $\zeta \in N_g N$.

3 Products of isometric immersions

We start this section by characterizing isometric immersions into a slice of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ as those for which either R or $I - R$ vanishes identically.

Proposition 8. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion. Then $f(M^m) \subset \mathbb{Q}_{k_1}^{n_1} \times \{x_2\}$ for some $\{x_2\} \in \mathbb{Q}_{k_2}^{n_2}$ if and only if $R = 0$.*

Proof: We have $f(M^m) \subset \mathbb{Q}_{k_1}^{n_1} \times \{x_2\}$ for some $\{x_2\} \in \mathbb{Q}_{k_2}^{n_2}$ if and only $\pi_2 \circ f_* = 0$, which is equivalent to $R = 0$. ∎

Next we show that products

$$f = f_1 \times f_2\colon M^m = M_1^{m_1} \times M_2^{m_2} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$$

of isometric immersions are characterized by having vanishing tensor S.

Lemma 9. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion. Then $\ker S$ splits orthogonally pointwise as*

$$\ker S = \ker R \oplus \ker(I - R).$$

Moreover, the following holds:

(*i*) *If* $\ker S$ *has constant dimension, then it is a smooth subbundle of* TM *and so are* $\ker R$ *and* $\ker(I-R)$*;*

(*ii*) *If* $S=0$ *then* $\ker R$ *and* $\ker(I-R)$ *are parallel subbundles of* TM*.*

Proof: The first assertion follows from the first equation in (3). Assume that $\ker S = (\text{Im } S^t)^\perp$ has constant dimension k on M^m. Given $x \in M^m$, let $\xi_1,\ldots,\xi_{m-k}$ be normal vectors at x such that $\{S^t\xi_i\}_{1\le i\le m-k}$ is linearly independent. Extend $\xi_1,\ldots,\xi_{m-k}$ to smooth normal vector fields on a neighborhood of x. Then $\{S^t\xi_i\}_{1\le i\le m-k}$ is still linearly independent on a (possibly smaller) neighborhood V of x, hence it spans the image of S^t on V. It follows that Im S^t, and hence $\ker S$, is a smooth distribution. Moreover, using the lower semicontinuity of the rank of both R and $I-R$ we easily obtain that $A_\ell = \{x \in M : \ker R(x) = \ell\}$ is an open subset of M^m for any $0 \le \ell \le m$. Hence $M^m = A_\ell$ for some $\ell \in \{0,\ldots,m\}$, that is, $\ker R$ has constant dimension ℓ on M^m. Arguing as before we conclude that $\ker R = \text{Im }(I-R)$ and $\ker(I-R) = \text{Im } R$ are also smooth. Finally, from (4) we obtain that $\nabla R = 0$ if $S=0$, and (*ii*) follows. ∎

Proposition 10. *Let* $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ *be an isometric immersion. Then* M^m *is locally a Riemannian product* $M^m = M_1^{m_1} \times M_2^{m_2}$ *and* $f = f_1 \times f_2$*, where* $f_i\colon M_i^{m_i} \to \mathbb{Q}^{n_i}_{k_i}$*,* $1 \le i \le 2$*, is an isometric immersion, if and only if* $S=0$ *and neither* $R=0$ *nor* $R=I$*. The "if" part holds globally if* M^m *is complete and simply connected.*

Proof: Assume that M^m is locally a Riemannian product $M^m = M_1^{m_1} \times M_2^{m_2}$ and $f = f_1 \times f_2$, where $f_i\colon M_i^{m_i} \to \mathbb{Q}^{n_i}_{k_i}$, $1 \le i \le 2$, is an isometric immersion. Then TM splits orthogonally as $TM = TM_1 \oplus TM_2$, with $f_*TM_i = f_{i*}TM_i \subset T\mathbb{Q}^{n_i}_{k_i}$, $1 \le i \le 2$. Hence $R=0$ on TM_1 and $R=I$ on TM_2. Thus $S^tS = R(I-R) = 0$, and therefore $S=0$.

Conversely, suppose that $S=0$ and that neither $R=0$ nor $R=I$. Then $\ker R$ and $\ker(I-R)$ are nontrivial parallel subbundles of TM, and TM splits orthogonally as $TM = \ker R \oplus \ker(I-R)$ by Lemma 9. That M^m splits locally as a Riemannian product $M^m = M_1 \times M_2$ then follows from the local version of de Rham theorem.

Since $S=0$, formula (5) becomes

$$\alpha_f(RX,Y) = T\alpha_f(X,Y)$$

for all $X, Y \in TM$. The right-hand-side is symmetric with respect to X and Y, so the same holds for the left-hand-side, i.e.,

$$\alpha_f(RX,Y) = \alpha_f(X,RY)$$

for all $X, Y \in TM$. It follows that $\alpha_f(X,Y) = 0$ if $X \in \ker R$ and $Y \in \ker(I-R)$.

Set $F = h \circ f$, where h is the inclusion of $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ into $\mathbb{R}^{N_1+N_2}_{\mu} = \mathbb{R}^{N_1}_{\sigma(k_1)} \times \mathbb{R}^{N_2}_{\sigma(k_2)}$ as in (11). Then the second formulas in (13) and (14) show that we also have $\alpha_F(X, Y) = 0$ if $X \in \ker R$ and $Y \in \ker(I - R)$. Define

$$V_1 = \text{span}\{F_*(x)X : x \in M^m,\ X \in \ker R(x)\}$$

and

$$V_2 = \text{span}\{F_*(x)X : x \in M^m,\ X \in \ker(I - R(x))\}.$$

Since $\tilde{\pi}_2(F_* \ker R) = \{0\} = \tilde{\pi}_1(F_* \ker(I - R))$, we have $V_1 \subset \mathbb{R}^{N_1}_{\sigma(k_1)}$ and $V_2 \subset \mathbb{R}^{N_2}_{\sigma(k_2)}$. As in the proof of Moore's Lemma [14], it follows that there exist isometric immersions $F_1\colon M_1 \to \mathbb{R}^{N_1}_{\sigma(k_1)}$ and $F_2\colon M_2 \to \mathbb{R}^{N_2}_{\sigma(k_2)}$ such that $F(x, y) = (F_1(x), F_2(y))$. Since

$$F(M_1 \times M_2) = F_1(M_1) \times F_2(M_2) \subset \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2},$$

we have that $F_i = j_i \circ f_i$ for some isometric immersions $f_i\colon M_i \to \mathbb{Q}^{n_i}_{k_i}$, $1 \leq i \leq 2$, where j_i is the inclusion of $\mathbb{Q}^{n_i}_{k_i}$ into $\mathbb{R}^{N_i}_{\sigma(k_i)}$.

The global assertion follows as before from the global version of de Rham Theorem. ∎

4 A reduction of codimension theorem

We say that an isometric immersion $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ *reduces codimension on the left by* ℓ if there exists a totally geodesic inclusion $j_1\colon \mathbb{Q}^{m_1}_{k_1} \to \mathbb{Q}^{n_1}_{k_1}$ with $n_1 - m_1 = \ell$ and an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}^{m_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ such that $f = (j_1 \times id) \circ \bar{f}$. Similarly we define what it means by f reducing codimension *on the right*. In this section we give necessary and sufficient conditions for an isometric immersion $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ to reduce codimension on the left or on the right. We start with the following observation.

Lemma 11. *Let $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an isometric immersion. Then*

$$S(TM)^\perp = U \oplus V, \quad \textit{where } U = \ker T \ \textit{ and } \ V = \ker(I - T).$$

Proof: It follows from the third equation in (3) that $\ker T(I - T) = \ker S^t = S(TM)^\perp$, hence the restriction of T to $S(TM)^\perp$ is the orthogonal projection onto V. ∎

The following result and its corollary are the analogues for isometric immersions into products of space forms of the well known criterion for reduction of codimension of isometric immersions into space forms (see, e.g., Proposition 4.1 and Corollary 4.2 in Chapter 4 of [6]; see also [8]). We restrict ourselves to stating the results for reduction of codimension *on the left*, for the case of reduction of codimension on the right is completely similar, just by replacing the vector subbundle U by V in what follows.

Given an isometric immersion $f\colon M \to N$ between Riemannian manifolds, its *first normal space* at $x \in M$ is the subspace $N_1(x)$ of $N_f M(x)$ spanned by the image of its second fundamental form $\alpha_f(x)$.

Theorem 12. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion. Then the following assertions are equivalent:*

(i) f reduces codimension on the left by ℓ;

(ii) There exists a subbundle L of rank ℓ of N_fM such that L is parallel in the normal connection and $L \subset U \cap N_1^\perp$.

Proof: Assume that there exists a totally geodesic inclusion $j_1\colon \mathbb{Q}_{k_1}^{m_1} \to \mathbb{Q}_{k_1}^{n_1}$ with $\ell = n_1 - m_1$ and an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}_{k_1}^{m_1} \times \mathbb{Q}_{k_2}^{n_2}$ such that $f = j \circ \bar{f}$, where $j = j_1 \times id$. Set

$$L = N_{j_1}\mathbb{Q}_{k_1}^{m_1} = N_j(\mathbb{Q}_{k_1}^{m_1} \times \mathbb{Q}_{k_2}^{n_2}) \subset N_fM.$$

Since j is totally geodesic, it follows from (20) (with $\bar{f}$, j and f playing the roles of f, g and F, respectively) that $N_1 \subset L^\perp$, hence $L \subset N_1^\perp$. Also because j is totally geodesic, we obtain from (22) that L is parallel in the normal connection. Finally, since π_2 vanishes on $L = N_{j_1}\mathbb{Q}_{k_1}^{m_1}$, it follows that $L \subset U$.

