THE VANISHING EULER CHARACTERISTIC OF AN ISOLATED DETERMINANTAL SINGULARITY

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ABSTRACT. Let $(X,0)$ be a complex analytic isolated determinantal singularity. We will define the vanishing Euler characteristic of $(X,0)$ and the Milnor number of a holomorphic function germ with an isolated singularity on X, $f:(X,0)\to\mathbb{C}$.

1. INTRODUCTION

The Milnor number is an important invariant associated to some complex analytic variety germs $(X,0)$, like curves ([4]), isolated hypersurface singularities ([22]) and isolated complete intersection singularities, or ICIS, ([20]). This number $\mu(X,0)$ is related to certain geometric properties of these germs. In fact, it can be computed in terms of the vanishing Euler characteristic

$$\mu(X,0)=(-1)^{\dim X}(\chi(F)-1),$$

where F is the Milnor fibre of $(X,0)$.

A kind of analytic variety germ that can be naturally considered as a generalization of an ICIS is an isolated determinantal singularity, or IDS, that is, germs of varieties in $\mathbb{C}^N$ given as zeros of certain minors of matrices whose elements are function germs in $\mathcal{O}_N$ under appropriate hypothesis on the codimension. Determinantal varieties appear in a natural way in singularity theory. An example is the singular locus, $S(f)$, of a map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, which is given by the maximal minors of the Jacobian matrix of f. We find recent papers dedicated to the study of invariants related to the Milnor number of an IDS, see [5, 8, 7, 26].

Here we define the vanishing Euler charasteristic for an IDS $(X,0)$ of type $(m,n;s)$ in $\mathbb{C}^N$ (see definition 2.1) such that either $s=1$ or $N<(n-s+2)(m-s+2)$ and the defining matrix has rank $s-1$ outside the origin. To do this, we first exhibit an special smoothing of X and show that the Euler characteristic of its generic fibers is constant. Then we define the vanishing Euler characteristic of X by

$$\nu(X,0)=(-1)^{\dim X}(\chi(X_A)-1),$$

where X_A is the generic fiber of the smoothing. We show that this number satisfies some properties which hold for the Milnor number of an ICIS, such as the relations with polar multiplicities [16] and the Lê-Greuel formula [3, 18].

We remark that the vanishing Euler characteristic of an IDS is defined in [5] as $\chi(X_t)-1$ where X_t is a stabilization of $(X,0)$ with respect to the $\mathcal{K}_M$-equivalence (see [5] for the definition). It follows from theorem 3.4 that their definition coincides, up to the sign, with that of ours.

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As an application, we show that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an $\mathcal{A}$-finitely determined germ with $n < 2(|n-p|+2)$, then $\nu(S(f)) = (-1)^{\dim S(f)}(\chi(S(f_s)) - 1)$, where f_s is a stabilization of f.

In the last part, we define the Milnor number of a function germ $f : (X, 0) \to \mathbb{C}$ with isolated singularity on an IDS $(X, 0)$. We also include an appendix about Morse theory with some results that we need.

2. Determinantal singularities

Let $0 < s \leq m \leq n$ be integer numbers. We denote by $M_{m,n} = M_{m,n}(\mathbb{C})$ the set of complex matrices of size $m \times n$, by $M^s_{m,n}$ the subset consisting of the matrices with rank less than s and by Σ^s the subset consisting of those with rank equal to s.

The set $M^s_{m,n}$ is an irreducible algebraic subvariety of $M_{m,n}$ with codimension $(m-s+1)(n-s+1)$ and it is called a *generic determinantal variety*. The singular set of $M^s_{m,n}$ is $M^{s-1}_{m,n}$. In fact, the family $\mathcal{S} = \{\Sigma^i\}_{0 \leq i \leq s-1}$ provides a Whitney stratification of $M^s_{m,n}$ (see [1]).

Let $F : (\mathbb{C}^N, 0) \to M_{m,n}$ be a holomorphic map germ defined by $F(x) = (f_{ij}(x))$ with $f_{ij} \in \mathcal{O}_N$, the ring of holomorphic function germs from $(\mathbb{C}^N, 0)$ to $\mathbb{C}$.

Definition 2.1. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be the variety germ defined by $(X, 0) = F^{-1}(M^s_{m,n})$. We say that $(X, 0)$ is a *determinantal singularity* of type $(m, n; s)$ in $(\mathbb{C}^N, 0)$ if the dimension of $(X, 0)$ is equal to

$$N - (m-s+1)(n-s+1).$$

We consider $(X, 0)$ with the analytic structure defined by F and $M^s_{m,n}$, that is, given by the minors of size s of F (not necessarily reduced). Note that if $s = 1$, then $(X, 0)$ is a complete intersection. In general, $(X, 0)$ is Cohen-Macaulay by the Eagon-Hochster theorem [6].

Example 2.2. Let $F : (\mathbb{C}^4, 0) \to M_{2,3}$ be the map germ given by

$$F(x, y, z, w) = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,3})$. Then,

$$(X, 0) = V(xz - y^2, yw - z^2, xw - yz)$$

has dimension $2 = 4-(2-2+1)(3-2+1)$. Therefore $(X, 0)$ is a determinantal singularity of type $(2, 3; 2)$ in $(\mathbb{C}^4, 0)$.

Example 2.3. Let $F : (\mathbb{C}^5, 0) \to M_{2,4}$ be the map germ given by

$$F(x, y, z, w, v) = \begin{pmatrix} x & y & z & w \\ y & z & w & v \end{pmatrix}$$

and consider $(X, 0) = F^{-1}(M^2_{2,4})$. Then,

$$(X, 0) = V(-w^2 + zv, -zw + yv, -yw + xv, z^2 - yw, yz - xw, -y^2 + xz)$$

has dimension $2 = 5-(2-2+1)(4-2+1)$. Therefore $(X, 0)$ is a determinantal singularity of type $(2, 4; 2)$ in $(\mathbb{C}^5, 0)$.

We introduce now the concept of isolated determinantal singularity, which is slightly different from that of [7]. For technical reasons, we restrict ourselves from now on to the following cases:

(*) $$\text{either } s = 1 \text{ or } N < (m - s + 2)(n - s + 2).$$

We remark that the condition (*) is included here because we need to use transversality properties. We do not know if our results are still true without this hypothesis.