Now we prove the converse. Set $F = h \circ f$, where h is as in (11). For any $\xi \in N_fM$, using (12) we obtain

$$\tilde{\nabla}_X h_*\xi = h_*\nabla_X^\perp\xi - F_*A_\xi X + \alpha_h(f_*X, \xi) = h_*\nabla_X^\perp\xi - F_*A_\xi X - \langle SX, \xi\rangle\vartheta, \tag{23}$$

with ϑ as in (15). Given $\xi \in L$, using that L is parallel in the normal connection and that $L \subset U \cap N_1^\perp$ it follows from (23) that

$$\tilde{\nabla}_X h_*\xi = h_*\nabla_X^\perp\xi \in h_*L. \tag{24}$$

This shows that $W := h_*L$ is a constant subspace in $\mathbb{R}_\mu^{N_1+N_2}$. Moreover, we have that $\tilde{\pi}_2 h_*\xi = h_*\pi_2\xi = h_*T\xi = 0$, because $\pi_2|_{S(TM)^\perp} = T$ and $\xi \in U = \ker T$. Hence $\tilde{\pi}_2|_W = 0$.

Claim: $\tilde{\pi}_1(F(M^m)) \subset \mathbb{Q}_{k_1}^{n_1} \cap W^\perp$.

Given $\xi \in L$ and $X \in TM$, we have using (24) that

$$X\langle\tilde{\pi}_1 \circ F, h_*\xi\rangle = \langle\tilde{\pi}_1 F_*X, h_*\xi\rangle = -\langle SX, \xi\rangle = 0,$$

since $L \subset U \subset S(TM)^\perp$. Hence $\tilde{\pi}_1(F(M^m)) \subset \tilde{\pi}_1(F(x_0)) + W^\perp$ for any fixed $x_0 \in M^m$. But $\tilde{\pi}_1(F(x_0)) \in W(x_0)^\perp = W^\perp$, hence $\tilde{\pi}_1(F(M^m)) \subset \mathbb{Q}_{k_1}^{n_1} \cap W^\perp$ as claimed. ∎

Corollary 13. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion. Assume that $U \cap N_1^\perp$ is a vector subbundle of N_fM with rank ℓ satisfying*

$$\nabla^\perp(U \cap N_1^\perp) \subset N_1^\perp. \tag{25}$$

Then f reduces codimension on the left by ℓ.

Proof: By Theorem 12, it suffices to prove that

$$\nabla^\perp(U \cap N_1^\perp) \subset U. \tag{26}$$

But this follows from (6), for it implies for all $\xi \in U \cap N_1^\perp$ and $X \in TM$ that

$$T\nabla_X^\perp \xi = \nabla_X^\perp T\xi = 0. \quad \blacksquare$$

In case $U \cap N_1^\perp$ is a vector subbundle of the normal bundle of an isometric immersion $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, the next result gives necessary and sufficient conditions for (25) to hold in terms of its normal curvature tensor and mean curvature vector field. It is the version for submanifolds of products of space forms of a theorem by Dajczer for submanifolds of space forms (see [5] or Theorem 4.4 in [6]).

Theorem 14. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion. Assume that $U \cap N_1^\perp$ is a vector subbundle of N_fM. Then $\nabla^\perp(U \cap N_1^\perp) \subset N_1^\perp$ if and only if the following conditions hold:*

(*i*) $\nabla^\perp \mathcal{R}^\perp|_{U\cap N_1{}^\perp} = 0$,

(*ii*) $\nabla^\perp(U \cap N_1^\perp) \subset \{H\}^\perp$.

Proof: Assume that $\nabla^\perp(U \cap N_1^\perp) \subset N_1^\perp$. Then (*ii*) is clear. By the Ricci equation (9), for any $\xi \in U \cap N_1{}^\perp \subset S(TM)^\perp \cap N_1^\perp$ we have

$$\mathcal{R}^\perp(X,Y)\xi = \alpha(X, A_\xi Y) - \alpha(A_\xi X, Y) + (k_1+k_2)(SX \wedge SY)\xi = 0.$$

Using that $\nabla_Z^\perp \xi \in U \cap N_1^\perp$ by (26) and the assumption, we obtain

$$\begin{aligned}\left(\nabla_Z^\perp \mathcal{R}^\perp\right)(X,Y,\xi) = \nabla_Z^\perp \mathcal{R}^\perp(X,Y)\xi - \mathcal{R}^\perp(\nabla_Z X, Y)\xi - \mathcal{R}^\perp(X, \nabla_Z Y)\xi - \\ -\mathcal{R}^\perp(X,Y)\nabla_Z^\perp \xi = 0.\end{aligned}$$

Hence $\nabla^\perp \mathcal{R}^\perp|_{U\cap N_1{}^\perp} = 0$.

We now prove the converse. Let $\xi \in U \cap N_1^\perp$. We must prove that $\nabla_Z^\perp \xi \in N_1^\perp$ for all $Z \in TM$. We obtain from (*i*) that $\mathcal{R}^\perp(X,Y)\nabla_Z^\perp \xi = 0$. Since $\nabla_Z^\perp \xi \in U \subset S(TM)^\perp$ by (26), the Ricci equation (9) yields

$$\left[A_{\nabla_Z^\perp \xi}, A_{\nabla_W^\perp \xi}\right] = 0 \text{ for all } Z, W \in TM.$$

Hence, at any $x \in M$ there exists an orthonormal basis $\{E_1(x), \cdots, E_m(x)\}$ of T_xM that diagonalizes simultaneously the family of operators $\left\{A_{\nabla_X^\perp \xi} : X \in T_xM\right\}$. Then

$$\langle \alpha(E_i, E_j), \nabla_{E_k}^\perp \xi \rangle = \langle A_{\nabla_{E_k}^\perp \xi} E_i, E_j \rangle = 0, \text{ if } i \neq j.$$

Using that $\xi \in U \cap N_1^\perp \subset S(TM)^\perp \cap N_1^\perp$, we obtain from the Codazzi equation (10) that

$$A_{\nabla^\perp_X \xi} Y = A_{\nabla^\perp_Y \xi} X \ \text{ for all } \ X, Y \in TM,$$

which implies that $A_{\nabla^\perp_{E_i}\xi} E_j = 0$ for $i \neq j$. Hence

$$\langle \alpha(E_j, E_j), \nabla^\perp_{E_k}\xi \rangle = \langle A_{\nabla^\perp_{E_k}\xi} E_j, E_j \rangle = 0$$

if $k \neq j$. Finally, using condition (ii) and the above we obtain

$$\langle \alpha(E_j, E_j), \nabla^\perp_{E_j}\xi \rangle = n\langle H, \nabla^\perp_{E_j}\xi \rangle = 0.$$

Hence $\nabla^\perp_Z \xi \in N_1^\perp$ for all $Z \in TM$. ∎

Remark 15. Given an isometric immersion $f\colon M^m \to \mathbb{Q}^n_k \times \mathbb{R}$, let $\frac{\partial}{\partial t}$ be a unit vector field tangent to the second factor. Then, a tangent vector field Z on M^m and a normal vector field η along f are defined by

$$\frac{\partial}{\partial t} = f_* Z + \eta.$$

The tensors R, S and T associated to f are given by

$$RX = \langle X, Z \rangle Z, \quad SX = \langle X, Z \rangle \eta \quad \text{and} \quad T\xi = \langle \xi, \eta \rangle \eta.$$

Then $U = \ker T = \{\eta\}^\perp$, hence $U \cap N_1^\perp = (N_1 + \operatorname{span}\{\eta\})^\perp$. Thus, condition (25) is equivalent to

$$\nabla^\perp N_1 \subset N_1 + \operatorname{span}\{\eta\}.$$

Therefore, in this case Corollary 13 and Theorem 14 reduce to Lemma 6 and Theorem 7 of [13], respectively.

5 Weighted sums of isometric immersions

Given $a, b \in \mathbb{R}^*$ with $a^2 + b^2 = 1$, set $\tilde{k}_1 = a^2 k_1$ and $\tilde{k}_2 = b^2 k_2$. Let $f_i\colon M^m \to \mathbb{Q}^{n_i}_{\tilde{k}_i} \subset \mathbb{R}^{N_i}_{\sigma(k_i)}$ be isometric immersions, $1 \leq i \leq 2$. Then $f = (af_1, bf_2)\colon M^m \to \mathbb{R}^{N_1+N_2}_\mu$ is an isometric immersion that takes values in $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$. We call f the *weighted sum of* f_1 *and* f_2 with weights a and b. The normal space of f (in $\mathbb{R}^{N_1+N_2}_\mu$) is the orthogonal sum

$$N_f M = N_{f_1} M \oplus N_{f_2} M \oplus W,$$

where $W = \operatorname{span}\{-bf_{1*}X + af_{2*}X \,:\, X \in TM\}$. We have

$$(\pi_2 \circ f_*)X = bf_{2*}X = f_* b^2 X + ab(-bf_{1*}X + af_{2*}X),$$

with $-bf_{1*}X + af_{2*}X \in N_f M$, hence $R = b^2 I$ and $SX = ab(-bf_{1*}X + af_{2*}X)$ for any $X \in TM$. In particular, we have $S(TM) = W$.

Example 16 Let $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 > 0$ and set

$$a := \sqrt{\frac{k_2}{k_1 + k_2}},\ b := \sqrt{\frac{k_1}{k_1 + k_2}} \text{ and } \epsilon = \sigma(k_1) = \sigma(k_2).$$

Given $T_i \in O_\epsilon(n+1)$, $1 \leq i \leq 2$, define G: $\mathbb{R}^{n+1}_\epsilon \to \mathbb{R}^{2n+2}_{2\epsilon} = \mathbb{R}^{n+1}_\epsilon \oplus \mathbb{R}^{n+1}_\epsilon$ by

$$G(x) = (aT_1(x), bT_2(x)).$$

Then $G(\mathbb{Q}^n_k) \subset \mathbb{Q}^n_{k_1} \times \mathbb{Q}^n_{k_2}$, $k = k_1k_2/(k_1 + k_2)$, thus $G|_{\mathbb{Q}^n_k} = h \circ g$ for some isometric immersion g: $\mathbb{Q}^n_k \to \mathbb{Q}^n_{k_1} \times \mathbb{Q}^n_{k_2}$, where h is as in (11). Since $G(\mathbb{Q}^n_k) = V \cap \mathbb{Q}^{2n+1}_k$, where $V = G(\mathbb{R}^{n+1}_\epsilon)$, it follows that $\bar{h} \circ g$ is totally geodesic, where $\bar{h}$: $\mathbb{Q}^n_{k_1} \times \mathbb{Q}^n_{k_2} \to \mathbb{Q}^{2n+1}_k$ is the inclusion. Hence g is totally geodesic. Moreover, $R^g = b^2 I$ by the preceding discussion.

The following result shows that weighted sums of isometric immersions are characterized by the fact that the tensor R is a multiple of the identity tensor.

Proposition 17 *Let f: $M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an isometric immersion. Then the following assertions are equivalent:*

(i) there exist isometric immersions f_i: $M^m \to \mathbb{Q}^{n_i}_{\tilde{k}_i} \subset \mathbb{R}^{N_i}_{\sigma(k_i)}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1 \cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$ for some $\theta \in (0, \pi/2)$, such that $f = (\cos\theta f_1, \sin\theta f_2)$;

(ii) $R = \sin^2\theta I$ for some $\theta \in (0, \pi/2)$.

Proof: We have that $R = \sin^2\theta I$ for some $\theta \in (0, \pi/2)$ if and only if the tensor $L = \pi_2 \circ f_*$ satisfies

$$L^t L = R = \sin^2\theta I.$$

This is equivalent to $\pi_2 \circ f_*$ being a similarity of ratio $\sin\theta$. In turn, this holds if and only if there exist isometric immersions f_i: $M^m \to \mathbb{Q}^{n_i}_{\tilde{k}_i} \subset \mathbb{R}^{n_i+1}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1 \cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$, such that $\pi_1 \circ f = \cos\theta f_1$ and $\pi_2 \circ f = \sin\theta f_2$. ∎

6 A further theorem on reduction of codimension

We derive in this section necessary and sufficient conditions for the image of an isometric immersion f: $M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ to be contained in the totally geodesic submanifold of Example 16. This will be used in the proof of Theorem 1 in the next section. We need some preliminary results.

Lemma 18. *Let f: $M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an isometric immersion. Then $\nabla R = 0$ if and only if $S(TM) \subset N_1^\perp$.*

Proof: By (4), we have that $\nabla R = 0$ if and only if

$$A_{SY}X + S^t\alpha(X,Y) = 0$$

for all $X, Y \in TM$. This is equivalent to

$$\langle\alpha(X,Z), SY\rangle = -\langle\alpha(X,Y), SZ\rangle$$

for all $X, Y, Z \in TM$. Hence, $\nabla R = 0$ if and only if the trilinear form β given by

$$\beta(X,Y,Z) = \langle\alpha(X,Y), SZ\rangle$$

is skew-symmetric in the last two variables. Since β is symmetric in the first two variables, it follows from the next lemma, called the *Braid Lemma* (see Section 9.5.4.9 of [2]), that this is the case if and only if β vanishes. ∎

Lemma 19 *Let β: $V \times V \times V \to W$ be a trilinear map. If β is symmetric in the first two variables and skew-symmetric in the last two, then $\beta = 0$.*

Proof: For any $X, Y, Z \in V$ we have

$$\begin{aligned}\beta(X,Y,Z) &= -\beta(X,Z,Y) = -\beta(Z,X,Y) = \beta(Z,Y,X) = \beta(Y,Z,X) = -\beta(Y,X,Z)\\ &= -\beta(X,Y,Z),\end{aligned}$$

hence $\beta = 0$. ∎

Lemma 20. *Let f: $M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an isometric immersion. If L is a subbundle of $S(TM)^\perp$ such that*

$$(T - \lambda I)L \subset L^\perp \text{ for some } \lambda \in \mathbb{R},$$

then ($\lambda \in [0,1]$ and) $\pi_2|_L$ is a similarity of ratio λ.

Proof: Since $L \subset S(TM)^\perp$ and $T|_{S(TM)^\perp} = \pi_2|_{S(TM)^\perp}$, we have for all $\xi, \eta \in L$ that

$$\langle\pi_2\xi, \pi_2\eta\rangle = \langle\pi_2\xi, \eta\rangle = \langle T\xi, \eta\rangle = \lambda\langle\xi, \eta\rangle. \quad ∎$$

Lemma 21. *Let f: $M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$, $k_1k_2 \neq 0$, be an isometric immersion with $\Phi = 0$. Then $k_1k_2 > 0$ and $R = b^2 I$, with $b := \sqrt{\frac{k_1}{k_1+k_2}}$.*

Proof: Since

$$0 = \Phi = k_1 I - (k_1 + k_2)R,$$

if $k_1 + k_2 = 0$ then $k_1 = 0$, in contradiction with the assumption that $k_1k_2 \neq 0$. Hence $k_1 + k_2 \neq 0$ and $R = \lambda I$ with $\lambda = \frac{k_1}{k_1+k_2} \in (0,1)$. In particular, we have that $\frac{k_1k_2}{(k_1+k_2)^2} = \lambda(1-\lambda) > 0$, hence $k_1k_2 > 0$. ∎

Theorem 22. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1k_2 \neq 0$, be an isometric immersion. Then the following assertions are equivalent:*

(*i*) *$k_1k_2 > 0$ and there exist $\ell \geq 0$, an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}_k^{m+\ell}$, with $k = k_1k_2/(k_1+k_2)$, and a totally geodesic inclusion $j\colon \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ such that*

$$f = j \circ g \circ \bar{f},$$

where $g\colon \mathbb{Q}_k^{m+\ell} \to \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell}$ is the totally geodesic embedding of Example 16.

(*ii*) *$\Phi = 0$ and there exists a subbundle L^ℓ of $S(TM)^\perp$ such that $N_1 \subset L$, L is parallel in the normal connection and $(T - b^2 I)L \subset L^\perp$, where $b := \sqrt{\frac{k_1}{k_1+k_2}}$.*

Proof: Let us prove that (*ii*) implies (*i*). By Lemma 21, we have $k_1k_2 > 0$, hence $\sigma(k_1) = \sigma(k_2) := \epsilon \in \{0, 1\}$. Let $h\colon \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2} \to \mathbb{R}_{2\epsilon}^N$, $N = n_1 + n_2 + 2$, be the canonical inclusion and set $F = h \circ f$.

Claim 1: $V := F_*TM \oplus h_*L \oplus \operatorname{span}\{F\}$ is a constant subspace of $\mathbb{R}_{2\epsilon}^N$.

To prove Claim 1, it suffices to show that the orthogonal complement $V^\perp$ of V in $\mathbb{R}_{2\epsilon}^N$ is a constant subspace. We have

$$V^\perp = h_*L^\perp \oplus \operatorname{span}\{\vartheta\},$$

where ϑ is as in (15). By the assumption and the first equation in (16) we have

$$A_\vartheta^F = \Phi = 0.$$

Using this and the second equation in (16) we obtain

$$\tilde{\nabla}_X \vartheta = -F_* A_\vartheta^F X + {}^F\nabla_X^\perp \vartheta = (k_1 + k_2) h_* SX \in h_* L^\perp \subset V^\perp,$$

because $S(TM) \subset L^\perp$ by assumption. On the other hand, using that L is parallel in the normal connection and that $L^\perp \subset N_1^\perp$ we have

$$\tilde{\nabla}_X h_* \xi = h_* \bar{\nabla}_X \xi + \alpha_h(f_* X, \xi) = h_* \nabla_X^\perp \xi - \langle SX, \xi \rangle \vartheta \in V^\perp$$

for all $\xi \in L^\perp$, where $\bar{\nabla}$ stands for the connection in $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$. Thus Claim 1 is proved.