Definition 2.4. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type $(m, n; s)$ defined by $(X, 0) = F^{-1}(M^s_{m,n})$, satisfying condition (*). We say $(X, 0)$ is an *isolated determinantal singularity* or *IDS* if X is smooth at x and $\operatorname{rank} F(x) = s - 1$, for all $x \neq 0$ in a neighbourhood of the origin.

An IDS has isolated singularity in the usual sense, but we have the extra condition about the rank of F. We shall need such condition in the next section in order to define the vanishing Euler characteristic. Because of condition (*), our definition is equivalent to the concept of essentially isolated determinantal singularity (EIDS) in the sense of [7]. This will be clear in the following lemma.

Lemma 2.5. *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type $(m, n; s)$ defined by $(X, 0) = F^{-1}(M^s_{m,n})$, satisfying condition (*). Then $(X, 0)$ is an IDS if and only if F is transverse to the stratification $\mathcal{S}$ on a punctured neighbourhood of the origin.*

Proof. Take $x \in X$, $x \neq 0$ in a neighbourhood of the origin. If $s = 1$, then obviously $\operatorname{rank} F(x) = 0$. Otherwise, $N < (m-s+2)(n-s+2)$ and the transversality to the strata Σ^i for $0 \leq i < s-1$ is equivalent to $F(x) \notin \Sigma^i$. But this is equivalent to $\operatorname{rank} F(x) = s-1$.

Since $F(x) \in \Sigma^{s-1}$ and Σ^{s-1} is open in $M^s_{m,n}$, then

$$(X, x) = F^{-1}(M^s_{m,n}, F(x)) = F^{-1}(\Sigma^{s-1}, F(x)).$$

Hence, F is transverse to Σ^{s-1} at x if and only if (X, x) is smooth by [2, Lemma 4.2]. □

Remark 2.6. If $s = 1$, then condition (*) is satisfied automatically and in this case $(X, 0)$ is an IDS if and only if it is an ICIS.

In general, if $(X, 0)$ is an IDS of dimension > 0, then it is reduced, since it is Cohen-Macaulay and satisfies Serre's R0 condition, see for instance [13].

3. The vanishing Euler characteristic of an IDS

Throughout this section we suppose that $(X, 0) \subset (\mathbb{C}^N, 0)$ is an IDS of type $(m, n; s)$ defined by $(X, 0) = F^{-1}(M^s_{m,n})$. We want to associate a vanishing Euler characteristic to $(X, 0)$. To do this, we first construct a special determinantal smoothing of $(X, 0)$. We take a small enough representative $X = F^{-1}(M^s_{m,n})$, where $F : B \to M_{m,n}$ is defined on an small enough open ball $B = B_\epsilon$ centered at the origin in $\mathbb{C}^N$. For each matrix A in $M_{m,n}$, we denote

$$\begin{array}{rccc} F_A : & B & \to & M_{m,n} \\ & x & \mapsto & F(x) + A \end{array}$$

and $X_A := F_A^{-1}(M^s_{m,n})$.

Lemma 3.1. *There exists a nonempty Zariski open subset $W \subset M_{m,n}$ such that:*

1. *X_A is smooth and $\operatorname{rank}(F(x) + A) = s - 1$, for all $x \in X_A$ and A in W;*
2. *the Euler characteristic of X_A, $\chi(X_A)$, does not depend on $A \in W$.*

Proof. We take $B \subset \mathbb{C}^N$ in such a way that X is smooth at x and $\operatorname{rank} F(x) = s-1$, for all $x \in B \setminus \{0\}$. We denote by $\tilde{C} \subset M_{m,n} \times U$ the subset of pairs (A, x) such that either x is a singular point of X_A or $\operatorname{rank}(F(x) + A) < s-1$. We also denote $C = \pi_1(\tilde{C})$ where $\pi_1 : M_{m,n} \times U \to M_{m,n}$ is the projection onto the first factor. Then, we put $W = M_{m,n} \setminus C$.

Note that $\tilde{C}$ is a closed analytic subset of $M_{m,n} \times U$. In fact,

$$\tilde{C} = S(X) \cup V(I_{s-1}(F(x) + A)),$$

where $S(X)$ is the singular set of X and $I_r(M)$ denotes the ideal generated by the minors of size r of a matrix M. We consider now the restriction of the projection $\pi_1|_{\tilde{C}} : \tilde{C} \to M_{m,n}$. We have that $(\pi_1|_{\tilde{C}})^{-1}(0) = \{(0,0)\}$. By shrinking the neighbourhood B if necessary, we may assume that $\pi_1|_{\tilde{C}}$ is a finite map and its image $C = \pi_1(\tilde{C})$ is also analytic (see for instance [12, Theorem 2 - page 53]). Hence, W is Zariski open.

Let us denote

$$\begin{array}{rccc} \phi: & M_{m,n} \times B & \to & M_{m,n} \\ & (A, x) & \mapsto & F(x) + A. \end{array}$$

This map is a submersion, since for each $x \in B$, the map $A \to \phi(A, x)$ is a translation and hence, a diffeomorphism. Then, ϕ is transverse to $\mathcal{S}$. By the transversality lemma [15, Lemma 4.6], we deduce that $\phi_A = F_A$ is transverse to $\mathcal{S}$ for almost all matrices A. By using the same arguments of the proof of lemma 2.5, we have that $(A, x) \notin \tilde{C}$ if and only if F_A is transverse to $\mathcal{S}$ at x. Thus, $A \in W$ if and only if F_A is transverse to $\mathcal{S}$. In particular, W cannot be empty.