Since $F_*TM \subset V$, we have that $F(M) \subset V$, because V contains the position vector at any point.

Claim 2: $\tilde{\pi}_2|_V$ is a similarity of ratio b.

Since $L \subset S(TM)^\perp$ and $(T - b^2 I)L \subset L^\perp$ by assumption, it follows from Lemma 20 that $\tilde{\pi}_2|_{h_*L}$ is a similarity of ratio b. We also have that

$$\langle \tilde{\pi}_2 F, \tilde{\pi}_2 F \rangle = \frac{1}{k_2} = b^2 \frac{k_1 + k_2}{k_1 k_2} = b^2 \langle F, F \rangle$$

and that $\tilde{\pi}_2|_{F_*TM}$ is a similarity of ratio b, because $R = b^2 I$ by Lemma 21. The proof of Claim 2 is completed by noticing that

$$\langle \tilde{\pi}_2 F, \tilde{\pi}_2 F_* X\rangle = \langle \tilde{\pi}_2 F, F_* X\rangle = 0,$$

$$\langle \tilde{\pi}_2 F, \tilde{\pi}_2 h_* \xi\rangle = \langle \tilde{\pi}_2 F, h_* \xi\rangle = 0$$

and

$$\langle \tilde{\pi}_2 F_* X, \tilde{\pi}_2 h_* \xi\rangle = \langle SX, \xi\rangle = 0$$

for any $\xi \in L$.

Let ℓ be the rank of L, set $a = \sqrt{\frac{k_2}{k_1+k_2}}$ and denote by $\mathbb{R}^{m+\ell+1}_\epsilon$ the image of both $\tilde{\pi}_1|_V$ and $\tilde{\pi}_2|_V$ in $\mathbb{R}^{n_1+1}_\epsilon$ and $\mathbb{R}^{n_2+1}_\epsilon$, respectively. By Claim 2, we have that $T_1 = a^{-1}\tilde{\pi}_1|_V$ and $T_2 = b^{-1}\tilde{\pi}_2|_V$ are linear isometries onto $\mathbb{R}^{m+\ell+1}_\epsilon$, and

$$V = \{G(X) := (aT_1(X), bT_2(X)) \,:\, X \in \mathbb{R}^{m+\ell+1}_\epsilon\}.$$

Let $g = G|_{\mathbb{Q}^{m+\ell}_k}\colon \mathbb{Q}^{m+\ell}_k \to \mathbb{Q}^{m+\ell}_{k_1} \times \mathbb{Q}^{m+\ell}_{k_2}$ be the totally geodesic embedding in Example 16 with $g(\mathbb{Q}^{m+\ell}_k) = V \cap \mathbb{Q}^{2m+2\ell+1}_k$.

Since $f(M) \subset g(\mathbb{Q}^{m+\ell}_k) \subset \mathbb{Q}^{m+\ell}_{k_1} \times \mathbb{Q}^{m+\ell}_{k_2} \subset \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$, it follows that $f = j \circ \tilde{f}$, where $j\colon \mathbb{Q}^{m+\ell}_{k_1} \times \mathbb{Q}^{m+\ell}_{k_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ is a totally geodesic inclusion and $\tilde{f}\colon M \to \mathbb{Q}^{m+\ell}_{k_1} \times \mathbb{Q}^{m+\ell}_{k_2}$ is an isometric immersion with $\tilde{f}(M) \subset g(\mathbb{Q}^{m+\ell}_k)$. Therefore, $\tilde{f} = g \circ \bar{f}$ for some isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}^{m+\ell}_k$.

For the converse, define $L = j_* g_* N_{\bar{f}} M$. From formula (20), with $\bar{f}$, $j \circ g$ and f playing the roles of f, g and F in that formula, respectively, it follows that $N^f_1 \subset L$, because $j \circ g$ is totally geodesic. That L is parallel in the normal connection follows from the first formula in (21), also because $j \circ g$ is totally geodesic.

By the first formula in (18), with $\bar{f}$, $j \circ g$ and f playing the roles of f, g and F in that formula, respectively, and taking into account the first formula in Lemma 6 and the fact that $R^g = b^2 I$ for the totally geodesic embedding g, it follows that

$$\langle T^f (j \circ g)_* \xi, (j \circ g)_* \eta\rangle = \langle R^{j \circ g} \xi, \eta\rangle = \langle R^g \xi, \eta\rangle = b^2 \langle \xi, \eta\rangle$$

for all $\xi, \eta \in N_{\bar{f}} M$, hence

$$\langle (T - b^2 I)(j \circ g)_* \xi, (j \circ g)_* \eta\rangle = 0$$

for all $\xi, \eta \in N_{\bar{f}} M$.

Finally, to see that $\Phi = 0$, recall from (16) that $\Phi = A^F_\vartheta$, where $F = h \circ f$. Write $h = k \circ \bar{h}$, where $\bar{h}\colon \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2} \to \mathbb{Q}^{n_1+n_2+1}_k$ and $k\colon \mathbb{Q}^{n_1+n_2+1}_k \to \mathbb{R}^{n_1+n_2+2}_{2\epsilon}$ are inclusions. Using that $\vartheta \in k_* N_{\bar{h}} (\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2})$ and that $\bar{h} \circ j \circ g$ is totally geodesic we obtain from (20), with $\bar{f}$ and $h \circ j \circ g$ playing the roles of f and g in that formula, respectively, that

$$\langle A^F_\vartheta X, Y\rangle = \langle \alpha_{h \circ j \circ g}(\bar{f}_* X, \bar{f}_* Y), \vartheta\rangle = \langle k_* \alpha_{\bar{h} \circ j \circ g}(\bar{f}_* X, \bar{f}_* Y), \vartheta\rangle = 0. \quad \blacksquare$$

An isometric immersion $f\colon M^m \to N^n$ between Riemannian manifolds is said to be 1-*regular* if its first normal spaces have constant dimension on M^m.

Corollary 23. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1k_2 \neq 0$, be a 1-regular isometric immersion. Assume that $\Phi = 0$ and that N_1 is parallel in the normal connection. Set $\ell = \text{rank } N_1$. Then $k_1k_2 > 0$, $n_i \geq m + \ell$ for $1 \leq i \leq 2$, and there exist an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}_k^{m+\ell}$, with $k = k_1k_2/(k_1 + k_2)$, and a totally geodesic inclusion $j\colon \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ such that*

$$f = j \circ g \circ \bar{f},$$

where $g\colon \mathbb{Q}_k^{m+\ell} \to \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell}$ is the totally geodesic embedding of Example 16.

Proof: Since $R = b^2I$, with $b := \sqrt{\frac{k_1}{k_1+k_2}}$, then $S(TM) \subset N_1^\perp$ by Lemma 18. By the assumption that N_1 (hence $N_1^\perp$) is a parallel subbundle of N_fM with respect to the normal connection, we have that $(\nabla_X S)Y \in N_1^\perp$ for all $X, Y \in TM$. Then (5) implies that

$$(T - b^2I)N_1 \subset N_1^\perp,$$

and the statement follows from Theorem 22. ∎

7 Parallel submanifolds

We now use the results of the previous sections to prove Theorems 1 and 2 in the introduction.

Lemma 24. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be a parallel isometric immersion. Then the following holds:*

(*i*) *If $k_1 \neq 0$ and $k_2 = 0$, then at any point $x \in M^m$ either $S = 0$ or there exist a unit vector $B \in T_xM$ and $\lambda \in (0, 1)$ such that*

$$SX = \langle X, B\rangle SB \quad \text{and} \quad (I - R)X = \lambda\langle X, B\rangle B. \tag{27}$$

(*ii*) *If $k_1k_2 \neq 0$ then either S or Φ vanishes everywhere on M^m.*

Proof: Given $x \in M^m$, from Codazzi equation (8) we obtain

$$\langle \Phi X, Z\rangle SY = \langle \Phi Y, Z\rangle SX, \quad \text{for all } X, Y, Z \in T_xM. \tag{28}$$

If $\Phi \neq 0$, then either $S = 0$ or the first equation in (27) holds for a unit vector $B \in T_xM$ spanning $(\ker S)^\perp$. Replacing that equation into (28) yields

$$\Phi X = \langle X, B\rangle \Phi B = \mu\langle X, B\rangle B$$

for some $\mu \neq 0$ and for all $X \in T_xM$, where in the last equality we have used that Φ is self adjoint.

Suppose that $k_2 = 0$. Then $\Phi = k_1(I - R)$, hence $S = 0$ wherever $\Phi = 0$. Therefore (i) follows with $\lambda = \mu/k_1$.

Assume now that $k_1k_2 \neq 0$. Then we get a contradiction by assuming that both Φ and S are nonvanishing at x. In fact, in this case we would have that $\{B\}^\perp \subset \ker\Phi \cap \ker S$, hence it suffices to prove that

$$W := \ker\Phi \cap \ker S = \{0\} \tag{29}$$

if $k_1k_2 \neq 0$. Otherwise we would have $k_1I|_W - (k_1 + k_2)R|_W = 0$. Thus $k_1 + k_2 \neq 0$ and $R|_W = \frac{k_1}{k_1+k_2}I|_W$, hence the first equation in (3) would give

$$0 = S^tS|_W = R(I - R)|_W = \frac{k_1k_2}{(k_1 + k_2)^2}I|_W,$$

a contradiction.

Now let

$$A := \{x \in M^m \,:\, S|_{T_xM} \neq 0\} \;\text{ and }\; B := \{x \in M^m \,:\, \Phi|_{T_xM} \neq 0\}.$$

Clearly, both A and B are open subsets of M^m. We have just proved that $A^c \cup B^c = M^m$, where $A^c = M^m \setminus A$ and $B^c = M^m \setminus B$. On the other hand, it follows from (29) that $A^c \cap B^c = \emptyset$. Thus, either $A^c = M^m$ or $B^c = M^m$. ∎

7.1 Proof of Theorem 1.

By Lemma 24, either S or Φ vanishes identically on M^m. Suppose first that S is identically zero. If either $R = 0$ or $R = I$ everywhere, then case (i) in the statement holds by Proposition 8. Otherwise, we obtain from Proposition 10 that M^m is locally a Riemannian product $M_1 \times M_2$ and $f = f_1 \times f_2$, where $f_i\colon M_i \to \mathbb{Q}_{k_i}^{n_i}$, $1 \leq i \leq 2$, is an isometric immersion. Since f is parallel, it is easily seen that the same must hold for f_i, $1 \leq i \leq 2$.

Assume now that $\Phi = 0$. First observe that parallelism of the second fundamental form implies that f is 1-regular and that N_1 is parallel in the normal connection. Then Corollary 23 applies. We obtain that $k_1k_2 > 0$, $n_i \geq m + \ell$ for $1 \leq i \leq 2$, where $\ell = \operatorname{rank} N_1$, and that there exist an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}_k^{m+\ell}$, with $k = k_1k_2/(k_1 + k_2)$, and a totally geodesic inclusion $j\colon \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell} \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ such that $f = j \circ g \circ \bar{f}$, where $g\colon \mathbb{Q}_k^{m+\ell} \to \mathbb{Q}_{k_1}^{m+\ell} \times \mathbb{Q}_{k_2}^{m+\ell}$ is the totally geodesic embedding of Example 16. Again, since f is parallel, the same must hold for $\bar{f}$. ∎

7.2 The case $k_2 = 0$

The following reduction of codimension theorem for parallel submanifolds of symmetric spaces was obtained by Dombrowski [7].

Theorem 25. *Let N be a symmetric space. If $f\colon M \to N$ is a parallel isometric immersion and if for some point $x \in M$ the second osculating space $\mathcal{O}_x f = f_* T_x M \oplus N_1(x)$ is contained in some curvature invariant subspace V of $T_{f(x)}N$, then $f(M) \subset \bar{N}$, where $\bar{N}$ denotes the totally geodesic submanifold $\exp^N_{f(x)}(V)$.*

We will make use of the following consequence of the preceding theorem.

Corollary 26. *Let $f\colon M \to N$ be a parallel isometric immersion into a symmetric space. Assume that there exists an open subset $\mathcal{U} \subset M$ such that $f(\mathcal{U})$ is contained in a totally geodesic submanifold $\bar{N} \subset N$. Then $f(M) \subset \bar{N}$.*

Proof of Theorem 2: Assume first that $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$, $k_1 \neq 0$, is a parallel isometric immersion such that $S = 0$ everywhere. If either $R = 0$ or $R = I$, then f is as in (i) by Proposition 8. Otherwise, it is given as in (ii) by Proposition 10.

Suppose now that $S \neq 0$ on an open subset $\mathcal{U} \subset M$. By Lemma 24, the tensor R is given on $\mathcal{U}$ by (27) for some unit vector field B and some smooth real function λ with values in $(0, 1)$. Notice that the first equation in (27) can also be written as

$$S^t \eta = \langle SB, \eta \rangle B$$

for any $\eta \in N_f M$. In particular, $\ker S^t = \{SB\}^\perp$ splits orthogonally as $\ker S^t = U \oplus V$, with $U = \ker T$ and $V = \ker(I - T)$. Arguing as in the proof of Lemma 9, both U and V have constant rank.

Equation (5) is equivalent to

$$(\nabla_X S^t)\xi = A_{T\xi} X - R A_\xi X \tag{30}$$

for all $X \in TM$, $\xi \in N_f M$. For $\xi \in U = \ker T$ it yields

$$\langle SB, \nabla^\perp_X \xi \rangle B = S^t \nabla^\perp_X \xi = -(\nabla_X S^t)\xi = R A_\xi X = A_\xi X - \lambda \langle A_\xi X, B \rangle B \tag{31}$$

for all $X \in TM$. Therefore

$$A_\xi X = \rho \langle X, B \rangle B \tag{32}$$

for some $\rho \in C^\infty(\mathcal{U})$. In particular, if $\xi \in U$ is orthogonal to $\zeta := (\alpha(B, B))_U$, then $\xi \in N_1^\perp$.

Now, given $X \in TM$ and $\xi \in N_f M$, we have

$$\pi_2(f_* X + \xi) = f_*(RX + S^t \xi) + SX + T\xi,$$

hence $\pi_2(f_*X+\xi)=0$ if and only if

$$X-\lambda\langle X,B\rangle B=RX=-S^t\xi=-\langle SB,\xi\rangle B \tag{33}$$

and

$$T\xi=-SX=-\langle X,B\rangle SB. \tag{34}$$

Write $X=\mu B+Y$, with $\langle Y,B\rangle=0$, and $\xi=\beta SB+\eta_U+\eta_V$, with $\eta_U\in U$ and $\eta_V\in V$. Then (33) becomes

$$\beta|SB|^2B=-Y+(\lambda-1)\mu B,$$

hence $Y=0$ and $\beta\lambda=-\mu$, using that $|SB|^2=\lambda(1-\lambda)$. On the other hand, (34) gives

$$\beta TSB+\eta_V=-\mu SB,$$

hence

$$\beta\lambda SB+\eta_V=-\mu SB,$$

where we have used that $TSB=S(I-R)B=\lambda SB$. We obtain that $\eta_V=0$ and conclude that the kernel of π_2 is the subspace spanned by U and the vector λf_*B-SB.

If the vector field $\zeta=(\alpha(B,B))_U$ vanishes everywhere on $\mathcal{U}$, then $U\subset N_1^\perp$. Thus $U\cap N_1^\perp=U$ has constant rank on $\mathcal{U}$, and since N_1 is parallel in the normal connection, condition (25) is trivially satisfied. We claim that f reduces codimension on the left by n_1-1 on $\mathcal{U}$. To prove our claim, it is equivalent to show that if f does not reduce codimension on the left then $n_1\le 1$. By Corollary 13, we must have $U=\{0\}$. Then the claim follows from that fact that the kernel of π_2 is the one-dimensional subspace spanned by λf_*B-SB, which implies that $n_2\ge n_1+n_2-1$. Therefore $f(\mathcal{U})$ is contained in a totally geodesic submanifold $\mathbb{Q}^1_{k_1}\times\mathbb{R}^{n_2}$ of $\mathbb{Q}^{n_1}_{k_1}\times\mathbb{R}^{n_2}$, and hence $f(M^m)\subset\mathbb{Q}^1_{k_1}\times\mathbb{R}^{n_2}$ by Corollary 26. We conclude that f is as in (iii).

To finish the proof of Theorem 2, it remains to show that if there exists an open subset $\mathcal{U}\subset M$ where $S\neq 0$ and the vector field $\zeta=(\alpha(B,B))_U$ is nowhere vanishing, then f is given as in (iv). This is the more delicate part of the proof.

First notice that in this case $U\cap N_1^\perp=U\cap\{\zeta\}^\perp$, hence we have again that $U\cap N_1^\perp$ has constant rank on $\mathcal{U}$ and satisfies condition (25), since N_1 is parallel. We now argue that f reduces codimension on the left by n_1-2 on $\mathcal{U}$. In fact, assuming as before that f does not reduce codimension on the left, we obtain from Corollary 13 that $U\cap\{\zeta\}^\perp=\{0\}$, that is, $U=\operatorname{span}\{\zeta\}$. It follows that $n_1\le 2$, because the kernel of π_2 is now spanned by λf_*B-SB and ζ, which implies that $n_2\ge n_1+n_2-2$. Therefore $f(\mathcal{U})$ is contained in a totally geodesic submanifold $\mathbb{Q}^2_{k_1}\times\mathbb{R}^{n_2}$ of $\mathbb{Q}^{n_1}_{k_1}\times\mathbb{R}^{n_2}$, and hence $f(M^m)\subset\mathbb{Q}^2_{k_1}\times\mathbb{R}^{n_2}$ by Corollary 26.