Finally, we consider again the restriction of the projection $\pi_1 : \phi^{-1}(\Sigma^{s-1}) \to W$, For any $A \in W$, ϕ_A is transverse to Σ^{s-1}. Hence, A is a regular value of π_1 (see [10], proof of Theorem 2.2). Therefore, π_1 is a submersion and hence, a fibration over the connected set W. The fibers are all homotopic and $\chi(X_A) = \chi(\pi_1^{-1}(A))$ does not depend on A in W. □

If $(X, 0) \subset (\mathbb{C}^N, 0)$ is an ICIS (when $s = 1$), X_A can be seen as the Milnor fiber of $(X, 0)$. In this case, X_A has the homotopy type of a bouquet of spheres and the Milnor number of $(X, 0)$ is the number of such spheres (see for instance [20]). Then, in this case the Milnor number coincides with the so-called vanishing Euler characteristic, that is

$$\mu(X, 0) = \beta_d(X_A) = (-1)^d(\chi(X_A) - 1),$$

where $d = \dim(X, 0)$ and $\beta_d(X_A)$ denotes the dth Betti number of X_A. Inspired by this fact and supported by the previous lemma, we make the following definition now in the general case of an IDS.

Definition 3.2. We define the *vanishing Euler characteristic* of the IDS $(X, 0)$ by

$$\nu(X, 0) := (-1)^{\dim X}(\chi(X_A) - 1),$$

where $A \in W$ and W is given in lemma 3.1.

We shall see now that this formula for the vanishing Euler characteristic is also valid when we consider a general determinantal smoothing X_t instead of the special smoothing X_A.

Definition 3.3. A *determinantal deformation* of $(X, 0)$ is a map germ $H : (\mathbb{C}^N \times \mathbb{C}, 0) \to M_{m,n}$ such that $H(x, 0) = F(x)$ for all $x \in \mathbb{C}^N$. We denote $F_t(x) = H(x, t)$ and $X_t = F_t^{-1}(M^s_{m,n})$.

We say that H defines a *determinantal smoothing* of $(X,0)$ if in addition X_t is smooth and $\operatorname{rank}(F_t(x)) = s-1$ for all $x \in X_t$ and all $t \neq 0$ small enough.

Theorem 3.4. *Let $H : (\mathbb{C}^N \times \mathbb{C}, 0) \to M_{m,n}$ be a determinantal smoothing of $(X,0)$. Then, for all $t \neq 0$ small enough,*

$$\nu(X,0) = (-1)^{\dim X}(\chi(X_t) - 1).$$

Proof. We take a representative $H : B \times D \to M_{m,n}$, where B, D are small enough open balls centered at the origin in $\mathbb{C}^N, \mathbb{C}$ respectively and such that X_t is smooth and $\operatorname{rank}(F_t(x)) = s-1$ for all $x \in X_t$ and all $t \in D \setminus \{0\}$. We also take $W \subset M_{m,n}$ the nonempty Zariski open set given by lemma 3.1.

We construct a new deformation as the sum of two deformations X_t and X_A. Given $A \in M_{m,n}$ and $t \in D$, we denote

$$X_{(A,t)} = (F_t + A)^{-1}(M^s_{m,n}).$$

Then, by using the same arguments as in the proof of lemma 3.1, we can show that there is a nonempty Zariski open subset $W_0 \subset M_{m,n} \times D$ such that

(1) $X_{(A,t)}$ is smooth and $\operatorname{rank}(F_t(x) + A) = s-1$, for all $x \in X_{(A,t)}$ and for all $(A,t) \in W_0$.
(2) $\chi(X_{(A,t)})$ does not depend on $(A,t) \in W_0$.

In fact, W_0 is obtained as the complement of $C = \pi_1(\tilde{C})$, where $\pi_1(A,t,x) = (A,t)$ and $\tilde{C} \subset M_{m,n} \times D \times B$ is the subset of triples (A,t,x) such that either x is a singular point of $X_{(A,t)}$ or $\operatorname{rank}(F_t(x) + A) < s-1$. Since the map

$$\begin{array}{rcl} \phi : M_{m,n} \times D \times B & \to & M_{m,n} \\ (A,t,x) & \mapsto & F_t(x) + A \end{array}$$

is obviously a submersion, we conclude there is nothing left to prove.

Consider now $A \in W$ and $t \in D \setminus \{0\}$. It is clear that $(A,0), (0,t) \in W_0$. Then,

$$\begin{array}{rcl} \nu(X,0) & = & (-1)^{\dim X}(\chi(X_A) - 1) \\ & = & (-1)^{\dim X}(\chi(X_{(A,0)}) - 1) \\ & = & (-1)^{\dim X}(\chi(X_{(0,t)}) - 1) \\ & = & (-1)^{\dim X}(\chi(X_t) - 1). \end{array}$$

□

The following corollary is a well known property for a 0-dimensional ICIS which gives the Milnor number in terms of the length of its local ring (see [20]).

Corollary 3.5. *Let $(X,0)$ be an IDS of dimension 0. Then,*

$$\nu(X,0) = \dim_{\mathbb{C}} \mathcal{O}_{X,0} - 1.$$

Proof. Let $H : (\mathbb{C}^N \times \mathbb{C}, 0) \to M_{m,n}$ be a determinantal smoothing of $(X,0)$. We denote by $(\mathcal{X},0) = H^{-1}(M^s_{m,n})$ the total space of the deformation. Then $\dim(\mathcal{X},0) = 1$ and $(\mathcal{X},0)$ is also an IDS and hence reduced. We consider the restriction of the projection onto the second factor $\pi_2 : (\mathcal{X},0) \to (\mathbb{C},0)$, which is a finite map germ whose generic fiber is $\pi_2^{-1}(t) = X_t$. The number of points of X_t, $\#X_t$, is the degree of π_2 which can be computed by Samuel's formula as

$$\deg(\pi_2) = e(\langle t \rangle, \mathcal{O}_{\mathcal{X},0}),$$

where $e(I,R)$ denotes the Hilbert-Samuel multiplicity of an ideal I in a local Noetherian ring R [23]. Moreover, since $(\mathcal{X},0)$ is Cohen-Macaulay we have that

$$e(\langle t\rangle,\mathcal{O}_{\mathcal{X},0})=\dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathcal{X},0}}{\langle t\rangle}=\dim_{\mathbb{C}}\mathcal{O}_{X,0}.$$

Finally, by theorem 3.4,

$$\nu(X,0)=\chi(X_t)-1=\#X_t-1=\dim_{\mathbb{C}}\mathcal{O}_{X,0}-1.$$

□

Buchweitz and Greuel defined in [4] the Milnor number of a reduced curve $(X,0)$ and showed that if $(X,0)$ is smoothable and $\Pi:\mathcal{X}\to D$ is a good representative of a smoothing of $(X,0)$ then, for $t\in D\setminus\{0\}$,

$$\mu(X,0)=1-\chi(X_t).$$

If $(X,0)$ is an IDS with dimension 1, that is, $(X,0)$ is a curve, then $(X,0)$ is smoothable and by theorem 3.4, for any determinantal smoothing of $(X,0)$,

$$\nu(X,0)=1-\chi(X_t).$$

Hence, we have the following corollary.

Corollary 3.6. *If $(X,0)$ is an IDS with dimension 1, the vanishing Euler characteristic of $(X,0)$ is equal to the Milnor number defined by Buchweitz and Greuel in [4].*

It is usual to define the Milnor number of a variety germ $(X,0)$ of dimension d as the dth Betti number $\beta_d(X_t)$ of the generic fibre X_t of a smoothing of $(X,0)$ (whenever this is independent of the smoothing), that is,

$$\mu(X,0):=\beta_d(X_t).$$

We remark that in general $\beta_d(X_t)$ is not equal to $\nu(X,0)$ for an IDS $(X,0)$ of dimension d, since the smoothing X_t may also present homology in other dimensions (see table 5 of [5]). However, if $(X,0)$ is an IDS of dimension 2,

$$\begin{aligned}\nu(X,0)&=\chi(X_t)-1\\&=\beta_0(X_t)-\beta_1(X_t)+\beta_2(X_t)-1\\&=-\beta_1(X_t)+\beta_2(X_t)\end{aligned}$$

Since $(X,0)$ is normal, by [14, Theorem 2], $\beta_1(X_t)=0$. Thus, we have the following consequence.

Corollary 3.7. *If $(X,0)$ is an IDS with dimension 2, the vanishing Euler characteristic of $(X,0)$ is equal to the Milnor number:*

$$\nu(X,0)=\beta_2(X_t)=\mu(X,0).$$

The above property has been proved independently by Pereira and Ruas in [26] in the case of a determinantal surface $(X,0)$ in $(\mathbb{C}^4,0)$ with isolated singularity.

3.1. **The vanishing Euler characteristic of the singular locus of a map germ.**

An important example of determinantal variety is the singular locus of a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, given by the points $x \in \mathbb{C}^n$ where the differential $Df(x)$ does not have maximal rank.

Assume for instance, that $n \leq p$ and $f = (f_1, ..., f_p)$. The singular locus of f is

$$S(f) = \{x \in \mathbb{C}^n : \operatorname{rank}(Df(x)) < n\} = Df^{-1}(M^n_{n,p}),$$

where $Df : (\mathbb{C}^n, 0) \to M_{n,p}$ is the jacobian matrix $Df(x) = (\frac{\partial f_i}{\partial x_j}(x))$, with $x = (x_1, ..., x_n)$. If f is $\mathcal{A}$-finitely determined, then $S(f)$ has the expected dimension $2n - p + 1$ and hence it is a determinantal singularity.

If, in addition, we assume $n < 2(p - n + 2)$ then we have condition (*) and $S(f)$ is an IDS. In fact, f is stable outside the origin by the Mather-Gaffney determinacy criterion [27] and hence, Df is transverse to the singular stratification Σ^i, $i = 1, \dots, n$. It makes sense to consider the vanishing Euler characteristic $\nu(S(f), 0)$. Let f_s be a stabilization of f (or more generally, a generic deformation of f), then $S(f_s)$ is smooth and it defines a determinantal smoothing of $S(f)$. By theorem 3.4,

$$\nu(S(f), 0) = (-1)^{2n-p-1}(\chi(S(f_s)) - 1).$$

Analogously, if $n > p$,

$$S(f) = \{x \in \mathbb{C}^n : \operatorname{rank}(Df(x)) < p\} = Df^{-1}(M^p_{n,p})$$

is a determinantal singularity of type $(n, p; p)$ if f is $\mathcal{A}$-finitely determined. Suppose that $n < 2(n - p + 2)$, we have condition (*) and $S(f)$ is an IDS. Then,

$$\nu(S(f), 0) = (-1)^{p-1}(\chi(S(f_s) - 1),$$

where f_s is a stabilization of f.

4. Relation with polar multiplicities and Euler obstruction

We suppose again throughout this section that $(X, 0) \subset (\mathbb{C}^N, 0)$ an IDS of type $(m, n; s)$ defined by $(X, 0) = F^{-1}(M^s_{m,n})$. We want to give a recursive formula in order to compute the vanishing Euler characteristic of $(X, 0)$.

We recall first the ICIS case. If $(X, 0) = V(\phi_1, ..., \phi_k)$ and $p : (\mathbb{C}^N, 0) \to \mathbb{C}$ is a function such that $(X \cap p^{-1}(0), 0)$ is again an ICIS, then we have the Lê-Greuel formula [3, 18]:

$$\mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_N}{\langle \phi_1, ..., \phi_k, J(p, \phi_1, ..., \phi_k)\rangle} - \mu(X \cap p^{-1}(0), 0).$$

The number $\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi_1, ..., \phi_k, J(p, \phi_1, ..., \phi_k)\rangle}$ can be interpreted as the Milnor number of the restriction $p|_X : (X, 0) \to \mathbb{C}$ (as pointed out by Nuño-Ballesteros and Tomazella in [24]). Here we show an analogue result for IDS. For this, we need the following lemma.

Lemma 4.1. *There exist a linear function $p : \mathbb{C}^N \to \mathbb{C}$ and a nonempty Zariski open subset $W \subset M_{m,n}$ such that X_A is smooth and $p|_{X_A}$ is a Morse function for all $A \in W$.*

Proof. We take a representative X in a small enough open ball B centered at the origin in $\mathbb{C}^N$ and such that X is smooth outside the origin. Let $A_0 \in M_{m,n}$ be a matrix such that X_{A_0} is smooth. Given $a = (a_1, ..., a_N) \in \mathbb{C}^N$ we denote by $p_a : \mathbb{C}^N \to \mathbb{C}$ the linear function $p_a(x_1, ..., x_N) = a_1x_1 + ... + a_Nx_N$. By lemma A.2, taking $f \equiv 0$, we can choose a point $a \in \mathbb{C}^N$ such that $p_a|_{X_{A_0}}$ and $p_a|_{X\setminus\{0\}}$ are both Morse functions.