Lemma 27. *Let $f\colon M^m\to\mathbb{Q}^2_k\times\mathbb{R}^n$, $k\neq 0$, be a parallel isometric immersion with $S\neq 0$ everywhere that does not reduce codimension on the left. Then M^m is locally (globally, if M^m is complete and simply connected) a Riemannian product $M^m=\mathbb{R}\times N^{m-1}$ and $f=\gamma\times\tilde{f}$, where $\gamma\colon\mathbb{R}\to\mathbb{Q}^2_k\times\mathbb{R}$ is a full extrinsic circle and $\tilde{f}\colon N^{m-1}\to\mathbb{R}^{n-1}$ is a parallel isometric immersion.*

Proof: By (27) we have

$$SX = \langle X, B\rangle SB \ \text{ and } \ (I-R)X = \lambda\langle X, B\rangle B,$$

where B is a locally defined smooth unit vector field and λ is a nonvanishing smooth function.

Claim 1: $\text{span}\{B\}$ and $\{B\}^\perp$ are totally geodesic distributions.

By the assumption that f does not reduce codimension on the left we have

$$N_fM = \text{span}\{\xi\} \oplus \text{span}\{SB\} \oplus V,$$

where $V = \ker(I-T)$ and ξ is a unit vector field spanning $U = \ker T$ for which (32) holds with $\rho \neq 0$ on any open subset. From (31) and (32) we obtain

$$\langle \nabla^\perp_X \xi, SB\rangle = \rho(1-\lambda)\langle X, B\rangle.$$

On the other hand, using that $T\xi = 0$ and $S^t\xi = 0$, it follows from (6) that

$$-T\nabla^\perp_X\xi = (\nabla^\perp_X T)\xi = -SA_\xi X - \alpha(X, S^t\xi) = -\rho\langle X, B\rangle SB,$$

hence

$$\langle \nabla^\perp_X \xi, \eta\rangle = 0$$

for any $\eta \in V$. It follows that

$$\nabla^\perp_X\xi = |SB|^{-2}\langle \nabla^\perp_X\xi, SB\rangle SB = \frac{\rho}{\lambda}\langle X, B\rangle SB. \tag{35}$$

By equation (30) for $\xi = SB$, i.e.,

$$\nabla_X S^tSB - S^t\nabla^\perp_X SB = A_{TSB}X - RA_{SB}X,$$

we obtain that

$$\lambda\langle \nabla_X Y, B\rangle = \langle A_{SB}X, Y\rangle$$

for all $X \in TM$ and $Y \in \{B\}^\perp$. In particular,

$$(A_{SB}B)_{\{B\}^\perp} = -\lambda\nabla_B B. \tag{36}$$

Since f is parallel, we have

$$\nabla_Y A_\xi X = A_\xi \nabla_Y X + A_{\nabla^\perp_Y\xi}X.$$

Using (32), the left-hand-side of the preceding equation becomes

$$Y(\rho)\langle X, B\rangle B + \rho\langle \nabla_Y X, B\rangle B + \rho\langle X, \nabla_Y B\rangle B + \rho\langle X, B\rangle \nabla_Y B,$$

whereas the right-hand-side is

$$\rho\langle\nabla_Y X, B\rangle + \frac{\rho}{\lambda}\langle Y, B\rangle A_{SB}X,$$

where we have used (35). Therefore,

$$Y(\rho)\langle X, B\rangle B + \rho\langle X, \nabla_Y B\rangle B + \rho\langle X, B\rangle\nabla_Y B = \frac{\rho}{\lambda}\langle Y, B\rangle A_{SB}X.$$

For $\langle X, B\rangle = 0 = \langle Y, B\rangle$ we obtain

$$\langle\nabla_Y X, B\rangle = 0.$$

Hence $\{B\}^\perp$ is totally geodesic. For $\langle X, B\rangle = 0$ and $Y = B$ we get

$$A_{SB}X = \lambda\langle\nabla_B B, X\rangle B, \tag{37}$$

which implies that

$$(A_{SB}B)_{\{B\}^\perp} = \lambda\nabla_B B. \tag{38}$$

Comparing (36) and (38) yields

$$\nabla_B B = 0, \tag{39}$$

and the proof of the Claim 1 is completed.

Let $h\colon \mathbb{Q}_k^2 \times \mathbb{R}^n \to \mathbb{R}_{\sigma(k)}^{n+3}$ be the inclusion and set $F = h \circ f$.

Claim 2 : The second fundamental form of F satisfies $\alpha_F(B, X) = 0$ for any $X \in \{B\}^\perp$. We have from (12) and (20) that

$$\alpha_F(X, Y) = h_*\alpha_f(X, Y) - k\langle(I - R)X, Y\rangle\tilde{\pi}_1 \circ F = h_*\alpha_f(X, Y) - k\lambda\langle X, B\rangle\langle Y, B\rangle\tilde{\pi}_1 \circ F,$$

hence it suffices to prove that $\alpha_f(B, X) = 0$ for any $X \in \{B\}^\perp$, or equivalently, that $\{B\}^\perp$ is invariant under A_ζ for any $\zeta \in N_fM$.

We have from (32) that $\{B\}^\perp$ is invariant under A_ξ, whereas (37) and (39) imply that the same holds for A_{SB}. Thus, it remains to show that this holds for any $\eta \in V$.

By (5) and (27) we have

$$\nabla_X^\perp SB - S\nabla_X B = T\alpha(X, B) - \alpha(X, RB) = (T - I)\alpha(X, B) + \lambda\alpha(X, B).$$

Taking the inner product with $\eta \in V = \ker(I - T)$ yields

$$\langle\nabla_X^\perp SB, \eta\rangle = \lambda\langle A_\eta X, B\rangle. \tag{40}$$

On the other hand, since f is parallel we have

$$\nabla_Y A_{SB}X = A_{SB}\nabla_Y X + A_{\nabla_Y^\perp SB}X$$

for all $X, Y \in TM$. Taking into account that $\{B\}^\perp \subset \ker A_{SB}$ by (37) and (39), and that $\{B\}^\perp$ is totally geodesic, we obtain

$$A_{\nabla^\perp_Y SB} X = 0 \tag{41}$$

for all $X, Y \in \{B\}^\perp$. Define $\psi\colon \{B\}^\perp \to V$ by $\psi(Y) = \nabla^\perp_Y SB$. Then $\{B\}^\perp \subset \ker A_\eta$ for any $\eta \in \mathrm{Im}\ \psi$ by (41), whereas (40) implies that $A_\eta(\{B\}^\perp) \subset \{B\}^\perp$ for any η in the orthogonal complement of Im ψ in V. Therefore $A_\eta\{B\}^\perp \subset \{B\}^\perp$ for any $\eta \in V$, and Claim 2 is proved.

It follows from Claim 1 and the local version of de Rham theorem that M^m splits locally as a Riemannian product $M^m = \mathbb{R} \times N^{m-1}$. Moreover, the splitting is global if M^m is complete and simply connected by the global version of de Rham Theorem.

Define

$$\mathbb{R}^{n-\ell} = \mathrm{span}\{F_*(x)X \,:\, x \in M^m,\, X \in \{B\}^\perp(x)\}.$$

Since $\{B\}^\perp \subset \ker(I - R)$ we have that $\tilde{\pi}_1(F_*\{B\}^\perp) = \{0\}$, hence $\mathbb{R}^{n-\ell} \subset \mathbb{R}^n$. By Claim 2 and Moore's lemma [14], there exist a unit speed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^{\ell+3} = (\mathbb{R}^{n-\ell})^\perp$ and a full isometric immersion $\tilde{f}\colon N^{m-1} \to \mathbb{R}^{n-\ell}$ such that $F(x, y) = (\gamma(x), \tilde{f}(y))$.

Since $F(M^m) \subset \mathbb{Q}^2_k \times \mathbb{R}^n$ and f is parallel, then $\gamma(\mathbb{R}) \subset \mathbb{Q}^2_k \times \mathbb{R}^\ell$ and both $\tilde{f}$ and γ are parallel. It follows from Corollary 13 that $\gamma(\mathbb{R})$ is contained in a totally geodesic $\mathbb{Q}^2_k \times \mathbb{R} \subset \mathbb{Q}^2_k \times \mathbb{R}^\ell$. Moreover, since $S \neq 0$ and f does not reduce codimension on the left, γ must be full in $\mathbb{Q}^2_k \times \mathbb{R}$. ∎

We now complete the proof of Theorem 2. Let $f\colon M^m \to \mathbb{Q}^2_k \times \mathbb{R}^n$, $k \neq 0$, be a parallel isometric immersion such that $S \neq 0$ and the vector field $\zeta = (\alpha(B, B))_U$ is nowhere vanishing on an open subset $\mathcal{U} \subset M$. In view of Lemma 27, it is enough to show that S can not vanish on any open subset.

Since the first factor in $\mathbb{Q}^2_k \times \mathbb{R}^n$ has dimension two, then rank $(I - R) \in \{0, 1, 2\}$. If $R = I$ on an open subset of M^m, then f is as in (i) by Proposition 8 and Corollary 26. Hence $R = I$ everywhere, contradicting the fact that $S \neq 0$ on $\mathcal{U}$. If $S = 0$ and rank $(I - R) = 2$ on a maximal open subset $\mathcal{V}$, then this also holds on $\bar{\mathcal{V}}$, since the eigenvalues of $(I - R)$ are either 0 or 1 on $\mathcal{V}$. But then rank $(I - R) = 2$ on an open neighborhood of $\bar{\mathcal{V}}$, contradicting the fact that rank $(I - R) = 1$ at points where $S \neq 0$ by Lemma 24.