Define $\tilde{C} \subset B \times M_{m,n}$ as the subset of pairs (x, A) such that either x is a singular point of X_A or x is a degenerate critical point of $p_a|_{X_A}$. Then $\tilde{C}$ is analytic and the restriction of the projection onto the second factor $\pi_2 : \tilde{C} \to M_{m,n}$ is a finite map. Thus, the image $C = \pi_2(\tilde{C})$ is also analytic and $W = M_{m,n} \setminus C$ is Zariski open. Since $A_0 \in W$, W is nonempty. □

Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function. Then $(X \cap p^{-1}(0), 0)$ is also an IDS of type $(m, n; s)$ in the hyperplane $p^{-1}(0)$. Since $p^{-1}(0)$ has dimension $N - 1$, the condition $(*)$ is also satisfied. Thus, it makes sense to consider the vanishing Euler characteristic $\nu(X \cap p^{-1}(0), 0)$.

Lemma 4.2. *Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function, $A \in M_{m,n}$ a generic matrix and $c \in \mathbb{C} \setminus \{0\}$ small enough. Then,*

$$\begin{aligned}\nu(X \cap p^{-1}(0), 0) &= (-1)^{d-1}(\chi(X \cap p^{-1}(c)) \\ &= (-1)^{d-1}(\chi(X_A \cap p^{-1}(0)) \\ &= (-1)^{d-1}(\chi(X_A \cap p^{-1}(c)),\end{aligned}$$

where $d = \dim(X, 0)$.

Proof. By taking appropriate linear coordinates, we may assume that $p(x_1, \dots, x_N) = x_N$. For each $c \in \mathbb{C}$, we denote

$$F_c(x_1, \dots, x_{N-1}) = F(x_1, \dots, x_{N-1}, c).$$

We have the following identifications:

1. $(X \cap p^{-1}(0), 0)$ is biholomorphic to the IDS $(Y, 0) \subset (\mathbb{C}^{N-1}, 0)$ given by $(Y, 0) = F_0^{-1}(M^s_{m,n})$.
2. $X_A \cap p^{-1}(0)$ is biholomorphic to $Y_A = (F_0 + A)^{-1}(M^s_{m,n})$, the special determinantal smoothing of $(Y, 0)$.
3. $X \cap p^{-1}(c)$ is biholomorphic to $Y_c = (F_c)^{-1}(M^s_{m,n})$, a determinantal smoothing of $(Y, 0)$.
4. $X_A \cap p^{-1}(c)$ is biholomorphic to $Y_{(A,c)} = (F_c + A)^{-1}(M^s_{m,n})$, the sum of the two deformations, as considered in the proof of theorem 3.4.

Then, the result is a direct consequence of theorem 3.4. □

Theorem 4.3. *Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function and $A \in M_{m,n}$ a generic matrix. Then,*

$$\#\Sigma(p|_{X_A}) = \nu(X, 0) + \nu(X \cap p^{-1}(0), 0),$$

where $\#\Sigma(p|_{X_A})$ denotes the number of critical points of $p|_{X_A}$.

Proof. We choose a generic matrix $A \in M_{m,n}$ such that X_A is smooth, $p|_{X_A}$ is a Morse function and let $c \in \mathbb{C}$ be a regular value of $p|_{X_A}$. By theorem A.5 and lemma 4.2,

$$\begin{aligned}\#\Sigma(p|_{X_A}) &= (-1)^{d+1}(\chi(p^{-1}(c) \cap X_A) - \chi(X_A)) \\ &= (-1)^{d+1}(\chi(p^{-1}(c) \cap X_A) - 1 + 1 - \chi(X_A)) \\ &= (-1)^{d+1}((-1)^{d-1}\mu(X \cap p^{-1}(0), 0) + (-1)^{d+1}\mu(X, 0)) \\ &= \nu(X, 0) + \nu(X \cap p^{-1}(0), 0).\end{aligned}$$

□

Perez and Saia showed in [16], that if $(X,0)$ is an ICIS of dimension d,

$$1+(-1)^d\mu(X,0)=\sum_{i=0}^{d}(-1)^i m_i(X,0).$$

Here $m_i(X,0)$, $0\leq i\leq d-1$, denotes the ith polar multiplicity of $(X,0)$ and $m_d(X,0)$ is the dth polar multiplicity as defined by Gaffney in [9] only for ICIS:

$$m_d(X,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_{X,0}}{I_{N-d+1}(J(f_1,\ldots,f_{N-d},p))},$$

where $(X,0)\subset(\mathbb{C}^N,0)$ is defined by the zeros of $(f_1,\ldots,f_{N-d})$ and $p:\mathbb{C}^N\to\mathbb{C}$ a generic linear projection. Moreover, this number is equal to $\#\Sigma p|_{X_t}$ with X_t the Milnor fibre of $(X,0)$. Again, we will show an analogous result for a general IDS.

Definition 4.4. Define the *dth polar multiplicity* of $(X,0)$ by

$$m_d(X,0):=\#\Sigma(p|_{X_A}),$$

where $p:\mathbb{C}^N\to\mathbb{C}$ is a generic linear map, $A\in M_{m,n}$ is a generic matrix and d is the dimension of $(X,0)$.

Then, by theorem 4.3,

$$m_d(X,0)=\nu(X,0)+\nu(X\cap p^{-1}(0),0).$$

Hence, $m_d(X,0)$ does not depend on the matrix A. We need to show that $m_d(X,0)$ does not depend on the chosen linear function p. By Lê and Teissier ([19]), we have

$$\mathrm{Eu}(X,0)=\chi(X\cap p^{-1}(t))=(-1)^{d-1}\nu(X\cap p^{-1}(0),0)+1,$$

where $\mathrm{Eu}(X,0)$ is the local Euler obstruction. Then,

$$m_d(X,0)=\nu(X,0)+(-1)^{d-1}\,\mathrm{Eu}(X,0)+1,$$

is also well defined. Moreover, we have the following result.