Therefore, if $S = 0$ on a maximal open subset $\mathcal{V}$, there is a smooth unit vector field B such that $(I - R)X = \langle X, B\rangle B$ for any $X \in T\mathcal{V}$. Let $x \in \bar{\mathcal{V}}$. Then there exist an open neighborhood W of x and a smooth nowhere vanishing function λ on W such that $(I - R)X = \lambda\langle X, B\rangle B$ for any $X \in TW$. The proof of Lemma 27 shows that there exists a product neighborhood $I \times Z$ of x, an extrinsic circle $\gamma\colon I \to \mathbb{Q}^2_k \times \mathbb{R}$ and a parallel isometric immersion $\bar{f}\colon Z \to \mathbb{R}^{n-1}$ such that $f|_{I\times Z} = \gamma \times \bar{f}$. Since $(I \times Z) \cap \mathcal{V} \neq \emptyset$, there exists an open interval $J \subset I$ such that $\gamma(J) \subset \mathbb{Q}^2_k$, hence we must have $\gamma(I) \subset \mathbb{Q}^2_k$ by Corollary 26. But then $S = 0$ on W, hence on $\mathcal{V} \cup W$, contradicting the maximality of $\mathcal{V}$ with respect to this property. ∎

8 Umbilical submanifolds of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$

Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an umbilical isometric immersion. Denote by η its mean curvature vector. Then formulas (4)–(6) become, respectively,

$$(\nabla_X R)Y = \langle SY, \eta\rangle X + \langle X, Y\rangle S^t\eta, \tag{42}$$

$$(\nabla_X S)Y = \langle X, Y\rangle T\eta - \langle X, RY\rangle\eta \tag{43}$$

and

$$(\nabla_X T)\xi = -\langle \xi, \eta\rangle SX - \langle X, S^t\xi\rangle\eta. \tag{44}$$

The Gauss, Codazzi and Ricci equations (7), (8) and (9), respectively, take the form

$$R(X, Y) = k_1(X \wedge Y - X \wedge RY - RX \wedge Y) + (k_1 + k_2)RX \wedge RY + \|\eta\|^2 X \wedge Y, \tag{45}$$

$$\langle Y, Z\rangle\nabla_X^\perp\eta - \langle X, Z\rangle\nabla_Y^\perp\eta = \langle \Phi X, Z\rangle SY - \langle \Phi Y, Z\rangle SX \tag{46}$$

and

$$R^\perp(X, Y)\xi = (k_1 + k_2)(SX \wedge SY)\xi, \tag{47}$$

whereas the equivalent form (10) of the Codazzi equation becomes

$$\langle \xi, \nabla_Y^\perp\eta\rangle X - \langle \xi, \nabla_X^\perp\eta\rangle Y = \langle SX, \xi\rangle\Phi Y - \langle SY, \xi\rangle\Phi X. \tag{48}$$

8.1 Case $S = 0$

Our approach to proving Theorem 4 is based on an analysis of the various possible structures of the tensor S. The next lemma takes care of the simplest case, in which S vanishes identically.

Lemma 28. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an umbilical nontotally geodesic isometric immersion. Assume that $S = \{0\}$ everywhere. Then f is as in part (i) of Theorem 4.*

Proof: Let $x \in M^m$ be a point where the mean curvature vector η is nonzero and let $\mathcal{U} \subset M^m$ be the maximal connected open neighborhood of x where η does not vanish. It follows from (43) that $R = \lambda I$ on $\mathcal{U}$, where $\lambda \in C^\infty(\mathcal{U})$ is given by $\lambda = |\eta|^{-2}\langle T\eta, \eta\rangle$. On the other hand, R is an orthogonal projection at every $x \in \mathcal{U}$ by the first equation in (3), hence its eigenvalues are either 0 or 1. Thus, either $R = 0$ or $R = I$ on $\mathcal{U}$. Assuming the first possibility, we conclude from Proposition 8 that $f|_{\mathcal{U}}$ is as in the statement. In particular, $|\eta|$ is constant on $\mathcal{U}$. If $\mathcal{U} \neq M$, then $|\eta|$ would be nonzero on some open subset containing $\mathcal{U}$, contradicting the maximality of $\mathcal{U}$ with respect to this property. ∎

8.2 Case $\ker S = 0$

Our next step is to consider the other extreme case, in which $\ker S = \{0\}$ at some point.

Lemma 29. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an umbilical isometric immersion with $m \geq 3$ and $k_1 + k_2 \neq 0$. Assume that $\ker S = \{0\}$ at some point $x \in M^m$. Then there exist umbilical isometric immersions $f_i\colon M^m \to \mathbb{Q}_{\tilde{k}_i}^{n_i}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1 \cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$ for some $\theta \in (0, \pi/2)$, such that $f = (\cos\theta f_1, \sin\theta f_2)$.*

Proof: Let $\mathcal{U} \subset M$ be the maximal connected open subset containing x where $\ker S = \{0\}$. Let $X_1, \ldots, X_m$ be an orthonormal diagonalizing frame for R, and let $\lambda_1, \ldots, \lambda_m$ be the corresponding eigenvalues. From Codazzi equation (46) we obtain

$$\nabla^{\perp}_{X_i}\eta = ((k_1 + k_2)\lambda_j - k_1)SX_i$$

for all $1 \leq i, j \leq m$ with $i \neq j$. Using that $m \geq 3$, the preceding equation implies that

$$(k_1 + k_2)\lambda_j = (k_1 + k_2)\lambda_\ell$$

for all $1 \leq j, \ell \leq m$. Since $k_1 + k_2 \neq 0$, it follows that there exists $\lambda \in C^\infty(\mathcal{U})$ such that $\lambda_i = \lambda$ for all $1 \leq i \leq m$. From (42) we obtain

$$X_i(\lambda)X_j + (\lambda I - R)\nabla_{X_i}X_j = \langle SX_j, \eta\rangle X_i,$$

for all $1 \leq i, j \leq m$ with $i \neq j$, hence $X_i(\lambda) = 0$ for all $1 \leq i \leq m$. Thus $R = \lambda I$ on $\mathcal{U}$, where λ is a constant in $(0, 1)$. But this implies that $R = \lambda I$ on the closure $\bar{\mathcal{U}}$ of $\mathcal{U}$, thus $\ker S = \{0\}$ also on $\bar{\mathcal{U}}$, and hence on an open neighborhood of $\mathcal{U}$, contradicting the maximality of $\mathcal{U}$ with respect to this property. We conclude that $R = \lambda I$ on M. The proof is completed by Proposition 17. ∎

8.3 Case $\dim\ker S = k \in (0, \dim M)$.

In this subsection we study umbilical isometric immersions $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ for which the kernel of the tensor S has constant dimension $k \in (0, m)$ on M^m.

Lemma 30. *Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1 + k_2 \neq 0$, be an umbilical isometric immersion. Assume that $\ker S$ has constant dimension $k \in (0, m)$ on M^m. Then $k = m - 1$. Moreover, either $\ker S = \ker R$ on M^m or $\ker S = \ker(I - R)$ on M^m.*

Proof: Since $\ker S$ has constant dimension, the same holds for $\ker R$ and $\ker(I - R)$. Moreover, since $\ker S$ is nontrivial, the same must hold for either $\ker R$ or $\ker(I - R)$. We will show that in the first case we must have $k = m - 1$ and $\ker S = \ker R$. By a similar argument one shows that ($k = m - 1$ and) $\ker S = \ker(I - R)$ in the second case.

Applying (46) for $X \in \ker R$ and $Y = Z$ orthogonal to X gives

$$\nabla^\perp_X \eta = 0 \quad \text{for all} \ \ X \in \ker R, \tag{49}$$

whereas for $Z = X \in \ker R$ and $Y \in (\ker R)^\perp \neq \{0\}$ it yields

$$\nabla^\perp_Y \eta = -k_1 SY \quad \text{for all} \ \ Y \in (\ker R)^\perp. \tag{50}$$

Choose $0 \neq Y \in (\ker S)^\perp$. Then, applying (48) for $\xi = SY$ and using that $k_1+k_2 \neq 0$ we obtain for any $X \in (\ker R)^\perp$ that

$$RX = \frac{\langle SX, SY\rangle}{|SY|^2} RY.$$

Thus R has rank one, and the remaining of the statement follows from $\ker R \subset \ker S$. ∎

In the next result we assume that $\ker S$ has constant dimension $m-1$ and that $\ker S = \ker R$ everywhere, the case $\ker S = \ker(I-R)$ being completely similar.

Lemma 31. *Let $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an umbilical isometric immersion such that $\ker S$ has constant dimension $m-1$ and $\ker S = \ker R$ everywhere. Then f reduces codimension on the left by either $n_1 - m$ or $n_1 - m - 1$ and on the right by $n_2 - 1$.*

Proof: To prove the statement, it is equivalent to show that if f reduces codimension neither on the left nor on the right then $n_2 = 1$ and n_1 is either m or $m+1$.