Theorem 4.5. *Let $(X,0)$ be an IDS of dimension d. Then,*

$$\mathrm{Eu}(X,0)+(-1)^d m_d(X,0)=1+(-1)^d\nu(X,0).$$

Example 4.6. Consider again $(X,0)$ the determinantal surface in $(\mathbb{C}^4,0)$ defined by the 2×2-minors of the matrix

$$F(x,y,z,w)=\begin{pmatrix} x & y & z\\ y & z & w\end{pmatrix}.$$

Take

$$A=\frac{1}{100}\begin{pmatrix} 6 & -8 & 5\\ 1 & 8 & 7\end{pmatrix}$$

and $p=2x-3y+4z-w$. Making computations with Mathematica, we see that

$$m_2(X,0)=\#\Sigma(p|_{X_A})=3.$$

Since $Y=X\cap p^{-1}(0)$ is a curve, following [24] we can compute its Milnor number by means of the Milnor number $\mu(g|_Y)$ and the local degree $\deg(g|_Y)$ of a function $g:\mathbb{C}^4\to\mathbb{C}$ on the curve Y: $\mu(g|_Y)=\mu(Y,0)+\deg(g|_Y)-1$. Taking $g=x-2y+5z+5w$, we have that

$$\mu(g|_Y)=\#\Sigma(g|_{X_A\cap p^{-1}(0)})=4,\quad \deg(g|_Y)=\#\{X_A\cap p^{-1}(0)\cap g^{-1}(0)\}=3.$$

Therefore, $\nu(Y,0)=\mu(Y,0)=4-3+1=2$. Hence,

$$\mathrm{Eu}(X,0)=1-\nu(Y,0)=-1$$

and

$$\nu(X,0)=m_2(X,0)-\nu(Y,0)=3-2=1.$$

Example 4.7. Consider again $(X,0)$ the determinantal surface in $(\mathbb{C}^5,0)$ defined by the 2×2-minors of the matrix

$$F(x,y,z,w,v)=\begin{pmatrix} x & y & z & w \\ y & z & w & v \end{pmatrix}.$$

We follow the same procedure of example 4.6. Take

$$A=\frac{1}{100}\begin{pmatrix} -1 & 9 & -7 & -4 \\ -4 & 7 & -8 & -5 \end{pmatrix}$$

and $p=4x+6y-5z+8w-8v$. Making computations with Mathematica, we see that

$$m_2(X,0)=\#\Sigma(p|_{X_A})=4.$$

We choose $g=x+2y-5z+5w-8v$, then

$$\begin{aligned} \mu(g|_Y) &= \#\Sigma(g|_{X_A\cap p^{-1}(0)})=6, \\ \deg(g|_Y) &= \#\{X_A\cap p^{-1}(0)\cap g^{-1}(0)\}=4, \\ \nu(Y,0) &= \mu(Y,0)=6-4+1=3, \\ \mathrm{Eu}(X,0) &= 1-\nu(Y,0)=-2, \\ \nu(X,0) &= m_2(X,0)-\nu(Y,0)=4-3=1. \end{aligned}$$

5. The Milnor number of a function on an IDS

Let $(X,0)\subset(\mathbb{C}^N,0)$ be an IDS and let $f:(\mathbb{C}^N,0)\to\mathbb{C}$ be a holomorphic function germ such that $f|_X:(X,0)\to\mathbb{C}$ has isolated singularity. In this section we define the Milnor number of f with respect to X. To do this, we need again a genericity result.

Assume that $(X,0)$ has type $(m,n;s)$ and is defined by $(X,0)=F^{-1}(M^s_{m,n})$. Given $(a,A)\in\mathbb{C}^N\times M_{m,n}$, we consider the function $f_a|_{X_A}:X_A\to\mathbb{C}$ defined by

$$f_a(x_1,\dots,x_N)=f(x_1,\dots,x_N)+a_1x_1+\dots a_Nx_N,$$

and $X_A=(F+A)^{-1}(M^s_{m,n})$.

Lemma 5.1. *There exists a nonempty Zariski open subset $W\subset\mathbb{C}^N\times M_{m,n}$ such that for all $(a,A)\in W$, X_A is smooth and $f_a|_{X_A}$ is a Morse function. Moreover, the number of critical points of $f_a|_{X_A}$ is independent of $(a,A)\in W$.*

Proof. The proof of the first part of the lemma will be omitted. The arguments are similar to those used in the proof of lemma 4.1.

To see the second part, we consider the set germ $(C,0)$ in $(\mathbb{C}^{2N+mn},0)$ consisting of triples (x,a,A) such that either x is a singular point of X_A or x is a critical point of $f_a|_{X_A}$. Then $(C,0)$ is analytic and the germ of the projection $\pi:(C,0)\to(\mathbb{C}^{N+mn},0)$ given by $\pi(x,a,A)=(a,A)$ is a finite map germ. We have that the number of critical points of $f_a|_{X_A}$ is equal to the local degree of π which is independent of (a,A). □

By lemma 5.1, we can make the following definition.

Definition 5.2. We define the *Milnor number* of $f|_X : (X,0) \to \mathbb{C}$ as
$$\mu(f|_X) = \#\Sigma\left(f_a|_{X_A}\right),$$
with $(a, A) \in W$.

Definition 5.3. We define the *vanishing Euler characteristic* of the fibre $(X \cap f^{-1}(0), 0)$ by
$$\nu(X \cap f^{-1}(0), 0) = (-1)^{\dim X - 1}(\chi(X_A \cap f_a^{-1}(c)) - 1),$$
where $(a, A, c) \in \mathbb{C}^N \times M_{m,n} \times \mathbb{C}$ are generic values such that X_A is smooth, $f_a|_{X_A}$ is a Morse function and c is a regular value of $f_a|_{X_A}$.

We need to show that this number is well defined, that is, $\chi(X_A \cap f_a^{-1}(c))$ does not depend on the choices of a, A and c. To see this, let $\epsilon > 0$ such that $f_a|_{X_A}$ has not critical points on $X_A \cap S_\epsilon$ for all matrix A in a neighborhood small enough of the origin in $M_{m,n}$. By theorem A.5, being $M = X_A \cap B(0, \epsilon)$, if c is a regular value of $f_a|_{X_A}$, we have:
$$\begin{aligned}
\#\Sigma\left(f_a|_{X_A}\right) &= (-1)^{\dim X}(\chi(X_A) - \chi(X_A \cap f_a^{-1}(c))) \\
&= (-1)^{\dim X}(\chi(X_A) - 1 + 1 - \chi(X_A \cap f_a^{-1}(c))) \\
&= (-1)^{\dim X}(\chi(X_A) - 1) + (-1)^{\dim X - 1}(\chi(X_A \cap f_a^{-1}(c)) - 1) \\
&= \nu(X, 0) + \nu(X \cap f^{-1}(0), 0).
\end{aligned}$$
Then, $\nu(X \cap f^{-1}(0), 0)$ is well defined and we have a Lê-Greuel type formula.

Theorem 5.4. *Given a function $f|_X : (X,0) \to \mathbb{C}$ with isolated singularity on an IDS $(X, 0)$, we have:*
$$\mu(f|_X) = \nu(X, 0) + \nu(X \cap f^{-1}(0), 0).$$

Remark 5.5.
(1) If $(X, 0)$ is an IDS of dimension two, $X \cap f^{-1}(0)$ is a space curve and $X_A \cap f_a^{-1}(c)$ defines a smoothing of it. Thus,
$$\nu(X \cap f^{-1}(0), 0) = 1 - \chi(X_A \cap f_a^{-1}(c)) = \mu(X \cap f^{-1}(0), 0),$$
the Milnor number of the curve as defined by Buchweitz and Greuel in [4].
(2) If 0 is a regular point of $f : (\mathbb{C}^N, 0) \to \mathbb{C}$, we can consider $X \cap f^{-1}(0)$ as a determinantal variety in the smooth manifold $f^{-1}(0)$. In this case, our definition of vanishing Euler characteristic coincides with the definition of vanishing Euler characteristic of an IDS.

Appendix A. Morse Theory

In this appendix, we will present the results about Morse Theory we have cited throughout the paper. Most of them are already known, in these cases we omit the proof and indicate some reference.

The following theorem, whose proof can be found in [11, Theorem 2.2.3], will be useful to ensure that a holomorphic function with an isolated singularity can be deformed to a Morse function.

Theorem A.1. *Let Z be a closed subanalytic subset of an analytic manifold M. Let P be a finite dimensional smooth manifold and let $F : P \times M \to \mathbb{R}$ be a smooth function. For each $\alpha \in P$, we denote $f_\alpha(x) = F(\alpha, x)$. Define $\phi : P \times M \to T^*M$ by $\phi(\alpha, x) = df_\alpha(x)$. If ϕ is a submersion, then for almost any $\alpha \in P$, $f_\alpha|_Z$ is a Morse function on Z.*

Lemma A.2. *Let $V \subset \mathbb{C}^n$ be a smooth subanalytic subset and let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Denote $f_a(x_1 ..., x_n) := f(x_1, ..., x_n) + a_1 x_1 + ... + a_n x_n$ if $a = (a_1 ..., a_n) \in \mathbb{C}^n$. Then $f_a|_V$ is a Morse function for almost all point $a \in \mathbb{C}^n$.*

Proof. Define $F : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by $F(a, x) = f_a(x)$. Then, following the notation of lemma A.1, $\phi : \mathbb{C}^n \times \mathbb{C}^n \to T^*\mathbb{C}^n \equiv \mathbb{C}^{2n}$ is given by $\phi(a, x) = (x, \frac{\partial f}{\partial x_1} + a_1, ..., \frac{\partial f}{\partial x_n} + a_n)$. It is obvious that ϕ is a submersion. Then, by theorem A.1, $f_a|_V$ is a Morse function for almost all $a \in \mathbb{C}^n$. □

A.1. **Generalized Morse Theory.** In [25], Palais and Smale developed a generalization of Morse theory, which we will describe now.

Let M be a C^2 complete Riemannian manifold without boundary modeled on a separable Hilbert space and let $f : M \to \mathbb{R}$ be a C^2 function. Palais and Smale considered pairs (M, f) as above which satisfy at least the following extra condition:

(C) If S is a subset of M on which $|f|$ is bounded but on which $||\nabla(f)||$ is not bounded away from zero, then there is a critical point of f in the closure of S.

Palais and Smale showed the following theorem.

Theorem A.3. *Let (M, f) satisfy condition (C) and assume that all critical points of f are nondegenerate. Then*

1. *For any real numbers $a < b$ there are only finitely many critical points of f in $f^{-1}([a, b])$, hence the critical values of f are isolated.*
2. *Let a and b be regular values of f and suppose that among the critical points of f in $f^{-1}([a, b])$ there are r having finite index. Let the indices of these critical points be $d_1, \dots, d_r$. Then $f^{-1}((-\infty, b])$ has the homotopy type of $f^{-1}((-\infty, a])$ with r cells of dimensions $d_1, \dots, d_r$ attached.*

We are always working with germs of function and of varieties, the following theorem ensures that we can find a representative which satisfies the condition (C).

Theorem A.4. *Let $X \subset \mathbb{R}^n$ be a smooth manifold and let $M = X \cap B(0, \epsilon)$ for some $\epsilon > 0$. Let $f : X \to \mathbb{R}$ be a Morse function without critical points on $X \cap S_\epsilon$. Then (M, f) satisfies (C).*

Proof. First, we observe that, by the Whitney embedding theorem, we can consider M embedded in $\mathbb{R}^{2 \dim M + 1}$ by a proper embedding ϕ. Then, $\phi(M)$ is closed an hence complete. By considering M with the metric induced by ϕ, we have that M is also complete.

Let (x_i) be a sequence in M on which $|f|$ is bounded and $||\nabla f(x_i)|| \to 0$. We can see (x_i) as a sequence in $\overline{M}$, which is compact. Then, there exists a subsequence (x_{i_k}) of (x_i) which converges to some x_0 in $\overline{M}$.