By the assumption, there exist a locally defined smooth unit vector field B and a smooth nowhere vanishing function λ such that

$$SX = \langle X, B\rangle SB \quad \text{and} \quad RX = \lambda\langle X, B\rangle B$$

for all $X \in TM$. Hence $S^t\xi = \langle SB, \xi\rangle B$ for all $\xi \in N_fM$. We obtain from (42) that

$$X(\lambda)\langle Y, B\rangle B + \lambda\langle Y, \nabla_X B\rangle B + \lambda\langle Y, B\rangle \nabla_X B = \langle SB, \eta\rangle(\langle Y, B\rangle X + \langle X, Y\rangle B). \tag{51}$$

For $X = Y = B$, the preceding equation gives $B(\lambda) = 2\langle SB, \eta\rangle$ and $\nabla_B B = 0$. Hence $(\ker R)^\perp = \text{span}\{B\}$ is a totally geodesic distribution. Applying (51) for $Y \in \ker R = \{B\}^\perp$ yields

$$\langle \nabla_X Y, B\rangle = -\frac{\langle SB, \eta\rangle}{\lambda}\langle X, Y\rangle.$$

Thus $\ker R$ is an umbilical distribution with mean curvature vector field $\varphi = \mu B$, where $\mu = -\langle SB, \eta\rangle/\lambda$, i.e.,

$$(\nabla_X Y)_{(\ker R)^\perp} = \langle X, Y\rangle \varphi \quad \text{for all} \ \ X, Y \in \ker R.$$

Applying (43) for a unit vector field $Y = X \in \{B\}^\perp$ gives

$$T\eta = -\langle \nabla_X X, B\rangle SB = -\mu SB. \tag{52}$$

Then, for $Y = B$ it yields

$$\nabla^\perp_X SB = -\langle X, B\rangle(\mu SB + \lambda\eta) \tag{53}$$

for any $X \in TM$. In particular,

$$\nabla^\perp_X SB = 0 \ \text{ for any } \ X \in \{B\}^\perp. \tag{54}$$

On the other hand, for $X \in \{B\}^\perp$ and $Y = B$ we obtain from (51) that

$$X(\lambda) = 0 \ \text{ for any } \ X \in \{B\}^\perp. \tag{55}$$

It follows from (49), (54) and (55) that

$$X(\mu) = 0 \ \text{ for any } \ X \in \{B\}^\perp,$$

hence $\ker R$ is a spherical distribution, i.e., $\nabla_X\varphi \in \ker R$ for all $X \in \ker R$.

By the local version of Hiepko's theorem [10], there exists locally an isometry

$$\psi\colon I \times_\rho N^{m-1} \to M^m$$

from a warping product manifold, where $I \subset \mathbb{R}$ is an open interval and $\rho \in C^\infty(I)$ is the warping function, that maps the leaves of the product foliation induced by the factors I and N^{m-1} into the leaves of $\text{span}\{B\}$ and $\{B\}^\perp$, respectively.

Our next step is to prove that the vector fields η and SB are either linearly independent everywhere or linearly dependent everywhere. For that we consider the function $h\colon M^m \to \mathbb{R}$ given by

$$h(x) = |\eta(x)|^2|SB(x)|^2 - \langle \eta(x), SB(x)\rangle^2,$$

which vanishes precisely at the points where η and SB are linearly dependent. Using (49), (50) and (53), a straightforward computation gives

$$X(h) = -2\langle X, B\rangle\mu h \ \text{ for all } \ X \in TM. \tag{56}$$

Therefore $\tilde{h} = h \circ \psi$ depends only on I and we can write (56) as $\tilde{h}'(t) = -2\mu(t)\tilde{h}(t)$, where we also write μ for $\mu \circ \psi$. Hence

$$\tilde{h}(t) = \tilde{h}(t_0)\exp\left(-2\int_{t_0}^t \mu(s)\,ds\right)$$

for any fixed $t_0 \in I$. We obtain that each point of M^m has an open neighborhood V where h is either identically zero or is nowhere vanishing. Therefore, the set $h^{-1}(0)$ is both open and closed, hence h is either identically zero or nowhere vanishing in M^m.

Let $U = \ker T$ and $V = \ker(I - T)$. We prove that $\eta \in U \oplus \text{span}\{SB\}$. We have

$$\eta = \eta_U + \eta_V + \frac{\langle \eta, SB \rangle}{|SB|^2} SB = \eta_U + \eta_V + \frac{\langle \eta, SB \rangle}{\lambda(1-\lambda)} SB,$$

where η_U and η_V are the components of η in U and V, respectively. Then, using that $TSB = S(I - R)B = (1 - \lambda)SB$ we obtain

$$T\eta = \eta_V + \frac{\langle \eta, SB \rangle}{\lambda} SB = \eta_V - \mu SB.$$

Comparing with (52) gives $\eta_V = 0$, as we claimed.

Given $\zeta \in U \cap \{\eta\}^\perp = U \cap N_1^\perp$, it follows from (49) and (50) that

$$\langle \nabla_X^\perp \zeta, \eta \rangle = -\langle \zeta, \nabla_X^\perp \eta \rangle = 0,$$

hence $\nabla_X^\perp \zeta \in \{\eta\}^\perp$. Similarly, we have $\nabla_X^\perp \zeta \in \{\eta\}^\perp$ for any $\zeta \in V = V \cap \{\eta\}^\perp = V \cap N_1^\perp$.

Since we are assuming that f reduces codimension neither on the left nor on the right, it follows from Corollary 13 that $V = \{0\} = U \cap \{\eta\}^\perp$. Hence, the codimension $n_1 + n_2 - m$ is either 1 or 2, according as η_U is zero or not.

Now,

$$\begin{aligned} \pi_1 \eta &= -f_* S^t \eta + (I - T)\eta = -\langle \eta, SB \rangle f_* B + \eta_U + \frac{\langle \eta, SB \rangle}{\lambda(1-\lambda)} SB - \frac{\langle \eta, SB \rangle}{\lambda} SB \\ &= -\frac{\langle \eta, SB \rangle}{1-\lambda} ((1-\lambda) f_* B - SB) + \eta_U. \end{aligned}$$

On the other hand,

$$\pi_1 f_* X = f_*(I - R)X - SX = f_* X$$

if $X \in \ker R$, and

$$\pi_1 f_* B = f_*(I - R)B - SB = (1 - \lambda) f_* B - SB.$$

It follows that

$$\pi_1(f_* TM \oplus \text{span}\{\eta\}) = f_* \ker R \oplus \text{span}\{(1-\lambda) f_* B - SB\} \oplus \text{span}\{\eta_U\},$$

hence n_1 is either m or $m + 1$, according as η_U is zero or not. ∎

8.4 Proof of Theorem 4:

We use the following consequence of part (ii) of Proposition 1 of [15]:

Proposition 32. *Let N^n, $n \geq 3$, be a Riemannian manifold, let $x \in N^n$, let L^ℓ be an ℓ-dimensional subspace of T_xN^n, $2 \leq \ell < n$, and let H^* be a vector in T_xN^n orthogonal to L^ℓ. Assume that there exist a totally umbilical isometric immersion $f\colon \bar{N}^\ell \to N^n$ of a connected Riemannian manifold $\bar{N}^\ell$ and $\bar{x} \in \bar{N}^\ell$ such that $f(\bar{x}) = x$, $f_*T_{\bar{x}}\bar{N}^\ell = L^\ell$ and $H^f(\bar{x}) = H^*$. Suppose also that there exist a complete Riemannian manifold $\tilde{N}^{\ell+1}$, a totally geodesic isometric immersion $g\colon \tilde{N}^{\ell+1} \to N^n$ and $\tilde{x} \in \tilde{N}^{\ell+1}$ such that $g(\tilde{x}) = x$ and $g_*T_{\tilde{x}}\tilde{N}^{\ell+1} = L^\ell \oplus span\{H^*\}$. Then $f(\bar{N}^\ell) \subset g(\tilde{N}^{\ell+1})$.*

Let $f\colon M^m \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be as in Theorem 4. Since f is not totally geodesic, it follows from Proposition 32 that there exists no open subset of M^m where the mean curvature vector η of f vanishes.

If S vanishes everywhere on M^m, then f is as in either (i) by Lemma 28. If $\ker S = \{0\}$ at some point $x \in M^m$, then f is as in (ii) by Lemma 29. Then, we can assume that there exists an open subset $\mathcal{U} \subset M^m$ where $\ker S$ has constant dimension $k \in (0, m)$ and η is nowhere vanishing. By Lemma 30, we must have $k = m - 1$. Moreover, either $\ker S = \ker R$ on $\mathcal{U}$ or $\ker S = \ker(I - R)$ on $\mathcal{U}$. Assume, say, that the first possibility holds. Then, Lemma 31 implies that, on $\mathcal{U}$, f reduces codimension on the left by either $n_1 - m$ or $n_1 - m - 1$ and on the right by $n_2 - 1$. It follows from Proposition 32 that the same must hold everywhere on M^m. Since $S \neq 0$ and $\eta \neq 0$ on $\mathcal{U}$, the codimension of f can not be further reduced either on the left or on the right. Hence f is as in (iii). Similarly, if $\ker S = \ker(I-R)$ on $\mathcal{U}$, we conclude that f is as in (iii) after interchanging the factors. ∎

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