Since f is smooth, $||\nabla f(.)||$ is a continuous function and hence, $||\nabla f(x_0)|| = 0$. Therefore, x_0 is a critical point of f. From the hypothesis, $x_0 \notin X \cap S_\epsilon = \partial M$. Then $x_0 \in M$, i.e., there exist a critical point of f in the closure of $\{x_1, x_2, ...\}$. Hence, (M, f) satisfies (C). □

In [17], Kaveh showed a theorem which relates the Euler characteristic of an algebraic variety and of the fibers of a generic linear function on it to the number of critical points of this function. Inspired by this result, we show the following theorem.

Theorem A.5. *Let $X \subset \mathbb{C}^n$ be a complex analytical manifold of dimension d and let $M = X \cap B(0, \epsilon)$ for some $\epsilon > 0$. Let $f : X \to \mathbb{C}$ be a holomorphic Morse function* *without critical points on* $X \cap S_\epsilon$. *Then,*

$$\chi(f^{-1}(c) \cap M) = \chi(M) + (-1)^{d+1} \# \Sigma f_{|_M},$$

where c *is a regular value of* $f|_M$ *and* $\#\Sigma f|_M$ *is the number of critical points of* $f|_M$.

To show this result, we need the following lemma.

Lemma A.6. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ. We write* $f = f_1 + i f_2$, *where* $f_1 = \Re(f)$ *is the real part of* f *and* $f_2 = \Im(f)$ *is its imaginary part.*

1. f *has a critical point at the origin if and only if* f_1 *has a critical point at the origin.*
2. *If* f *has a critical point at the origin and the hessian matrix of* f *has rank* r *then the hessian matrix of* f_1 *has rank* $2r$ *and index* r.
3. f *has a nondegenerate critical point at the origin if and only if* f_1 *has a nondegenerate critical point at the origin (with index* n*).*

Proof. We write f as a power series:

$$f(z) = \sum_{j=1}^{n} a_j z_j + \sum_{j,k=1}^{n} a_{jk} z_j z_k + \dots$$

where $a_j, a_{jk} \in \mathbb{C}$ and the dots denote terms of higher degree. If 0 is a regular point of f, after a linear change of coordinates, we can assume that f is of the form:

$$f(z) = z_1 + \sum_{j,k=1}^{n} a_{jk} z_j z_k + \dots$$

If we make $z_j = x_j + i y_j$, we get:

$$f(z) = x_1 + i y_1 + \sum_{j,k=1}^{n} a_{jk}(x_j + i y_j)(x_k + i y_k) + \dots$$

Then,

$$f_1(z) = x_1 + \sum_{j,k=1}^{n} b_{jk} x_j y_j + \dots$$

for some $b_{jk} \in \mathbb{R}$. So, 0 is a regular point of f_1.

Conversely, if 0 is a critical point of f, we have

$$f(z) = \sum_{j,k=1}^{n} a_{jk} z_j z_k + \dots$$

therefore

$$f_1(z) = \sum_{j,k=1}^{n} b_{jk} x_j y_j + \dots$$

and 0 is a critical point of f_1. This shows (1).

Let us see (2). Assume now that 0 is a critical point of f. By the linear classification of quadratics forms, after a linear change of coordinates, we can suppose

$$f(z) = z_1^2 + \dots + z_r^2 + \dots$$

where r is the rank of the hessian matrix. Making $z_j = x_j + i y_j$, we have

$$f(z) = x_1^2 - y_1^2 + \dots + x_r^2 - y_r^2 + 2i(x_1 y_1 + \dots + x_r y_r) + \dots$$

Therefore,

$$f_1(z) = x_1^2 - y_1^2 + \cdots + x_r^2 - y_r^2 + \ldots$$

Then the hessian matrix of f_1 has rank $2r$ and index r.

Finally, part (3) is immediate from (2). □

Proof of theorem A.5. We consider M as real manifold of dimension $2d$. By the previous lemma, f_1 has not critical points on $X \cap S_\epsilon$ and $f_1 : X \to \mathbb{R}$ is a Morse function. Then, (M, f_1) is a pair which satisfies the hypothesis of theorem A.4.

Let $a < b$ be regular values of $f_1|_M$ such that b is bigger than the maximum of f on $\overline{M}$ and the set of critical values of $f_1|_M$ is inside of (a, b). Such a and b exist because the set of critical values of $f_1|_M$ is finite. In fact, lets suppose that f_1 has an infinite number of critical values. Then f_1 has an infinite number of critical points on M. Let (x_i) be a sequence of critical points in M. Particularly, (x_i) is a sequence in $\overline{M}$, which is compact. Then, there exists a subsequence (x_{i_k}) of (x_i) convergent. By the curve selection lemma, there exists a curve of critical points of f_1 in M, therefore the set of critical points of $f_1|_M$ is not discrete. This contradicts the fact that $f_1|_M$ is a Morse function.

Then, from theorem A.3 we have

$$\begin{aligned} M = f_1|_M^{-1}(-\infty, \infty) &= f_1|_M^{-1}(-\infty, b] \\ &= f_1^{-1}(-\infty, a] \text{ with } \#\Sigma f_1|_M \text{ cells of dimension } d \text{ attached.} \end{aligned}$$

Hence,

$$\chi(M) = \chi(f_1|_M^{-1}(-\infty, a]) + (-1)^d \#\Sigma f_1|_M.$$

We observe that $M \cap f_1^{-1}(-\infty, a]$ is a locally closed subset of $\mathbb{C}^n$, $f|_{M\cap f_1^{-1}(-\infty,a]}$ is a submersion and $f|_{\overline{M}\cap f_1^{-1}(-\infty,a]}$ is a proper application, then by the Thom first isotopy lemma, $f|_M$ is a fibration on $f_1^{-1}(-\infty, a]$. Therefore,

$$\chi(f_1|_M^{-1}(-\infty, a]) = \chi(f|_M^{-1}(a))\chi(\mathbb{C}) = \chi(f|_M^{-1}(a)).$$

Since $f|_M$ is a fibration on the subset of $\mathbb{C}$ of its regular values, the homotopy type of $f|_M^{-1}(c)$ is independent of the regular value c of f. Thus,

$$\chi(M) = \chi(f|_M^{-1}(c)) + (-1)^d \#\Sigma f_1|_M,$$

for any regular value c of f. Then,

$$\chi(f^{-1}(c) \cap M) = \chi(M) + (-1)^{d+1} \#\Sigma f|_M.$$

□

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